Reference

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10.1017/mag.2018.71

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102.30 A surprising 3-D result involving a hexagon

In a recent posting (June 2017) on *Facebook*, the first author presented a relatively unknown theorem regarding the concurrency of lines connecting the midpoints of the opposite sides of a plane hexagon with opposite sides parallel, but not necessarily equal (see [1]). To prove this theorem, it was necessary to employ projective geometry methods as using simple Euclidean methods do not suffice. In response to this posting, John Berry (a mathematical friend for many years from York, UK) asked the question on *Facebook* whether this result would also be true in 3-D space.

Obviously these lines will be concurrent for a spatial 3-D hexagon with opposite sides equal *and* parallel as the concurrency follows immediately from the joint central point of (reflective) symmetry of the three parallelograms formed in space. However, the question arose whether one could construct a spatial hexagon with opposite sides parallel, but not equal.

Experimentally exploring this possibility using the dynamic 3-D software *Cabri* 3D by construction and measurement (e.g. [2]), it turned out much to our surprise, that in 3-D space, if a hexagon has three pairs of opposite sides parallel, then they are equal also. This seemed rather counter-intuitive to us as we had expected to be able to produce a 3D analogue to a plane hexagon with opposite sides parallel, but not necessarily equal, especially since one normally has more degrees of freedom in space.

Below is presented a Euclidean proof of this result and its point (reflective) symmetric property (as described below), followed by a vector proof.

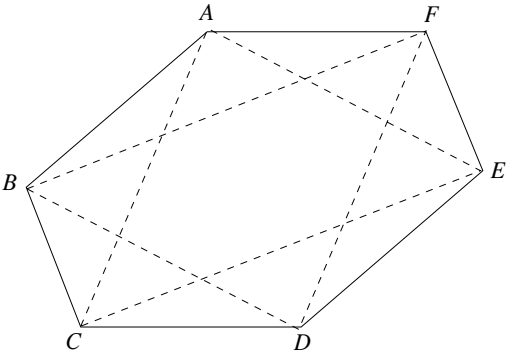


FIGURE 1

https://doi.org/10.1017/mag.2018.72 Published online by Cambridge University Press

Proof 1:

Given *ABCDEF* as a 3-D hexagon (i.e. not all six points lie in a common plane) with *AB* parallel to *ED*, *BC* parallel to *FE* and *CD* parallel to *AF*. (See Figure 1 which shows a parallel projection onto the plane). The six straight lines then lie pairwise parallel in 3 different directions in space. Since *AB* is parallel to *ED* and *BC* is parallel to *FE*, the planes *ABC* and *DEF* on which they respectively lie are also parallel to each other. Therefore *AC* must be parallel to *FD*, but since the straight lines *CD* and *FA* are given as parallel to each another, it follows that *FACD* is a parallelogram. Similarly, the quadrilaterals *ABDE* and *BCEF* are parallelograms.

Since *FACD* is a parallelogram, diagonals *AD* and *FC* have a common midpoint *M*. But *ABDE* is also a parallelogram, and hence *M* must also be the midpoint of *BE*. Hence, the three diagonals of the non-planar 3-D parallelo-hexagon are concurrent at *M*, which is a centre of point symmetry. An object has a centre of 'point symmetry' *P* if it is invariant under a point reflection around that point *P*.

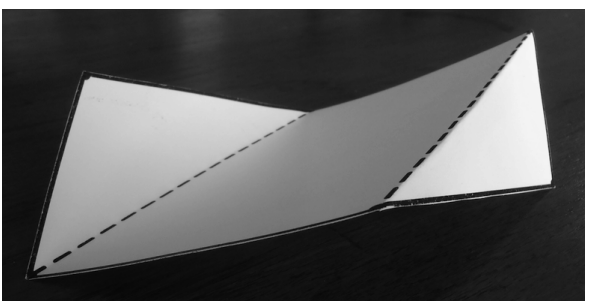


FIGURE 2

Note that the first proof was partly inspired by the construction of a non-planar 3-D parallelo-hexagon by paper folding as shown by the photograph in Figure 2.

Proof 2: Let $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$, $\mathbf{E}$ and $\mathbf{F}$ be the position vectors of the vertices. Because of parallelism, there are real x, y and z non-zero with $\mathbf{E}-\mathbf{D}=x(\mathbf{B}-\mathbf{A})$, $\mathbf{F}-\mathbf{E}=y(\mathbf{C}-\mathbf{B})$ and $\mathbf{A}-\mathbf{F}=z(\mathbf{D}-\mathbf{C})$. Note firstly that since the spatial hexagon is non-planar, the vectors $\{(\mathbf{B}-\mathbf{A}),(\mathbf{C}-\mathbf{B}),(\mathbf{D}-\mathbf{C})\}$ are linearly independent. In addition, since the spatial hexagon is closed, it follows that

$$(\mathbf{B}-\mathbf{A})+(\mathbf{C}-\mathbf{B})+(\mathbf{D}-\mathbf{C})+x(\mathbf{B}-\mathbf{A})+y(\mathbf{C}-\mathbf{B})+z(\mathbf{D}-\mathbf{C})=\mathbf{0}.$$

Thus, $x=y=z=-1$ and thus $\mathbf{E}-\mathbf{D}=\mathbf{A}-\mathbf{B}$, $\mathbf{F}-\mathbf{E}=\mathbf{B}-\mathbf{C}$ and $\mathbf{A}-\mathbf{F}=\mathbf{C}-\mathbf{D}$, i.e. opposite sides are equal.

Let $\mathbf{M}$ be the centre of $\mathbf{AD}$, that is, $\mathbf{M}=\frac{1}{2}(\mathbf{A}+\mathbf{D})$. Then $\mathbf{M}=\frac{1}{2}(\mathbf{B}+\mathbf{E})$ and further $\mathbf{M}=\frac{1}{2}(\mathbf{C}+\mathbf{F})$.

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A generalisation for a 'spatial' $2n$-gon in the n-dimensional space is now immediately obvious from this vector proof, giving a good example of a proof illustrating not only the 'explanatory' function of proof, but its 'discovery' function as well (see [3]). Clearly this proof does not work in the plane, since the requirement of linear independence does not hold.

Additionally, though there is no 3-D analogue for a hexagon in the plane with opposite sides parallel, but not necessarily equal, it became apparent with the advantage of hindsight, that the 3-D parallelo-hexagon was the 3-D analogue of a 2-D parallelogram. However, unlike the 2-D parallelogram, the converse of the above result, namely that a 3-D hexagon having three pairs of opposite sides equal would imply opposite sides parallel, is not true in 3-D space. A counterexample is most easily constructed by considering the paper-folding example in Figure 2, and having both folded 'flaps' (*ABC* and *DEF* in Figure 1) instead folded towards the *same side* of the original figure. Though opposite sides will still be equal from such a folding, the two pairs of sides involved will no longer be correspondingly parallel.

Also note that though the 3-D parallelo-hexagon has a point of (reflective) symmetry like the parallelogram, it does *not*, like the parallelogram, have half-turn symmetry (as the 3-D parallelo-hexagon can be rotated by a half-turn in say a direction perpendicular to *FACD* without mapping onto itself). An object has a point P of 'half-turn symmetry' if it is invariant under a *half-turn* (rotation of 180°) around that point P. It therefore also provides a good pedagogic example of showing the non-equivalence in space of the two different, but related, concepts of 'point symmetry' and 'half-turn symmetry' (which are easy to prove equivalent in the plane, and easily illustrated by, for example, the symmetries of a parallelogram, parallelo-hexagon or a cubic function).

Concluding remarks

This little episode illustrates how some experimental exploration with 3-D dynamic geometry software (as well as some paper folding) helped lead to an unanticipated conjecture (at least for us), which provides a nice problem with which to challenge high school learners or even prospective mathematics teachers, as it is relatively easy to prove. Available computing technologies are now making it much more feasible and easy to bring 3-D investigations into the classroom.

Acknowledgement

Thanks also to Kate Mackrell from Kingston, Ontario, Canada who similarly experimentally confirmed our finding with *Cabri* 3D. Readers are invited to establish experimentally the validity of the conjecture with other available software with dynamic 3-D facilities.

https://doi.org/10.1017/mag.2018.72 Published online by Cambridge University Press

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   (An interactive, dynamic sketch illustrating the concurrency result mentioned at the start can be accessed online at:

   http://dynamicmathematicslearning.com/parahex.html)
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10.1017/mag.2018.72

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102.31 Cyclic gridpoint polygons

Consider a standard circle, centre the origin, with equation $x^2 + y^2 = r^2$, where $x, y, r \in \mathbb{R}$ and $r \geqslant 0$. As r increases from 0, the circle passes through 'gridpoints', where x and y are both integers for specific values of r. Hereafter we shall confine our attention to these circles, which are given by the equation $x^2 + y^2 = k^2$, where x, y, k are integers and k is necessarily the largest number (and the length of the hypotenuse) in a primitive Pythagorean triple. For each value if $r = k$, we call the polygon formed by joining the gridpoints on the circle a 'cyclic gridpoint polygon' (CGP).

Dissect a CGP into strips parallel to one of the axes (say the y-axis). These are either triangles or trapezia. Because the vertices are gridpoints, the dimensions needed to calculate the area of each strip are integers. Furthermore, by symmetry about the x-axis, the lengths of their bases and tops (with reference to the formulae for the area of a triangle and a trapezium) are even. The area of each strip, therefore, and hence the area of the whole CGP, is integral. Let this be A_k for the case $r = k$.

By making r the product of an arbitrarily large number of primitive Pythagorean hypotenuses, $c_1, c_2, c_3, \ldots$ we can construct a CGP with as many sides as we wish. This is by virtue of the fact that the ratio $a : b$ of every primitive Pythagorean triple (a, b, k) is distinct and primitive Pythagorean triples are infinite in number. Figure 1 shows what happens within a sector defined by the gridpoints (4, 3) and (4, −3) as we increase the number of Pythagorean hypotenuses. In fact the set of CGP vertices on the outer circle is the union of all the sets of CGP vertices on the inner

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