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On self-similar measures with absolutely continuous projections and dimension conservation in each direction

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Abstract. Relying on results due to Shmerkin and Solomyak, we show that outside a zero-dimensional set of parameters, for every planar homogeneous self-similar measure ν, with strong separation, dense rotations and dimension greater than 1, there exists $q > 1$ such that $\{P_z\nu\}_{z\in S} \subset L^q(\mathbb{R})$. Here S is the unit circle and $P_z w = \langle z, w\rangle$ for $w \in \mathbb{R}^2$. We then study such measures. For instance, we show that ν is dimension conserving in each direction and that the map $z \to P_z\nu$ is continuous with respect to the weak topology of $L^q(\mathbb{R})$.

Key words: self-similar sets and measures, absolute continuity, dimension conservation
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1. *Introduction*

Denote by $\mathbb{D}$ the open unit disc in $\mathbb{C}$ and let $0 \neq \lambda \in \mathbb{D}$ be such that $\arg\lambda \notin \pi\mathbb{Q}$. Consider a homogeneous iterated function system (IFS) on $\mathbb{C}$,

$$\{\varphi_i(w) = \lambda w + a_i\}_{i\in\Lambda}, \tag{1.1}$$

with the strong separation condition (SSC), and a self-similar measure

$$\nu = \sum_{i\in\Lambda} p_i \cdot \varphi_i\nu. \tag{1.2}$$

This is among the most basic planar self-similar measures. Hence it is a natural question in fractal geometry to study the dimension and continuity of the projections $\{P_z\nu\}_{z\in S}$ and slices

$$\{\{\nu_{z,w}\}_{w\in\mathbb{C}} : z \in S\}.$$

Here S is the unit circle of $\mathbb{C}$, $P_z w = \mathrm{Re}(z \cdot \overline{w})$ for $w \in \mathbb{C}$, and $\{\nu_{z,w}\}_{w\in\mathbb{C}}$ is the disintegration of ν with respect to $P_z^{-1}(\mathcal{B})$, where $\mathcal{B}$ is the Borel σ-algebra.

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Dimensionwise, the behaviour of the projections is as regular as possible. Indeed, Hochman and Shmerkin [**HS**] have proven that $P_z\nu$ is exact dimensional, with

$$\dim P_z\nu = \min\{1, \dim \nu\},$$

for each $z \in S$. A version of this, for self-similar sets with dense rotations, was first proven by Peres and Shmerkin [**PS**]. Considering the absolute continuity of the projections, Shmerkin and Solomyak [**SS1**] have shown, assuming $\dim \nu > 1$, that the set

$$E = \{z \in S : P_z\nu \text{ is singular}\}$$

has zero Hausdorff dimension.

Let us now discuss the concept of dimension conservation and the dimension of slices. A Borel probability measure μ on $\mathbb{C}$ is said to be dimension conserving (DC), with respect to the projection P_z, if

$$\dim_H \mu = \dim_H P_z\mu + \dim_H \mu_{z,w} \quad \text{for } \mu\text{-a.e. } w \in \mathbb{C},$$

where $\dim_H$ stands for Hausdorff dimension. The measure μ is always DC with respect to P_z for almost every (a.e.) $z \in S$. This follows from results, valid for general measures, regarding the typical dimension of projections (see [**HK**]) and slices (see [**JM**]). Falconer and Jin [**FJ1**] have shown that μ is DC, with respect to P_z for all $z \in S$, whenever μ is self-similar with a finite rotation group. An analogous statement, for self-similar sets with the SSC, was first proven by Furstenberg [**Fur**]. Another related result for sets is due to Falconer and Jin [**FJ2**]. They showed that if $K \subset \mathbb{C}$ is self-similar, with $\dim K > 1$ and a dense rotation group, then for every $\epsilon > 0$ there exists $N_\epsilon \subset S$, with $\dim_H N_\epsilon = 0$, such that for $z \in S \setminus N_\epsilon$ the set

$$\{x \in \mathbb{R} : \dim_H(K \cap P_z^{-1}\{x\}) > \dim K - 1 - \epsilon\}$$

has positive length.

Taking these results into account, one might expect the sets E, defined above, and

$$F = \{z \in S : \nu \text{ is not DC with respect to } P_z\}$$

to be empty whenever the dimension of ν exceeds 1. Nevertheless, in [**Rap2**] the author has constructed an example, of a measure ν as above, for which E and F are non-empty and even residual. Moreover, in this example there exists a G_δ-subset of directions $z \in S$ for which $\nu_{z,w}$ is discrete, and hence has dimension 0, for ν-a.e. $w \in \mathbb{C}$.

In this paper we shall show that, despite the last result, it is typically the case that E and F are empty. More precisely, we shall prove the following theorem. A sharper version of it is stated in §3.

THEOREM. *There exists a set $\mathcal{E} \subset \mathbb{D}$, with $\dim_H \mathcal{E} = 0$, such that the following holds. Let $\lambda \in \mathbb{D} \setminus \mathcal{E}$ be such that $\arg \lambda \notin \pi\mathbb{Q}$, let ν be a self-similar measure with the SSC as in (1.2), and assume that $\dim_H \nu > 1$. Then there exists $1 < q < \infty$ such that:*

(1) *$P_z\nu \in L^q(\mathbb{R})$ for each $z \in S$;*
(2) *the map which takes $z \in S$ to $P_z\nu \in L^q(\mathbb{R})$ is continuous with respect to the weak topology of $L^q(\mathbb{R})$;*
(3) *for each $z \in S$ the measure $\nu_{z,w}$ has exact dimension $\dim_H \nu - 1$ for ν-a.e. $w \in \mathbb{C}$.*

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Note that by parts (1) and (3) it follows that ν is DC, with respect to P_z, for all $z \in S$.

Part (1) follows almost directly from results which are due to Shmerkin and Solomyak [**SS1**] and [**SS2**] and Shmerkin [**Sh1**] and [**Sh2**]. Our main contribution is the derivation of parts (2) and (3). For this we first show that, assuming part (1), the collection $\{P_z\nu\}_{z\in S}$ is bounded in $(L^q(\mathbb{R}), \|\cdot\|_q)$. The reflexivity of $L^q(\mathbb{R})$, for $1 < q < \infty$, plays an important role here.

We obtain a much stronger form of continuity in (2) if instead of $\dim_H \nu > 1$ we assume that the correlation dimension of ν exceeds 1. More precisely, in Theorem 3.4 it is shown that, under this stronger assumption, there exists $\gamma > 0$ such that $P_z\nu$ lies in the 2, γ-Sobolev space for each $z \in S$ and that the map $z \to P_z\nu$ is continuous with respect to the corresponding Sobolev norm.

The boundedness of $\{P_z\nu\}_{z\in S}$ in $L^q(\mathbb{R})$ gives also a result regarding the measure class of the projections. Denote by K the self-similar set corresponding to the IFS (1.1). By extending an argument given in [**MS**] and [**PSS**], we show that, under the assumptions of the theorem above, the measures $P_z\nu$ and $\mathcal{L}|_{P_z(K)}$ are equivalent for all $z \in S$. Here $\mathcal{L}$ is the Lebesgue measure on $\mathbb{R}$.

Finally, our results also apply to self-similar sets. We recall the following definition due to Furstenberg [**Fur**]. A subset $A \subset \mathbb{C}$ is said to be DC, with respect to the projection P_z, if for some $\delta \geq 0$,

$$\delta + \dim_H\{x \in \mathbb{R} : \dim_H(A \cap P_z^{-1}\{x\}) \geq \delta\} \geq \dim_H A,$$

where the dimension of the empty set is defined to be $-\infty$. From our results on measures we shall obtain the following corollary. A sharper version of it is given in §3. Note its close connection with the previously mentioned result from [**FJ2**].

COROLLARY. *Let $\mathcal{E} \subset \mathbb{D}$ be the zero-dimensional set from the theorem above. Let $\lambda \in \mathbb{D} \setminus \mathcal{E}$ be such that* $\arg\lambda \notin \pi\mathbb{Q}$, *let K be a self-similar set with the SSC corresponding to an IFS as in (1.1), and assume that $s = \dim_H K > 1$. Then there exists $c > 0$ such that for every $z \in S$,*

$$\mathcal{L}\{x \in \mathbb{R} : \dim_H(K \cap P_z^{-1}\{x\}) = s - 1\} > c.$$

In particular, K is DC with respect to P_z for all $z \in S$.

The rest of this paper is organized as follows. In §2 we give the necessary definitions. In §3 we state our results. In §4 we show that the projections $P_z\nu$ all belong to the appropriate function space, and establish the continuity of the map $z \to P_z\nu$. In §5 we prove the result regarding the slices of measures. In §6 we establish our results for self-similar sets. Finally, in §7 we prove the statement regarding the measure class of the projections.

2. *Preliminaries*

2.1. *Self-similar sets and measures.* Write $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and $S = \{z \in \mathbb{C} : |z| = 1\}$. Given a metric space X, which will always be $\mathbb{C}$ or $\mathbb{R}$, denote by $\mathcal{M}(X)$ the collection

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of all compactly supported Borel probability measures on X. For a finite index set Λ write

$$\mathbb{P}_\Lambda = \left\{(p_i)_{i\in\Lambda} \in (0,\,1)^\Lambda : \sum_{i\in\Lambda} p_i = 1\right\}.$$

Given $p \in \mathbb{P}_\Lambda$, $(a_i)_{i\in\Lambda} = \mathbf{a} \in \mathbb{C}^\Lambda$ and $\lambda \in \mathbb{D}$, let $\nu^p_{\lambda,\mathbf{a}} \in \mathcal{M}(\mathbb{C})$ be the self-similar measure corresponding to the IFS

$$\mathcal{F}_{\lambda,\mathbf{a}} = \{f^i_{\lambda,\mathbf{a}}(z) = \lambda z + a_i\}_{i\in\Lambda} \tag{2.1}$$

and p, that is, $\nu^p_{\lambda,\mathbf{a}}$ is the unique member of $\mathcal{M}(\mathbb{C})$ such that

$$\nu^p_{\lambda,\mathbf{a}} = \sum_{i\in\Lambda} p_i \cdot f^i_{\lambda,\mathbf{a}}\nu^p_{\lambda,\mathbf{a}}.$$

Denote by $K_{\lambda,\mathbf{a}} \subset \mathbb{C}$ the attractor of $\mathcal{F}_{\lambda,\mathbf{a}}$, that is, $K_{\lambda,\mathbf{a}}$ is the unique non-empty compact subset of $\mathbb{C}$ with

$$K_{\lambda,\mathbf{a}} = \bigcup_{i\in\Lambda} f^i_{\lambda,\mathbf{a}}(K_{\lambda,\mathbf{a}}). \tag{2.2}$$

Since $p > 0$, we have that $K_{\lambda,\mathbf{a}} = \operatorname{supp}(\nu^p_{\lambda,\mathbf{a}})$. We say that the SSC holds if the union in (2.2) is disjoint.

2.2. *Projections and disintegrations.* For $z,\, w \in \mathbb{C}$ write $\langle z,\, w\rangle = \operatorname{Re}(z \cdot \overline{w})$; then $\langle\cdot,\,\cdot\rangle$ is the standard inner product on $\mathbb{C}$ when it is identified with $\mathbb{R}^2$. For $z \in S$ let $P_z : \mathbb{C} \to \mathbb{R}$ be given by $P_z w = \langle z,\, w\rangle$ for $w \in \mathbb{C}$. Note that, given $\mu \in \mathcal{M}(\mathbb{C})$, the measure $P_z\mu$ is, up to affine equivalence, the pushforward of μ by the orthogonal projection onto the line $z \cdot \mathbb{R}$.

Denote by $\mathcal{B}$ the Borel σ-algebra of $\mathbb{R}$ or $\mathbb{C}$. For $\mu \in \mathcal{M}(\mathbb{C})$ and $z \in S$ let $\{\mu_{z,w}\}_{w\in\mathbb{C}} \subset \mathcal{M}(\mathbb{C})$ be the disintegration of μ with respect to $P_z^{-1}(\mathcal{B})$, as defined in [**EW**, Theorem 5.14]. This means that $\mu_{z,w}$ is supported on $P_z^{-1}(P_z w)$ for μ-a.e. $w \in \mathbb{C}$, and that for each bounded $\mathcal{B}$-measurable $f : \mathbb{C} \to \mathbb{R}$,

$$\int f\, d\mu_{z,w} = E_\mu(f \mid P_z^{-1}(\mathcal{B}))(w) \quad \text{for } \mu\text{-a.e. } w \in \mathbb{C}.$$

In the last equality, the right-hand side is the conditional expectation of f given $P_z^{-1}(\mathcal{B})$ with respect to μ.

2.3. *Dimension of measures.* Let X be $\mathbb{C}$ or $\mathbb{R}$. For $E \subset X$ denote by $\dim_H E$ the Hausdorff dimension of E. Given $\mu \in \mathcal{M}(X)$, write $\dim_H \mu$ for the Hausdorff dimension of μ, which is defined by

$$\dim_H \mu = \inf\{\dim_H E : E \subset X \text{ is a Borel set with } \mu(E) > 0\}.$$

For $x \in X$ and $\delta > 0$ let $B(x,\,\delta)$ be the closed ball in X with centre x and radius δ. It is said that μ has exact dimension $s \geq 0$ if

$$\lim_{\delta\downarrow 0} \frac{\log \mu(B(x,\,\delta))}{\log \delta} = s \quad \text{for } \mu\text{-a.e. } x \in X.$$

It is well known that in this case $s = \dim_H \mu$ (see [**Fal2**, Ch. 10]).

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For $1 < q < \infty$ denote by $D_q(\mu)$ the lower L^q dimension of μ, which is defined by

$$D_q(\mu) = \liminf_{\delta \downarrow 0} \frac{\log \int (\mu(B(x, \delta)))^{q-1} \, d\mu(x)}{(q-1) \log \delta}.$$

It is always true that $\dim_H \mu \geq D_q(\mu)$ and $D_{q'}(\mu) \geq D_q(\mu)$ for all $1 < q' < q$. A proof of these standard facts can be found in [**FLR**]. The number $D_2(\mu)$ is called the lower correlation dimension of μ.

For $p \in \mathbb{P}_\Lambda$ write $\|p\|_q^q = \sum_{i \in \Lambda} p_i^q$. It is not hard to verify that for $p \in \mathbb{P}_\Lambda$, $\mathbf{a} \in \mathbb{C}^\Lambda$ and $\lambda \in \mathbb{D}$, such that the SSC is satisfied,

$$D_q(\nu_{\lambda,\mathbf{a}}^p) = \frac{\log \|p\|_q^q}{(q-1) \log |\lambda|}, \tag{2.3}$$

$$\dim_H \nu_{\lambda,\mathbf{a}}^p = \frac{\sum_{i \in \Lambda} p_i \log p_i}{\log |\lambda|}, \tag{2.4}$$

and

$$\dim_H K_{\lambda,\mathbf{a}} = \frac{\log |\Lambda|}{-\log |\lambda|}. \tag{2.5}$$

2.4. *Function spaces.* Let $\mathcal{L}$ be the Lebesgue measure on $\mathbb{R}$. For $1 \leq p \leq \infty$ denote by $\|\cdot\|_p$ the L^p-norm on $L^p(\mathbb{R})$. Given $g \in L^2(\mathbb{R})$, write $\widehat{g}$ for the Fourier transform of g, which is again a member of $L^2(\mathbb{R})$.

For $\gamma \in [0, \infty)$ denote by $H_\gamma(\mathbb{R})$ the 2, γ-Sobolev space of $\mathbb{R}$, that is,

$$H_\gamma(\mathbb{R}) = \left\{ g \in L^2(\mathbb{R}) : \int |\widehat{g}(\xi)|^2 (1 + |\xi|^2)^\gamma \, d\xi < \infty \right\}.$$

For $g \in H_\gamma(\mathbb{R})$ write

$$\|g\|_{(\gamma)} = \left(\int |\widehat{g}(\xi)|^2 (1 + |\xi|^2)^\gamma \, d\xi \right)^{1/2}.$$

Note that $(H_\gamma(\mathbb{R}), \|\cdot\|_{(\gamma)})$ is a Hilbert space, $H_\gamma(\mathbb{R}) \subset H_{\gamma'}(\mathbb{R})$ and $\|g\|_{(\gamma')} \leq \|g\|_{(\gamma)}$ for each $0 \leq \gamma' \leq \gamma$, and $H_0(\mathbb{R}) = L^2(\mathbb{R})$.

Given $\mu \in \mathcal{M}(\mathbb{R})$ and a Banach space B of functions on $\mathbb{R}$, we write $\mu \in B$ when μ is absolutely continuous and its density belongs to B.

2.5. *The exceptional set of parameters.* All of our results will be valid for self-similar measures $\nu_{\lambda,\mathbf{a}}^p$ and sets $K_{\lambda,\mathbf{a}}$ for which $\lambda \in \mathbb{D}$ lies outside a zero-dimensional set $\mathcal{E}$. We shall now define this exceptional set of parameters.

For X which is $\mathbb{R}$ or $\mathbb{C}$ and $\mu \in \mathcal{M}(X)$ write $\widehat{\mu}$ for the Fourier transform of μ, that is, if $X = \mathbb{R}$,

$$\widehat{\mu}(\xi) = \int e^{i\xi x} \, d\mu(x) \quad \text{for } \xi \in \mathbb{R},$$

and if $X = \mathbb{C}$,

$$\widehat{\mu}(\xi) = \int e^{i\langle \xi, w \rangle} \, d\mu(w) \quad \text{for } \xi \in \mathbb{C}.$$

By [**SS2**, Theorem D] there exists a set $\mathcal{E}' \subset \mathbb{D}$, with $\dim_H \mathcal{E}' = 0$, such that the following holds. Let $\lambda \in \mathbb{D} \setminus (\mathcal{E}' \cup \mathbb{R})$, Λ a finite non-empty set, $(a_i)_{i \in \Lambda} = \mathbf{a} \in \mathbb{C}^\Lambda$ with

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not all of the a_i equal, and $p \in \mathbb{P}_\Lambda$. Then there are $C, \gamma > 0$ (depending on λ, $\mathbf{a}$, p) such that

$$|\widehat{\nu^p_{\lambda,\mathbf{a}}}(\xi)| \le C|\xi|^{-\gamma} \quad \text{for all } \xi \in \mathbb{C}.$$

Write

$$\mathcal{E} = \bigcup_{k\ge 1} \{\lambda \in \mathbb{D} : \lambda^k \in \mathcal{E}'\};$$

then $\dim_H \mathcal{E} = 0$.

3. *Statement of results*

The following theorem is our main result. Parts (1) and (2) are proven in §4, part (3) is proven in §5, and part (4) is proven in §7.

THEOREM 3.1. *Let $\lambda \in \mathbb{D} \setminus \mathcal{E}$ be such that $\arg\lambda \notin \pi\mathbb{Q}$, Λ a finite non-empty set, $\mathbf{a} \in \mathbb{C}^\Lambda$ such that $\mathcal{F}_{\lambda,\mathbf{a}}$ satisfies the SSC, and $p \in \mathbb{P}_\Lambda$ with $D_q(\nu^p_{\lambda,\mathbf{a}}) > 1$ for some $1 < q < \infty$. Set $\nu = \nu^p_{\lambda,\mathbf{a}}$. Then:*

(1) *$P_z\nu \in L^q(\mathbb{R})$ for all $z \in S$;*
(2) *$z \to P_z\nu$ is continuous as a map from S to $L^q(\mathbb{R})$ with respect to the weak topology of $L^q(\mathbb{R})$;*
(3) *for every $z \in S$ the measure $\nu_{z,w}$ has exact dimension $\dim_H \nu - 1$ for ν-a.e. $w \in \mathbb{C}$;*
(4) *for every $z \in S$ the measures $P_z\nu$ and $\mathcal{L}|_{P_z(K_{\lambda,\mathbf{a}})}$ are equivalent.*

By differentiation it follows that for every probability vector $p \in \mathbb{P}_\Lambda$,

$$\lim_{q\downarrow 1} \frac{\log \|p\|_q^q}{(q-1)} = \sum_{i\in\Lambda} p_i \log p_i.$$

Hence by (2.3) and (2.4) we get the following direct corollary of Theorem 3.1.

COROLLARY 3.2. *Let $\lambda \in \mathbb{D} \setminus \mathcal{E}$ be such that $\arg\lambda \notin \pi\mathbb{Q}$, Λ a finite non-empty set, $\mathbf{a} \in \mathbb{C}^\Lambda$ such that $\mathcal{F}_{\lambda,\mathbf{a}}$ satisfies the SSC, and $p \in \mathbb{P}_\Lambda$ with $\dim_H \nu^p_{\lambda,\mathbf{a}} > 1$. Then there exists $1 < q < \infty$ such that parts (1) to (4) in Theorem 3.1 are valid for $\nu = \nu^p_{\lambda,\mathbf{a}}$.*

Remark. The following observation might be of interest. Let $\mathcal{F}_{\lambda,\mathbf{a}}$ be an IFS as in Theorem 3.1 and $p_1, p_2 \in \mathbb{P}_\Lambda$ with $p_1 \ne p_2$. Set $\nu_1 = \nu^{p_1}_{\lambda,\mathbf{a}}$ and $\nu_2 = \nu^{p_2}_{\lambda,\mathbf{a}}$, and assume

$$\dim_H \nu_1, \dim_H \nu_2 > 1.$$

Then ν_1 and ν_2 are mutually singular. This follows from the fact that they are both ergodic with respect to an appropriate map from $K_{\lambda,\mathbf{a}}$ onto itself (see §5) and since $\nu_1 \ne \nu_2$. On the other hand, by part (4) of Theorem 3.1, $P_z\nu_1$ is equivalent to $P_z\nu_2$ for each $z \in S$.

From Theorem 3.1 we obtain the following result for self-similar sets. Its proof is given in §6. For $s \ge 0$ denote by $\mathcal{H}^s$ and $\mathcal{P}^s$ the s-dimensional Hausdorff and packing measures, respectively. It is well known that $0 < \mathcal{H}^s(K) < \infty$ whenever K is a self-similar set with the SSC and dimension s. Given $z \in S$ and $x \in \mathbb{R}$, we write $K_{z,x} = K \cap P_z^{-1}\{x\}$.

COROLLARY 3.3. *Let $\lambda \in \mathbb{D} \setminus \mathcal{E}$ be such that $\arg\lambda \notin \pi\mathbb{Q}$, Λ a finite non-empty set, and $\mathbf{a} \in \mathbb{C}^\Lambda$ such that $\mathcal{F}_{\lambda,\mathbf{a}}$ satisfies the SSC. Set $K = K_{\lambda,\mathbf{a}}$ and $s = \dim_H K$, and assume $s > 1$. Then:*

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(1) *for each* $z \in S$

$$\dim_H K_{z,P_z w} = s-1 \quad \textit{and} \quad \mathcal{P}^{s-1}(K_{z,P_z w}) > 0 \quad \textit{for } \mathcal{H}^s\textit{-a.e. } w \in K;$$

(2) *there exists* $c > 0$ *such that for every* $z \in S$,

$$\mathcal{L}\{x \in \mathbb{R} : \dim_H K_{z,x} = s-1 \quad \textit{and} \quad \mathcal{P}^{s-1}(K_{z,x}) > 0\} > c.$$

Remark. In [**Rap1**, Theorem 2.4] the author has shown that, under the assumptions of the last corollary, for Lebesgue a.e. $z \in S$,

$$\mathcal{P}^{s-1}(K_{z,P_z w}) > 0 \quad \text{for } \mathcal{H}^s\text{-a.e. } w \in K.$$

In Corollary 3.3 this is established for all $z \in S$.

Remark. If the last corollary were true with $\mathcal{H}^{s-1}$ instead of $\mathcal{P}^{s-1}$, then by [**Rap1**, Corollary 2.3] it would follow that $P_z(\mathcal{H}^s|_{K_{\lambda,\mathbf{a}}}) \in L^\infty(\mathbb{R})$ for all $z \in S$ and that

$$\sup_{z\in S} \|P_z(\mathcal{H}^s|_{K_{\lambda,\mathbf{a}}})\|_\infty < \infty.$$

Unfortunately, currently we are unable to prove this stronger statement.

If in Theorem 3.1 we assume that the correlation dimension $D_2(\nu^p_{\lambda,\mathbf{a}})$ exceeds 1 then we get the following result. It guarantees the continuity of the map $z \to P_z\nu$ with respect to the norm topology, which is of course much stronger than continuity with respect to the weak topology. The proof is given in §4.

THEOREM 3.4. *Let* $\lambda \in \mathbb{D} \setminus \mathcal{E}$ *be such that* $\arg\lambda \notin \pi\mathbb{Q}$, Λ *a finite non-empty set,* $\mathbf{a} \in \mathbb{C}^\Lambda$ *such that* $\mathcal{F}_{\lambda,\mathbf{a}}$ *satisfies the SSC, and* $p \in \mathbb{P}_\Lambda$ *with* $D_2(\nu^p_{\lambda,\mathbf{a}}) > 1$. *Set* $\nu = \nu^p_{\lambda,\mathbf{a}}$. *Then there exists* $\gamma > 0$ *(which depends on* λ, $\mathbf{a}$, p*) such that:*

(1) $P_z\nu \in H_\gamma(\mathbb{R})$ *for all* $z \in S$;

(2) $z \to P_z\nu$ *is continuous as a map from* S *to* $(H_\gamma(\mathbb{R}), \|\cdot\|_{(\gamma)})$.

Remark. Note that part (2) of the last theorem implies that $z \to P_z\nu$ is continuous as a map from S to $(L^2(\mathbb{R}), \|\cdot\|_2)$. This in turn implies the continuity of this map with respect to the total variation norm on $\mathcal{M}(\mathbb{R})$.

Let $\lambda \in \mathbb{D}$ and $\mathbf{a} \in \mathbb{C}^\Lambda$ be such that $\mathcal{F}_{\lambda,\mathbf{a}}$ satisfies the SSC, and write $s = \dim_H K_{\lambda,\mathbf{a}}$ and $p' = (1/|\Lambda|, \ldots, 1/|\Lambda|)$. It is not hard to see that there exists $0 < c < \infty$ such that

$$\mathcal{H}^s|_{K_{\lambda,\mathbf{a}}} = c \cdot \nu^{p'}_{\lambda,\mathbf{a}}.$$

Also, for each $1 < q < \infty$ it follows by (2.3) and (2.5) that

$$D_q(\nu^{p'}_{\lambda,\mathbf{a}}) = \frac{\log|\Lambda|}{-\log|\lambda|} = \dim_H K_{\lambda,\mathbf{a}}.$$

From these facts we get the following direct corollary of Theorems 3.1 and 3.4. It shows that when $s > 1$ the projections of $\mathcal{H}^s|_{K_{\lambda,\mathbf{a}}}$ are very regular.

COROLLARY 3.5. *Let* $\lambda \in \mathbb{D} \setminus \mathcal{E}$ *be such that* $\arg\lambda \notin \pi\mathbb{Q}$, Λ *a finite non-empty set, and* $\mathbf{a} \in \mathbb{C}^\Lambda$ *such that* $\mathcal{F}_{\lambda,\mathbf{a}}$ *satisfies the SSC. Set* $s = \dim_H K_{\lambda,\mathbf{a}}$ *and assume* $s > 1$. *Then for* $\nu = \mathcal{H}^s|_{K_{\lambda,\mathbf{a}}}$ *and every* $1 < q < \infty$, *parts (1)–(4) of Theorem 3.1 are valid. Additionally, parts (1) and (2) of Theorem 3.4 are valid for some* $\gamma > 0$.

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4. *Proof of parts (1) and (2) of Theorems 3.1 and 3.4*

The following two results will be needed. Let X be $\mathbb{C}$ or $\mathbb{R}$ and for $\mu, \nu \in \mathcal{M}(X)$ write $\mu * \nu$ for their convolution.

THEOREM 4.1. [**SS1**, Theorem 4.4] *Let $1 < q < \infty$, $\mu \in \mathcal{M}(\mathbb{R})$ with $D_q(\mu) = 1$, and $\nu \in \mathcal{M}(\mathbb{R})$ such that there exist $C, \gamma > 0$ with $|\widehat{\nu}(\xi)| \leq C|\xi|^{-\gamma}$ for all $\xi \in \mathbb{R}$. Then $\mu * \nu \in L^q(\mathbb{R})$.*

LEMMA 4.2. [**Sh1**, Lemma 2.1] *Let $\mu \in \mathcal{M}(\mathbb{R})$ with $D_2(\mu) = 1$ and $\nu \in \mathcal{M}(\mathbb{R})$ such that there exist $C, \gamma > 0$ with $|\widehat{\nu}(\xi)| \leq C|\xi|^{-\gamma}$ for all $\xi \in \mathbb{R}$. Then $\mu * \nu \in H_{\gamma/4}(\mathbb{R})$.*

In the next proposition we prove the first part of Theorems 3.1 and 3.4. The proof follows an idea introduced in [**Sh1**]. It also relies on a result from [**Sh2**] regarding the L^q dimensions of projections of planar self-similar measures.

PROPOSITION 4.3. *Let $\lambda \in \mathbb{D} \setminus \mathcal{E}$ be such that $\arg \lambda \notin \pi\mathbb{Q}$, Λ a finite non-empty set, $(a_i)_{i\in\Lambda} = \mathbf{a} \in \mathbb{C}^\Lambda$ such that $\mathcal{F}_{\lambda,\mathbf{a}}$ satisfies the SSC, and $p \in \mathbb{P}_\Lambda$. Set $\nu = \nu^p_{\lambda,\mathbf{a}}$. Then:*

(1) *if $1 < q < \infty$ is such that $D_q(\nu) > 1$ then $P_z\nu \in L^q(\mathbb{R})$ for all $z \in S$;*

(2) *if $D_2(\nu) > 1$ then there exists $\gamma > 0$ such that $P_z\nu \in H_\gamma(\mathbb{R})$ for all $z \in S$.*

Proof. Without loss of generality we can assume that $a_i = 0$ for some $i \in \Lambda$. Suppose that $1 < q < \infty$ is such that $D_q(\nu) > 1$. Let $k \geq 2$ be such that $(1 - 1/k)D_q(\nu) > 1$. For $(i_1, \ldots, i_{k-1}) = \boldsymbol{i} \in \Lambda^{k-1}$ write $p_{\boldsymbol{i}} = p_{i_1} \cdot \cdots \cdot p_{i_{k-1}}$ and

$$g_{\boldsymbol{i}}(w) = \lambda^k w + \sum_{j=1}^{k-1} a_{i_j}\lambda^j \quad \text{for } w \in \mathbb{C}.$$

Since $\mathcal{F}_{\lambda,\mathbf{a}}$ satisfies the SSC and $a_i = 0$ for some $i \in \Lambda$, it is easy to see that $\{g_{\boldsymbol{i}}\}_{\boldsymbol{i}\in\Lambda^{k-1}}$ also satisfies the SSC. Let $\mu \in \mathcal{M}(\mathbb{C})$ be the self-similar measure corresponding to $\{g_{\boldsymbol{i}}\}_{\boldsymbol{i}\in\Lambda^{k-1}}$ and $\{p_{\boldsymbol{i}}\}_{\boldsymbol{i}\in\Lambda^{k-1}}$, that is,

$$\mu = \sum_{\boldsymbol{i}\in\Lambda^{k-1}} p_{\boldsymbol{i}} \cdot g_{\boldsymbol{i}}\mu.$$

By (2.3),

$$\begin{aligned} D_q(\mu) &= \frac{\log \|(p_{\boldsymbol{i}})_{\boldsymbol{i}\in\Lambda^{k-1}}\|_q^q}{(q-1)\log|\lambda^k|} \\ &= \frac{(k-1)\log\|p\|_q^q}{k(q-1)\log|\lambda|} = \left(1 - \frac{1}{k}\right) D_q(\nu) > 1. \end{aligned}$$

Hence by [**Sh2**, Theorem 8.2],

$$D_q(P_z\mu) = 1 \quad \text{for all } z \in S. \tag{4.1}$$

Recall the definition of the sets $\mathcal{E}$ and $\mathcal{E}'$ from §2.5. Since $\lambda \notin \mathcal{E}$ and $\arg\lambda \notin \pi\mathbb{Q}$ we have $\lambda^k \notin \mathcal{E}' \cup \mathbb{R}$. Hence there exist $C, \gamma > 0$ such that

$$|\widehat{\nu^p_{\lambda^k,\mathbf{a}}}(\xi)| \leq C|\xi|^{-\gamma} \quad \text{for all } \xi \in \mathbb{C}.$$

Let $z \in S$, then a direct computation shows that

$$\widehat{P_z\nu^p_{\lambda^k,\mathbf{a}}}(t) = \widehat{\nu^p_{\lambda^k,\mathbf{a}}}(tz) \quad \text{for all } t \in \mathbb{R},$$

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and so

$$|\widehat{P_z \nu^p_{\lambda^k,\mathbf{a}}}(t)| \le C|t|^{-\gamma} \quad \text{for all } t \in \mathbb{R}. \tag{4.2}$$

Note that $\nu = \mu * \nu^p_{\lambda^k,\mathbf{a}}$, hence $P_z\nu = P_z\mu * P_z\nu^p_{\lambda^k,\mathbf{a}}$. From this, (4.1), (4.2), and Theorem 4.1, it now follows that $P_z\nu \in L^q(\mathbb{R})$, which completes the proof of the first part.

The proof of the second part is similar, except that at the end of the proof one needs to use Lemma 4.2 instead of Theorem 4.1. □

We now turn to the proof of the second part of Theorems 3.1 and 3.4. Throughout this section the pair $(B, \|\cdot\|_B)$ will denote $(L^q(\mathbb{R}), \|\cdot\|_q)$ for some $1 < q < \infty$ or $(H_\gamma(\mathbb{R}), \|\cdot\|_{(\gamma)})$ for some $0 \le \gamma < \infty$. We write $C_c(\mathbb{R})$ for the collection of all compactly supported continuous functions on $\mathbb{R}$.

LEMMA 4.4. *Let $\nu \in \mathcal{M}(\mathbb{C})$ and $C > 0$ be given, and set*

$$F = \left\{ z \in S : P_z\nu \in B \text{ and } \left\| \frac{dP_z\nu}{d\mathcal{L}} \right\|_B \le C \right\}.$$

Then F is a closed subset of S.

Proof. For every $z \in F$ there exists $g_z \in B$ with $dP_z\nu = g_z\, d\mathcal{L}$ and $\|g_z\|_B \le C$. Let $z_1, z_2, \ldots \in F$ and $z \in S$ be such that $z_k \xrightarrow{k} z$ as $k \to \infty$. Note that $\{g_{z_k}\}_{k\ge1}$ is a bounded sequence in the reflexive Banach space B. Hence, by moving to a subsequence without changing notation, we may assume that there exists $g \in B$ such that

$$g_{z_k} \xrightarrow{k} g \text{ weakly in } B \text{ as } k \to \infty. \tag{4.3}$$

Additionally, by (4.3)

$$\|g\|_B \le \liminf_{k\to\infty} \|g_{z_k}\|_B \le C.$$

Given $h \in C_c(\mathbb{R})$, we get from (4.3) that

$$\int h(x)g(x)\,dx = \lim_{k\to\infty} \int h(x)g_{z_k}(x)\,dx = \lim_{k\to\infty} \int h(P_{z_k}w)\,d\nu(w).$$

Hence, by bounded convergence and since $P_{z_k}w \xrightarrow{k} P_z w$ for all $w \in \mathbb{C}$,

$$\int h(x)g(x)\,dx = \int h(P_z w)\,d\nu(w) = \int h(x)\,dP_z\nu(x).$$

This shows $dP_z\nu = g\,d\mathcal{L}$, which implies that $z \in F$ and completes the proof of the lemma. □

For $b \in \mathbb{R}$ and $c > 0$ let

$$\tau_b(x) = x - b \quad \text{and} \quad M_c(x) = cx \quad \text{for } x \in \mathbb{R}.$$

In what follows, Λ stands for some finite non-empty index set. Recall the definition of the IFS $\mathcal{F}_{\lambda,\mathbf{a}}$ from (2.1), which consists of maps $f_i = f^i_{\lambda,\mathbf{a}}$ for $i \in \Lambda$. Given a word $i_1 \ldots i_k = \boldsymbol{i} \in \Lambda^*$, write $f_{\boldsymbol{i}} = f_{i_1} \circ \cdots \circ f_{i_k}$, and for $(p_i)_{i\in\Lambda} = p \in \mathbb{P}_\Lambda$ set $p_{\boldsymbol{i}} = p_{i_1} \cdot \cdots \cdot p_{i_k}$.

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LEMMA 4.5. *Let $\alpha \in S$, $0 < r < 1$, $\mathbf{a} \in \mathbb{C}^\Lambda$ and $p \in \mathbb{P}_\Lambda$. Write $\lambda = r\alpha$, $\nu = \nu^p_{\lambda,\mathbf{a}}$ and $f_i = f^i_{\lambda,\mathbf{a}}$ for every $i \in \Lambda$. Assume that $P_z\nu$ is absolutely continuous for each $z \in S$ and write $g_z = dP_z\nu/d\mathcal{L}$. Let $z \in S$ and $k \geq 1$, and for each $\boldsymbol{i} \in \Lambda^k$ set $b_{\boldsymbol{i}} = P_z f_{\boldsymbol{i}}(0)$. Then for $\mathcal{L}$-a.e. $x \in \mathbb{R}$,*

$$g_z(x) = \sum_{\boldsymbol{i}\in\Lambda^k} p_{\boldsymbol{i}} r^{-k} \cdot g_{\alpha^{-k}z}(\tau_{r^{-k}b_{\boldsymbol{i}}} M_{r^{-k}} x).$$

Proof. For $h \in C_c(\mathbb{R})$,

$$\int h(x) g_z(x)\, dx = \int h(P_z w)\, d\nu(w) = \sum_{\boldsymbol{i}\in\Lambda^k} p_{\boldsymbol{i}} \int h(P_z w)\, df_{\boldsymbol{i}}\nu(w).$$

For $\boldsymbol{i} \in \Lambda^k$,

$$\begin{aligned}\int h(P_z f_{\boldsymbol{i}} w)\, d\nu(w) &= \int h(b_{\boldsymbol{i}} + r^k P_{\alpha^{-k}z} w)\, d\nu(w)\\ &= \int h(b_{\boldsymbol{i}} + r^k x)\, dP_{\alpha^{-k}z}\nu(x) = \int h(b_{\boldsymbol{i}} + r^k x) g_{\alpha^{-k}z}(x)\, dx\\ &= \int h(x) g_{\alpha^{-k}z}(r^{-k}x - r^{-k}b_{\boldsymbol{i}}) r^{-k}\, dx.\end{aligned}$$

By combining these equalities,

$$\int h(x) g_z(x)\, dx = \int h(x) \sum_{\boldsymbol{i}\in\Lambda^k} p_{\boldsymbol{i}} r^{-k} \cdot g_{\alpha^{-k}z}(\tau_{r^{-k}b_{\boldsymbol{i}}} M_{r^{-k}} x)\, dx.$$

Now since this holds for every $h \in C_c(\mathbb{R})$ the lemma follows. □

LEMMA 4.6. *There exists $\beta \in \mathbb{R}$ such that for every $g \in B$, $b \in \mathbb{R}$ and $c \geq 1$,*

$$\|g \circ \tau_b \circ M_c\|_B \leq c^\beta \cdot \|g\|_B.$$

Proof. Let $g \in B$, $b \in \mathbb{R}$ and $c \geq 1$. If $B = L^q(\mathbb{R})$ for $1 < q < \infty$, then

$$\|g \circ \tau_b \circ M_c\|_B = c^{-1/q} \|g\|_B.$$

If $B = H_\gamma(\mathbb{R})$ for $0 \leq \gamma < \infty$, then

$$\begin{aligned}\|g \circ \tau_b \circ M_c\|_B^2 &= \int |(g \circ \tau_b \circ M_c)^\wedge(\xi)|^2 (1 + |\xi|^2)^\gamma\, d\xi\\ &= \int |c^{-1} \cdot e^{ib\xi/c} \cdot \widehat{g}(\xi/c)|^2 (1 + |\xi|^2)^\gamma\, d\xi\\ &= c^{-1} \int |\widehat{g}(\xi)|^2 (1 + |c\xi|^2)^\gamma\, d\xi \leq c^{2\gamma-1} \|g\|_B^2,\end{aligned}$$

which proves the lemma. □

The following key proposition will be used several times below. Recall that $(B, \|\cdot\|_B)$ denotes $(L^q(\mathbb{R}), \|\cdot\|_q)$ for some $1 < q < \infty$ or $(H_\gamma(\mathbb{R}), \|\cdot\|_{(\gamma)})$ for some $0 \leq \gamma < \infty$.

PROPOSITION 4.7. *Let $\lambda \in \mathbb{D}$ be such that $\arg\lambda \notin \pi\mathbb{Q}$, $\mathbf{a} \in \mathbb{C}^\Lambda$ and $p \in \mathbb{P}_\Lambda$. Set $\nu = \nu^p_{\lambda,\mathbf{a}}$ and assume that $P_z\nu \in B$ for all $z \in S$. Then $\{P_z\nu\}_{z\in S}$ is a bounded subset of $(B, \|\cdot\|_B)$.*

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Proof. Set $f_i = f^i_{\lambda,\mathbf{a}}$ for every $i \in \Lambda$. By assumption, for every $z \in S$ there exists $g_z \in B$ with $dP_z\nu = g_z\, d\mathcal{L}$. For $n \geq 1$ let

$$F_n = \{z \in S : \|g_z\|_B \leq n\};$$

then F_n is closed in S by Lemma 4.4.

From $S = \bigcup_{n\geq 1} F_n$ and Baire's theorem, it follows that there exist $n \geq 1$ and an open non-empty subset V of S with $V \subset F_n$. Let $\alpha \in S$ and $0 < r < 1$ be such that $\lambda = r\alpha$. Since

$$\arg\alpha = \arg\lambda \notin \pi\mathbb{Q},$$

there exists $N \geq 1$ such that for each $z \in S$ there exists $1 \leq k \leq N$ with $\alpha^{-k}z \in V$. Fix $z \in S$ and let $1 \leq k \leq N$ be such that $\alpha^{-k}z \in V$.

For $\boldsymbol{i} \in \Lambda^k$ write $b_{\boldsymbol{i}} = P_z f_{\boldsymbol{i}}(0)$. By Lemma 4.5 it follows that for $\mathcal{L}$-a.e. $x \in \mathbb{R}$,

$$g_z(x) = \sum_{\boldsymbol{i}\in\Lambda^k} p_{\boldsymbol{i}} r^{-k} \cdot g_{\alpha^{-k}z}(\tau_{r^{-k}b_{\boldsymbol{i}}} M_{r^{-k}} x).$$

Hence by Lemma 4.6 and since $\alpha^{-k}z \in V \subset F_n$,

$$\begin{aligned}\|g_z\|_B &\leq \sum_{\boldsymbol{i}\in\Lambda^k} p_{\boldsymbol{i}} r^{-k} \cdot \|g_{\alpha^{-k}z} \circ \tau_{r^{-k}b_{\boldsymbol{i}}} \circ M_{r^{-k}}\|_B \\ &\leq r^{-k(1+\beta)} \cdot \|g_{\alpha^{-k}z}\|_B \leq n \cdot \max\{r^{-N(1+\beta)}, 1\},\end{aligned}$$

which completes the proof of the proposition. □

Part (2) of Theorems 3.1 and 3.4 will follow directly from the following two lemmas and Propositions 4.3 and 4.7.

LEMMA 4.8. *Let $\nu \in \mathcal{M}(\mathbb{C})$, $1 < q < \infty$ and $C > 0$. Assume that $P_z\nu \in L^q(\mathbb{R})$, with $\|P_z\nu\|_q \leq C$, for each $z \in S$. Then $z \to P_z\nu$ is continuous as a map from S to $L^q(\mathbb{R})$, with respect to the weak topology of $L^q(\mathbb{R})$.*

Proof. By assumption, for every $z \in S$ there exists $g_z \in L^q(\mathbb{R})$ with $dP_z\nu = g_z\, d\mathcal{L}$. Moreover,

$$\|g_z\|_q \leq C \quad \text{for all } z \in S. \tag{4.4}$$

Let $z \in S$ and $\{z_{0,k}\}_{k\geq 1} \subset S$ be such that $z_{0,k} \xrightarrow{k} z$. It suffices to show that

$$g_{z_{0,k}} \xrightarrow{k} g_z \text{ weakly in } L^q(\mathbb{R}) \quad \text{as } k \to \infty. \tag{4.5}$$

Let $\{z_{1,k}\}_{k\geq 1}$ be a subsequence of $\{z_{0,k}\}_{k\geq 1}$. From (4.4) and since $L^q(\mathbb{R})$ is reflexive, it follows that there exist a subsequence $\{z_{2,k}\}_{k\geq 1}$ of $\{z_{1,k}\}_{k\geq 1}$ and $g \in L^q(\mathbb{R})$ such that $g_{z_{2,k}} \xrightarrow{k} g$ weakly in $L^q(\mathbb{R})$ as $k \to \infty$.

Since $z_{2,k} \xrightarrow{k} z$, for each $h \in C_c(\mathbb{R})$,

$$\begin{aligned}\int h(x)g(x)\,dx &= \lim_{k\to\infty} \int h(x) g_{z_{2,k}}(x)\,dx \\ &= \lim_{k\to\infty} \int h(P_{z_{2,k}}w)\,d\nu(w) = \int h(P_z w)\,d\nu(w) = \int h(x)g_z(x)\,dx.\end{aligned}$$

Hence $g = g_z$, and so $g_{z_{2,k}} \xrightarrow{k} g_z$ weakly as $k \to \infty$. Since $\{z_{1,k}\}_{k\geq 1}$ was an arbitrary subsequence of $\{z_{0,k}\}_{k\geq 1}$ this gives (4.5) and completes the proof of the lemma. □

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LEMMA 4.9. *Let $\nu \in \mathcal{M}(\mathbb{C})$ and $\gamma, C > 0$. Assume that $P_z\nu \in H_\gamma(\mathbb{R})$, with $\|P_z\nu\|_{(\gamma)} \le C$, for all $z \in S$. Let $\gamma' \in (0, \gamma)$. Then $z \to P_z\nu$ is continuous as a map from S to $(H_{\gamma'}(\mathbb{R}), \|\cdot\|_{(\gamma')})$.*

Proof. By assumption, for every $z \in S$ there exists $g_z \in H_\gamma(\mathbb{R})$ with $dP_z\nu = g_z\, d\mathcal{L}$. Moreover,

$$\|g_z\|_{(\gamma)} \le C \quad \text{for all } z \in S. \tag{4.6}$$

Let $z \in S$ and $\{z_{0,k}\}_{k\ge1} \subset S$ be such that $z_{0,k} \overset{k}{\to} z$. It suffices to show that

$$\|g_z - g_{z_{0,k}}\|_{(\gamma')} \overset{k}{\to} 0 \quad \text{as } k \to \infty. \tag{4.7}$$

Let $\{z_{1,k}\}_{k\ge1}$ be a subsequence of $\{z_{0,k}\}_{k\ge1}$. There exists a bounded interval $J \subset \mathbb{R}$ with

$$\operatorname{supp}(P_z\nu) \subset J \quad \text{for all } z \in S.$$

From this, from (4.6), and by Rellich's theorem (see, for instance, [**Fol**, Theorem 9.22]), there exist a subsequence $\{z_{2,k}\}_{k\ge1}$ of $\{z_{1,k}\}_{k\ge1}$ and $g \in H_{\gamma'}(\mathbb{R})$ such that $\|g - g_{z_{2,k}}\|_{(\gamma')} \overset{k}{\to} 0$.

As in the proof of the previous lemma, it can be shown that $g = g_z$ and so $\|g_z - g_{z_{2,k}}\|_{(\gamma')} \overset{k}{\to} 0$. Since $\{z_{1,k}\}_{k\ge1}$ was an arbitrary subsequence of $\{z_{0,k}\}_{k\ge1}$ this gives (4.7) and completes the proof of the lemma. □

Proof of part (2) of Theorems 3.1 and 3.4. Let $\lambda \in \mathbb{D} \setminus \mathcal{E}$ be such that $\arg\lambda \notin \pi\mathbb{Q}$, $\mathbf{a} \in \mathbb{C}^\Lambda$ such that $\mathcal{F}_{\lambda,\mathbf{a}}$ satisfies the SSC, and $p \in \mathbb{P}_\Lambda$ with $D_q(\nu^p_{\lambda,\mathbf{a}}) > 1$ for some $1 < q < \infty$. Set $\nu = \nu^p_{\lambda,\mathbf{a}}$. Then by part (1) of Proposition 4.3 it follows that $P_z\nu \in L^q(\mathbb{R})$ for all $z \in S$. From this and Proposition 4.7 we get that $\{P_z\nu\}_{z\in S}$ is a bounded subset of $(L^q(\mathbb{R}), \|\cdot\|_q)$. By Lemma 4.8 it now follows that $z \to P_z\nu$ is continuous as a map from S to $L^q(\mathbb{R})$, with respect to the weak topology of $L^q(\mathbb{R})$, which completes the proof of part (2) of Theorem 3.1.

Part (2) of Theorem 3.4 follows in a similar manner, except that one needs to use part (2) of Proposition 4.3 and Lemma 4.9. □

5. *Proof of part (3) of Theorem 3.1*

Throughout this section fix $\alpha \in S$, $0 < r < 1$, Λ a finite non-empty index set, $\mathbf{a} \in \mathbb{C}^\Lambda$ and $p \in \mathbb{P}_\Lambda$. Write $\lambda = r\alpha$, $K = K_{\lambda,\mathbf{a}}$, $\nu = \nu^p_{\lambda,\mathbf{a}}$, and $f_i = f^i_{\lambda,\mathbf{a}}$ for every $i \in \Lambda$. Assume that $\mathcal{F}_{\lambda,\mathbf{a}}$ satisfies the SSC. Assume further that $P_z\nu$ is absolutely continuous for each $z \in S$ and write $g_z = dP_z\nu/d\mathcal{L}$.

Given $i_1 \dots i_k = \boldsymbol{i} \in \Lambda^*$, recall that

$$f_{\boldsymbol{i}} = f_{i_1} \circ \cdots \circ f_{i_k} \quad \text{and} \quad p_{\boldsymbol{i}} = p_{i_1} \cdot \cdots \cdot p_{i_k},$$

and set $K_{\boldsymbol{i}} = f_{\boldsymbol{i}}(K)$. For $w \in K$ and $k \ge 1$ denote by $\boldsymbol{i}_k(w)$ the unique word of length k over Λ with $w \in K_{\boldsymbol{i}_k(w)}$. Let $T : K \to K$ be such that $Tw = f^{-1}_{i_1(w)}(w)$ for $w \in K$. It is easy to verify that the system (K, T, ν) is measure preserving and ergodic.

LEMMA 5.1. *Let $z \in S$ and $k \ge 1$. Then for ν-a.e. $w \in K$,*

$$\nu_{z,w}(K_{\boldsymbol{i}_k(w)}) = \frac{g_{\alpha^{-k}z}(P_{\alpha^{-k}z}T^k w)}{g_z(P_z w)} \cdot p_{\boldsymbol{i}_k(w)} r^{-k}.$$

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Proof. From [**FH**, Lemma 3.3] we get that for each $\eta \in S$ and Borel set $A \subset \mathbb{C}$,

$$\nu_{\eta,w}(A) = \lim_{\delta \downarrow 0} \frac{\nu(P_\eta^{-1}(B(P_\eta w, \delta)) \cap A)}{\nu(P_\eta^{-1}(B(P_\eta w, \delta)))} \quad \text{for } \nu\text{-a.e. } w \in \mathbb{C}. \tag{5.1}$$

By [**M**, Theorem 2.12] it follows that for each $\eta \in S$,

$$g_\eta(P_\eta w) = \lim_{\delta \downarrow 0} \frac{P_\eta \nu(B(P_\eta w, \delta))}{2\delta} \quad \text{for } \nu\text{-a.e. } w \in \mathbb{C}. \tag{5.2}$$

From (5.1) and (5.2) we get that for ν-a.e. $w \in K$,

$$\begin{aligned} \nu_{z,w}(K_{\boldsymbol{i}_k(w)}) &= \lim_{\delta \downarrow 0} \frac{\nu(K_{\boldsymbol{i}_k(w)} \cap P_z^{-1}(B(P_z w, \delta)))}{\nu(P_z^{-1}(B(P_z w, \delta)))} \\ &= \lim_{\delta \downarrow 0} \frac{2\delta}{\nu(P_z^{-1}(B(P_z w, \delta)))} \cdot \frac{\nu(K_{\boldsymbol{i}_k(w)} \cap P_z^{-1}(B(P_z w, \delta)))}{2\delta} \\ &= g_z(P_z w)^{-1} \lim_{\delta \downarrow 0} \frac{\nu(K_{\boldsymbol{i}_k(w)} \cap P_z^{-1}(B(P_z w, \delta)))}{2\delta}. \end{aligned} \tag{5.3}$$

For each $\delta > 0$ set

$$F_\delta = P_{\alpha^{-k} z}^{-1}(B(P_{\alpha^{-k} z} f_{\boldsymbol{i}_k(w)}^{-1}(w), \delta r^{-k}))$$

and for $\eta \in S$ write $\eta^\perp = e^{i\pi/2}\eta$. We have

$$\begin{aligned} P_z^{-1}(B(P_z w, \delta)) &= w + z^\perp \mathbb{R} + B(0, \delta) \\ &= f_{\boldsymbol{i}_k(w)} \circ f_{\boldsymbol{i}_k(w)}^{-1}(w + z^\perp \mathbb{R} + B(0, \delta)) \\ &= f_{\boldsymbol{i}_k(w)}(f_{\boldsymbol{i}_k(w)}^{-1}(w) + (\alpha^{-k} z)^\perp \mathbb{R} + B(0, \delta r^{-k})) \\ &= f_{\boldsymbol{i}_k(w)}(F_\delta). \end{aligned}$$

From this, from (5.3) and since ν is self-similar,

$$\begin{aligned} \nu_{z,w}(K_{\boldsymbol{i}_k(w)}) &= g_z(P_z w)^{-1} \lim_{\delta \downarrow 0} \frac{\nu(K_{\boldsymbol{i}_k(w)} \cap f_{\boldsymbol{i}_k(w)}(F_\delta))}{2\delta} \\ &= g_z(P_z w)^{-1} \lim_{\delta \downarrow 0} \frac{1}{2\delta} \sum_{\boldsymbol{i} \in \Lambda^k} p_{\boldsymbol{i}} \cdot \nu(f_{\boldsymbol{i}}^{-1}(f_{\boldsymbol{i}_k(w)}(K \cap F_\delta))). \end{aligned} \tag{5.4}$$

Since $\mathcal{F}_{\lambda,\mathbf{a}}$ satisfies the SSC,

$$K \cap f_{\boldsymbol{i}}^{-1}(f_{\boldsymbol{i}_k(w)}(K)) = \emptyset \quad \text{for each } \boldsymbol{i} \in \Lambda^k \setminus \{\boldsymbol{i}_k(w)\}.$$

Hence by (5.4) and (5.2),

$$\begin{aligned} \nu_{z,w}(K_{\boldsymbol{i}_k(w)}) &= g_z(P_z w)^{-1} \lim_{\delta \downarrow 0} p_{\boldsymbol{i}_k(w)} r^{-k} \cdot \frac{\nu(K \cap F_\delta)}{2\delta r^{-k}} \\ &= g_z(P_z w)^{-1} \cdot p_{\boldsymbol{i}_k(w)} r^{-k} \cdot g_{\alpha^{-k} z}(P_{\alpha^{-k} z} f_{\boldsymbol{i}_k(w)}^{-1}(w)) \\ &= \frac{g_{\alpha^{-k} z}(P_{\alpha^{-k} z} T^k w)}{g_z(P_z w)} \cdot p_{\boldsymbol{i}_k(w)} r^{-k}, \end{aligned}$$

which completes the proof of the lemma. □

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Part (3) of Theorem 3.1 follows directly from part (1) combined with the following proposition.

PROPOSITION 5.2. *Let* $1 < q < \infty$, *assume that* $g_z \in L^q(\mathbb{R})$ *for all* $z \in S$, *and that* $\arg \lambda \notin \pi\mathbb{Q}$. *Then for every* $z \in S$ *the measure* $\nu_{z,w}$ *has exact dimension* $\dim_H \nu - 1$ *for* ν*-a.e.* $w \in \mathbb{C}$.

Proof. By Proposition 4.7 there exists $C > 0$ with $\|g_z\|_q \le C$ for all $z \in S$. Fix $z \in S$. Then since $\mathcal{F}_{\lambda,\mathbf{a}}$ satisfies the SSC it suffices to show that for ν-a.e. $w \in K$,

$$\lim_{k\to\infty} \frac{\log \nu_{z,w}(K_{\boldsymbol{i}_k(w)})}{\log r^k} = \dim_H \nu - 1.$$

For every $k \ge 1$ set $z_k = \alpha^{-k} z$ and

$$A_k = \{w \in K : g_{z_k}(P_{z_k} w) \ge k^{2/(q-1)}\}.$$

We have

$$\begin{aligned}\nu(A_k) &= P_{z_k}\nu\{x \in \mathbb{R} : (g_{z_k}(x))^{q-1} \ge k^2\} \\ &\le \frac{1}{k^2} \cdot \int (g_{z_k}(x))^{q-1}\, dP_{z_k}\nu(x) = \frac{\|g_{z_k}\|_q^q}{k^2} \le \frac{C^q}{k^2}.\end{aligned}$$

From this and since (K, T, ν) is measure preserving,

$$\sum_{k=1}^{\infty} \nu(T^{-k}(A_k)) = \sum_{k=1}^{\infty} \nu(A_k) < \infty.$$

Now from the Borel–Cantelli lemma it follows that for ν-a.e. $w \in K$,

$$g_{z_k}(P_{z_k} T^k w) \le k^{2/(q-1)} \quad \text{for all large enough } k \ge 1. \tag{5.5}$$

For $k \ge 1$ set

$$B_k = \{w \in K : g_{z_k}(P_{z_k} w) \le k^{-2}\},$$

and let $J \subset \mathbb{R}$ be a bounded interval with $\operatorname{supp}(P_\eta \nu) \subset J$ for all $\eta \in S$. We have

$$\begin{aligned}\nu(B_k) &= P_{z_k}\nu\{x \in \mathbb{R} : g_{z_k}(x) \le k^{-2}\} \\ &= \int_J 1_{\{g_{z_k}(x) \le k^{-2}\}} g_{z_k}(x)\, dx \le \frac{\mathcal{L}(J)}{k^2},\end{aligned}$$

and so $\sum_{k=1}^{\infty} \nu(T^{-k}(B_k)) < \infty$. From (5.5) and the Borel–Cantelli lemma it now follows that for ν-a.e. $w \in K$,

$$k^{-2} \le g_{z_k}(P_{z_k} T^k w) \le k^{2/(q-1)} \quad \text{for all large enough } k \ge 1. \tag{5.6}$$

Denote by $H(p)$ the entropy of p. Then $\dim_H \nu = H(p)/-\log r$ by (2.4). From the Shannon–McMillan–Breiman theorem, applied to the ergodic system (K, T, ν) and partition $\{f_i(K)\}_{i\in\Lambda}$, it follows that for ν-a.e. $w \in K$,

$$\lim_{k\to\infty} -\frac{\log p_{\boldsymbol{i}_k(w)}}{k} = \lim_{k\to\infty} -\frac{\log \nu(K_{\boldsymbol{i}_k(w)})}{k} = H(p).$$

From this, Lemmas 5.1 and (5.6), we get that for ν-a.e. $w \in K$,

$$\begin{aligned}\lim_{k\to\infty} \frac{\log \nu_{z,w}(K_{\boldsymbol{i}_k(w)})}{\log r^k} &= \lim_{k\to\infty} \frac{1}{k \log r} \cdot \log\left(\frac{g_{z_k}(P_{z_k} T^k w)}{g_z(P_z w)} \cdot p_{\boldsymbol{i}_k(w)} r^{-k}\right) \\ &= \lim_{k\to\infty} \frac{1}{k \log r} \cdot \log(p_{\boldsymbol{i}_k(w)} r^{-k}) = \frac{H(p)}{-\log r} - 1 = \dim_H \mu - 1,\end{aligned}$$

which completes the proof of the proposition. □

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6. *Proof of Corollary 3.3*

Let $\lambda \in \mathbb{D} \setminus \mathcal{E}$ be such that $\arg \lambda \notin \pi\mathbb{Q}$, Λ a finite non-empty set, and $\mathbf{a} \in \mathbb{C}^\Lambda$ such that $\mathcal{F}_{\lambda,\mathbf{a}}$ satisfies the SSC. Set $K = K_{\lambda,\mathbf{a}}$, let $s = \dim_H K$ and assume $s > 1$. Recall that for $z \in S$ and $x \in \mathbb{R}$ we write $K_{z,x} = K \cap P_z^{-1}\{x\}$.

Let $(p_i)_{i\in\Lambda} = p \in \mathbb{P}_\Lambda$ be such that $p_i = |\Lambda|^{-1}$ for each $i \in \Lambda$, and set $\nu = \nu^p_{\lambda,\mathbf{a}}$. Note that $\nu = C_0 \cdot \mathcal{H}^s|_K$ for some normalizing constant $0 < C_0 < \infty$. By (2.3) and (2.5),

$$D_2(\nu) = \frac{\log \|p\|_2^2}{\log|\lambda|} = \frac{\log |\Lambda|^{-1}}{\log|\lambda|} = \dim_H K > 1.$$

Hence, by part (1) of Theorem 3.1 and Proposition 4.7, it follows that $\{P_z\nu\}_{z\in S}$ is a bounded subset of $(L^2(\mathbb{R}), \|\cdot\|_2)$. Let $C > 0$ be such that $\|g_z\|_2 \le C$ for all $z \in S$, where $g_z = dP_z\nu/d\mathcal{L}$.

Fix $z \in S$ and set

$$B = \{w \in K : \dim_H K_{z,P_z w} = s - 1\}.$$

By part (3) of Theorem 3.1,

$$\dim_H \nu_{z,w} = \dim_H \nu - 1 = s - 1 \quad \text{for } \nu\text{-a.e. } w \in K.$$

Additionally, $\nu_{z,w}$ is supported on $K_{z,P_z w}$ for ν-a.e. $w \in K$. This shows that $\dim_H K_{z,P_z w} \ge s - 1$ for ν-a.e. $w \in K$. From [**Fal1**, Theorem 5.8] it follows that $\dim_H(K_{z,x}) \le s - 1$ for $\mathcal{L}$-a.e. $x \in \mathbb{R}$. Since $P_z\nu \ll \mathcal{L}$ this shows that $\dim_H K_{z,P_z w} \le s - 1$ for ν-a.e. $w \in K$, which gives $\nu(B) = 1$.

Write

$$A = \{w \in K : \mathcal{P}^{s-1}(K_{z,P_z w}) > 0\},$$

and recall the map $T : K \to K$ from the previous section. Let $\alpha \in S$ and $0 < r < 1$ be such that $\lambda = r\alpha$, and for each $k \ge 1$ set $z_k = \alpha^{-k} z$. Fix $n \ge 1$, for each $k \ge 1$ let

$$A_{n,k} = \{w \in K : g_{z_k}(P_{z_k} T^k w) \le n\},$$

and write

$$A_n = \bigcap_{N\ge 1} \bigcup_{k\ge N} A_{n,k}.$$

Since (K, T, ν) is measure preserving,

$$\begin{aligned}\nu(K \setminus A_{n,k}) &= P_{z_k}\nu\{x \in \mathbb{R} : g_{z_k}(x) > n\} \\ &\le \frac{1}{n}\int g_{z_k}(x)\, dP_{z_k}\nu(x) = \frac{\|g_{z_k}\|_2^2}{n} \le \frac{C^2}{n}.\end{aligned}$$

Hence,

$$\nu(A_n) = \lim_{N\to\infty} \nu\Big(\bigcup_{k\ge N} A_{n,k}\Big) \ge \liminf_{N\to\infty} \nu(A_{n,N}) \ge 1 - \frac{C^2}{n}.$$

Write

$$D = \{w \in K : \nu_{z,w}(A_n) > 0\}.$$

Then $\nu(D) \ge 1 - C^2/n$.

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For $k \geq 1$ and $w \in K$ let $K_{\boldsymbol{i}_k(w)}$ be as defined in the previous section. From Lemma 5.1 we get that for ν-a.e. $w \in K$ and $\nu_{z,w}$-a.e. $\eta \in A_n$,

$$\begin{aligned}\liminf_{k\to\infty} \frac{\nu_{z,w}(K_{\boldsymbol{i}_k(\eta)})}{|\lambda|^{k(s-1)}} &= \liminf_{k\to\infty} \frac{\nu_{z,\eta}(K_{\boldsymbol{i}_k(\eta)})}{|\lambda|^{k(s-1)}} \\ &= \liminf_{k\to\infty} \frac{g_{z_k}(P_{z_k}T^k\eta)\cdot|\Lambda|^{-k}|\lambda|^{-k}}{g_z(P_z\eta)|\lambda|^{k(s-1)}} \\ &\leq \frac{n}{g_z(P_z\eta)} \lim_{k\to\infty} \frac{|\Lambda|^{-k}}{|\lambda|^{ks}} = \frac{n}{g_z(P_z w)},\end{aligned}$$

where in the last equality we have used $s = \log|\Lambda|/-\log|\lambda|$. Now since $\mathcal{F}_{\lambda,\mathbf{a}}$ satisfies the SSC, we get that for ν-a.e. $w \in K$ there exists $0 < C_{w,n} < \infty$ with

$$\liminf_{\delta\downarrow 0} \frac{\nu_{z,w}(B(\eta,\delta))}{(2\delta)^{s-1}} \leq C_{w,n} \quad \text{for } \nu_{z,w}\text{-a.e. } \eta \in A_n.$$

From this and by [**M**, Theorem 6.11] it follows that for ν-a.e. $w \in D$,

$$\mathcal{P}^{s-1}(K_{z,P_z w}) \geq C_{w,n}^{-1} \cdot \nu_{z,w}(A_n) > 0,$$

and so $w \in A$. This gives $\nu(A) \geq \nu(D) \geq 1 - C^2/n$, which shows $\nu(A) = 1$ since $n \geq 1$ can be taken to be arbitrarily large. We have thus shown $\nu(A \cap B) = 1$. From this and since the measures ν and $\mathcal{H}^s|_K$ are equivalent, we obtain part (1) of the corollary.

Write

$$F = \big\{x \in \mathbb{R} : \dim_H(K_{z,x}) = s-1 \text{ and } \mathcal{P}^{s-1}(K_{z,x}) > 0\big\}.$$

From $\nu(A \cap B) = 1$ we get $P_z\nu(F) = 1$. Hence by the Cauchy–Schwarz inequality,

$$1 = P_z\nu(F) = \int 1_F(x) g_z(x)\, dx \leq \|1_F\|_2 \cdot \|g_z\|_2 \leq C\|1_F\|_2,$$

which gives $\mathcal{L}(F) \geq C^{-2}$. This completes the proof of part (2) of the corollary with $c = C^{-2}$. □

7. *Proof of part (4) of Theorem 3.1*

The following proof is an adaptation of the argument in [**PSS**, Proposition 3.1].

Proof of part (4) of Theorem 3.1. Let $\lambda \in \mathbb{D} \setminus \mathcal{E}$ be such that $\arg\lambda \notin \pi\mathbb{Q}$, Λ a finite non-empty set, $\mathbf{a} \in \mathbb{C}^\Lambda$ such that $\mathcal{F}_{\lambda,\mathbf{a}}$ satisfies the SSC, and $p \in \mathbb{P}_\Lambda$ with $D_q(\nu^p_{\lambda,\mathbf{a}}) > 1$ for some $1 < q < \infty$. Set $K = K_{\lambda,\mathbf{a}}$, $\nu = \nu^p_{\lambda,\mathbf{a}}$, and $f_i = f^i_{\lambda,\mathbf{a}}$ for $i \in \Lambda$. By part (1) of Theorem 3.1 it follows that $P_z\nu \in L^q(\mathbb{R})$ for each $z \in S$. Hence it suffices to show that $\mathcal{L}|_{P_z(K)} \ll P_z\nu$ for each $z \in S$.

By Proposition 4.7 there exists $C > 0$ with $\|g_z\|_q \leq C$ for all $z \in S$, where $g_z = dP_z\nu/d\mathcal{L}$. From

$$\dim_H K \geq \dim_H \nu \geq D_q(\nu) > 1$$

and part (2) of Corollary 3.3, it follows that there exists $c > 0$ with

$$\mathcal{L}(P_z(K)) > c \quad \text{for all } z \in S. \tag{7.1}$$

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Write

$$\beta = \sup\left\{\frac{\mathcal{L}(A)}{\mathcal{L}(P_z(K))} : z \in S,\ A \subset P_z(K) \text{ is Borel, and } P_z\nu(A) = 0\right\}.$$

Assume by contradiction that $\beta = 1$. Given $\epsilon > 0$, there exists $z \in S$ and a Borel set $A \subset P_z(K)$ with $P_z\nu(A) = 0$ and $\mathcal{L}(A)/\mathcal{L}(P_z(K)) > 1 - \epsilon$. We have

$$\mathcal{L}(P_z(K) \setminus A) < \epsilon \cdot \mathcal{L}(P_z(K)),$$

hence by Hölder's inequality,

$$\begin{aligned} 1 = P_z\nu(P_z(K) \setminus A) &= \int_{P_z(K)\setminus A} g_z \, d\mathcal{L} \\ &\le \|g_z\|_q \cdot (\mathcal{L}(P_z(K) \setminus A))^{1-q^{-1}} < C \cdot (\epsilon \cdot \mathcal{L}(P_z(K)))^{1-q^{-1}}. \end{aligned}$$

For a sufficiently small $\epsilon > 0$ this clearly yields a contradiction, and so we must have $\beta < 1$.

Fix $z \in S$ and let $A_0 \subset P_z(K)$ be a Borel set with $P_z\nu(A_0) = 0$. It suffices to prove that $\mathcal{L}(A_0) = 0$. We will show that for each $x \in \mathbb{R}$,

$$\limsup_{\delta \downarrow 0} \frac{\mathcal{L}((x-\delta, x+\delta) \cap A_0)}{2\delta} < 1, \tag{7.2}$$

and so that A_0 has no Lebesgue density points. From this and Lebesgue's density theorem it follows that we must have $\mathcal{L}(A_0) = 0$. If $x \notin P_z(K)$ then $(x-\delta, x+\delta) \cap A_0 = \emptyset$ for all sufficiently small $\delta > 0$, and so (7.2) is clearly satisfied.

Fix $x \in P_z(K)$ and $0 < \delta < c/2$. By (7.1) there exists $m \ge 1$ and $i_1 \dots i_m = \boldsymbol{i} \in \Lambda^m$ such that $x \in P_z f_{\boldsymbol{i}}(K) \subset (x-\delta, x+\delta)$ and

$$P_z f_{i_1 \dots i_{m-1}}(K) \not\subseteq (x-\delta, x+\delta), \tag{7.3}$$

where we recall that $f_{\boldsymbol{i}} = f_{i_1} \circ \cdots \circ f_{i_m}$. Let $\alpha \in S$ and $0 < r < 1$ be such that $\lambda = r\alpha$ and set $a = -P_z f_{\boldsymbol{i}}(0)$. Then

$$P_z f_{\boldsymbol{i}} = \tau_a M_{r^m} P_{\alpha^{-m} z}, \tag{7.4}$$

where recall that $\tau_a(y) = y - a$ and $M_{r^m}(y) = r^m y$ for $y \in \mathbb{R}$. By (7.3),

$$r^m \cdot \operatorname{diam}(K) = r \cdot \operatorname{diam}(f_{i_1 \dots i_{m-1}}(K)) \ge r\delta.$$

From this, (7.4) and (7.1),

$$\mathcal{L}(P_z f_{\boldsymbol{i}}(K)) = r^m \cdot \mathcal{L}(P_{\alpha^{-m} z}(K)) > \frac{r\delta c}{\operatorname{diam}(K)} = b\delta, \tag{7.5}$$

with $b = rc/\operatorname{diam}(K)$.

Since ν is self-similar we have $P_z f_{\boldsymbol{i}}\nu \ll P_z\nu$. From this and since $P_z\nu(A_0 \cap P_z f_{\boldsymbol{i}}(K)) = 0$,

$$\begin{aligned} 0 &= P_z f_{\boldsymbol{i}}\nu(A_0 \cap P_z f_{\boldsymbol{i}}(K)) = \tau_a M_{r^m} P_{\alpha^{-m} z}\nu(A_0 \cap \tau_a M_{r^m} P_{\alpha^{-m} z}(K)) \\ &= P_{\alpha^{-m} z}\nu(M_{r^{-m}} \tau_{-a}(A_0) \cap P_{\alpha^{-m} z}(K)). \end{aligned}$$

Now by the definition of β,

$$\mathcal{L}(M_{r^{-m}} \tau_{-a}(A_0) \cap P_{\alpha^{-m} z}(K)) \le \beta \cdot \mathcal{L}(P_{\alpha^{-m} z}(K)).$$

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Hence,

$$\mathcal{L}(A_0 \cap \tau_a M_{r^m} P_{\alpha^{-m}z}(K)) \le \beta \cdot \mathcal{L}(\tau_a M_{r^m} P_{\alpha^{-m}z}(K)),$$

which gives

$$\mathcal{L}(A_0 \cap P_z f_i(K)) \le \beta \cdot \mathcal{L}(P_z f_i(K)).$$

From this and (7.5) we get

$$\begin{aligned}\mathcal{L}((x-\delta, x+\delta) \setminus A_0) &\ge \mathcal{L}(P_z f_i(K) \setminus A_0)\\ &= \mathcal{L}(P_z f_i(K)) - \mathcal{L}(P_z f_i(K) \cap A_0)\\ &\ge (1-\beta)\mathcal{L}(P_z f_i(K)) \ge (1-\beta) b\delta.\end{aligned}$$

Since $(1-\beta)b > 0$ does not depend on δ this gives (7.2), which completes the proof. □

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