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ON AN IDENTITY ASSOCIATED WITH JACOBI'S THETA FUNCTIONS

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Abstract

Recently, Yang ['On modular equations for the Jacobi theta functions', Preprint] discovered an identity that involves the sum of squares of the theta functions $\vartheta_j(0|\tau)\vartheta_j(0|N\tau)$ for $j = 2, 3, 4$. In this article, we prove this identity and its generalisations. Our study also leads us to a four-variable extension of an identity discovered by Chan, Chua and Solé ['Quadratic iterations to π associated to elliptic functions to the cubic and septic base', *Trans. Amer. Math. Soc.* **355**(4) (2002), 1505–1520].

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1. Introduction

Throughout this work, $q = e^{\pi i\tau}$, where the imaginary part of τ is positive. The Jacobi theta functions are defined by

$$\vartheta_1(z|\tau) = -i\sum_{n=-\infty}^{\infty}(-1)^n q^{(2n+1)^2/4}e^{(2n+1)iz},$$

$$\vartheta_2(z|\tau) = \sum_{n=-\infty}^{\infty} q^{(2n+1)^2/4}e^{(2n+1)iz},$$

$$\vartheta_3(z|\tau) = \sum_{n=-\infty}^{\infty} q^{n^2}e^{2niz},$$

$$\vartheta_4(z|\tau) = \sum_{n=-\infty}^{\infty}(-1)^n q^{n^2}e^{2niz}.$$

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Let N a positive integer such that $N \equiv 3 \pmod 4$ and let

$$a_N(q) = \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+((N+1)/4)n^2}. \tag{1.1}$$

It is known that

$$a_N(q) = \vartheta_2(0|\tau)\vartheta_2(0|N\tau) + \vartheta_3(0|\tau)\vartheta_3(0|N\tau). \tag{1.2}$$

This identity can be proved by splitting (1.1) into two sums according to the parity of n. Recently, Yang [9] discovered that for $N \equiv 3 \pmod 4$,

$$2a_N^2(q^2) = \sum_{i=2}^{4} \vartheta_i^2(0|\tau)\vartheta_i^2(0|N\tau). \tag{1.3}$$

Although (1.2) involves only $\vartheta_2(0|\tau)\vartheta_2(0|N\tau)$ and $\vartheta_3(0|\tau)\vartheta_3(0|N\tau)$, Yang's identity for $a_N^2(q^2)$ involves not only the square of $\vartheta_2(0|\tau)\vartheta_2(0|N\tau)$ and $\vartheta_3(0|\tau)\vartheta_3(0|N\tau)$, but also the square of $\vartheta_4(0|\tau)\vartheta_4(0|N\tau)$.

We show that (1.3) holds not only for $N \equiv 3 \pmod 4$, but also for all odd positive integers N. We present two proofs of (1.3). In Section 2, we prove (1.3) using only Jacobi's theta functions of one variable. We define *permanent modular equations* and show that (1.3) is indeed a permanent modular equation of composite degrees $1, 2, \ell$ and 2ℓ. In Section 3, we present a generalisation of (1.3) which involves Jacobi's theta functions of two variables. We also present several analogues of (1.3). In Section 4, we present a generalisation of an identity [2, Theorem 5.1] found recently by Chan *et al*. Our discovery is motivated by the method presented in Section 3.

2. A first proof of (1.3) and a permanent modular equation

We begin with some well-known identities associated with $\vartheta_j(0|\tau)$, $j = 2, 3, 4$.

LEMMA 2.1. *The following identities hold:*

$$2\vartheta_2(0|2\tau)\vartheta_3(0|2\tau) = \vartheta_2^2(0|\tau), \tag{2.1}$$

$$\vartheta_3^2(0|2\tau) - \vartheta_3^2(0|\tau) = -\vartheta_2^2(0|2\tau), \tag{2.2}$$

$$\vartheta_3^2(0|2\tau) - \vartheta_2^2(0|\tau) = \vartheta_4^2(0|2\tau), \tag{2.3}$$

$$2\vartheta_3^2(0|2\tau) - \vartheta_3^2(0|\tau) = \vartheta_4^2(0|\tau). \tag{2.4}$$

Identities (2.1)–(2.4) play important roles in the study of the Gauss–Brent–Salamin algorithm and Ramanujan's series for $1/\pi$. (For more details, see [1].) Their proofs can be found in [1, page 58, (5.70)], [1, page 22, (2.26)], [1, page 22, (2.27)] and [1, Lemma 2.3], respectively.

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PROOF OF (1.3). By (1.2),

$$\begin{aligned}
&2\Bigg(\sum_{m,n=-\infty}^{\infty} q^{2(m^2+mn+((N+1)/4)n^2)}\Bigg)^2 \\
&\quad = 2[\vartheta_3(0|2\tau)\vartheta_3(0|2N\tau) + \vartheta_2(0|2\tau)\vartheta_2(0|2N\tau)]^2 \\
&\quad = 2\vartheta_3^2(0|2\tau)\vartheta_3^2(0|2N\tau) + 2\vartheta_2^2(0|2\tau)\vartheta_2^2(0|2N\tau) \\
&\qquad + 4\vartheta_3(0|2\tau)\vartheta_3(0|2N\tau)\vartheta_2(0|2\tau)\vartheta_2(0|2N\tau) \\
&\quad = 2\vartheta_3^2(0|2\tau)\vartheta_3^2(0|2N\tau) + 2\vartheta_2^2(0|2\tau)\vartheta_2^2(0|2N\tau) + \vartheta_2^2(0|\tau)\vartheta_2^2(0|N\tau), \qquad (2.5)
\end{aligned}$$

after two applications of (2.1). Applying (2.2) twice, the sum of the first two products in (2.5) is given by

$$\begin{aligned}
&2\vartheta_3^2(0|2\tau)\vartheta_3^2(0|2N\tau) + 2\vartheta_2^2(0|2\tau)\vartheta_2^2(0|2N\tau) \\
&\quad = 2\vartheta_3^2(0|2\tau)\vartheta_3^2(0|2N\tau) + 2[\vartheta_3^2(0|2\tau) - \vartheta_3^2(0|\tau)][\vartheta_3^2(0|2N\tau) - \vartheta_3^2(0|N\tau)] \\
&\quad = [2\vartheta_3^2(0|2\tau) - \vartheta_3^2(0|\tau)][2\vartheta_3^2(0|2N\tau) - \vartheta_3^2(0|N\tau)] + \vartheta_3^2(0|\tau)\vartheta_3^2(0|N\tau) \\
&\quad = \vartheta_4^2(0|\tau)\vartheta_4^2(0|N\tau) + \vartheta_3^2(0|\tau)\vartheta_3^2(0|N\tau), \qquad (2.6)
\end{aligned}$$

after applying (2.4) twice. Substituting (2.6) back into (2.5), we obtain (1.3) and this completes the proof. □

Let

$$\lambda(\tau) = \frac{\vartheta_2^4(0|\tau)}{\vartheta_3^4(0|\tau)} \quad \text{and} \quad \beta_s = \lambda(s\tau).$$

A *modular equation of degree s associated with* $\lambda(\tau)$ is an algebraic relation satisfied by β_1 and β_s. For example, the identity [1, Theorem 2.6]

$$\beta_2 = \frac{\left(1 - \sqrt{1-\beta_1}\right)^2}{\left(1 + \sqrt{1-\beta_1}\right)^2} \qquad (2.7)$$

is a modular equation of degree 2. A *modular equation of composite degrees 1,m,n,mn associated with* $\lambda(\tau)$ is an algebraic relation involving $\beta_1, \beta_m, \beta_n$ and β_{mn}. An example of a modular equation of composite degrees $1, 3, 13, 39$, given by Ramanujan in his second letter to Hardy [7, page xxix, (21)], is

$$\frac{\{(1-\beta_1)(1-\beta_{39})\}^{1/8} - (\beta_1\beta_{39})^{1/8}}{\{(1-\beta_3)(1-\beta_{13})\}^{1/8} - (\beta_3\beta_{13})^{1/8}} = \frac{1 + \{(1-\beta_1)(1-\beta_{39})\}^{1/4} + (\beta_1\beta_{39})^{1/4}}{1 + \{(1-\beta_3)(1-\beta_{13})\}^{1/4} + (\beta_3\beta_{13})^{1/4}}. \qquad (2.8)$$

A modular equation of degree s, or of composite degrees $1, m, n, mn$, is called *a permanent modular equation* if the same relation is satisfied by more than one value of s, or by more than one pair of values (m, n), respectively. For example, in the same letter mentioned above, Ramanujan asserted that (2.8) is also true for $(m, n) = (5, 11)$ and $(7, 9)$. We end this section by indicating that (2.6) can be interpreted as a permanent

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modular equation of composite degrees $1, 2, \ell, 2\ell$ by using [1, (8.53)]

$$\frac{\vartheta_3^2(0|\tau)}{\vartheta_3^2(0|2\tau)} = \frac{2}{1+\sqrt{1-\beta_1}}.$$

The permanent modular equation is given by

$$(1+(\beta_2\beta_{2\ell})^{1/4})^2 = 2\frac{1+\sqrt{\beta_1\beta_\ell}+\sqrt{(1-\beta_1)(1-\beta_\ell)}}{(1+\sqrt{1-\beta_1})(1+\sqrt{1-\beta_\ell})}. \tag{2.9}$$

Identity (2.9) can be transformed to a relation involving only β_1 and β_ℓ by using the modular equation (2.7) of degree 2 to eliminate β_2 and $\beta_{2\ell}$. This results in a 'trivial' relation which is essentially the identity

$$\left(\sqrt{(1+x)(1+y)}+\sqrt{(1-x)(1-y)}\right)^2 = 2\left(1+\sqrt{(1-x^2)(1-y^2)}+xy\right).$$

3. Second proof of (1.3)

Our main theorem in this section is the following result.

THEOREM 3.1. *Let*

$$A_{\alpha,\beta}(u,v|\tau) = \sum_{m,n=-\infty}^{\infty} q^{\alpha m^2+\alpha mn+\beta n^2} e^{2iu(2m+n)} e^{2ivn}, \tag{3.1}$$

where α and β are integers satisfying $0<\alpha<4\beta$, so that the double sum converges. Then, for any integer M and even integer N satisfying $0<M<N$,

$$\begin{aligned} &2A_{2M,N/2}(u+v,w+x|\tau)A_{2M,N/2}(u-v,w-x|\tau)\\ &\quad=\sum_{i=1}^{4}\vartheta_i(2u|M\tau)\vartheta_i(2v|M\tau)\vartheta_i(2w|(N-M)\tau)\vartheta_i(2x|(N-M)\tau). \end{aligned} \tag{3.2}$$

COROLLARY 3.2. *Identity (1.3) is true for all odd positive integers N.*

PROOF. First, we observe that the quadratic form $2(m^2+mn+((N+1)/4)n^2)$ may be written as $2m^2+2mn+((N+1)/2)n^2$. So, we may let $M=1$ and replace N by $N+1$ after setting $u=v=w=x=0$ in (3.2). Since $\vartheta_1(0|\tau)$ is zero, identity (1.3) follows immediately. □

In addition to establishing (1.3), Theorem 3.1 also leads to other analogues of (1.3). We give a sample of such identities as follows:

$$\begin{aligned} &2\Big(\sum_{m,n=-\infty}^{\infty}(-1)^n q^{2(m^2+mn+((N+1)/4)n^2)}\Big)^2\\ &\quad=-\vartheta_2^2(0|\tau)\vartheta_2^2(0|N\tau)+\vartheta_3^2(0|\tau)\vartheta_3^2(0|N\tau)+\vartheta_4^2(0|\tau)\vartheta_4^2(0|N\tau), \end{aligned}$$

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$$2\Bigg(\sum_{m,n=-\infty}^{\infty} q^{2((m+1/2)^2+(m+1/2)n+((N+1)/4)n^2)}\Bigg)^2$$
$$= \vartheta_2^2(0|\tau)\vartheta_2^2(0|N\tau) + \vartheta_3^2(0|\tau)\vartheta_3^2(0|N\tau) - \vartheta_4^2(0|\tau)\vartheta_4^2(0|N\tau),$$
$$2\Bigg(\sum_{m,n=-\infty}^{\infty} (-1)^n q^{2((m+1/2)^2+(m+1/2)n+((N+1)/4)n^2)}\Bigg)^2$$
$$= -\vartheta_2^2(0|\tau)\vartheta_2^2(0|N\tau) + \vartheta_3^2(0|\tau)\vartheta_3^2(0|N\tau) - \vartheta_4^2(0|\tau)\vartheta_4^2(0|N\tau).$$

We next prove Theorem 3.1. To establish Theorem 3.1, we need several identities involving Jacobi's theta functions defined in Section 1. These identities are extensions of identities in Lemma 2.1 and they can all be found in [6, Ch. 1].

LEMMA 3.3. *The following identities hold:*

$$\vartheta_3(u+v|2\tau)\vartheta_2(u-v|2\tau) - \vartheta_2(u+v|2\tau)\vartheta_3(u-v|2\tau) = \vartheta_1(u|\tau)\vartheta_1(v|\tau), \tag{3.3}$$

$$\vartheta_3(u+v|2\tau)\vartheta_2(u-v|2\tau) + \vartheta_2(u+v|2\tau)\vartheta_3(u-v|2\tau) = \vartheta_2(u|\tau)\vartheta_2(v|\tau), \tag{3.4}$$

$$\vartheta_3(u+v|2\tau)\vartheta_3(u-v|2\tau) - \vartheta_3(u|\tau)\vartheta_3(v|\tau) = -\vartheta_2(u+v|2\tau)\vartheta_2(u-v|2\tau), \tag{3.5}$$

$$\vartheta_3(u+v|2\tau)\vartheta_3(u-v|2\tau) - \vartheta_2(u+v|\tau)\vartheta_2(u-v|\tau) = \vartheta_4(u|2\tau)\vartheta_4(v|2\tau), \tag{3.6}$$

$$2\vartheta_3(u+v|2\tau)\vartheta_3(u-v|2\tau) - \vartheta_3(u|\tau)\vartheta_3(v|\tau) = \vartheta_4(u|\tau)\vartheta_4(v|\tau). \tag{3.7}$$

PROOF. Identities (3.3)–(3.6) are identities (1.4.7), (1.4.9), (1.4.10) and (1.4.12) in [6], respectively. Identity (3.7) follows immediately after combining (3.5) and (3.6). This completes the proof of Lemma 3.3. □

We are now ready to prove Theorem 3.1.

PROOF OF THEOREM 3.1. From [2, (5.7)] and the proof of [2, Theorem 5.2],

$$A_{\alpha,\beta}(u,v|\tau) = \vartheta_3(2u|\alpha\tau)\vartheta_3(2v|(4\beta-\alpha)\tau) + \vartheta_2(2u|\alpha\tau)\vartheta_2(2v|(4\beta-\alpha)\tau). \tag{3.8}$$

Therefore,

$$\begin{aligned}
&2A_{2M,N/2}(u+v,w+x|\tau)A_{2M,N/2}(u-v,w-x|\tau)\\
&\quad= 2[\vartheta_3(2u+2v|2M\tau)\vartheta_3(2w+2x|(2N-2M)\tau)\\
&\qquad+ \vartheta_2(2u+2v|2M\tau)\vartheta_2(2w+2x|(2N-2M)\tau)]\\
&\qquad\times[\vartheta_3(2u-2v|2M\tau)\vartheta_3(2w-2x|(2N-2M)\tau)\\
&\qquad+ \vartheta_2(2u-2v|2M\tau)\vartheta_2(2w-2x|(2N-2M)\tau)]\\
&\quad=: 2[AB+CD][EF+GH]\\
&\quad= 2ABEF + 2CDGH + 2ABGH + 2CDEF,
\end{aligned} \tag{3.9}$$

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where

$$\begin{aligned} A &= \vartheta_3(2(u+v)|2M\tau), \quad B = \vartheta_3(2(w+x)|2(N-M)\tau),\\ C &= \vartheta_2(2(u+v)|2M\tau), \quad D = \vartheta_2(2(w+x)|2(N-M)\tau),\\ E &= \vartheta_3(2(u-v)|2M\tau), \quad F = \vartheta_3(2(w-x)|2(N-M)\tau),\\ G &= \vartheta_2(2(u-v)|2M\tau), \quad H = \vartheta_2(2(w-x)|2(N-M)\tau). \end{aligned}$$

Further, let

$$\begin{aligned} J &= \vartheta_3(2u|M\tau)\vartheta_3(2v|M\tau), \quad K = \vartheta_3(2v|M\tau),\\ L &= \vartheta_3(2w|(N-M)\tau), \quad P = \vartheta_3(2x|(N-M)\tau). \end{aligned}$$

Then, by two applications of (3.5), the sum of the first two products in (3.9) becomes

$$\begin{aligned} 2ABEF + 2CDGH &= 2ABEF + 2[AE - JK][BF - LP]\\ &= [2AE - JK][2BF - LP] + JKLP\\ &= \vartheta_4(2u|M\tau)\vartheta_4(2v|M\tau)\vartheta_4(2w|(N-M)\tau)\vartheta_4(2x|(N-M)\tau)\\ &\quad + \vartheta_3(2u|M\tau)\vartheta_3(2v|M\tau)\vartheta_3(2w|(N-M)\tau)\vartheta_3(2x|(N-M)\tau), \end{aligned} \tag{3.10}$$

after applying (3.7) twice.

For the sum of the next two products in (3.9), we find that

$$\begin{aligned} &2ABGH + 2CDEF\\ &\quad = [AG - CE][BH - DF] + [AG + CE][BH + DF]\\ &\quad = \vartheta_1(2u|M\tau)\vartheta_1(2v|M\tau)\vartheta_1(2w|(N-M)\tau)\vartheta_1(2x|(N-M)\tau)\\ &\qquad + \vartheta_2(2u|M\tau)\vartheta_2(2v|M\tau)\vartheta_2(2w|(N-M)\tau)\vartheta_2(2x|(N-M)\tau), \end{aligned} \tag{3.11}$$

after two applications each of (3.3) and (3.4). Substituting (3.10) and (3.11) back into (3.9), we obtain (3.2) and this completes the proof of Theorem 3.1. □

4. A generalisation of an identity of Chan, Chua and Solé

In [5, Theorem 1.1], Chan, Chua and Solé discovered the identity

$$\begin{aligned} \Bigg(\sum_{m,n=-\infty}^{\infty} q^{2(m^2+mn+2n^2)}\Bigg)^2 &= \Bigg(\sum_{m,n=-\infty}^{\infty} (-1)^{m+n} q^{2(m^2+mn+2n^2)}\Bigg)^2\\ &\quad + \Bigg(\sum_{m,n=-\infty}^{\infty} q^{2((m+1/2)^2+(m+1/2)n+2n^2)}\Bigg)^2 \end{aligned} \tag{4.1}$$

and gave a proof by considering 7-modular lattices from codes over $\mathbf{F}_2 \times \mathbf{F}_2$. Shortly after the discovery of (4.1), they [4, Lemma 2.2(ii)] found an infinite family of identities, parametrised by d, extending (4.1), given by

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$$\Bigg(\sum_{m,n=-\infty}^{\infty} q^{2(m^2+mn+dn^2)}\Bigg)^2 = \Bigg(\sum_{m,n=-\infty}^{\infty} (-1)^{m+n} q^{2(m^2+mn+dn^2)}\Bigg)^2 + \Bigg(\sum_{m,n=-\infty}^{\infty} q^{2((m+1/2)^2+(m+1/2)n+dn^2)}\Bigg)^2. \tag{4.2}$$

Recently, motivated by the study of 15-modular lattices, Chan *et al.* [3] discovered the identity

$$\Bigg(\sum_{m,n=-\infty}^{\infty} q^{2m^2+mn+2n^2}\Bigg)^2 = \Bigg(\sum_{m,n=-\infty}^{\infty} (-1)^{m+n} q^{2m^2+mn+2n^2}\Bigg)^2 + \Bigg(2\sum_{m,n=-\infty}^{\infty} q^{2(2(m+1/2)^2+(m+1/2)n+2n^2)}\Bigg)^2. \tag{4.3}$$

They proved (4.3) by establishing an infinite family of identities [3, Theorem 1.1], parametrised by b and d, extending (4.3), given by

$$\Bigg(\sum_{m,n=-\infty}^{\infty} q^{dm^2+bmn+dn^2}\Bigg)^2 = \Bigg(\sum_{m,n=-\infty}^{\infty} (-1)^{m+n} q^{dm^2+bmn+dn^2}\Bigg)^2 + \Bigg(2\sum_{m,n=-\infty}^{\infty} q^{2(d(m+1/2)^2+b(m+1/2)n+dn^2)}\Bigg)^2, \tag{4.4}$$

where $1 \le b \le d-1$. Motivated by (4.4), they further generalised (4.2) by introducing another family of identities depending on the parameters b and d [3, (1.6)], given by

$$\Bigg(\sum_{m,n=-\infty}^{\infty} q^{2(bm^2+bmn+dn^2)}\Bigg)^2 = \Bigg(\sum_{m,n=-\infty}^{\infty} (-1)^{m+n} q^{2(bm^2+bmn+dn^2)}\Bigg)^2 + \Bigg(\sum_{m,n=-\infty}^{\infty} q^{2(b(m+1/2)^2+b(m+1/2)n+dn^2)}\Bigg)^2. \tag{4.5}$$

Both (4.4) and (4.5) turn out to admit two variable extensions [2, Theorems 5.1 and 5.2]. For example, the two variable extension of (4.5) is given by

$$A^2_{\alpha,\beta}(u,v|\tau) - C^2_{\alpha,\beta}(u,b|\tau) = A^2_{\alpha,\beta}(u,v+\pi/2|\tau) - C^2_{\alpha,\beta}(u,v+\pi|\tau), \tag{4.6}$$

where $A_{\alpha,\beta}(u,v|\tau)$ is defined in (3.1) and

$$C_{\alpha,\beta}(u,v|\tau) = \sum_{m,n=-\infty}^{\infty} q^{\alpha m^2+\alpha mn+\beta n^2+\alpha m+\alpha n/2+\alpha/4} e^{2iu(2m+n+1)} e^{2ivn}. \tag{4.7}$$

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The discovery of (4.6) was motivated by the generalisation of Borweins' cubic theta function identity discovered by Schultz [8, Corollary 4.3], [2, (1.4)]. We now present a two variable extension of (4.6).

THEOREM 4.1. *Let $A_{\alpha,\beta}(u,v|\tau)$ and $C_{\alpha,\beta}(u,v|\tau)$ be defined in (3.1) and (4.7), respectively. Suppose $0<\alpha<4\beta$, so that the double series converge. Then,*

$$\begin{aligned}&A_{\alpha,\beta}(u,w-u|\tau)A_{\alpha,\beta}(v,x-v|\tau)+C_{\alpha,\beta}(v,w-u+\pi/2|\tau)C_{\alpha,\beta}(u,x-v+\pi/2|\tau)\\&\quad=A_{\alpha,\beta}(u,w-u+\pi/2|\tau)A_{\alpha,\beta}(v,x-v+\pi/2|\tau)+C_{\alpha,\beta}(v,w-u|\tau)C_{\alpha,\beta}(u,x-v|\tau).\end{aligned}\tag{4.8}$$

PROOF. Since the proof is a straightforward computation, we omit the details and present only the key steps.

Identity (3.8) allows us to express $A_{\alpha,\beta}(u,v|\tau)$ in terms of $\vartheta_2(\cdot|\tau)$ and $\vartheta_3(\cdot|\tau)$. Together with the easy facts that

$$\vartheta_2(u+\pi|\tau)=-\vartheta_2(u|\tau)\quad\text{and}\quad\vartheta_3(u+\pi|\tau)=\vartheta_3(u|\tau),\tag{4.9}$$

one can show that

$$\begin{aligned}&A_{\alpha,\beta}(u,w-u|\tau)A_{\alpha,\beta}(v,x-v|\tau)-A_{\alpha,\beta}(u,w-u+\pi/2|\tau)A_{\alpha,\beta}(v,x-v+\pi/2|\tau)\\&\quad=2\vartheta_3(2u|\alpha\tau)\vartheta_3(2(w-u)|(4\beta-\alpha)\tau)\vartheta_2(2v|\alpha\tau)\vartheta_2(2(x-v)|(4\beta-\alpha)\tau)\\&\qquad+2\vartheta_2(2u|\alpha\tau)\vartheta_2(2(w-u)|(4\beta-\alpha)\tau)\vartheta_3(2v|\alpha\tau)\vartheta_3(2(x-v)|(4\beta-\alpha)\tau).\end{aligned}\tag{4.10}$$

Next, from [2, (5.7)] and the proof of [2, Theorem 5.2], we see that

$$C_{\alpha,\beta}(u,v|\tau)=\vartheta_2(2u|\alpha\tau)\vartheta_3(2v|(4\beta-\alpha)\tau)+\vartheta_3(2u|\alpha\tau)\vartheta_2(2v|(4\beta-\alpha)\tau).$$

This, together with (4.9), then enables one to show that

$$C_{\alpha,\beta}(v,w-u|\tau)C_{\alpha,\beta}(u,x-v|\tau)-C_{\alpha,\beta}(v,w-u+\pi/2|\tau)C_{\alpha,\beta}(u,x-v+\pi/2|\tau)$$

takes the same form as the right-hand side of (4.10). Thus, we obtain (4.8) and this completes the proof of the theorem. □

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