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A GENERALIZATION OF THE BURRIDGE–GUERRE NONPARAMETRIC UNIT ROOT TEST

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In this note the nonparametric unit root test of Burridge and Guerre (1996, *Econometric Theory*, 12, 705–723), which is based on the standardized number of crossings of a level of a random walk, is extended in two ways, allowing for a deterministic trend in the process and more general innovations. The test has a well-known standard limit distribution. Monte Carlo experiments revealed the good finite-sample properties of the proposed test.

1. MOTIVATION

Testing for unit roots has been among the most widely researched topics in econometrics for the last 20 years. Burridge and Guerre (1996) observed that the number of level crossings in a variable would be larger if no unit root was present rather than otherwise and, hence, proposed a unit root test based on the standardized number of level crossings of a random walk without deterministic terms. They showed that the asymptotic distribution of this statistic relates to a scaled local time of a Brownian motion and found that the scale factor depends on the long-run variance of the process. In the present note we extend the proposal of Burridge and Guerre in two directions. First, we consider the process with time trend and prove that the modified standardized number of level crossings converges to a Rayleigh distribution. Second, more general structures of autocorrelated disturbances are also taken into account. Monte Carlo experiments show the good sample properties of this nonparametric unit root test.

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2. RESULTS

Let $x_t = \mu_t + u_t$ be an I(1) process, where $\mu_t = \alpha + \beta t$, $u_t = u_{t-1} + \varepsilon_t$, and the sequence ε_t satisfies the conditions in Akonom (1993), namely, (a) $\varepsilon_t = \sum_{j=0}^{\infty} \pi_j e_{t-j}$; (b) e_t is a sequence of independent and identically distributed (i.i.d.) random variables with zero mean and constant variance, σ_e^2, and $E(|e_t|^p) < \infty$ for some $p > 2$; (c) $\pi_0 = 1$, $\sum_{j=0}^{\infty} \pi_j = \pi(1) \neq 0$, and $\sum_{j=0}^{\infty} j|\pi_j| < \infty$. Moreover, we also impose, as Burridge and Guerre (1996) did, (d) there exists n such that for $t \geq n$, $t^{-1/2} \sum_{j=1}^{t} \varepsilon_j$ has bounded probability density function. We denote the long-run variance of ε_t as $\sigma^2 = \sigma_e^2 \pi(1)^2$.

Let $x_t^{(0)} = x_t$, $x_t^{(1)} = x_t - x_1 - ct$, where $c = (x_T - x_1)/T$ and T is the sample size. Thus, $x_t^{(1)}$ is a detrended I(1) process. Define the normalized number of crossing levels as

$$K_T^{(i)}(z) = T^{-1/2} \sum_{t=1}^{T} (\mathbf{1}[x_{t-1}^{(i)} \leq z, x_t^{(i)} > z] + \mathbf{1}[x_{t-1}^{(i)} > z, x_t^{(i)} \leq z]),$$

for $i = 0,1$, where $\mathbf{1}[\cdot]$ is an indicator function that takes value one when the condition between brackets holds true and zero otherwise. Hence, $T^{1/2} K_T^{(i)}(z)$ counts the number of times $x_t^{(i)}$ crosses the level z. Burridge and Guerre (1996) considered the case where $\mu_t = 0$ for all t and found that, when $\{\varepsilon_t\}_{t \geq 1}$ is an i.i.d. sequence of zero-mean finite-variance random variables, then

$$K_T^{(0)}(z) \Rightarrow \frac{E|\varepsilon_1|}{\sigma} l_0(z,1) \stackrel{d}{=} \frac{E|\varepsilon_1|}{\sigma} |N(0,1)|,$$

where $l_0(z,t)$, $t \in [0,1]$, is a local time process (see, e.g., Revuz and Yor, 1991) and $\stackrel{d}{=}$ stands for equal distribution. Hence, $K_T^{(0)}(z)$ does not depend on z. Given this result, they suggested the following statistic to test the null of a unit root:

$$\varphi^{(0)} = \frac{\sqrt{T^{-1} \sum_{t=1}^{T} (\Delta x_t^{(0)})^2}}{\widehat{MAD}} K_T^{(0)}(0),$$

where $\widehat{MAD} = T^{-1} \sum_{t=1}^{T} |\Delta x_t^{(i)}|$ and $\Delta x_t^{(i)} = x_t^{(i)} - x_{t-1}^{(i)}$. The statistic $\varphi^{(0)}$ has an $|N(0,1)|$ limit distribution when $\{\varepsilon_t\}_{t \geq 1}$ is an i.i.d. sequence of zero-mean finite-variance random variables and $\mu_t = 0$ for all t. Under the alternative hypothesis of stationarity, $\varphi^{(0)}$ diverges. We herein extend the previous statistic in two ways, allowing for trended processes and also for more general innovations:

$$\eta^{(i)} = \frac{\hat{\sigma}}{\widehat{MAD}} K_T^{(i)}(0),$$

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$i = 0,1$, where $\hat{\sigma}$ is a consistent estimator of the square root of the long-run variance of $\{\varepsilon_t\}_{t\geq 1}$. In the simulation experiments that follow we used the procedure suggested by Sul, Phillips, and Choi (2005) with the Bartlett window to estimate $\hat{\sigma}$. The following theorem shows the limit distribution of the previous test for trending processes.

THEOREM 1. *$\{\varepsilon_t\}_{t\geq 1}$ satisfies the conditions (a)–(d) given previously. Then,*

$$\eta^{(1)} \Rightarrow R,$$

where R is a random variable with (standard) Rayleigh distribution, that is, $f_R(x) = \mathbf{1}(x > 0)x\exp(-\frac{1}{2}x^2)$. Moreover, if $\mu_t = 0$, then $\eta^{(0)} \Rightarrow |N(0,1)|$.

Proof. See the Appendix.

Table 1 shows estimates of the critical values of $\eta^{(0)}$ and $\eta^{(1)}$ obtained from 50,000 replications in addition to the asymptotic distribution. As can be seen, these values approximate the limit values for large sample sizes.

Finally, we compare our test $\eta^{(1)}$ with the DF^{GLS} test proposed by Elliott, Rothenberg, and Stock (1996) with lags selected according to the modified Akaike information criterion (AIC) of Ng and Perron (2001). The implementation of unit root tests often requires the selection of an autoregressive truncation lag. These authors use a local asymptotic framework in which a class of modified information criterion (MIC) in conjunction with generalized least squares (GLS) detrending results in a class of tests with good size and power. For the $\eta^{(1)}$ statistic, an initial bandwidth of $l = [12(T/100)^{1/4}]$ was used for the computation of the long-run variance using the procedure suggested by Sul et al. (2005).

Table 2 shows the empirical size of the two tests for MA(1) autocorrelated disturbances, a situation where unit root tests usually have size problems. The

TABLE 1. Critical values of the level crossing tests

	T	50	75	100	200	300	500	1,000	2,000	$\lvert N(0,1)\rvert$
$\eta^{(0)}$	0.9	1.45	1.50	1.52	1.56	1.59	1.60	1.61	1.62	1.64
	0.95	1.73	1.78	1.82	1.85	1.89	1.91	1.91	1.94	1.96
	0.975	1.99	2.03	2.08	2.12	2.16	2.17	2.19	2.21	2.22
	0.99	2.26	2.36	2.37	2.44	2.48	2.51	2.53	2.54	2.57
										Rayleigh
$\eta^{(1)}$	0.9	1.91	1.98	2.07	2.08	2.10	2.11	2.12	2.13	2.14
	0.95	2.16	2.23	2.36	2.38	2.39	2.40	2.41	2.43	2.44
	0.975	2.38	2.46	2.63	2.63	2.66	2.66	2.68	2.69	2.71
	0.99	2.64	2.73	2.92	2.95	2.96	2.97	2.99	3.01	3.03

Note: Computed using 50,000 replications. Here x_t is a Gaussian random walk.

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TABLE 2. Empirical size

θ	$\eta^{(1)}$	DF^{GLS}
−0.9	0.036	0.030
−0.7	0.047	0.038
−0.5	0.057	0.043
−0.3	0.061	0.055
0	0.051	0.049
0.3	0.046	0.053
0.5	0.032	0.066
0.7	0.030	0.095
0.9	0.021	0.335

Note: Computed using 3,000 replications. Data generating process (DGP): $x_t = x_{t-1} + u_t$, $u_t = \varepsilon_t - \theta\varepsilon_{t-1}$, $\varepsilon_t \sim iidN(0,1)$; $T = 200$.

$\eta^{(1)}$ test presents very good size in all cases, whereas in the DF^{GLS} test problems arise for large values of the moving average parameter. Next, in Table 3 is shown the empirical power for stationary AR(1) alternatives. As observed, the DF^{GLS} test happens to show better power, which could be explained by the fact that this statistic is designed to be powerful against local alternatives.

3. CONCLUSIONS

In this note we have proposed a new nonparametric unit root test for trending processes that is based on normalized level crossings. The test is a generalization of the one proposed in Burridge and Guerre (1996), allowing for trended processes and more general forms of autocorrelated disturbances. This new test has an asymptotic distribution that is a standard Rayleigh distribution. We have

TABLE 3. Empirical power

ρ	$\eta^{(1)}$	DF^{GLS}
0.8	0.6357	0.8007
0.85	0.5497	0.8000
0.9	0.4023	0.6763
0.95	0.1657	0.2843

Note: Computed using 3,000 replications. DGP: $x_t = \rho x_{t-1} + \varepsilon_t$, $\varepsilon_t \sim iidN(0,1)$; $T = 200$. For the DF^{GLS} test 5% critical values computed from 50,000 replications were used.

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also shown that the test has good finite-sample properties and presents, for some but not all parametrizations of the DGP, better size properties than the DF^{GLS} test considered in Ng and Perron (2001). Thus, the $\eta^{(1)}$ test proposed in this note can be considered a good complement to other unit root tests.

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APPENDIX

We shall make use of the following lemmas to prove Theorem 1.

LEMMA 2. *Let x_t be an I(1) process that satisfies the previous conditions (a)–(d) with σ^2 being the long-run variance of ε_t. Let $x_t^{(0)} = x_t$ and $x_t^{(1)} = x_t - x_1 - ct$, where $c = (x_T - x_1)/T$. Then, if $\mu_t = 0$*

$$T^{-1/2} x_{[rT]}^{(i)} \Rightarrow \sigma W^{(i)}(r),$$

for $i = 0,1$ and $r \in [0,1]$, where $\Rightarrow$ stands for weak convergence of the associated probability measures, $[\cdot]$ is the integer part, $W^{(0)}(r) \equiv W(r)$ is a standard Brownian motion, and $W^{(1)}(r) \equiv W(r) - rW(1)$ is a standard Brownian bridge. If $\mu_t = \alpha + \beta t$, then

$$T^{-1/2} x_{[rT]}^{(1)} \Rightarrow \sigma W^{(1)}(r).$$

Proof. See, Akonom (1993) for $\mu_t = 0$ and $i = 0$. For $i = 1$ arguments similar to those in Phillips and Schmidt (1992) apply. ■

LEMMA 3. *Under the same assumptions as in the previous lemma, if $\mu_t = 0$*

$$K_T^{(i)}(z) \Rightarrow \frac{E|\varepsilon_1|}{\sigma} l_i(0,1),$$

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where $l_0(z,t)$, $t \in [0,1]$, is the local time of a Brownian motion in the vicinity of z and $l_1(z,t)$ is that of a Brownian bridge. Moreover, if $\mu_t = \alpha + \beta t$

$$K_T^{(1)}(z) \Rightarrow \frac{E|\varepsilon_1|}{\sigma} l_1(0,1).$$

Proof. See Burridge and Guerre (1996) for the result regarding $l_0(z,t)$ with i.i.d. ε_t. Using Theorems 3 and 4 in Akonom (1993) in the proof of Burridge and Guerre the results extend to the case of nonindependent sequences such as those considered under the Akonom assumptions. Proof of $l_1(z,t)$ follows similar steps. ■

LEMMA 4. *(a) $l_0(0,1) \stackrel{d}{=} |N(0,1)|$ and (b) $l_1(0,1) \stackrel{d}{=} R$, where R presents a standard Rayleigh distribution with a density function $f(r) = \mathbf{1}(r > 0) r \exp(-r^2/2)$.*

Proof. See Revuz and Yor (1991, p. 250). ■

LEMMA 5. *Under the conditions stated previously, $\widehat{MAD} = T^{-1} \sum |\Delta x_t^{(i)}| \underset{p}{\rightarrow} E|\varepsilon_1|$.*

Proof. First, note that $c = (x_T - x_1)/T$ is a consistent estimator of β and that

$$T^{-1} \sum_{t=1}^{T} |\Delta x_t^{(i)}| = T^{-1} \sum_{t=1}^{T} \Delta x_t^{(i)} \mathbf{1}_t(\Delta x_t^{(i)} > 0) - T^{-1} \sum_{t=1}^{T} \Delta x_t^{(i)} \mathbf{1}_t(\Delta x_t^{(i)} < 0).$$

Let us consider first the term $T^{-1} \sum_{t=1}^{T} \Delta x_t^{(i)} \mathbf{1}_t(\Delta x_t^{(i)} > 0)$. Under the set of assumptions (a)–(d) stated earlier, $\Delta x_t^{(i)} \mathbf{1}_t(\Delta x_t^{(i)} > 0)$ is a covariance stationary process, and it follows that (see, e.g., Bierens, 2004, Thm. 7.3)

$$p \lim_{T \to \infty} T^{-1} \sum_{t=1}^{T} \{\Delta x_t^{(i)} \mathbf{1}_t(\Delta x_t^{(i)} > 0) - E[\Delta x_t^{(i)} \mathbf{1}_t(\Delta x_t^{(i)} > 0)]\} = 0.$$

Similar arguments can be used for the term $T^{-1} \sum_{t=1}^{T} \Delta x_t^{(i)} \mathbf{1}_t(\Delta x_t^{(i)} < 0)$. ■

Proof of Theorem 1. Given that $\widehat{MAD}$ and $\hat{\sigma}$ are consistent estimators, the theorem follows immediately from the three previous lemmas. ■

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