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C^r surface diffeomorphisms with no maximal entropy measure

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Abstract. For any $1 \leq r < \infty$, we build on the disk, and therefore on any manifold, a C^r-diffeomorphism with no measure of maximal entropy.

1. *Introduction*

1.1. *Measures of maximal entropy.* Consider a diffeomorphism f of a compact manifold M with non-zero topological entropy $h_{\mathrm{top}}(f) > 0$, i.e., having an exponentially large number of orbits of a given length distinguishable with a fixed precision (see §1.2 for definitions). A natural question is whether there exist invariant measures μ 'describing almost all orbits' in the sense that their entropy $h(f, \mu)$ is maximal and equal to $h_{\mathrm{top}}(f)$. Such measures, if they exist, include ergodic ones, which we will call *maximal (entropy) measures*.

The variational principle states that continuity is enough to ensure the existence of measures with entropy arbitrarily close to $h_{\mathrm{top}}(f)$. Sufficient conditions for achieving this value include uniform hyperbolicity and, by a classical result of Newhouse [**13**], C^∞ smoothness (or a combination [**7**]). Newhouse's theorem is sharp: for every $r < \infty$, there exist C^r smooth dynamical systems on compact manifolds with no maximal measures.

The first counter-examples are due to Misiurewicz [**12**], who built C^r diffeomorphisms of the 4-torus with no measure of maximal entropy for all $r < \infty$. This author has given simple examples for maps on the interval [**4**] (see also [**14**]) and for continuous, piecewise affine maps of the square [**5**]. However, the case of C^r diffeomorphisms in dimensions two and three has remained open. The present paper relies on a construction of Newhouse to provide such counter-examples.

1.2. *Definitions.* For a real number $r \geq 0$, one says that a map $f : M \to N$ is C^r smooth if it is $[r]$ times† differentiable with differential $D^{[r]}f$ Hölder continuous with exponent $r - [r]$ (or just continuous if $r = [r]$, i.e., if $r \in \mathbb{Z}_+$, the set of non-negative integers). Let

† Here, $[r] := \max\{s \in \mathbb{Z} : s \leq r\}$, where $\mathbb{Z}$ is the set of integers.

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$D := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 4\}$. $\mathrm{Diff}_0^r(D)$ denotes the set of C^r diffeomorphisms of D which coincide with the identity on a neighborhood of the boundary of D. $\mathrm{Diff}_0^r(D)$ is endowed with the C^r topology defined by the C^r norm on any finite C^r atlas of D.

We recall† that the *topological entropy* of a continuous map $f : M \to M$ of a compact metric space is

$$h_{\mathrm{top}}(f) := \lim_{\epsilon \to 0} h_{\mathrm{top}}(f, \epsilon) \quad \text{with } h_{\mathrm{top}}(f, \epsilon) := \limsup_{n \to \infty} \frac{1}{n} \log r_f(\epsilon, n)$$

and

$$r_f(\epsilon, n) := \max\{\#S : S \subset M \text{ and } \forall x, y \in S\, x = y \text{ or } d_{f,n}(x, y) \geq \epsilon\}$$

using the Bowen–Dinaburg distances

$$d_{f,n}(x, y) := \max\{d(f^k x, f^k y) : 0 \leq k < n\} \quad \text{for } n \geq 1.$$

One can loosely say that the topological entropy counts the number of orbits.

There is a *measure-theoretic entropy*: to each measure‡ μ is associated a number $h(f, \mu) \in [0, \infty]$, which 'counts the number of orbits seen' by the measure μ. The *variational principle* relates the two notions:

$$h_{\mathrm{top}}(f) = \sup_{\mu \in \mathrm{Prob}(f)} h(f, \mu). \tag{1}$$

Maximal entropy measures are exactly those achieving this supremum (if they exist) and can be interpreted as those seeing almost all the orbits (on a logarithmic scale).

Finally, any measure μ has an ergodic decomposition: $\mu = \int \mu_c \, \hat{\mu}(dc)$ with μ_c an ergodic measure for $\hat{\mu}$-almost every (a.e.) c, and the entropy function is *harmonic* over the set $\mathrm{Prob}(f)$ of invariant probability measures with the vague topology: $h(f, \mu) = \int h(f, \mu_c) \, \hat{\mu}(dc)$. Thus: (i) to exclude the existence of invariant probability measures with maximal entropy, it is enough to exclude that of ergodic ones; (ii) the supremum in the variational principle (1) can be restricted to ergodic measures.

We will need the following two bounds on the entropy of a measure. First, say that a measure is *quasi-periodic* if it is supported by a periodic orbit or inside a topological circle which is conjugate to a rotation. The entropy of a quasi-periodic measure is zero. Second, if the map is C^1, the *Ruelle inequality* states (in the case of surface diffeomorphisms) that for an ergodic measure μ,

$$h(f, \mu) \leq \lambda^u(f, \mu)^+ \quad \text{with } \lambda^u(f, \mu) \text{ the } \mu\text{-a.e. value of } \max_{v \in T_x^1 M} \lambda(x, v), \tag{2}$$

where

$$s^+ := \max(s, 0), \qquad \lambda(x, v) := \liminf_{n \to \infty} \frac{1}{n} \log \|(f^n)'(x) \cdot v\|$$

and

$$T_x^1 M := \{v \in T_x M : \|v\|_x = 1\}.$$

† We refer to [**16**] or [**11**] for background on the entropy theory of continuous maps of compact metric spaces.
‡ From now on, measure means invariant probability Borel measure.

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Here, $\lambda(x, v)$ is the Lyapunov exponent of $v \in T_x M$, whereas $\lambda^u(f, \mu)$ is the maximal Lyapunov exponent of μ. The Oseledets theorem (see, e.g., [**11**]) ensures the existence of such a μ-a.e. constant value.

Let $\mathrm{Lip}(f) := \sup_{x \neq y} d(fx, fy)/d(x, y)$ be the Lipschitz constant of f and set

$$\lambda(f) := \lim_{n \to \infty} \frac{1}{n} \log \mathrm{Lip}(f^n).$$

Clearly, $\lambda(x, v) \leq \lambda(f)$ (in fact, $\lambda(f) = \sup_{(x,v) \in T^1 M} \lambda(x, v)$, but we shall not use this fact and so we omit the rather routine proof). Hence, $h_{\mathrm{top}}(f) \leq \lambda(f)$.

1.3. *Main results.* Let $\mathcal{P}'(f)$ be the set of ergodic, invariant probability measures for f which are not quasi-periodic. Let $\lambda^u(\mathcal{P}'(f)) := \sup_{\nu \in \mathcal{P}'(f)} \lambda^u(f, \nu)$.

THEOREM 1. *There exists $f_0 \in \mathrm{Diff}_0^\infty(D)$ with $h_{\mathrm{top}}(f_0) = 0$ and the following properties. For any number $1 \leq r < \infty$ and any neighborhood $\mathcal{U}_0$ of f_0 in $\mathrm{Diff}_0^r(D)$, there exists $f \in \mathcal{U}_0$ such that:*

(1) $h_{\mathrm{top}}(f) = (1/r)\lambda(f_0) > 0$;
(2) $\lambda^u(\mathcal{P}'(f)) = (1/r)\lambda(f_0)$ *and this supremum is not achieved;*
(3) f *has no measure of maximal entropy.*

Using Ruelle's inequality (2) and the zero entropy of quasi-periodic measures, we obtain the following corollary.

COROLLARY 1. *On any manifold of dimension at least two, for any real number $1 \leq r < \infty$, there exists a C^r-diffeomorphism with finite topological entropy and no measure of maximal entropy.*

We did not expect the above estimates: Ruelle's inequality is typically strict† and, together with the following fact (proved in §5), this would suggest that exponents provide too crude a control.

FACT 1. *In the setting of Theorem 1 with $1 < r < \infty$,*

$$\limsup_{f \to^{C^\infty} f_0} \lambda^u(\mathcal{P}'(f)) = \lambda(f_0) > \limsup_{f \to^{C^r} f_0} h_{\mathrm{top}}(f) = \frac{\lambda(f_0)}{r}.$$

Our constructions also apply in the setting of partial hyperbolicity (see, e.g., [**2**]). It has been shown in that setting, that, when the center bundle is one-dimensional (or even a dominated sum of one-dimensional subbundles), there always is a measure of maximal entropy [**9**]. Our examples show that this does not extend to center dimension two.

COROLLARY 2. *On the d-torus, for any $d \geq 4$, there exists a strongly partially hyperbolic diffeomorphism with center dimension two and no measure of maximal entropy.*

1.4. *Comments.*

Comparison with interval maps. It is instructive to compare the examples presented here with those built for interval maps (see [**4, 14**]). They are very similar in being based on a

† The Ruelle inequality is an equality (Pesin formula) if and only if the measure is of Sinai–Ruelle–Bowen type.

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sequence of horseshoes homoclinically related to a given hyperbolic fixed point and with topological entropies increasing to, but not achieving, the limit $h_{\mathrm{top}}(f)(1/r)\log\Lambda$, where Λ is the expanding eigenvalue at the fixed point.

There is, however, a significant difference which accounts for the relative difficulty (and delay) in building the diffeomorphisms of this paper. In [**4**], there is a sequence of perturbations, each supported by some interval I_n, $n \geq 1$, and giving rise to an N_n-to-one horseshoe which is a forward-invariant disjoint union of intervals $\bigcup_{k=0}^{T_n-1} f^k(I_n)$ and entropy $h_{\mathrm{top}}(f) = (1/T_n)\log N_n$. The absence of a maximal entropy measure follows from the fact that this would be an ergodic one, therefore with support either included in a single horseshoe or disjoint from their union. But one can arrange so that $(1/T_n)\log N_n \nearrow (1/r)\log\Lambda$ while $f|\overline{[0,1]\setminus\bigcup_{n,k} f^k(I_n)}$ has smaller topological entropy (the construction in [**14**] is subtler, but still relies on an explicit topologically Markov description of the dynamics).

In the construction below, the supports of the perturbations do not define invariant cycles. Indeed, the orbit of their supports accumulate along the unstable manifold of the hyperbolic fixed point, hence they must intersect one another.

We note that this seems also to prevent a straightforward modification of these examples to construct infinitely many ergodic measures of maximal entropy, as was done on the interval by taking $(1/T_n)\log N_n$ constant.

Non-differentiable examples. Ergodic theory sometimes differs greatly between C^1 and higher differentiability (see, e.g., Pesin theory or Pugh's closing lemma). However, with regard to existence of measures with maximal entropy, the threshold seems to lie more between Lipschitz and non-Lipschitz homeomorphisms. The C^1 case is not distinguished, though the analysis of the example simplifies significantly (see Remark 4.2). We note that the Lipschitz case is similar to the case of interval maps. In particular, examples with infinitely many maximal measures are easily obtained (see Appendix A, where folklore examples are described for completeness). For piecewise affine *maps* of compact surfaces, there are examples with no maximal measures [**5**]. For piecewise affine *homeomorphisms* of compact surfaces, Newhouse (private communication) observed that maximal measures always exist and this author [**6**] showed that there are finitely many of them.

1.5. *Questions.*

Does 'large entropy' imply existence of a maximal entropy measure? Indeed, the examples built here satisfy $h_{\mathrm{top}}(f) \leq (1/r)\log \mathrm{Lip}(f)$. This is known in the setting of interval maps [**3, 7**].

For diffeomorphisms of three-dimensional compact manifolds, does the existence of a dominated splitting imply that of a maximal entropy measure? We have noted the result of [**9**], which shows that strong partial hyperbolicity (i.e., a splitting $E^s \oplus E^c \oplus E^u$ with no trivial subbundles, hence all of dimension one) implies existence. An intermediate question would be: does (weak) partial hyperbolicity (i.e., a dominated splitting of the type $E^1 \oplus E^2$ with E^1 uniformly expanding—or contracting—and $\dim E^2 = 2$) imply the existence of a maximal entropy measure?

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What about infinite multiplicity? Sarig's construction [**15**] of a countable state Markov shift for $C^{1+\alpha}$ smooth surface diffeomorphisms shows that, for such maps with positive topological entropy, the collection of ergodic measures of maximal entropy is at most countable. However, no examples of C^r diffeomorphisms of compact surfaces with infinitely many ergodic measures with positive maximal entropy are known for $r \geq 1$. We note that it is unknown if a bi-Lipschitz homeomorphism (or even a C^1 diffeomorphism) of a compact surface could have *uncountably* many such measures.

1.6. *Outline of the paper and notation.* We first state the useful properties of the diffeomorphism f_0 (§2). Then we explain the sequence of C^r perturbations used to create entropy for our examples f (§3). The main point is the control of the Lyapunov exponent of non-quasi-periodic orbits (§4). We conclude in §5 with the proofs of the main results.

Two appendices are provided for the convenience of the reader. Appendix A describes the situation for Lipschitz homeomorphisms and general homeomorphisms. Appendix B gives the rather easy details of the construction of the map f_0.

Notation. The notation $X := Y$ means that this is the definition of X. The usual Landau notation of $\mathcal{O}(\cdot)$ and $o(\cdot)$ is used. When we need to be more precise we will use various constants denoted by the same letter C. These constants are numbers that only need to be large enough for the claims to hold. Their eventual dependence on the parameters of our construction, K, L, p_0, ζ (see §§2 and 3), will always be signaled by writing $C(K)$, etc. They will never depend on the orbit or iterate under consideration. The use of $\pm C$ denotes a number $x \in [-C, C]$. Note that, in contrast to C, x can have arbitrary dependencies provided it remains in that interval.

The usual, canonical coordinates of $\mathbb{R}^2$ will be denoted by upper case letters such as $(X, Y) := (\pi_1(p), \pi_2(p))$ for $p \in \mathbb{R}^2$. We shall define maps with four-fold symmetry, i.e. commuting with: $\rho_0(X, Y) = (X, Y)$, $\rho_1(X, Y) = (Y, -X)$, $\rho_2(X, Y) = (-X, -Y)$, $\rho_3(X, Y) = (-Y, X)$.

We will denote by $\mathcal{Q} := [-1/2, -5/12]^2$ the (bottom, left) corner inside which we will make perturbations (duplicated by the symmetries ρ_i). Inside $\rho_i(\mathcal{Q})$, it will often be more convenient to use 'corner coordinates' centered at $\rho_i(-1/2, -1/2)$ (a hyperbolic fixed point in our construction): $(p_1, p_2) := (24\pi_1 \circ \rho_i^{-1}(p) + 12, 24\pi_2 \circ \rho_i^{-1}(p) + 12)$ for a point $p \in \rho_i(\mathcal{Q})$ and $(w_1, w_2) := (24\pi_1 \circ \rho_i'^{-1}(w), 24\pi_2 \circ \rho_i'^{-1}(w))$ for a vector w at such a point p. Unless otherwise specified, coordinates will mean corner coordinates.

We shall use the following closed sets centered at the origin in $\mathbb{R}^2$ defined in canonical coordinates: $D := \overline{B(0, 2)}$ and

$$Q_1 = [-\tfrac{1}{2}, \tfrac{1}{2}]^2 \supset Q_2 = [-\tfrac{5}{12}, \tfrac{5}{12}]^2 \supset Q_3 = [-\tfrac{1}{6}, \tfrac{1}{6}]^2 \supset Q_4 = [-\tfrac{1}{10}, \tfrac{1}{10}]^2.$$

Finally, $\mathrm{int}(E)$ will denote the interior of the set E.

2. *Homoclinic map* $f_0 : D \to D$

We start from a diffeomorphism f_0 on the disk D exhibiting a homoclinic loop at a strongly dissipative, hyperbolic fixed point and depending on two large parameters K and L. The properties we shall use follow. Let $\Lambda := 6/5 > 1$.

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PROPOSITION 2.1. *For all parameters $K > C$, $L > C$, there is a C^∞ smooth vector field such that its time 1 map $f_0 : D \to D$ satisfies the following properties.*

(i) *The map f_0 coincides with the identity near the boundary of D, $f_0 \circ \rho_i = \rho_i \circ f_0$ for $i = 1, 2, 3$, and, on $\mathcal{Q}$, $f_0(x, y) = (K^{-1}x, \Lambda y)$.*

(ii) *The asymptotic Lipschitz constant is $\lambda(f_0) = \log \Lambda$.*

(iii) *All omega-limit sets either are included in an invariant topological circle inside Q_4 or are single points.*

(iv) *The large square $\mathcal{Q}_1$ is invariant: $f_0(Q_1) = Q_1$. Moreover, for all $k \geq C(K, L)$, $f_0^k(Q_1 \setminus Q_3) \subset Q_1 \setminus Q_2$ in canonical coordinates. Moreover, the omega-limit set of any point in that set coincides with ∂Q_1.*

(v) *There is a neighborhood U of $\bigcup_{i=0}^{3} \rho_i([1, 2] \times \{0\}) \subset \partial \mathcal{Q}_1$ such that $f_0^k(U \setminus Q_1) \cap U = \emptyset$ for all $k \geq 1$.*

(vi) *For $p \in f_0(\mathcal{Q}) \setminus \mathcal{Q}$, the transition time to the next corner $\rho_1(\mathcal{Q})$, i.e.*

$$\Theta(p) := \min\{n \geq 1 : f_0^n(p) \in \rho_1(\mathcal{Q})\},$$

satisfies $\Theta(p) = C^{\pm 1}L \pm C^{\pm 1}$. Moreover, $T(p)$ is independent of p_1 (if (p_1, p_2) are the corner coordinates).

(vii) *There are C^∞ smooth functions $\tilde{\alpha} : [2, 2\Lambda] \to (0, \infty)$ and $\tilde{h} : [0, 2] \to [0, 2]$ (depending on K and L) such that, for any $p \in f_0(\mathcal{Q}) \setminus \mathcal{Q}$, if $q := f_0^{\Theta}(p) \in \rho_1(\mathcal{Q})$, then*

$$q_1 = \tilde{h}(p_2) \quad \text{and} \quad q_2 = \tilde{\alpha}(p_2)p_1,$$

with $\Theta := \Theta(p)$ from the previous item and (q_1, q_2) the corner coordinates within $\tau_1(\mathcal{Q})$. Moreover,

$$\tilde{\alpha}(p_2) = K^{-C^{\pm 1}}, \quad |\tilde{\alpha}'(p_2)| \leq C\tilde{\alpha}(p_2), \quad \tilde{h}'(p_2) = C^{\pm 1} \log K. \tag{3}$$

(viii) *There exists a constant $u < 1$, independent of L, such that, for all $p \in \mathcal{Q}$,*

$$(f_0^{\Theta}(p))_1 \leq u \cdot p_1.$$

For the sake of completeness an explicit construction is given in Appendix B. It is completely elementary and without difficulty. The only point we would like to stress is that the transition between two visits to the affine neighborhoods of the hyperbolic fixed points cannot exactly map the (previously) expanding direction to the (to be) contracting direction. Nevertheless, the angular error is small enough (proportional to the distance to the homoclinic loop ∂Q_1) that it does not spoil the estimates.

3. *Perturbations*

We are going to perturb the map $f_0 : D \to D$ to create a sequence of horseshoes with entropies strictly increasing to $(1/r) \log \Lambda$. The resulting map f will have entropy $h_{\text{top}}(f) = (1/r) \log \Lambda$ and no measure of maximal entropy. This would follow immediately *provided there were no other sources of entropy*, but we are not able to prove it directly. Fortunately, one can prove a strong enough bound on the exponents, see §4.

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3.1. *Construction.* We will make perturbations inside the four corners $\rho_i(\mathcal{Q})$, $i = 0, 1, 2, 3$, keeping the four-fold symmetry. Thus we only specify the perturbed maps inside $D^- := \{(X, Y) \in D : X \le 0, Y \le 0\}$ in canonical coordinates. Recall that in corner coordinates (which we use, unless specified otherwise), $\mathcal{Q}$ is identified with $[0, 2]^2$ and $f_0 : \mathcal{Q} \to D$ with the linear map defined by $A = \left(\begin{smallmatrix} K^{-1} & 0 \\ 0 & \Lambda \end{smallmatrix}\right)$.

The perturbations are indexed by integers $p \ge p_0$, with p_0 large (how large will be specified in the course of the paper). The pth perturbation will be supported around $I_p \times \{0\} := [1 + 1/p^2, 1 + 1/p^2 + 1/p^4] \times \{0\}$. It will be made of N_p 'wiggles' pushing upward the unstable manifold of the fixed point $(0, 0)$ by Λ^{-T_p} (the stable manifold is similarly pushed down). It will follow that T_p is approximately equal to the time to grow back these wiggles to size 1, hence to have their image intersecting them. We set

$$T_p := 10^p \quad \text{and} \quad N_p := \Lambda^{T_p/r}/p^5.$$

More precisely, the pth perturbation is supported in

$$R_p := [a_p, b_p] \times [-\ell_p/N_p, \ell_p/N_p] \subset D^- \text{ (corner coordinates)},$$

with $a_p := 1 + 1/p^2$, $b_p := a_p + \ell_p$, $\ell_p := 1/p^4$. For each $p \ge 2$, we define $f_p^{K,L} : D \to D$ (or simply f_p) by using the symmetries ρ_i and setting $f_p|D_-\backslash R_p = f_0$, $f_p|R_p = f_0 \circ g_p$ with

$$g_p : (x, y) \mapsto (x, y + \alpha_p(x, y)\Lambda^{-T_p}\{2 + \sin(\pi N_p(x - a_p)/\ell_p)\}) \tag{4}$$

and cut-off function

$$\alpha_p(x, y) = \alpha(N_p(x - a_p)/\ell_p)\alpha(N_p(b_p - x)/\ell_p)\beta(N_p y/\ell_p),$$

where

- $\alpha : \mathbb{R} \to [0, 1]$ is C^∞ smooth and non-decreasing such that $\alpha'(x) = o(\alpha(x)^{1-1/r})$, $\alpha(x) = 0$ for $x \le 0$ and $\alpha(x) = 1$ for $x \ge 1$;
- $\beta : \mathbb{R} \to [0, 1]$ is C^∞ smooth, such that $\beta(x) = 1$ for $|x| \le 1/2$ and $\beta(x) = 0$ for $|x| \ge 1$ and is monotone on $(-\infty, 0]$ and on $[0, \infty)$.

Finally, observe that $b_p < a_{p-1}$ so the rectangles R_p are pairwise disjoint (for $p - 1 \ge p_0 \ge 2$). Hence we can define $f_{\ge p_0}^{K,L} : D \to D$ (or simply $f_{\ge p_0}$) by setting:

- $f_{\ge p_0}(q) = f_p(q)$ if $q \in \rho_i(R_p)$ for some $p \ge p_0$, $i = 0, 1, 2, 3$;
- $f_{\ge p_0}(q) = f_0(q)$ if $q \in D\backslash\hat{R}$ with $\hat{R} := \bigcup_{i=0,1,2,3} \bigcup_{p \ge p_0} \rho_i(R_p)$.

The following key properties of the perturbed maps are essentially due to the fact that our perturbations push only 'upwards' (in $\mathcal{Q}$) with a reasonable scaling between the widths and the heights of wiggles ($N_p = o(\Lambda^{T_p/r}\ell_p)$).

LEMMA 3.1. *For all $K \ge C$, $L \ge C$, $p_0 \ge C$, the maps $f_{\ge p_0}^{K,L} : D \to D$ are C^r and satisfy:*

(1) $\lim_{p_0 \to \infty} \|f_{\ge p_0}^{K,L} - f_0^{K,L}\|_{C^r} = 0$;
(2) *Q_1 is mapped into itself;*
(3) *between any two visits to $\hat{R} \cap Q_1$ there is a visit to $Q_1\backslash\mathcal{Q}$;*
(4) *any orbit that does not enter Q_1 converges to a subset of an invariant topological circle or a fixed point.*

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Proof. We abbreviate $f^{K,L}_{\geq p_0}$ to f and $f^{K,L}_0$ to f_0.

Observe that the rectangles R_p are pairwise disjoint so that $f_{\geq p_0}$ is welldefined. Thanks to the cut-off functions, the maps f_p are C^∞ and the perturbations $\overline{\{q \in D : f_p(q) \neq f_0(q)\}} \subset \mathrm{int}(R_p)$ are pairwise disjoint. As f_0 itself is C^∞, it follows that the maps $f_{\geq p_0}$ are C^r smooth if

$$\lim_{p\to\infty} \|g_p - \mathrm{id}_{R_p}\|_{C^r} = 0. \tag{5}$$

More precisely,

$$\|f_{\geq p_0} - f_0\|_{C^r} = \sup_{p\geq p_0} \|f_p - f_0\|_{C^r}.$$

Thus, equation (5) will imply claim (1).

A simple computation shows that the partial derivatives of $g_n - \mathrm{id}$ of any order $s \leq r$ are bounded by $C(N_p/\ell_p)^s \Lambda^{-T_p} = o(\Lambda^{-(1-s/r)T_p}) = o(1)$, hence (5) is satisfied.

Next, we have $(f_{\geq p_0}(x, y))_1 = (f_0(x, y))_1 \geq 0$ and $(f_{\geq p_0}(x, y))_2 \geq (f_0(x, y))_2 \geq 0$. Together with item (iv) of Proposition 2.1, this shows claim (2).

Claim (3) follows from claim (2), the formula $f|\mathcal{Q}$: $(f(x, y))_1 \leq K^{-1}x$ for $f|Q$, and $K^{-1}(1 + 1/p_0^2 + 1/p_0^4) < 1$.

Recall the open set U defined in (v) of Proposition 2.1. As $p_0 \geq C$, we can assume $\hat{R} \subset U$. Hence, $f(\hat{R})\backslash Q_1 = f_0(\hat{R})\backslash Q_1 \subset f_0(U\backslash Q_1)$ so, by an easy induction, $f^k(\hat{R})\backslash Q_1 = f_0^k(\hat{R})\backslash Q_1$ does not meet $\hat{R}$ for all $k \geq 1$. Thus, f-orbits that do not enter Q_1 can visit $\hat{R}$ at most once, so eventually coincide with f_0-orbits outside of Q_1. Claim (4) now follows from (iii) of Proposition 2.1. □

LEMMA 3.2. *For all $K \geq C$, $L \geq C$, $p_0 \geq C$, the maps $f := f^{K,L}_{\geq p_0} : D \to D$ satisfy, for all $(x, y) \in \mathcal{Q}$,*

$$\left|\frac{\partial (f(x, y))_2}{\partial x}\right| \leq o((f(x, y)_2)^{1-1/r}).$$

Proof. On the one hand, for $(x, y) \in R_p$, $(g_p(x, y))_2 \geq \alpha_p(x, y)\Lambda^{-T_p}$. On the other hand,

$$\begin{aligned}\left|\frac{\partial \alpha_p(x, y)}{\partial x}\right| &\leq \frac{N_p}{\ell_p}\begin{cases}\alpha'(N_p(x - a_p)/\ell_p)\alpha(N_p(b_p - x)/\ell_p)\beta(N_p y/\ell_p)\\ 0\\ \alpha(N_p(x - a_p)/\ell_p)\alpha'(N_p(b_p - x)/\ell_p)\beta(N_p y/\ell_p)\end{cases}\\ &\leq \frac{N_p}{\ell_p} o(\alpha_p(x, y)^{1-1/r}),\end{aligned}$$

where we have used $|\alpha'(u)| = o(\alpha(u)^{1-1/r})$, $|\alpha(u)|, |\beta(u)| \leq 1$ and distinguished three cases depending on the position of x with respect to $a_p + \ell_p/N_p$ and $b_p - \ell_p/N_p$. Therefore,

$$\begin{aligned}\left|\frac{\partial g_p(x, y)_2}{\partial x}\right| &\leq \Lambda^{-T_p} \times \left(3\left|\frac{\partial \alpha_p(x, y)}{\partial x}\right| + \pi\frac{N_p}{\ell_p}\alpha_p(x, y)\right)\\ &\leq \Lambda^{-T_n}\frac{N_p}{\ell_p}\big(o(\alpha_p(x, y)^{1-1/r}) + \pi\alpha_p(x, y)\big)\\ &\leq o(\Lambda^{-(1-1/r)T_p})\big(o(\alpha_p(x, y)^{1-1/r}) + \pi\alpha_p(x, y)\big)\\ &\leq o(\Lambda^{-(1-1/r)T_p})\alpha_p(x, y)^{1-1/r} \quad (0 \leq \alpha_p(x, y) \leq 1).\end{aligned}$$

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Hence,

$$\left|\frac{\partial g_p(x,\,y)_2}{\partial x}\right| \le o((g_p(x,\,y)_2)^{1-1/r}).$$

Finally, observe that

$$|\partial g_p(x,\,y)_2/\partial x| = \Lambda^{-1}|\partial f(x,\,y)_2/\partial x|$$

and

$$(g_p(x,\,y)_2)^{1-1/r} = \Lambda^{-1}\Lambda^{1/r}(f(x,\,y)_2)^{1-1/r} \quad \text{and} \quad 0 \le f(x,\,y)_2 < 1$$

(it is bounded by $\Lambda\ell_{p_0}/N_{p_0} \le p_0\Lambda^{-(10^{p_0}/r)+1}$ with $p \ge p_0$, large). □

3.2. *Entropy of the horseshoes.* We consider all integers $p \ge p_0$ (which we will increase if necessary a finite number of times).

We show that $h_{\text{top}}(f_p^{T_p}|R_p) \ge \log N_p$. In particular, this will imply $h_{\text{top}}(f_{\ge p_0}) \ge (1/r)\log\Lambda$. Recall that $N_p := [\Lambda^{T_p/r}/p^5]$. We begin by exhibiting a *horseshoe* inside R_p. We use $\|\cdot\|_{\sup}$ to denote the supremum norm.

For $j = 2, \ldots, N_p - 2$, let

$$I_j := [a_p + (j - 1/4)\ell_p/N_p,\, a_p + (j + 1/4)\ell_p/N_p]$$

and let $\mathcal{G}_j$ be the set of curves that, in corner coordinates, are graphs of functions $\phi : I_j \to \mathbb{R}$ satisfying $\|\phi\|_{\sup} < \Lambda^{-T_p-1}/10$ and $\|\phi'\|_{\sup} < K^{-T_p}$. Observe that any such curve is contained in R_p as $\Lambda^{-T_p-1}/10 = o(\ell_p/N_p)$.

LEMMA 3.3. *Let $K \ge C$, $p_0 \ge C(K)$, and let f be $f^{K,L}_{p\ge p_0}$ or $f^{K,L}_p$ for some $p \ge p_0$.*

For each couple $j, k = 1, \ldots, N_p - 1$, for any $i = 0, 1, 2, 3$ and any $\Gamma \in \mathcal{G}_j$, there exists $\Gamma' \in \mathcal{G}_k$ such that $\rho_{i+1}^{-1} \circ f^{T_p}(\rho_i(\Gamma)) \supset \Gamma'$.

Proof. By symmetry, we can assume $i = 0$. We have that $\Gamma \in \mathcal{G}_j$ is the graph of some function $\phi : I_j \to \mathbb{R}$. Let $(x,\,y) := f_p(x_0,\,y_0) \in f_p(\Gamma)$. Observe that $|x| \le K^{-1}|x_0|$ and $|\cos(\pi N_p(x_0 - a_p)/\ell_p)| \ge \cos(\pi/4)$ and $\alpha_p(x_0,\,y_0) = 1$. Also $|y| \le |y_0| + 3\Lambda^{-T_p} \le 3\Lambda^{-T_p+1}$, y taking all values in $[2\Lambda^{-T_p},\, 3\Lambda^{-T_p}]$.

It follows that the slope of $f_p(\Gamma)$ is not extremely small:

$$\begin{aligned}\left|\frac{dy}{dx}\right| &\ge \frac{\pi}{\sqrt{2}}\Lambda^{-(1-1/r)T_p}/p - K^{-T_p}\\ &\ge C^{-1}(K\Lambda)\Lambda^{-(1-1/r)T_p}/p.\end{aligned}$$

An immediate induction shows that

$$f^k(f_p(\Gamma)) \subset [0,\,1] \times [0,\,3\Lambda^{-T_p+1+k}] \subset \mathcal{Q}$$

(in corner coordinates) for $1 \le k \le T_p + 9$. Thus, $f_p^{T_p+3}(\Gamma)$ contains a piece which is at a Hausdorff distance at most CK^{-T_p} to a vertical segment containing $\{0\} \times [2,\,3] \subset f(\mathcal{Q})\backslash\mathcal{Q}$ with large slope, at least

$$C^{-1}(K\Lambda)^{T_p}\Lambda^{-(1-1/r)T_p}/p = C^{-1}\Lambda^{T_p/r}K^{T_p}/p \to \infty \quad \text{as } p_0 \to \infty.$$

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Applying the (non-linear) transition f_0^{Θ}, which, according to Proposition 2.1 and its item (vi), has differential $\begin{pmatrix} \mathcal{O}(1) & K^{C^{\pm 1}}x \\ 0 & C^{\pm 1}\log K \end{pmatrix}$ with respect to corner coordinates respective to $\mathcal{Q}$ and $\rho_1(\mathcal{Q})$, we get a curve in the CK^{-T_p}-neighborhood of $\{-1/2\} \times [0, 1/6]$ with slope $|dx/dy| \le K^{-T_p}$. Such a curve contains the image by ρ_1 of some graph $\Gamma' \in \mathcal{G}_k$, proving the claim. □

Iterating the above lemma m times, one gets at least $(N_n - 1)^m$ orbits which are $(\frac{1}{2}\ell_p/N_p, mT_p)$-separated. Taking the limit $m \to \infty$, we obtain the following corollary.

COROLLARY 3.4. *For any $K \ge C$ and $L \ge C$, for any $\zeta > 0$, any $p_0 \ge C(K, L, \zeta)$, the map $f = f_{\ge p_0}^{K,L}$ is C^r and satisfies*

$$\|f - f_0\|_{C^r} < \zeta \quad \text{and} \quad h_{\text{top}}(f) \ge \sup_{p \ge p_0} \frac{1}{T_p} \log(N_p - 1) = \frac{1}{r} \log \Lambda.$$

4. *Lyapunov exponent*

We consider $f = f_{\ge p_0}^{K,L}$ for arbitrary parameters L, K, p_0 such that

$$K \ge C, \quad L \ge C(K) \quad \text{and} \quad p_0 \ge C(K, L)$$

for lower bounds that will be specified in the course of this section.

Each non-zero vector $(x, v) \in \bigcup_{x \in D} T_x^1 D$ defines a (lower) exponent

$$\lambda(x, v) := \liminf_{n \to \infty} \frac{1}{n} \log \|Df^n(x) \cdot v\|.$$

We define, in corner coordinates,

$$\Delta := \bigcup_{i=0}^{3} \rho_i([2K^{-1}, 2] \times (0, 2]). \tag{6}$$

PROPOSITION 4.1. *For all $K \ge C_1$, $L \ge C_2(K)$, and $p_0 \ge C_3(K, L)$, there is $\chi > 0$ such that, for any $x \in \text{int}(Q_1 \setminus Q_3)$ and all $v \in \mathbb{R}^2 \setminus \{0\}$,*

$$\lambda(x, v) < \frac{1}{r} \log \Lambda - \chi \cdot \liminf_{n \to \infty} \frac{1}{n} \#\{0 \le k < n : f^k x \in \Delta\}. \tag{7}$$

Remark 4.2. The case $r = 1$ of the bound (7) follows immediately from the fact that, for $x \in Q_1$, $\|f'(x) \cdot v\| \le \Lambda \|v\|$, and that equality holds for all forward iterates of x if and only if the orbit of x converges to one of the vertices of Q_1, i.e., if $x \in \partial Q_1$.

Standing Assumption 1. We have $K, L \ge C$ so that Proposition 2.1 applies.

4.1. *Division of the orbit.*

Let $x \in \text{int}(Q_1 \setminus Q_3)$ and $v \in \mathbb{R}^2 \setminus \{0\}$. We call the maximal intervals of integers $t \in \mathbb{Z}$ such that $f^t(x)$ belongs to $\mathcal{Q}$ *affine segments*. By replacing (x, v) with some iterate (which does not change $\lambda(x, v)$), we ensure $x \in \mathcal{Q} \setminus f(\mathcal{Q})$. We denote the affine segments contained in $\mathbb{Z}_+$ by $[t_0, s_1] < [t_1, s_2] < \cdots$ (there are infinitely many of them—see (iv) in Proposition 2.1).

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According to claim (3) of Lemma 3.1, the integers t_n are the only integers $t \geq 0$ for which $f^t(x)$ may belong to the support R_p of a perturbation (for some $p \geq p_0$). Note also that, for all $t \geq 0$, $f^t x \in \mathcal{Q} \backslash f(\mathcal{Q})$ if and only if $f^t x \in \Delta$. In particular,

$$\{t_0 < t_1 < t_2 < \cdots\} = \{t \geq 0 : f^t x \in \Delta\}.$$

We also define $\tau_n := t_n - t_{n-1}$, $\tau_n' := s_n - t_{n-1}$, $x(t) := f^t(x)$, $v(t) := (f^t)'(x) \cdot v$, $x_n := f^{t_n}(x)$ and $v_n := (f^{t_n})'(x) \cdot v$.

For points $x(t) \in \rho_i([-1/2, -5/12] \times [-1/2, 0))$ (in canonical coordinates), we use corner coordinates. They will be denoted by $(x(t)_1, x(t)_2)$. We use the same rotating coordinates for $v(t) \in T^1_{x(t)} D$ and denote their absolute value by $(v(t)_1, v(t)_2)$. We also define $\theta_n, \tilde{\theta}_n \in [0, \infty]$ by $(v_n)_2 = \theta_n \cdot (v_n)_1$ and $v(s_n)_1 = \tilde{\theta}_n \cdot v(s_n)_2$.

The following lemmas are straightforward applications of the previous estimates from Proposition 2.1 and Lemma 3.2 to the sequences $x(t)$, $v(t)$, $t \geq 0$.

LEMMA 4.3. *From $t_{n-1} + 1$ to s_n:*
(1) $x(s_n)_1 = K^{-\tau_n'+1} x(t_{n-1} + 1)_1$ *and* $x(s_n)_2 = \Lambda^{\tau_n'-1} x(t_{n-1} + 1)_2$;
(2) $v(s_n)_1 = K^{-\tau_n'+1} v(t_{n-1} + 1)_1$ *and* $v(s_n)_2 = \Lambda^{\tau_n'-1} v(t_{n-1} + 1)_2$.

LEMMA 4.4. *From t_n to $t_n + 1$, if $x_n \in R_p$:*
(1) $x(t_n + 1)_1 = K^{-1}(x_n)_1$ *and* $x(t_n + 1)_2 = \Lambda((x_n)_2 + (2 \pm 1)\alpha_p(x_n)\Lambda^{-T_p})$;
(2) $v(t_n + 1)_1 = K^{-1}(v_n)_1$ *and* $v(t_n + 1)_2 = \Lambda((v_n)_2 \pm C(x(t_n + 1)_2)^{1-1/r}(v_n)_1)$.

LEMMA 4.5. *From s_n to t_n:*
(1) $(x_n)_1 = \tilde{h}(x(s_n)_2)$ *and* $(x_n)_2 = \tilde{\alpha}(x(s_n)_2)x(s_n)_1$;
(2) $(v_n)_1 = \tilde{h}'(x(s_n)_2)v(s_n)_2$ *and* $(v_n)_2 = |\tilde{\alpha}(x(s_n)_2)v(s_n)_1 + \tilde{\alpha}'(x(s_n)_2)x(s_n)_1 v(s_n)_2|$.
From equation (3): $\tilde{\alpha}(p_2) = K^{-C^{\pm 1}}$, $|\tilde{\alpha}'(p_2)| \leq C\tilde{\alpha}(p_2)$ *and* $\tilde{h}'(p_2) = C^{\pm 1} \log K$.

The lengths of the affine segments cannot grow too fast.

LEMMA 4.6. *For every $n \geq 1$,*

$$K^{-\tau_n'} \Lambda^{\tau_{n+1}'} \leq K^C.$$

Proof. Using, Lemmas 4.3, 4.4, 4.5, 4.3, and 4.4, in that order,

$$\begin{aligned} x(s_{n+1})_2 &\geq \Lambda^{\tau_{n+1}'}(x_n)_2 = \Lambda^{\tau_{n+1}'} \tilde{\alpha}(x(s_n)_2)x(s_n)_1 \geq \Lambda^{\tau_{n+1}'} K^{-C} x(s_n)_1 \\ &\geq K^{-C} \Lambda^{\tau_{n+1}'} K^{-\tau_n'+1} x(t_{n-1} + 1)_1 \\ &\geq K^{-C} \Lambda^{\tau_{n+1}'} K^{-\tau_n'} (x_{n-1})_1. \end{aligned}$$

As $x(s_{n+1})_2 \leq C$ and $(x_{n-1})_1 \geq C^{-1}$, we get

$$C \geq K^{-C} \Lambda^{\tau_{n+1}'} K^{-\tau_n'}.$$ □

The affine segments are very long.

LEMMA 4.7. *For $p_0 \geq C(K)$, for all large enough n,*

$$\tau_n' \geq T_{p_0}/3r.$$

Proof. We split it into three claims.

Claim 1. If $x_n \in R_p$ with $p \geq p_0$, then $\tau_{n+1}' \geq T_p/2r - C \log p \geq T_{p_0}/2.5r$.

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Observe that $x_n \in R_p$ for some $p \geq p_0$ implies that $(x_n)_2 \leq p\Lambda^{-T_p/r}$ (the top of R_p) so $x(t_n+1)_2 \leq p\Lambda^{-T_p/r+1}$ (the top of $f(R_p)$) and, by Lemma 4.3, $\tau'_{n+1} \geq T_p/2r - C\log p \geq T_{p_0}/2r - C\log p_0$ (using $T_p := 10^p$), which proves the claim for $p_0 \geq C$.

Claim 2. We have that $\limsup_{n\to\infty} \tau'_n > T_{p_0}/3r$.

By Claim 1, $\limsup_{n\to\infty} \tau'_n \leq T_{p_0}/3r$ implies that the orbit of x eventually coincides with an orbit of f_0. Item (iv) of Proposition 2.1 then implies $\lim_{n\to\infty} \tau'_n = \infty$, a contradiction.

Claim 3. If $x_n \notin \bigcup_{p\geq p_0} R_p$, then $\tau'_{n+1} \geq \eta\tau'_n - C\log K$, where $\eta := \log K/\log\Lambda > 1$.

Using Lemmas 4.3, 4.5 and 4.3, as well as $x(s_{n+1})_2 = C^{\pm 1}$ and $(x_{n-1})_1 = C^{\pm 1}$,

$$\begin{aligned} C^{\pm 1} &= x(s_{n+1})_2 = \Lambda^{\tau'_{n+1}}(x_n)_2 = \Lambda^{\tau'_{n+1}} K^{-C^{\pm 1}} x(s_n)_1 \\ &= \Lambda^{\tau'_{n+1}} K^{-C^{\pm 1}} K^{-\tau'_n}(x_{n-1})_1 \\ &= K^{\pm C}\Lambda^{\tau'_{n+1}} K^{-\tau'_n}. \end{aligned}$$

Thus, $\tau'_{n+1} = \eta\tau'_n \pm C\log K$, hence the claim.

By taking $p_0 \geq C(K)$, we ensure

$$(\eta - 1)T_{p_0}/3r > 2C\log K \text{ and } C\log p_0 < T_{p_0}/6r.$$

From Claim 2, there is some $\tau'_{n_*} \geq T_{p_0}/3r$. Assuming by induction $\tau'_n \geq T_{p_0}/3r$, we obtain, for all $n \geq n_*$, applying Claim 1 or 3,

$$\tau'_{n+1} \geq \min((1 + (\eta-1)/2)\tau'_n,\, T_{p_0}/3) \geq T_{p_0}/3r. \qquad \square$$

We can therefore make the following assumption.

Standing Assumption 2. We have $p_0 \geq C(K)$ and, for all $n \geq 0$, $\tau'_n \geq T_{p_0}/3r$.

LEMMA 4.8. *We have*

$$\theta_n = K^{-C^{\pm 1}}(\tilde{\theta}_n + \mathcal{O}(x(s_n)_1)) = K^{-C^{\pm 1}}(\tilde{\theta}_n + \mathcal{O}(K^C\Lambda^{-\tau'_{n+1}})).$$

Observe that if the transition map was a translation, one would simply have $\tilde{\theta}_n = \theta_n$.

Proof. Lemma 4.5 yields

$$\begin{aligned} \theta_n = \frac{(v_n)_2}{(v_n)_1} &= \frac{\tilde{\alpha}(x(s_n)_2)v(s_n)_1 + \tilde{\alpha}'(x(s_n)_2)x(s_n)_1 v(s_n)_2}{\tilde{h}'(x(s_n)_2)v(s_n)_2} \\ &= \frac{\tilde{\alpha}(x(s_n)_2)}{\tilde{h}'(x(s_n)_2)}(\tilde{\theta}_n + \mathcal{O}(x(s_n)_1)) \\ &= \frac{K^{-C^{\pm 1}}}{\log K}(\tilde{\theta}_n + \mathcal{O}(x(s_n)_1)). \end{aligned}$$

Hence, $K^{-C^{\pm 1}}/\log K = K^{-C^{\pm 1}}$ and, using Lemmas 4.5, 4.4 and 4.3, we obtain

$$x(s_n)_1 = \mathcal{O}(K^C(x_n)_2) = \mathcal{O}(K^C x(t_n+1)_2) = \mathcal{O}(K^C\Lambda^{-\tau'_{n+1}}). \qquad \square$$

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4.2. *Expansion during one affine segment.* We bound the expansion during affine segments, distinguished according to the value of the tangent $\theta_n = (v_n)_2/(v_n)_1$ (in corner coordinates).

Definition 4.9. A time t_n is said to be *special* if $\theta_n > \Lambda^{-(1-1/r)\tau_{n+1}}$.

Note that a similar definition could have been made in terms of $\tilde{\theta}_n$ (see Lemma 4.8).

LEMMA 4.10. *For all $n \geq 1$, if t_{n-1} is not special, then*

$$|v_n| \leq C \log K \cdot \Lambda^{\tau'_n/r} |v_{n-1}|$$

($|\cdot|$ is the Euclidean norm).

Recall that constants such as C, $C(K)$, $C(K, L)$ do not depend on (x, v) or $n \geq 0$.

Proof. Using Lemmas 4.3 and 4.4 and $x(t_{n-1}+1)_2 \leq C\Lambda^{-\tau'_n}$,

$$\begin{aligned} |v(s_n)| &\leq v(s_n)_1 + v(s_n)_2 \leq K^{-\tau'_n}|v_{n-1}| + \Lambda^{\tau'_n - 1}(v(t_{n-1}+1))_2 \\ &\leq K^{-\tau'_n}|v_{n-1}| + \Lambda^{\tau'_n}(\theta_{n-1} + (x(t_{n-1}+1)_2)^{1-1/r})(v_{n-1})_1 \\ &\leq K^{-\tau'_n}|v_{n-1}| + \Lambda^{\tau'_n}(\theta_{n-1} + C\Lambda^{-(1-1/r)\tau'_n})|v_{n-1}|. \end{aligned}$$

Hence, as $\theta_{n-1} \leq \Lambda^{-(1-1/r)\tau'_n}$,

$$\begin{aligned} |v(s_n)| &\leq (1 + C\Lambda^{\tau'_n}\Lambda^{-(1-1/r)\tau'_n})|v_{n-1}| \\ &\leq (1 + C\Lambda^{\tau'_n/r})|v_{n-1}| \\ &\leq C\Lambda^{\tau'_n/r}|v_{n-1}| \end{aligned}$$

and, using Lemma 4.5,

$$|v_n| \leq C(\log K)|v(s_n)| \leq C(\log K)\Lambda^{\tau'_n/r}|v_{n-1}|. \qquad \square$$

LEMMA 4.11. *For all $n \geq 2$, if t_{n-1} is special, then*

$$|v_n| \leq K^C |v_{n-2}|.$$

Proof. Let n be such that t_{n-1} is special, i.e., $\theta_{n-1} > \Lambda^{-(1-1/r)\tau'_n}$. Using Lemma 4.8,

$$\begin{aligned} \frac{\tilde{\theta}_{n-1}}{\theta_{n-1}} &\geq K^{-C} - C\frac{K^C\Lambda^{-\tau'_n}}{\theta_{n-1}} \geq K^{-C} - \frac{K^C\Lambda^{-\tau'_n}}{\Lambda^{-(1-1/r)\tau'_n}} \\ &\geq K^{-C} - C\Lambda^{-\tau'_n/r} \geq K^{-C}, \end{aligned} \tag{8}$$

as $\tau'_n \geq T_{p_0}/3r$ is large. By definition, $(v_{n-1})_2 = \theta_{n-1}(v_{n-1})_1$ and $v(s_{n-1})_1 = \tilde{\theta}_{n-1}v(s_{n-1})_2$. Hence

$$(v_{n-1})_2 \leq \min(\theta_{n-1}, 1)|v_{n-1}| \quad \text{and} \quad |v(s_{n-1})| \leq \frac{2}{\min(\tilde{\theta}_{n-1}, 1)}v(s_{n-1})_1.$$

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From Lemma 4.5, $|v_n| \le C(\log K)|v(s_n)|$ and, using Lemmas 4.3, 4.4 and that t_{n-1} is special, we obtain

$$\begin{aligned}
|v(s_n)| &\le \Lambda^{\tau'_n-1} v(t_{n-1}+1)_2 + K^{-\tau'_n+1} v(t_{n-1}+1)_1 \\
&\le \Lambda^{\tau'_n}((v_{n-1})_2 + C(x(t_{n-1}+1)_2)^{1-1/r}(v_{n-1})_1) + K^{-\tau'_n}(v_{n-1})_1 \\
&\le \Lambda^{\tau'_n}((v_{n-1})_2 + C\Lambda^{-(1-1/r)\tau'_n}(v_{n-1})_1) + K^{-\tau'_n}(v_{n-1})_1 \\
&\le \Lambda^{\tau'_n}(v_{n-1})_2 + (C\Lambda^{\tau'_n/r} + K^{-\tau'_n})(v_{n-1})_1 \\
&\le (\Lambda^{\tau'_n} \cdot 2\min(\theta_{n-1}, 1) + C\Lambda^{\tau'_n/r} + K^{-\tau'_n})|v_{n-1}| \\
&\le C\Lambda^{\tau'_n}\min(\theta_{n-1}, 1)|v_{n-1}|.
\end{aligned}$$

We also have $|v_{n-1}| \le C(\log K)|v(s_{n-1})|$ and

$$|v(s_{n-1})| \le \frac{2}{\min(\tilde{\theta}_{n-1}, 1)} v(s_{n-1})_1 \le \frac{2K^{-\tau'_{n-1}}}{\min(\tilde{\theta}_{n-1}, 1)}|v_{n-2}|.$$

Collecting the inequalities above and using Lemma 4.6 and (8),

$$\begin{aligned}
|v_n| &\le C(\log K)^2 \frac{\min(\theta_{n-1}, 1)}{\min(\tilde{\theta}_{n-1}, 1)} \Lambda^{\tau'_n} K^{-\tau'_{n-1}} |v_{n-2}| \\
&\le K^C |v_{n-2}|.
\end{aligned}$$

□

4.3. *Proof of Proposition 4.1.* We observe that replacing (x, v) by $(f^t x, (f^t)'(v))$ does not affect the claim in Proposition 4.1. Thus we can continue to assume $t_0 = 0$ and to make the standing assumptions 1 and 2. These assumptions will allow the application of Corollary 3.4 and Lemmas 4.10 and 4.11.

If necessary we increase the lower bounds $L \ge C(K)$ and $p_0 \ge C(K)$ to ensure that (for any given $\epsilon > 0$, in fact $\epsilon := 1$ is enough for our purposes)

$$1000 \cdot C \log K < \Lambda^{C^{-1}L} \quad \text{and} \quad 1000^2 \cdot K^C < \Lambda^{2\epsilon T_{p_0}/(3r^2)}.$$

As $\tau_n = \tau'_n + C^{\pm 1}L$, Lemmas 4.10 and 4.11 imply, respectively,

$$|v_n| \le \frac{1}{1000}\Lambda^{\tau_n/r}|v_{n-1}| \quad \text{and} \quad |v_n| \le \frac{1}{1000^2}\Lambda^{\epsilon(\tau_n+\tau_{n-1})/r}|v_{n-2}|. \tag{9}$$

We shall use the following groupings of affine segments for (x, v) chosen as in the statement of the proposition.

Definition 4.12. A *special block* is an integer interval $[t_{n-2}, t_n) \subset \mathbb{Z}_+$ such that t_{n-1} is special. A *normal block* is a non-empty integer interval $[t_{n-1}, t_n) \subset \mathbb{Z}_+$ such that t_{n-1} is not special.

We prove Proposition 4.1 in two steps.

Claim 4.13. For any $N \ge 1$, there exists a partition of $[0, t_N)$ or $[t_1, t_N)$ into a disjoint union of special blocks and normal blocks.

To define the partition into blocks, let $a_1 := t_N$ and, inductively,

$$a_{k+1} := \min\{t_j : j \ge 0 \text{ such that } [t_j, a_k) \text{ is a normal or special block}\}.$$

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We set $I \geq 1$ to be the minimum such that a_{I+1} is not defined. For $k = 1, \ldots, I$, we let j_k denote the unique integer such that $a_k = t_{j_k}$. Let us check that

$$j_I \text{ is 0 or 1.} \tag{10}$$

Assume by contradiction $j_I \geq 2$. If t_{j_I-1} is special then $[t_{j_I-2}, t_{j_I}) \subset [0, t_N)$ is a special block and one could set $a_{I+1} = t_{j_I-2}$. Otherwise $[t_{j_I-1}, t_{j_I}) \subset [0, t_N)$ is a normal block and one could set $a_{I+1} = t_{j_I-1}$. In both cases, this contradicts the definition of I, proving (10).

Claim 4.14. There exists $C(x, v) < \infty$, independent of N, such that

$$|v_N| \leq C(x, v)\Lambda^{t_N/r}/1000^N. \tag{11}$$

Lemma 4.10 (or rather its rephrasing in equation (9)) states that if $[a_{k+1}, a_k)$ is a normal block, then $|v(a_k)| \leq (\Lambda^{(a_k-a_{k+1})/r}/1000)|v(a_{k+1})|$ and $j_k = j_{k+1} + 1$. Lemma 4.11 (same remark) shows that if $[a_{k+1}, a_k)$ is a special block then

$$|v(a_k)| \leq \Lambda^{(a_k-a_{k+1})/r}|v(a_{k+1})|/1000^2 \quad \text{and} \quad j_k = j_{k+1} + 2$$

in this case. We obtain

$$|v_N| \leq C(x, v) \prod_{i=1}^{I} \frac{\Lambda^{(a_i-a_{i+1})/r}}{1000^{j_i-j_{i+1}}} \cdot \frac{\Lambda^{a_I/r}}{1000^{a(I)}} \leq C(x, v)\Lambda^{t_N/r}/1000^N$$

with $C(x, v) := |v_{a(I)}|\Lambda^{-a_I/r}1000^{a(I)} < \infty$. Equation (11) is established.

To conclude the proof of Proposition 4.1, one observes that N is equal to the number of visits to Δ (recall equation (6)) before time t_N.

5. *Proof of the main results*

Proof of Theorem 1. We fix the parameters $K \geq C$, $L \geq C(K)$, as in Corollary 3.4 and Proposition 4.1. Let $f_0 := f_0^{K,L}$. Note that $h_{\text{top}}(f_0) = 0$ as f_0 is the time 1 map of a vector field on the two-dimensional disk.

Given a neighborhood $\mathcal{U}_0$ of f, it contains the ball $B(f_0, \zeta)$ in the C^r norm for some $\zeta > 0$. Now, for all $p_0 \geq C(K, L, \zeta)$, $f := f_{\geq p_0}^{K,L} \in \mathcal{U}_0$ by Corollary 3.4.

Let $\mu \in \mathcal{P}'(f)$. Observe that μ cannot satisfy $\mu(\text{int}(Q_1 \backslash Q_2)) = 0$ as it would be an invariant measure for f_0 (see claim (4) of Lemma 3.1) and therefore be quasi-periodic according to item (iii) of Proposition 2.1. By ergodicity and the invariance from claim (2) of Lemma 3.1, $\mu(\text{int}(Q_1 \backslash Q_2)) = 1$. Thus, the μ-a.e. orbit visits the set Δ (equation (6)), implying $\mu(\Delta) > 0$.

According to Proposition 4.1 (and the Birkhoff theorem applied to the characteristic function of Δ), $\lambda(x, v) < (1/r) \log \Lambda$ for μ-a.e. x and all non-zero $v \in \mathbb{R}^2$. In particular, the unstable Lyapunov exponent of μ is strictly less than $(1/r) \log \Lambda$. Using Ruelle's inequality,

$$h(f, \mu) \leq \lambda(f, \mu)^+ < \frac{1}{r} \log \Lambda. \tag{12}$$

The variational principle now implies

$$h_{\text{top}}(f) = \sup_{\mu \in \mathcal{P}'(f)} h(f, \mu) \leq \frac{1}{r} \log \Lambda,$$

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as quasi-periodic measures have zero entropy. Comparing with Corollary 3.4, we get $h_{\text{top}}(f) = (1/r)\log\Lambda$. Comparing with equation (12), we see that f admits no maximal measure.

The same variational principle gives the inequality $\lambda^u(\mathcal{P}'(f)) \geq h_{\text{top}}(f)$ so $\lambda^u(\mathcal{P}'(f)) = (1/r)\log\Lambda$. □

Proof of Corollary 1. Let M be any manifold of dimension $d \geq 2$. Let $K := D \times \mathbb{T}^{d-2}$ (recall D is a closed disk in $\mathbb{R}^2$) and $F(x, t) = (f(x), t)$. Observe that $\partial K = \partial D \times \mathbb{T}^{d-2}$, so F is a C^r diffeomorphism coinciding with the identity near the boundary of K; M contains a d-dimensional ball and therefore a diffeomorphic copy of K. Define a C^r diffeomorphism G by setting $G|M\backslash K \equiv \text{Id}$ and $G|K = F$. According to the variational principle for ergodic measures, $h_{\text{top}}(G) = h_{\text{top}}(F) > 0$.

Observe that any ergodic measure with maximal entropy for G must be carried by K. But then its projection by $K \to D$ is a measure for f with the same entropy, hence of maximal entropy for f, a contradiction: G has no measure of maximal entropy. □

Proof of Fact 1. Observe that $\lambda(f) = \inf_{n\geq 1}(1/n)\log \text{Lip}(f^n)$, hence $f \mapsto \lambda(f)$ is upper semicontinuous: $\limsup_{f\to C^1 f_0} \lambda(f) \leq \lambda(f_0)$. Thus,

$$\limsup_{f\to C^r f_0} h_{\text{top}}(f) \leq \lambda(f_0)/r$$

is a corollary of Yomdin's theory [**17**, Theorem 1.9] using $h_{\text{top}}(f_0) = 0$. The reverse inequality follows from Theorem 1. The second equality in Fact 1, $\limsup_{f\to C^r f_0} h_{\text{top}}(f) = \lambda(f_0)/r$, is proved.

For the first equality, observe that Lyapunov exponents are bounded by the logarithm of the Lipschitz constant, hence the following inequality is obvious:

$$\limsup_{f\to C^1 f_0} \lambda^u(\mathcal{P}'(f)) \leq \lambda(f_0).$$

To conclude, we show C^∞ perturbations f of f_0 with $\lambda^u(\mathcal{P}'(f)) \geq \lambda := \lambda(f_0)$. Let us sketch their construction.

To build such diffeomorphisms f, we make a perturbation around $[a_p, b_p] \times \{0\}$ as in §3 (recall $a_p = 1 + 1/p^2$, $b_p = a_p + 1/p^4$). We need to remain Λ^{-T_p}-close to the axis to get a large period $T_p = 10^p$. To obtain a large exponent, however, we need a relatively large slope. We get it by using transverse intersections of the (perturbed) unstable manifold with the (intact) stable manifold. We replace the map g_p of (4) from §3 by

$$\bar{g}_p(x, y) = (x, y + \phi_p(x, y)) \quad \text{where } \phi_p(x, y) := \bar{\alpha}_p(x, y)e^{-(\log p)^2}\cos(10\pi p^4 x)$$

with the cut-off $\bar{\alpha}_p(x, y) = \alpha(10p^4(x - a_p))\alpha(10p^4(b_p - x))\beta(p^4 y)$ (using α and β from §3). We thus obtain maps $\bar{f}_p$ and $\bar{f}_{\geq p}$.

Observe that $\|f_0 - \bar{f}_p\|_{C^r} \leq (Cp)^{4r}e^{-(\log p)^2}$ for any fixed, finite r. It follows that $\bar{f}_{\geq p}$ are C^∞ maps and $\lim_{p\to\infty}\|f_0 - \bar{f}_{\geq p}\|_{C^r} = 0$.

Consider a small neighborhood U_p of the graph of $x \mapsto \phi_p(x, 0)$ restricted to the values $x \in [a_p, b_p]$ such that $\phi_p(x, 0) \in [(5/12)\Lambda^{-T_p}, \Lambda^{-T_p}]$. The absolute value of the slope there is at least $p^4 e^{-(\log p)^2}\Lambda^{-T_p}$. The first return time to U_p is $T_p \pm C$. The intersection

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of U_p with $\bar{f}^{T_p \pm C}$ is easily seen to be hyperbolic. We thus obtain a horseshoe as the maximal invariant set for U_p under $\bar{f}^{T_p \pm C}$.

It is routine to check that the Lyapunov exponent of any ergodic measure supported by this horseshoe is at least

$$\frac{1}{T_p + C}(T_p \lambda - (\log p)^2 + 4 \log p - C) \to \lambda,$$

as $T_p = 10^p \geq p$. Thus $\lambda^u(\mathcal{P}'(\bar{f}_{\geq p_0})) = \lambda$ for any p_0. □

Proof of Corollary 2. Let $d \geq 2$. Theorem 1 gives a C^r diffeomorphism $f : \mathbb{T}^2 \to \mathbb{T}^2$ with no maximal measure. We then follow an observation of M. A. Rodriguez-Hertz and the author (unpublished). Let $\phi : \mathbb{T}^{d-2} \to \mathbb{T}^{d-2}$ be a hyperbolic automorphism. Set

$$\Lambda^u := \min\{\Lambda \in \mathrm{sp}(\phi) : |\Lambda| > 1\} \quad \text{and} \quad \Lambda^s := \max\{\Lambda \in \mathrm{sp}(\phi) : |\Lambda| < 1\}.$$

Let $N \geq 1$ be a large enough integer such that $(\Lambda^u)^N > \mathrm{Lip}(f)$ and $(\Lambda^s)^N < \mathrm{Lip}(f^{-1})^{-1}$. Define $F : \mathbb{T}^4 \to \mathbb{T}^4$ by $F(x, y) = (\phi^N(x), f(y))$. It is easy to check that F is partially hyperbolic.

If there was a measure μ with maximal entropy for $F = \phi^N \times f$, its projections $\pi_1\mu$, $\pi_2\mu$ would satisfy $h(F, \mu) \leq h(\phi^N, \pi_1\mu) + h(f, \pi_2\mu)$. As $h_{\mathrm{top}}(F) = h_{\mathrm{top}}(\phi^N) + h_{\mathrm{top}}(f)$, the projections would have to have a measure with maximal entropy for ϕ^N and f. Hence F, like f, has no measure of maximal entropy. □

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A. *Appendix. Non-differentiable examples*

A.1. *Lipschitz examples.*

Recall that a bi-Lipschitz transformation $h : X \to X$ of a metric space X is a bijection which, together with its inverse, is Lipschitz. The bi-Lipschitz constant is $\mathrm{biLip}(h) := \mathrm{Lip}(h) + \mathrm{Lip}(h^{-1})$, where $\mathrm{Lip}(h) := \sup_{x \neq y} d(h(x), h(y))/d(x, y)$. Things are much easier in this category.

THEOREM A.1. (Folklore) *On any compact surface M, there is:*

- *a bi-Lipschitz transformation $h_0 : M \to M$ with no measure of maximal entropy;*
- *a bi-Lipschitz transformation $h_\infty : M \to M$ with (countably) infinitely many ergodic measures of maximal entropy.*

Proof. We first build h_0. There is a family of bi-Lipschitz transformations T_n of the unit disk D satisfying:

- T_n is the identity outside of the disk of radius $1/2$;
- $\mathrm{biLip}(T_n) \leq C$ for some constant $C > 3$;
- $h_{\mathrm{top}}(T_1) < h_{\mathrm{top}}(T_2) < \cdots < \log 2$.

For instance, one can extract such a family from an isotopy† between the identity f_0 and Smale's horseshoe f_1 (see, e.g., [**11**, p. 83]) with entropy $\log 2$. Finally, $t \mapsto h(f_t)$ is continuous: extend each f_t into a C^∞-diffeomorphism of the compact 2-sphere,

† More precisely, a continuous path $t \in [0, 1] \mapsto f_t \in \mathrm{Diff}^\infty(D, \partial D)$, the subset of C^∞-diffeomorphisms of the open disk which coincides with the identity near its boundary.

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still depending continuously on t, and use entropy semicontinuity results of Katok and Yomdin [**10, 13, 17**].

Observe also that, for any bi-Lipschitz transformation $f : D \to D$ and any number $\rho > 0$, f_ρ, the self-map of $D_\rho := \{x \in \mathbb{R}^2 : |x| < r\}$ defined by $f_\rho(x) := \rho f(x/\rho)$, satisfies $\mathrm{biLip}(f_\rho) = \mathrm{biLip}(f)$.

Let $\chi : U \to V$, $U \subset \mathbb{R}^2$, $V \subset M$, be some bi-Lipschitz chart of M. Let $D_1, D_2, \ldots$ be an infinite sequence of pairwise disjoint disks included in U, with $D_i = B(x_i, \rho_i)$. The example $f : M \to M$ is defined as follows:

- $f(x) = x$ for $x \in M' := M \setminus \bigcup_{i \geq 1} \Delta_i$ with $\Delta_i := \chi(D_i) \subset M$;
- $f(x) = \chi_i \circ T_i \circ \chi_i^{-1}(x)$ if $x \in \Delta_i$ with $\chi_i : D \to \Delta_i$, $u \mapsto \chi(x_i + \rho_i u)$.

Note that each disk Δ_i is invariant. Hence any ergodic invariant probability measure satisfies exactly one of the following: $\mu(\Delta_i) = 1$ for some $i \geq 1$ or $\mu(M') = 1$. In the latter case, μ is the Dirac measure at some point and $h(f, \mu) = 0$. In the former case, $h(f, \mu) \leq h_{\mathrm{top}}(T_i)$. The variational principle implies that $h_{\mathrm{top}}(f) = \sup_{i \geq 1} h_{\mathrm{top}}(T_i)$. We see that $h(f, \mu) < h_{\mathrm{top}}(f)$ by the same reasoning: f has no maximal entropy measure.

We show that $f : M \to M$ is bi-Lipschitz using the χ-coordinates and the canonical Euclidean distance on $V \subset \mathbb{R}^2$. Let $x, y \in V$, $x \neq y$. We distinguish four cases.

Case 1. The case where $x, y \notin \bigcup_{i \geq 1} B(x_i, \rho_i/2)$. Then $fx = f^{-1}x = x$ and $fy = f^{-1}y = y$ and $d(fx, fy) = d(x, y)$.

Case 2. The case where $x \in B(x_i, \rho_i/2)$ and $y \in B(x_j, \rho_j/2)$ for some $i \neq j$. We compute $d(fx, fy) \leq \rho_i + \rho_j + d(x, y)$ and $d(x, y) \geq (\rho_i + \rho_j)/2$, so $d(fx, fy)/d(x, y) \leq 2 + 1 \leq 3$.

Case 3. The case where $x \in B(x_i, \rho_i/2)$ and $y \in B(x_i, \rho_i)$ for some $i \geq 1$. Use $\mathrm{biLip}(F|B(x_i, \rho_i)) = \mathrm{biLip}(T_i) \leq C$.

Case 4. The case where $x \in B(x_i, \rho_i/2)$ and $y \notin \bigcup_{j \geq 1} B(x_j, \rho_j/2) \cup B(x_i, \rho_i)$. We compute $d(fx, fy) \leq \rho_i + d(x, y)$ and $d(x, y) \geq \rho_i/2$, so $d(fx, fy) \leq 3d(x, y)$.

Thus $\mathrm{Lip}(f) \leq 3$. The same reasoning applies to $\mathrm{Lip}(f^{-1})$, so $\mathrm{biLip}(f) \leq 6$.

This concludes the construction of $h_0 := f$ with no measure of maximal entropy. To build a bi-Lipschitz transformation h_∞ with infinitely many ergodic invariant probability measures with maximal entropy, one may repeat the previous construction but using a sequence of maps such that $h_{\mathrm{top}}(T_n) > 0$ is constant, instead of strictly increasing (for instance by using the same map infinitely many times). □

6.2. *Homeomorphisms.* There exist topologically minimal homeomorphisms of the sphere S^2 with non-zero topological entropy and either no measure of maximal entropy, or infinitely many of them, both countably and uncountably many. This follows from a result of Béguin *et al* [**1**] and well-known examples of subshifts.

Indeed, according to [**1**], given any bi-measurable transformation T of a Cantor set and an irrational rotation $R : S^1 \to S^1$, there exists a homeomorphism h on the sphere S^2 and a bi-measurable conjugacy of restrictions of h and $T \times R$ to subsets of full measure with respect to any invariant probability measure. Well-known examples among subshifts of systems with non-zero topological entropy and infinitely many ergodic,

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invariant probability measures with maximal entropy (both countably and uncountably many, see [**8**]) can therefore be translated on the sphere.

B. *Appendix. Construction of the homoclinic map* f_0

In this section we build a C^∞ diffeomorphism f_0 and check that it fulfills the properties announced in Proposition 2.1. Here, f_0 will depend on two large parameters K and L specifying, respectively, the contraction and the time to pass from one corner to the next (see below).

Throughout this section, we use canonical coordinates, and $D \subset \mathbb{R}^2$ is the disk centered at $(0, 0)$ with radius 2. We set $\Lambda := 6/5$, $\lambda := \log \Lambda$, $\kappa := \log K$, $\eta := \log K / \log \Lambda > 1$ and let $\psi : \mathbb{R} \to [0, 1]$ be a C^∞ bump function: $\psi(t) = 1$ if and only if $t \le 0$, $\psi(t) = 0$ if and only if $t \ge 1$, and $\psi'(t) \le 0$ for all $t \in \mathbb{R}$. Also, $0 < \psi(t) < 1$ and $\psi'(t) < 0$ for $0 < t < 1$.

B.1. *Map* H_0 *on* $\{(X, Y) \in D : Y \le -1/6\}$. We start with a vector field V_0 on $\{(x, y) \in D : y \le -1/10\}$) defined by

$$V_0(X, Y) = \alpha_L(X, Y)(X_0(X), A_0(X)Y) \tag{B1}$$

with $\alpha_L(X, Y)$ a factor restricting the support to a compact subset of D and slowing down the flow on $\{\frac{1}{11} \le X \le \frac{1}{10}\}$ and

$$X_0(X) = \begin{cases} -\kappa(X + \frac{1}{2}) & \text{if } X \le \frac{1}{13}, \\ -\psi(156X - 12)\kappa(X + \frac{1}{2}) & \\ \quad -(1 - \psi(156X - 12))\lambda(\frac{1}{2} - X) & \text{if } \frac{1}{13} \le X \le \frac{1}{12}, \\ -\lambda(\frac{1}{2} - X) & \text{if } X \ge \frac{1}{12} \end{cases}$$

and

$$A_0(X) = \begin{cases} \lambda & \text{if } X \le -\frac{1}{11}, \\ \psi(132X + 12)\lambda & \\ \quad -(1 - \psi(132X + 12))\kappa & \text{if } -\frac{1}{11} \le X \le -\frac{1}{12}, \\ -\kappa & \text{if } X \ge -\frac{1}{12}. \end{cases}$$

Finally,

$$\alpha_L(X, Y) = (L^{-1} + (1 - L^{-1})\psi(|440X - 42| - 1))\psi(\tfrac{2}{7}(X^2 + Y^2) - \tfrac{1}{7}).$$

We define

$$H_0 : \{(X, Y) \in D : Y \le -1/6\} \to \{(X, Y) \in D : Y \le -1/10\}$$

to be the time 1 map of V_0. Observe that H_0 is well defined as

$$\Lambda \times (1/2 - 1/6) - 1/2 = -1/10. \tag{B2}$$

The non-linearities of H_0 appear in three vertical strips, namely, from right to left:

- $S_1 := \{\frac{1}{11} \le X \le \frac{1}{10}\}$: the slowing down by L^{-1};

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- $S_3 := \{\frac{1}{13} \le X \le \frac{1}{12}\}$: the transition from $X_0(X) = -\lambda(\frac{1}{2} - X)$ to $-\kappa(X + \frac{1}{2})$;
- $S_5 := \{-\frac{1}{11} \le X \le -\frac{1}{12}\}$: the transition from $A_0(X) = -\kappa$ to λ.

For convenience, we set $S_0 := \{X \ge \frac{1}{10}\}$, $S_2 := \{\frac{1}{12} \le X \le \frac{1}{11}\}$, $S_4 := \{-\frac{1}{12} \le X \le \frac{1}{13}\}$ and $S_6 := \{X \le -\frac{1}{11}\}$.

We also write the time 1 map of the flow as $(X, Y) \mapsto (h(X), h_X(Y))$ with maps $h : (-1/2, 1/2) \to (-1/2, 1/2)$ and $h_X : (-1/2, -1/6) \to (-1/2, -1/10)$.

B.2. *Contraction to* $\partial[-1/2, 1/2]^2$. The following, a simple consequence of the fact that the contraction comes first and is stronger than the expansion, will be used to prove item (iv) of Proposition 2.1. Recall that C denotes a constant independent of K, L, which may increase from one occurrence to the next.

LEMMA B.1. *For $K > C$ and any $L \ge 1$, splitting $[-1/2, 1/2] \times [-1/2, -1/6]$ into a right part and a left part,*

$$A_L := [-1/2, 1/11] \times [-1/2, -1/6], \quad A_R := [1/11, 1/2] \times [-1/2, -1/6],$$

we get,

$$(X, Y) \in A_L \implies H_0(X, Y) \in [-\tfrac{1}{2}, -\tfrac{5}{12}) \times [-\tfrac{1}{2}, 0] \tag{B3}$$

$$(X, Y) \in A_R \implies \begin{cases} \text{either } H_0(X, Y) \in [-\frac{1}{2}, \frac{1}{2}] \times [-\frac{1}{2}, -\frac{5}{12}), \\ \text{or } H_0(X, Y) \in [-\frac{1}{12}, \frac{1}{11}] \times [-\frac{1}{2}, Y[\subset A_L, \\ \text{or } \begin{cases} \frac{1}{11} \le h(X) < X - C^{-1}/L \\ \text{and } h_X(Y) \le -1/2 + K^{-\frac{9}{10}L^{-1}}(Y + 1/2) < Y. \end{cases} \end{cases} \tag{B4}$$

Proof of Lemma B.1. For any $(X, Y) \in [-1/2, 1/2] \times [-1/2, -1/6]$, let $(X(t), Y(t))$ be its orbit under the flow of the vector field V_0. We define $\Delta_{-1} := 0$ and, for $i = 0, \ldots, 6$, $\Delta_i := \sup\{t \ge 0 : x(t + \Delta_{i-1}) \in S_i\} \cup \{0\}$, the time spent in S_i. Observe that $\Delta_1 + \cdots + \Delta_6 = 1$ and $X_0(X) < 0$ for $-1/2 < X < 1/2$. We use the maps h, h_X from above.

The inclusion (B3) will be a consequence of the claim

$$h(\tfrac{1}{11}) < -\frac{5}{12}. \tag{B5}$$

(By (B2), $h_X(-1/6) \le -1/10 < 0$.) Let $(X, Y) \in \{\frac{1}{11}\} \times [-1/2, -1/6]$. Observe that $\Delta_0 = \Delta_1 = 0$,

$$\Delta_2 = \frac{1}{\lambda} \log \frac{5/12}{9/22} = \frac{1}{\lambda} \log \frac{55}{54} < 0.11 \quad \text{and} \quad \Delta_3 \le \frac{1/156}{(5/12)\lambda} < 0.09$$

(as $|X_0| \ge \lambda(1/2 - 1/12) = (5/12)\lambda$ on S_3). Hence $\Delta_4 + \Delta_5 + \Delta_6 \ge 1 - 0.20$ but

$$h(1/11) \le -1/2 + (1/13 + 1/2)K^{-(\Delta_4 + \Delta_5 + \Delta_6)}.$$

Thus, $h(\frac{1}{11}) \le -\frac{1}{2} + \frac{15}{26}K^{-0.80}$. Equation (B5) follows (recall $K > C$).

The inclusion (B4) will be a consequence of the following claim. For any $(X, Y) \in A_R$, one of the following holds:

(a) $h_X(Y) < -5/12$;

(b) $h_X(Y) \le -1/2 + (Y + 1/2)K^{-\frac{9}{10}L^{-1}} < Y$ and $\dfrac{1}{11} \le h(X) < X - \dfrac{C^{-1}}{L}$;

(c) $h_X(Y) < y \le -1/6$ and $h(X) \in [-1/12, 1/11]$.

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Observe that

$$h_X(Y) \le -1/2 + (Y + 1/2)K^{-\Delta_0 - \Delta_1/L - \Delta_2 - \Delta_3 - \Delta_4}\Lambda^{\Delta_5 + \Delta_6}$$

and consider the following cases.

Case 1: The case where $\Delta_0 + \Delta_2 + \Delta_3 + \Delta_4 \ge 1/10$. Thus, $\Delta_5 + \Delta_6 \le 9/10$ and $h_X(Y) \le -1/2 + (1/3)K^{-1/10}\Lambda^{9/10} < -5/12$ for $K > C$; (a) is satisfied.

In the remaining cases, we assume $\Delta_0 + \Delta_2 + \Delta_3 + \Delta_4 < 1/10$.

Case 2: The case where $\Delta_0 + \Delta_1 = 1$. Thus,

$$h_X(Y) \le -1/2 + (Y + 1/2)K^{-L^{-1}} < Y \le -1/6 \quad \text{and} \quad h(X) < X - C^{-1}/L.$$

Since $\Delta_2 = 0$ implies that $H_0(X, Y)$ is to the right of S_2, $h(X) \ge 1/11$, (b) is satisfied.

Case 3: The case where $\Delta_2 > 0$, $\Delta_5 + \Delta_6 = 0$. Thus, $h(X) \in [-1/12, 1/11]$ as $\Delta_2 > 0$ and $\Delta_5 = 0$. Also $h_X(Y) = -1/2 + (Y + 1/2)K^{-1/L} < Y \le -1/6$; (c) is satisfied

Case 4: The case where $\Delta_5 + \Delta_6 > 0$. As $X \ge 1/11$, $\Delta_2 = (1/\lambda)\log(55/54) > 0.10$ (see above) and therefore

$$h_X(Y) \le -1/2 + (Y + 1/2)K^{-0.10}\Lambda^{0.90} < -5/12,$$

for $K > C$ and (a) is satisfied. □

B.3. *Full map.* We first extend V_0 to $D\backslash Q_4$ with $Q_4 := [-1/10, 1/10]^2$ using the symmetries

$$V_1(X, Y) = \begin{cases} V_0(X, Y) & \text{if } Y \le \Lambda/3 - 1/2 = -1/10, \\ D\rho_1(V_0(\rho_1^{-1}(X, Y))) & \text{if } X \le \Lambda/3 - 1/2, \\ D\rho_2(V_0(\rho_2^{-1}(X, Y))) & \text{if } Y \ge -\Lambda/3 + 1/2, \\ D\rho_3(V_0(\rho_3^{-1}(X, Y))) & \text{if } X \ge -\Lambda/3 + 1/2, \end{cases}$$

where the ρ_i are the rotations defined before Proposition 2.1. Observe that, in the above definition of V_1, there are four regions on which two formulas are given: for instance, $V_0(X, Y)$ and $D\rho_1(V_0(\rho_1^{-1}(X, Y)))$ on the square $(X, Y) \in [-2, \Lambda/3 - 1/2] \times [-\Lambda/3 + 1/2, 2]$, but both are equal to $V_0(X, Y) = \psi(\frac{2}{7}(X^2 + Y^2) - \frac{1}{7})(-\lambda(1 - X), -\kappa Y)$; V_1 is properly defined as a C^∞ vector field on $D\backslash Q_4$. We denote by $H_1 : D\backslash Q \to D\backslash Q_4$ the time 1 flow of V_1 (it is the symmetric extension of H_0).

We extend V_1 to a C^∞ smooth field V over D such that $V|D\backslash Q_4 = V_1|D\backslash Q_4$ and then let $f_0 : D \to D$ be the time 1 map of V. If necessary, we modify this construction by replacing V with ϕV, where $\phi : D \to (0, 1]$ is a smooth map such that: (i) $\phi \equiv 1$ outside of Q_2; (ii) $\mathrm{Lip}(f_0|Q_4) < \Lambda$.

B.4. *Properties of f_0.* We check that the map f_0 has the claimed properties.

Proof of Proposition 2.1. Items (i) and (v) are clear from the construction. Item (ii), the bound on $\lambda(f_0)$, will be proved at the end.

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Item (iii) follows from applying the Poincaré–Bendixson theorem to the flow defined by V.

The first claim of item (iv) follows from Lemma B.1, the invariance of V under ρ_i, $i = 0, 1, 2, 3$, and an obvious induction. The statement about the omega-limit set follows from the fact that any V-orbit must visit the corners infinitely many times in their cyclic order and item (viii) proved below.

For $(X, Y) \in f_0(\mathcal{Q})\backslash\mathcal{Q}$, the second component of the vector field V depends only on Y, hence $\Theta(X, Y)$ depends only on Y, as claimed in item (vi). The bound on the transition time Θ in the same item follows from the fact that $-X_0(X) \geq c'/L$ for $-5/12 \leq X \leq 1/2 - (1/12)\Lambda^{-1} = 31/72$ and $-X_0(X) \leq c''/L$ for $41/440 \leq X \leq 43/440$, for some constants c', $c'' > 0$, that are independent of K and L. Item (vi) is proved.

We prove item (vii) by a computation. We first assume that $L = 1$. By the form of V in $[-1/2, -1/10] \times [-1/2, 1/2]$, the time t map of the flow has, in canonical coordinates, the form

$$(X_1, X_2) \mapsto (h^t(X_2)(X_1 + 1/2) - 1/2, h^t(X_2))$$

for t non-negative and small enough so that the image is still in the subset. In particular, writing $\Theta = \Theta(-1/2, X_2)$, we can define $\tilde{\alpha}(X_2) = h^{\Theta}(X_2)$ and $\tilde{h}(X_2) := h^{\Theta}(X_2)$.

Using (B1),

$$\tilde{h}(X_2) = \exp\left(\int_0^{\Theta} A_0(X(s))\, ds\right),$$

where $t \mapsto X(t)$ is the solution of $X'(t) = X_0(X(t))$ starting at $X(0) = X_2$. The first claim of item (vii) follows with $\tilde{\alpha}(X_2) := \exp(\int_0^{\Theta} A_0(X(s))\, ds)$.

We deduce the claimed estimates in the item by simple computations. First,

$$\tilde{h}'(X_2) = X_0(h^{\Theta}(X_2))/X_0(X_2) = C^{\pm 1} \log K.$$

Then

$$\tilde{\alpha}(X_2) = \exp\left(\int_0^{\Theta} A_0(X(s))\, ds\right) = K^{-C^{\pm 1}}.$$

Indeed, $A_0(X(s)) = -\log K$ for $X(s) \in [1/12, 2/5]$ and this occurs for a constant time. Now, $A_0(X) \leq \log \Lambda$ everywhere and the total time Θ is bounded (assuming, as we may in this estimate, that $L = 1$) implying that $\tilde{\alpha}(X_2) \leq K^{-C}$. Similarly, $A_0(X) \geq -\log K$ everywhere implies $\tilde{\alpha}(X_2) \geq K^{-C^{-1}}$, so $\tilde{\alpha}(X_2) = K^{-C^{\pm 1}}$.

We compute again

$$\begin{aligned}\tilde{\alpha}'(X_2) &= \tilde{\alpha}(X_2) \exp\left(\int_0^{\Theta} A_0'(X(s)) \frac{\partial X(s)}{\partial X(0)}\, ds\right) \\ &\leq C K^{-C^{\pm 1}},\end{aligned}$$

where $\partial x(s)/\partial x(0)$ is the derivative of $X(s)$ with respect to the initial condition at time 0. The last inequality uses that $A_0'(X) = 132\psi'(123X + 12)(\lambda + \kappa) \leq 0$ and $\partial X(s)/\partial X(0) > 0$; the above exponential is less than 1. We note that this implies

$$\tilde{\alpha}'(X) = \mathcal{O}(|\tilde{\alpha}(X)|).$$

The proof of item (vii) $L = 1$ is complete.

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We now extend the previous estimate to arbitrary values of $L \geq 1$. We express the transition map in terms of the passage map $P : \rho_i(C) \to \rho_{i+1}([0, 1] \times \{5/12\}), (X, Y) \mapsto \Phi^{\Theta^*_L(X,Y)}(X, Y)$ for the flow for parameter L. Notice that this passage map does not depend on L, though τ^*_L does (changing L only changes the speed at which the trajectories are described). We have $H^{\Theta(X,Y)}(X, Y) = \Phi^{\delta_L(X,Y)}(P(X, Y))$, where $\delta_L(X, Y) := \Theta_L(X, Y) - \Theta^*_L(X, Y) \in [0, 1]$. Observe that $\Theta_L(X, Y)$ and $\Theta^*_L(X, Y)$ depend only on X because $V(X, Y) = V(X, Z)$ for all $(X, Y), (X, Z) \in [-1/2, 1/2] \times [-1/2, -1/6]$. Using flow boxes, it is easily seen that $\delta_L(X, Y) = X + const$. The claimed uniformity follows from the boundedness of δ_L; item (vii) is proved.

Item (viii) is a consequence of the following easy estimates. Recall from the statement of Proposition 2.1 that $(X, Y) \in f_0(\mathcal{Q})\backslash\mathcal{Q}$ and that $T(X, Y)$ is its transition time to $\rho_1(\mathcal{Q})$. Also, $(X(t), Y(t))$ is the orbit starting from that point. Let t_1, t_2, t_3 be the unique times t such that $X(t) = 1/12, -1/12, -5/12$ respectively. Then:

- $X(0) \geq 1/2 - \Lambda \cdot (1/12) = 1/2 - 1/10$;
- $X(t_1) = 1/12 \leq 1/2 - \Lambda^{t_1}/10$, so $t_1 \geq \log(25/6)/\lambda$;
- $X(t_2) = -1/12 \geq -1/2 + K^{-(t_2-t_1)}7/12$, so $t_2 - t_1 \geq \log(7/5)/\kappa$;
- $X(t_3) = -5/12 = -1/2 + K^{-(t_3-t_2)}5/12$, so $t_3 - t_2 = \log 5/\kappa$.

Indeed, these imply, writing $y(t)$ for $Y(t) + 1/2$ and recalling $\eta := \log K/\log \Lambda > 1$:

- $y(t_1) = K^{-t_1} y(0) \leq (25/6)^{-\eta} y(0)$;
- $y(t_2) = K^{-(t_2-t_1)} y(t_1) \leq (5/7) y(t_1)$;
- $y(t_3) \leq \Lambda^{t_3-t_2} y(t_2) \leq 5^{1/\eta} y(t_2)$.

Finally, $y(\Theta) \leq \Lambda y(t_3)$ and

$$y(t_3) \leq 5^{1/\eta}(25/6)^{-\eta}(5/7) y(t_0) \leq \exp(1.61/\eta - 1.42\eta - 0.33) y(t_0).$$

This implies item (viii) of the proposition with $u := \exp(1.61/\eta - 1.42\eta - 0.33) < 1$ (recall that $\eta > 1$).

It remains to prove item (ii): $\lambda(f_0) = \log \Lambda$. Recall that $\mathrm{Lip}(f_0|Q_3) < \Lambda$. Now any orbit starting outside of Q_3 has a uniformly bounded number of iterates outside of a neighborhood of $\partial Q_1 \cup \{V = 0\}$ (recall item (iv)). An orbit in Q_1, close to the boundary, spends a fraction almost equal to 1 of its time in the corners $\rho_i(\mathcal{Q})$, $i = 0, 1, 2, 3$, where f_0 is affine. Other orbits outside of Q_3 converge to $\{V = 0\}$. The claim easily follows. □

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