Check for updates

J. Appl. Probab. **63,** 1168–1185 (2026)
*doi:*10.1017/jpr.2026.10073

ON REFLECTED LÉVY PROCESSES WITH COLLAPSE

ONNO J. BOXMA,* *EURANDOM and Eindhoven University of Technology*
OFFER KELLA,** *The Hebrew University of Jerusalem*
DAVID PERRY,*** *Holon Institute of Technology*

Abstract

We consider a Lévy process reflected at the origin with additional independent and identically distributed *collapses* that occur at Poisson epochs, where a *collapse* is a jump downward to a state which is a random fraction of the state just before the jump. We first study the general case, then specialize to the case where the Lévy process is spectrally positive, and, finally, we specialize further to the two cases where the Lévy process is a Brownian motion and a compound Poisson process with exponential jumps minus a linear slope.

Keywords: Reflected Lévy process; collapse; Lindley-style autoregressive recursions

2020 Mathematics Subject Classification: Primary 60G51
Secondary 60K25

1. Introduction

This paper is devoted to the study of Lévy processes which are reflected at the origin, and which have the special feature that at Poisson epochs they are subject to random collapses, jumps downward where the jump size is a random proportion of the state just before the jump. Such processes occur naturally in several different fields. We name a few as follows.

- **Population dynamics.** Population sizes may fluctuate according to birth-and-death processes, and experience disasters at random epochs. After appropriate scaling such a process may be approximated by a reflected Brownian motion subject to collapse. This case is given special attention in Example 6.1 of the current study.
- **Geophysics.** Pressure in Earth layers may fluctuate randomly between earthquakes or volcano eruptions (collapses).

Received 16 January 2025; accepted 30 December 2025.
* Postal address: EURANDOM and Department of Mathematics and Computer Science, Eindhoven University of Technology, the Netherlands. Email: o.j.boxma@tue.nl
** Postal address: Department of Statistics, The Hebrew University of Jerusalem; Jerusalem 9190501, Israel. Email: offer.kella@gmail.com
*** Postal address: Industrial Engineering and Technology Management, Holon Institute of Technology, Holon 5810201, Israel. Email: davidper@hit.ac.il

© The Author(s), 2026. Published by Cambridge University Press on behalf of Applied Probability Trust. This is an Open Access article, distributed under the terms of the Creative Commons Attribution-NonCommercial licence (https://creativecommons.org/licenses/by-nc/4.0/), which permits non-commercial re-use, distribution, and reproduction in any medium, provided the original article is properly cited. The written permission of Cambridge University Press or the rights holder(s) must be obtained prior to any commercial use.

https://doi.org/10.1017/jpr.2026.10073 Published online by Cambridge University Press

- **Transmission Control Protocol (TCP).** TCP adapts the window size (transmission rate) of data transfers to the congestion in the network. The window size along a path is increased until a signal is received that the path gets too congested; the window size is then proportionally reduced (additive increase multiplicative decrease). Stochastic processes with random collapses have been studied intensively in the analysis of the TCP of the Internet (see, among many others, [2, 14, 15, 26]), but typically those processes are nondecreasing (often linear) between collapses.
- **Perishable inventories.** In stochastic models of perishable inventory systems (PISs) with random input, the inventory typically increases due to production and decreases because of demand. However, in the sizable literature of such PISs (cf. the survey [11]) not much attention has been given to the modeling of incidents such as power outages, which may make a random proportion of the stored goods (e.g. food or blood) obsolete. See [21] for the case of PISs with *clearing*, i.e. collapses in which all the goods become obsolete.
- **Cash management**. We dwell a bit longer on this particular application area, as it may be less known than those above, whereas it formed one of our main motivations for the present study. The *peer-to-peer* (P2P) lending platform is the practice of lending money through online services that match lenders with borrowers. The P2P firm offers a platform where individuals or businesses can lend directly to other individuals or businesses without the need for a bank as a middleman. The firm takes brokerage fees for providing this match-making platform. Compared with investment and savings products from banks, borrowers can borrow money at lower interest rates, and lenders can earn higher returns. Although the P2P lending firm applies a strict screening system, it may happen that a borrower is in arrears. If that happens, the P2P firm initiates collection procedures against the debtor. P2P systems are exposed only to the exceptional volatility caused by a severe crash in which many borrowers simultaneously go bankrupt. Historical examples of such crashes are the dot.com bubble in 2000, the sub-prime crisis in 2008 that started with the crash of Lehman Brothers, and the corona crisis of 2020. In these cases, the stock market crash was accompanied by bankruptcies in the entire business sector of small and medium businesses and even large companies. In the events mentioned here the crisis hit the entire capital market, since all those economic entities could not meet their financial obligation. In [10] a stochastic analysis of such cash management processes with crashes/collapses is presented, with a compound Poisson process between collapses. However, the cash management process generated by big businesses together with many 'small' individuals can be more naturally approximated by an independent sum of a compound Poisson and a Brownian component, reflected at the origin, which is a special case of the more general reflected Lévy process considered in this paper.

For some background on growth collapse processes, where, unlike in the current paper, the processes are nondecreasing between collapse epochs, one may sample, among others, [7, 12, 16, 18, 20, 27] and references therein. See also [28, 29] for certain (spectrally positive) Lévy queues with workload modification at exponential times and some other features.

Globally, our *goals* are to study the stability conditions of reflected Lévy processes with random collapses, and to determine the stationary distribution (if it exists) when the Lévy process is spectrally positive (i.e. the Lévy process itself has no negative jumps).

https://doi.org/10.1017/jpr.2026.10073 Published online by Cambridge University Press

Our *main contributions* are as follows.

- A careful discussion of the conditions for the existence of a limiting/ergodic/stationary distribution of the reflected Lévy process with collapses.
- For the case of a spectrally positive Lévy process, we determine the Laplace–Stieltjes transform (LST) and moments of that limiting distribution.
- For the special case of a compound Poisson process with regularly varying jump sizes, we prove that the tail of the limiting distribution is just as heavy as the tail of the jump size distribution.
- For the special cases of Brownian motion and of a compound Poisson process with exponentially distributed jump sizes (corresponding to an $M/M/1$ queue with collapses) the LST of the limiting distribution is worked out in more detail, revealing a relation to incomplete Beta functions.

Remark 1. There are many possible extensions of our model; we mention a few. (i) The intervals between collapses could have a general distribution (cf. [7], where the resulting model is studied for the case of a compound Poisson process minus a drift). (ii) It would be interesting to study reflected Lévy processes in which workload decreases proportionally ('shot noise' with exponential decay or equivalently a 'linear dam'); cf. [18], where the relation between collapses and such a shot noise structure is explored. (iii) In PISs, it is often assumed that items have a finite lifetime [11]. In our setting, this could give rise to *doubly reflected* Lévy processes with collapses. (iv) Next to proportional collapses, one might also allow state-dependent jumps *upward*. In cash management and the crypto exchange market, such jumps might occur, although they seem milder than the collapses.

The remainder of the paper is organized as follows. In Section 2 we introduce the model under consideration: the Lévy process, the collapse procedure that is applied to it, the reflected version of the resulting process, and its local time at zero. We also show that the process level just before successive collapses satisfies a Lindley-type recursion of an autoregressive sequence. Section 3 considers this recursion in detail. It leads to the conclusion that, if a limiting distribution for this sequence exists, then the limiting random variable can be decomposed into two independent terms. In Section 4 we discuss the conditions under which the reflected Lévy process with general [independent and identically distributed (i.i.d.)] collapses at Poisson epochs has a limiting/ergodic/stationary distribution. By Poisson Arrivals See Time Averages (PASTA), this is also the limiting distribution of the process level just before collapses. Section 5 presents the derivation of this limiting distribution. The special cases of reflected Brownian motion with collapses, of an $M/M/1$ queue with collapses, and of a heavy-tailed $M/G/1$ queue with collapses are discussed in Section 6.

2. The general model

In what follows $x \vee y = \max(x, y)$, $x \wedge y = \min(x, y)$, $x^+ = x \vee 0$, and $x^- = (-x)^+$. In addition, $\sim$ denotes 'distributed' or 'distributed like' and $\approx$ denotes 'asymptotic to'.

Consider a càdlàg Lévy process $X = \{X_t \mid t \geq 0\}$ with $X_0 = 0$. Associated with each Lévy process is a triplet (c, σ^2, ν) where $c \in \mathbb{R}$, $\sigma \geq 0$, and ν is a (sigma-finite) measure, called the *Lévy measure* which satisfies $\nu(\{0\}) = 0$, $\int_{\mathbb{R}} x^2 \wedge 1 \, \nu(\mathrm{d}x) < \infty$. Any such Lévy process can be decomposed into a sum of independent processes, where one is a Brownian motion

https://doi.org/10.1017/jpr.2026.10073 Published online by Cambridge University Press

(possibly only a drift), another is a compound Poisson process (possibly zero) with jumps in $\mathbb{R} \setminus [-1, 1]$, and, finally, a convergent sum of centered compound Poisson processes of the form $\sum_{n=1}^{\infty} (X_{n,t} - t\mathrm{E}X_{n,1})$, where the absolute values of the jumps of the process X_n are in $(1/(n+1), 1/n]$. For background on Lévy processes and Lévy queues, see, e.g., [4, 13, 21, 25, 30].

In addition, let $\tau, \tau_1, \tau_2, \ldots$ be i.i.d. $\exp(\lambda)$ distributed random variables which are also independent of X and use the notation $S_0 = 0$ and $S_n = \sum_{i=1}^{n} \tau_i$ for $i \geq 1$, with renewal counting (Poisson) process $N_t = \sup\{n \mid S_n \leq t\}$. Finally, let $U, U_1, U_2, \ldots$ be i.i.d. with $\mathrm{P}(U \in [0, 1]) = 1$. These are thought of as random proportions and are independent of all the other random objects defined thus far.

For $t \geq 0$, denote $L_t = -\inf_{0 \leq s \leq t} X_s$, $W_t = X_t + L_t$, and, for $x \geq 0$,

$$\begin{aligned} L_t^x &= \left(-\inf_{0 \leq s \leq t} (x + X_s) \right)^+ = (L_t - x)^+, \\ W_t^x &= x + X_t + L_t^x = X_t + x \vee L_t = W_t + (x - L_t)^+. \end{aligned} \tag{1}$$

This is the reflection (Skorokhod) map associated with $\{x + X_t \mid t \geq 0\}$ where (L_t^x, W_t^x) are the unique càdlàg processes that jointly satisfy (see [17]):

(i) $L_0^x = 0$ and L_t^x is nondecreasing in t;

(ii) $W_t^x \geq 0$ for each $t \geq 0$; and

(iii) $\{t \mid W_t^x = 0\} \subset \{t \mid L_s^x < L_t^x \ \forall s \in [0, t)\}$.

Now denote $\sigma_s X = \{X_{t+s} - X_s \mid t \geq 0\}$ (which is independent of the history of X until time s) and let $(\sigma_s L_t^x, \sigma_s W_t^x)$ be the corresponding reflection map associated with $\sigma_s X$. Let $Z = \{Z_t \mid t \geq 0\}$ be defined as follows. Assume that we have already defined it on $[0, S_n)$. Then for $t \in [S_n, S_{n+1})$, $Z_t = \sigma_{S_n} W_{t-S_n}^{U_n Z_{S_n -}}$, where $Z_{t-} = \lim_{v \uparrow t} Z_v$. Namely, at time S_n, which we refer to as *collapse epochs*, the content modeled by the process Z collapses to a fraction U_n of its precollapse level and continues according to a reflected Lévy process until the next collapse epoch. Here Z_0 may have any distribution and is assumed to be independent of everything else.

Comment 1. In fact, because of the assumptions made until now, Z is a (nonnegative) Markov process with the following generator:

$$\begin{aligned} \mathcal{A}g(x) = cg'(x) + \frac{\sigma^2}{2} g''(x) + \int_{\mathbb{R}} \left(g(x+y) - g(x) - g'(x) y 1_{[-1,1]}(y) \right) \nu(\mathrm{d}y) \\ + \lambda(\mathrm{E}g(Ux) - g(x)) \end{aligned} \tag{2}$$

for twice continuously differentiable g with $g(x) = g(0)$ for $x \leq 0$ (hence, $g'(x) = g''(x) = 0$ for $x \leq 0$). In particular, for $x \leq 0$ it may be checked that

$$\mathcal{A}g(x) = \int_{(-x, \infty)} (g(x+y) - g(0)) \nu(\mathrm{d}y) = \int_0^{\infty} g'(y) \nu(-x+y)\, \mathrm{d}y. \tag{3}$$

This can be shown by applying the (local) martingales discussed in [24] or [19] but will not be needed for what follows.

https://doi.org/10.1017/jpr.2026.10073 Published online by Cambridge University Press

A famous result, which in particular may be found in [4, 13, 25, 30], is that W_τ and L_τ are independent. By (1) this implies the following decomposition:

$$\big(W_\tau^x, L_\tau^x\big) = (W_\tau, 0) + \big((x - L_\tau)^+, (L_\tau - x)^+\big), \tag{4}$$

where the two random vectors on the right are independent. In particular, denoting $w(\alpha) = \mathrm{E}\mathrm{e}^{-\alpha W_\tau}$, we have that, for $\alpha, \beta \geq 0$,

$$\mathrm{E}\mathrm{e}^{-\alpha W_\tau^x - \beta L_\tau^x} = w(\alpha)\,\mathrm{E}\mathrm{e}^{-\alpha(x-L_\tau)^+ - \beta(L_\tau - x)^+}. \tag{5}$$

With our setup, it follows from (4) that

$$Z_{S_n-} = \sigma_{S_{n-1}} W_{\tau_n} + (Z_{S_{n-1}-} U_n - \sigma_{S_{n-1}} L_{\tau_n})^+, \tag{6}$$

where $\sigma_{S_{n-1}} W_{\tau_n}$, $Z_{S_{n-1}-}$, U_n, $\sigma_{S_{n-1}} L_{\tau_n}$ are independent and $\sigma_{S_{n-1}} W_{\tau_n}$, U_n, $\sigma_{S_{n-1}} L_{\tau_n}$ are distributed like W_τ, U, L_τ, respectively.

We observe that by PASTA the ergodic distribution of the continuous time process Z is the same as that of the discrete time process Z_{S_n-} and, therefore, it is important to study this latter process. With $\zeta_n = Z_{S_n-}$, $V_n = \sigma_{S_{n-1}} W_{\tau_n}$, and $Y_n = \sigma_{S_{n-1}} L_{\tau_n}$ we therefore have the following Lindley-style version of an autoregressive sequence:

$$\zeta_n = V_n + (\zeta_{n-1} U_n - Y_n)^+, \tag{7}$$

where all four random variables on the right are independent with $V_n \sim W_\tau$, $U_n \sim U$, and $Y_n \sim L_\tau$. In the next section we consider this recursion in some detail.

3. Lindley-style autoregressive recursions

Consider the recursion (for now, deterministic),

$$z_n = v_n + (z_{n-1} u_n - y_n)^+, \quad n \geq 1, \tag{8}$$

where $v_{i-1}, y_i \geq 0$, $u_i \in [0, 1]$ for $i \geq 1$. In addition, let $z_0 = v_0$. This is the same as letting $z_{-1} = 0$, defining $u_0 \in [0, 1]$ arbitrarily and starting the recursion one index earlier. Furthermore, let $\Pi_0 = 1$ and $\Pi_n = \prod_{i=1}^n u_i$ for $n \geq 1$.

We first observe that if $z'_n = v_n + (z'_{n-1} u_n - y_n)^+$ and $z''_n = v_n + (z''_{n-1} u_n - y_n)^+$, for any two choices where $z''_0 \geq z'_0 \geq 0$, then it follows by induction that $z''_n \geq z'_n$ for all $n \geq 0$. Moreover:

- when $z''_{n-1} u_n \leq y_n$ we have that $z''_n - z'_n = 0 \leq \big(z''_{n-1} - z'_{n-1}\big) u_n$;
- when $z'_{n-1} u_n > y_n$ we have that $z''_n - z'_n = \big(z''_{n-1} - z'_{n-1}\big) u_n$;
- when $z'_{n-1} u_n \leq y_n < z''_{n-1} u_n$ we have that

$$z''_n - z'_n = z''_{n-1} u_n - y_n \leq z''_{n-1} u_n - y_n + y_n - z'_{n-1} u_n = \big(z''_{n-1} - z'_{n-1}\big) u_n. \tag{9}$$

Therefore, whenever $z''_0 \geq z'_0 \geq 0$, for all $n \geq 1$, $z''_n - z'_n \leq (z''_{n-1} - z'_{n-1}) u_n$ and, thus, for every $n \geq 0$,

$$0 \leq z''_n - z'_n \leq \big(z''_0 - z'_0\big)\Pi_n. \tag{10}$$

In particular, this implies the following.

Lemma 1. *If $\Pi_n \to 0$ as $n \to \infty$ then $z'_n - z_n \to 0$ as $n \to \infty$ for any choice of $z'_0 \geq 0$ (smaller, equal, or larger than $z_0 = v_0$).*

https://doi.org/10.1017/jpr.2026.10073 Published online by Cambridge University Press

We note that since we will be replacing u_i by i.i.d. random variables $U_i \sim U$ with support in [0, 1], then, unless $\mathrm{P}(U = 1) = 1$, it will always hold that $\prod_{i=1}^{n} U_i \to 0$ as $n \to \infty$.

We begin by assuming that $u_i > 0$ for all $i \geq 1$, so that $\Pi_n > 0$ for all $n \geq 1$. Denoting, for $n \geq 1$,

$$w_n = \frac{z_n - v_n}{\Pi_n}, \quad x_n = \frac{v_{n-1}}{\Pi_{n-1}} - \frac{y_n}{\Pi_n}, \tag{11}$$

we have that $w_0 = 0$ and, for $n \geq 1$, $w_n = (w_{n-1} + x_n)^+$, which is Lindley's recursion. Thus, with $s_0 = 0$ and $s_n = \sum_{i=1}^{n} x_i$ for $n \geq 1$ we have the well-known solution:

$$w_n = s_n - \min_{0 \leq k \leq n} s_k = \max_{0 \leq k \leq n} (s_n - s_{n-k}) \tag{12}$$

and, thus,

$$z_n = v_n + \max_{0 \leq k \leq n} ((s_n - s_{n-k})\Pi_n). \tag{13}$$

For $k = 0$ we have that $(s_n - s_{n-k})\Pi_n = 0$ whereas for $1 \leq k \leq n$ we have

$$\begin{aligned}(s_n - s_{n-k})\Pi_n &= \sum_{i=n-k+1}^{n} \left(\frac{v_{i-1}}{\Pi_{i-1}} - \frac{y_i}{\Pi_i} \right) \Pi_n \\ &= \sum_{i=1}^{k} \left(\frac{v_{n-i}}{\Pi_{n-i}} - \frac{y_{n-i+1}}{\Pi_{n-i+1}} \right) \Pi_n,\end{aligned} \tag{14}$$

recalling that $v_0 = 0$ (relevant for $i = k = n$). Now,

$$\frac{\Pi_n}{\Pi_{n-i}} = \prod_{j=n-i+1}^{n} u_j = \prod_{j=1}^{i} u_{n-j+1} \tag{15}$$

and, similarly, for $i = 1$, we have that $\Pi_n / \Pi_{n-1+1} = 1$ and for $2 \leq i \leq k \leq n$,

$$\frac{\Pi_n}{\Pi_{n-i+1}} = \prod_{j=n-i+2}^{n} u_j = \prod_{j=1}^{i-1} u_{n-j+1}. \tag{16}$$

This implies that we can write

$$(s_n - s_{n-k})\Pi_n = \sum_{i=1}^{k} v_{n-i} \prod_{j=1}^{i} u_{n-j+1} - \sum_{i=1}^{k} y_{n-i+1} \prod_{j=1}^{i-1} u_{n-j+1}, \tag{17}$$

where an empty product is defined to be one and an empty sum is zero. Therefore, we finally have that

$$z_n = v_n + \max_{0 \leq k \leq n} \left(\sum_{i=1}^{k} (v_{n-i} u_{n-i+1} - y_{n-i+1}) \prod_{j=1}^{i-1} u_{n-j+1} \right). \tag{18}$$

We emphasize the fact that for $n = 0$ we have that (since empty sums are zero), (18) implies that $z_0 = v_0$ as initially assumed. Finally, it is a straightforward exercise to show by induction that (18) holds also for the case where u_i are allowed to be zero. For each such i we have that $z_i = v_i$ and the recursion evidently restarts from that point.

https://doi.org/10.1017/jpr.2026.10073 Published online by Cambridge University Press

Now, if we replace $\{(v_{i-1}, u_i, y_i) \mid i \geq 1\}$ by three independent i.i.d. sequences $\{(V_{i-1}, U_i, Y_i) \mid i \geq 1\}$ of nonnegative random variables and $\{z_n \mid n \geq 0\}$ by the random variable ζ_n, then by taking each of the random vectors $V_0, \ldots, V_n, U_1, \ldots, U_n$, and $Y_1, \ldots, Y_n$ in reverse order it follows that

$$\zeta_n \sim V_0 + \max_{0 \leq k \leq n} \left(\sum_{i=1}^{k} (V_i U_i - Y_i) \prod_{j=1}^{i-1} U_j \right), \tag{19}$$

where an empty sum is zero and an empty product is one. Therefore, ζ_n is stochastically increasing and bounded above by $\sum_{i=0}^{n} V_i \prod_{j=1}^{i} U_j$ and, hence, if $\mathrm{E}V_0 < \infty$ and $\mathrm{P}(U_i = 1) < 1$, ζ_n converges in distribution to an almost surely (a.s.) finite random variable ζ which is distributed like an independent sum of two random variables. The first is distributed like V_0 and the second like

$$\sup_{k \geq 0} \left(\sum_{i=1}^{k} (V_i U_i - Y_i) \prod_{j=1}^{i-1} U_j \right). \tag{20}$$

This is a generalization of the setup (hence, the conclusions) considered in [6]. See also [8] for a discussion of recursions of the form $W_n = (W_{n-1} U_n + X_n)^+$.

4. Stability of the Continuous Time Process

An important question is what are the conditions on Z that ensure that there exists a limiting/ergodic/stationary distribution. If $\mathrm{P}(U = 1) = 1$ then we have a reflected Lévy process, which is well understood and has been discussed many times in the literature and textbooks, and thus we will assume that $\mathrm{P}(U = 1) < 1$. Since the collapse epochs occur according to a Poisson process with rate λ, it is clear that true collapses (with $U_i < 1$) occur according to a Poisson process with rate $\lambda \mathrm{P}(U < 1)$. Therefore, without loss of generality (w.l.o.g.), let us assume that $\mathrm{P}(U = 1) = 0$. If $\mathrm{P}(U = 0) > 0$ then after a geometrically distributed number of trials the process jumps to zero (often called *clearing*), which implies that it is regenerative and therefore a proper limiting ergodic distribution clearly exists. It remains to consider the case $\mathrm{P}(U = 0) = \mathrm{P}(U = 1) = 0$. In this case it is clear that $\prod_{i=1}^{\infty} U_i = 0$ a.s.

It is easy to check that if $x \leq y$ then $W_t^x \leq W_t^y$ for all $t \geq 0$ and that $W_t^y - W_t^x$ is nonincreasing in t. See [23] for results like this in the multidimensional setting. Therefore, setting Z^x to be the process Z when started from $Z_0 = x$ we have that

$$Z_{\tau_1}^y - Z_{\tau_1}^x = \left(W_{\tau_1 -}^y - W_{\tau_1 -}^x \right) U_1 \leq (y - x) U_1, \tag{21}$$

where the left-hand side is nonnegative. Hence, by induction it also follows that

$$0 \leq Z_{S_n}^y - Z_{S_n}^x \leq (y - x) \prod_{i=1}^{n} U_i, \tag{22}$$

where the right-hand side vanishes a.s. as $n \to \infty$. Therefore, we also have that

$$0 \leq Z_t^y - Z_t^x \leq (y - x) \prod_{i=1}^{N_t} U_i \tag{23}$$

which vanishes a.s. as $t \to \infty$. Therefore, it follows that if a limiting distribution exists, it does not depend on the initial distribution of Z_0.

https://doi.org/10.1017/jpr.2026.10073 Published online by Cambridge University Press

Since $W_t^x = X_t + L_t \vee x \le x + X_t + L_t = x + W_t$ it follows that Z_{S_n} is stochastically bounded by $x\prod_{i=1}^n U_i + \sum_{i=1}^n V_i \prod_{j=i}^n U_j$ where all the variables on the right-hand side are independent and $V_i \sim W_\tau$. By results from [18] this implies that $\{Z_{S_n} \mid n \ge 0\}$ and, hence, $\{Z_t \mid t \ge 0\}$ are bounded above by a process that has a stationary version. Therefore, tightness of these processes is assured. It also implies that $\{Z_t \mid t \ge 0\}$ has a stationary version by following the (Loynes-type) construction from [18] which is applied there to the case where W is nondecreasing.

Therefore, we can apply PASTA to conclude that the (unique) ergodic distribution of Z is the same as that of the discrete time process $\zeta_n = Z_{S_n-}$, where we recall ζ_n, ζ from Sections 2 and 3. With $Z^* = \zeta$ having the steady-state distribution of ζ_n (and, hence, also of the process Z), we have that

$$Z^* \sim W_\tau^{Z^*U} \sim W_\tau + (Z^*U - L_\tau)^+, \tag{24}$$

where Z^*, W_τ, U, L_τ on the right are independent. To summarize, all that is needed for stability and independence of initial conditions for the continuous time process is that $\mathrm{P}(U_1 = 1) < 1$ and that $\mathrm{E}W_\tau < \infty$, for which it suffices that $\int_{(1,\infty)} x\nu(\mathrm{d}x) < \infty$.

5. The case where X is spectrally positive

When X is spectrally positive, that is, $\nu(-\infty, 0] = 0$, then with

$$\varphi(\alpha) = \log \mathrm{E}\mathrm{e}^{-\alpha X_1} = -c\alpha + \frac{\sigma^2}{2}\alpha^2 + \int_{(0,\infty)} \left(\mathrm{e}^{-\alpha x} - 1 + \alpha x 1_{(0,1]}(x)\right)\ \nu(\mathrm{d}x), \tag{25}$$

it may be concluded (e.g. applying the martingale from [22], as is done in [3, Theorem 3.10, p. 259]) that if X is not a subordinator and α_λ is the unique positive root of $\varphi(\alpha) = \lambda$, then $L_\tau \sim \exp(\alpha_\lambda)$ and

$$w(\alpha) = \mathrm{E}\mathrm{e}^{-\alpha W_\tau} = \frac{1 - \frac{\alpha}{\alpha_\lambda}}{1 - \frac{\varphi(\alpha)}{\lambda}}. \tag{26}$$

Therefore, it follows either by [3, Theorem 3.10, p. 259] or from (5) here that, for $\alpha \ne \alpha_\lambda$,

$$\mathrm{E}\mathrm{e}^{-\alpha W_\tau^x - \beta L_\tau^x} = w(\alpha)\,\frac{\mathrm{e}^{-\alpha x} - \frac{\alpha+\beta}{\alpha_\lambda+\beta}\mathrm{e}^{-\alpha_\lambda x}}{1 - \frac{\alpha}{\alpha_\lambda}}, \tag{27}$$

whereas by l'Hôpital's rule and continuity, for $\alpha = \alpha_\lambda$ the right-hand side is

$$\frac{x + \frac{1}{\alpha_\lambda+\beta}}{\varphi'(\alpha_\lambda)}\lambda\mathrm{e}^{-\alpha_\lambda x}. \tag{28}$$

In particular, for $\alpha \ne \alpha_\lambda$,

$$\mathrm{E}\mathrm{e}^{-\alpha W_\tau^x} = w(\alpha)\frac{\mathrm{e}^{-\alpha x} - \frac{\alpha}{\alpha_\lambda}\mathrm{e}^{-\alpha_\lambda x}}{1 - \frac{\alpha}{\alpha_\lambda}} = \frac{\mathrm{e}^{-\alpha x} - \frac{\alpha}{\alpha_\lambda}\mathrm{e}^{-\alpha_\lambda x}}{1 - \frac{\varphi(\alpha)}{\lambda}}, \tag{29}$$

where the term to the right of $w(\alpha)$ is the LST of $(x - L_\tau)^+$. The case where X is a subordinator is analyzed in [7, 18] and is thus ignored here.

From (24) and (29) it therefore follows that if we set $f(\alpha) = \mathrm{E}\mathrm{e}^{-\alpha Z^*}$ then

$$f(\alpha) = \frac{\mathrm{E}f(\alpha U) - \frac{\alpha}{\alpha_\lambda}\mathrm{E}f(\alpha_\lambda U)}{1 - \frac{\varphi(\alpha)}{\lambda}}. \tag{30}$$

https://doi.org/10.1017/jpr.2026.10073 Published online by Cambridge University Press

Noting that for $n \geq 1$ we have, with $\delta_{1n} = 0$ for $n \geq 2$ and $\delta_{11} = 1$, that

$$\left(1 - \frac{\varphi(\alpha)}{\lambda}\right) f^{(n)}(\alpha) - \frac{1}{\lambda} \sum_{k=0}^{n-1} \binom{n}{k} f^{(k)}(\alpha) \varphi^{(n-k)}(\alpha) = \mathrm{E} U^n f^{(n)}(\alpha U) - \frac{\mathrm{E} f(\alpha_\lambda U)}{\alpha_\lambda} \delta_{1n} \tag{31}$$

from which the following recursion is immediate, once we argue that $\mathrm{E}(Z^*)^n < \infty$ whenever $\varphi^{(n)}(0)$ is finite.

Lemma 2. *Assume that* $\mathrm{P}(U = 1) < 1$ *and for* $n \geq 0$ *denote* $m_n = \mathrm{E}(Z^*)^n = (-1)^n f^{(n)}(0)$*, and*

$$c_n = (-1)^n \varphi^{(n)}(0) = \begin{cases} 0 & n = 0 \\ c + \int_{(1,\infty)} x \nu(\mathrm{d}x) & n = 1 \\ \sigma^2 + \int_{(0,\infty)} x^2 \nu(\mathrm{d}x) & n = 2 \\ \int_{(0,\infty)} x^n \nu(\mathrm{d}x) & n \geq 3. \end{cases} \tag{32}$$

Then, for all $n \geq 0$*,* $m_n < \infty$ *when* $c_n < \infty$ *and for* $n \geq 2$

$$m_n = \frac{1}{1 - \mathrm{E} U^n} \frac{1}{\lambda} \sum_{k=0}^{n-1} \binom{n}{k} m_k c_{n-k}, \tag{33}$$

where $m_0 = 1$ *and*

$$m_1 = \frac{1}{1 - \mathrm{E} U} \left(\frac{c_1}{\lambda} + \frac{\mathrm{E} f(\alpha_\lambda U)}{\alpha_\lambda} \right). \tag{34}$$

Proof. Since (33) and (34) are obtained by letting $\alpha \downarrow 0$ in (31) and rearranging terms, provided that all the terms are finite, it remains to show that when $c_n < \infty$ then $m_n < \infty$, hence $c_k, m_k < \infty$ for $k \leq n$. Recall (19) which implies that ζ_n is stochastically bounded by the process $V_0 + \sum_{i=1}^n V_i \prod_{j=1}^i U_j$ and, thus, $Z^* \leq_{\mathrm{st}} V_0 + \sum_{i=1}^\infty V_i \prod_{j=1}^i U_j$, where in this case $V_i \sim \mathrm{W}_\tau$. One may apply Minkowski's inequality to argue that if $\mathrm{E} V_0^n = \mathrm{E} W_\tau^n < \infty$ then

$$\| V_0 + \sum_{i=1}^n V_i \prod_{j=1}^i U_j \|_n \leq \| V_0 \|_n \sum_{i=0}^n \| U_1 \|_n^i \leq \frac{\| V_0 \|}{1 - \| U \|_n}, \tag{35}$$

where $\| Y \|_n = (\mathrm{E}|Y|^n)^{1/n}$, so that by monotone convergence we also have that $\mathrm{E}(V_0 + \sum_{i=1}^\infty V_i \prod_{j=1}^i U_j)^n < \infty$ and, thus, $\mathrm{E}(Z^*)^n < \infty$. To see that $\mathrm{E} W_\tau^n < \infty$ whenever $c_n < \infty$ we recall that $X_\tau = W_\tau - L_\tau$ where W_τ, L_τ are independent and $L_\tau \sim \exp(\alpha_\lambda)$ has moments of all orders. Since whenever $c_n < \infty$, $\mathrm{E} X_t^n$ is a polynomial in t of order at most n, which is easy to check by first computing factorial moments via $\log \mathrm{E} \mathrm{e}^{-\alpha X_t} = \varphi(\alpha) t$, and since $\tau \sim \exp(\lambda)$ has finite moments of all orders, then $\mathrm{E} X_\tau^n < \infty$ and therefore necessarily also $\mathrm{E} W_\tau^n < \infty$, which completes the argument. □

It turns out that when $U \sim \mathrm{Uniform}(0, 1)$ the problem of identifying $\mathrm{E} \mathrm{e}^{-\alpha Z^*}$ for all $\alpha \geq 0$ (in particular, $\mathrm{E} f(\alpha_\lambda U)/\alpha_\lambda$), becomes tractable. For this case $\mathrm{E} U^n = (n+1)^{-1}$.

Theorem 1. *Assume that* $U \sim \mathrm{Uniform}(0, 1)$ *and that* X *is a finite mean spectrally positive Lévy process, which is not a subordinator. Then, with*

$$b = \left(\int_0^{\alpha_\lambda} \mathrm{e}^{-\int_0^x \frac{\varphi(y)}{y(\lambda - \varphi(y))} \mathrm{d}y} \mathrm{d}x \right)^{-1}, \tag{36}$$

https://doi.org/10.1017/jpr.2026.10073 Published online by Cambridge University Press

we have that

$$f(\alpha)=\mathrm{E}\mathrm{e}^{-\alpha Z^*}=\begin{cases}\frac{b}{1-\frac{\varphi(\alpha)}{\lambda}}\int_\alpha^{\alpha_\lambda}\mathrm{e}^{-\int_\alpha^x\frac{\varphi(y)}{y(\lambda-\varphi(y))}\,\mathrm{d}y}\,\mathrm{d}x, & \alpha\in[0,\alpha_\lambda),\\ \frac{b}{\frac{1}{\alpha_\lambda}+\frac{\varphi'(\alpha_\lambda)}{\lambda}}, & \alpha=\alpha_\lambda,\\ \frac{b}{\frac{\varphi(\alpha)}{\lambda}-1}\int_{\alpha_\lambda}^{\alpha}\mathrm{e}^{-\int_x^\alpha\frac{\varphi(y)}{y(\varphi(y)-\lambda)}\,\mathrm{d}y}\,\mathrm{d}x, & \alpha\in(\alpha_\lambda,\infty).\end{cases}\tag{37}$$

Moreover, whenever $X_t=J_t-dt$ where J is a nonzero pure jump subordinator and $d>0$ then $\mathrm{P}(Z^*=0)=\lambda b/2d$. *In all other cases* $\mathrm{P}(Z^*=0)=0$.

Proof. We first note that when $U\sim\mathrm{Uniform}(0,1)$ then with $F(\alpha)=\int_0^\alpha f(x)dx$, $b=2F(\alpha_\lambda)/\alpha_\lambda^2$ and $a(\alpha)=\left(\alpha\left(1-\frac{\varphi(\alpha)}{\lambda}\right)\right)^{-1}$, (30) becomes

$$f(\alpha)=F'(\alpha)=\frac{F(\alpha)-\frac{\alpha^2}{2}b}{\alpha\left(1-\frac{\varphi(\alpha)}{\lambda}\right)}=a(\alpha)F(\alpha)-\frac{\alpha^2}{2}a(\alpha)b.\tag{38}$$

From this it immediately follows via L'Hôpital's rule on the right-hand side and the fact that $\varphi(\alpha_\lambda)=\lambda$, that

$$f(\alpha_\lambda)=\frac{f(\alpha_\lambda)-\alpha_\lambda b}{-\alpha_\lambda\frac{\varphi'(\alpha_\lambda)}{\lambda}},\tag{39}$$

so that the case $\alpha=\alpha_\lambda$ in (37) is immediate.

From (38) it now follows that, for $0<\alpha<\beta<\alpha_\lambda$ and for $\alpha_\lambda<\alpha<\beta$,

$$F(\beta)\mathrm{e}^{-\int_\alpha^\beta a(y)\,\mathrm{d}y}=F(\alpha)-b\int_\alpha^\beta\frac{x^2}{2}a(x)\mathrm{e}^{-\int_\alpha^x a(y)\,\mathrm{d}y}\,\mathrm{d}x.\tag{40}$$

Integration by parts gives

$$\int_\alpha^\beta\frac{x^2}{2}a(x)\mathrm{e}^{-\int_\alpha^x a(y)\,\mathrm{d}y}\,\mathrm{d}x=\frac{\alpha^2}{2}-\frac{\beta^2}{2}\mathrm{e}^{-\int_\alpha^\beta a(y)\,\mathrm{d}y}+\int_\alpha^\beta x\mathrm{e}^{-\int_\alpha^x a(y)\,\mathrm{d}y}\,\mathrm{d}x.\tag{41}$$

Since for a sufficiently small neighborhood of α_λ we have that

$$\frac{\varphi(\alpha)-\lambda}{\alpha-\alpha_\lambda}\in(\varphi'(\alpha_\lambda)-\epsilon,\varphi'(\alpha_\lambda)+\epsilon),\tag{42}$$

where $0<\epsilon<\varphi'(\alpha_\lambda)$ and $\varphi'(\alpha_\lambda)$ is finite and positive, then for every $\alpha\in(0,\alpha_\lambda)$ we necessarily have that $\int_\alpha^{\alpha_\lambda}a(x)\,\mathrm{d}x=\infty$. Therefore, letting $\beta\uparrow\alpha_\lambda$ in (40), implies that

$$F(\alpha)=b\left(\frac{\alpha^2}{2}+\int_\alpha^{\alpha_\lambda}x\mathrm{e}^{-\int_\alpha^x a(y)\,\mathrm{d}y}\,\mathrm{d}x\right).\tag{43}$$

Differentiating both sides leads to

$$f(\alpha)=\mathrm{E}\mathrm{e}^{-\alpha Z^*}=ba(\alpha)\int_\alpha^{\alpha_\lambda}x\mathrm{e}^{-\int_\alpha^x a(y)\,\mathrm{d}y}\,\mathrm{d}x.\tag{44}$$

https://doi.org/10.1017/jpr.2026.10073 Published online by Cambridge University Press

We now recall that $a(\alpha) = (\alpha(1-\varphi(\alpha)/\lambda))^{-1}$ and note that for $\alpha < x < \alpha_\lambda$ we have that $x/\alpha = \mathrm{e}^{\int_\alpha^x y^{-1}\,\mathrm{d}y}$, implying that

$$a(\alpha)x\mathrm{e}^{-\int_\alpha^x a(y)\,\mathrm{d}y} = \frac{1}{1-\frac{\varphi(\alpha)}{\lambda}}\,\mathrm{e}^{\int_\alpha^x\left(\frac{1}{y}-a(y)\right)\mathrm{d}y} = \frac{1}{1-\frac{\varphi(\alpha)}{\lambda}}\,\mathrm{e}^{-\int_\alpha^x \frac{\varphi(y)}{y(\lambda-\varphi(y))}\,\mathrm{d}y}, \tag{45}$$

so that together with (44), the case $\alpha \in [0, \alpha_\lambda)$ in (37) follows.

Recalling, by assumption, that $\varphi'(0) = -\mathrm{E}X_1$ is finite then $\int_0^x \frac{\varphi(y)}{y(\lambda-\varphi(y))}\,\mathrm{d}y$ is convergent for any $x \in (0, \alpha_\lambda)$ so that, since $\varphi(\alpha) \to 0$ as $\alpha \downarrow 0$, we finally obtain (36).

For the case $\alpha > \alpha_\lambda$ we observe that in (40) we can multiply both sides by $\mathrm{e}^{\int_\alpha^\beta a(y)\,\mathrm{d}y}$ and note that for $y > \alpha_\lambda$

$$a(y) = \frac{1}{y\left(1-\frac{\varphi(y)}{\lambda}\right)} = -\frac{1}{y\left(\frac{\varphi(y)}{\lambda}-1\right)}. \tag{46}$$

The rest is a repetition of the proof for the case $\alpha \in [0, \alpha_\lambda)$, where eventually we let $\alpha \downarrow \alpha_\lambda$ and replace β in the resulting expression by α.

We finally turn to $\mathrm{P}(Z^* = 0)$. Note that

$$\int_{\alpha_\lambda}^{\alpha} \mathrm{e}^{-\int_x^\alpha \frac{\varphi(y)}{y(\varphi(y)-\lambda)}\,\mathrm{d}y}\,\mathrm{d}x = \alpha\int_{\alpha_\lambda/\alpha}^{1} \mathrm{e}^{-\int_x^1 \frac{\varphi(\alpha y)}{y(\varphi(\alpha y)-\lambda)}\,\mathrm{d}y}\,\mathrm{d}x, \tag{47}$$

and that by bounded convergence

$$\lim_{\alpha\to\infty}\int_{\alpha_\lambda/\alpha}^{1} \mathrm{e}^{-\int_x^1 \frac{\varphi(\alpha y)}{y(\varphi(\alpha y)-\lambda)}\,\mathrm{d}y}\,\mathrm{d}x = \int_0^1 \mathrm{e}^{-\int_x^1 \frac{1}{y}\,\mathrm{d}y}\,\mathrm{d}x = \int_0^1 x\,\mathrm{d}x = \frac{1}{2}. \tag{48}$$

Therefore,

$$\mathrm{P}(Z^* = 0) = \frac{b}{2}\lim_{\alpha\to\infty}\frac{\alpha}{\frac{\varphi(\alpha)}{\lambda}-1} = \frac{\lambda b}{2}\lim_{\alpha\to\infty}\frac{\alpha}{\varphi(\alpha)}. \tag{49}$$

If our Lévy process is a subordinator minus a positive drift d then

$$\varphi(\alpha) = d\alpha - \int_{(0,\infty)}\left(1-\mathrm{e}^{-\alpha x}\right)\nu(\mathrm{d}x) = \alpha\left(d - \int_0^\infty \mathrm{e}^{-\alpha x}\nu(x,\infty)\,\mathrm{d}x\right). \tag{50}$$

In this case the mean is finite if and only if $\int_0^\infty \nu(x,\infty)\,\mathrm{d}x = \int_{(0,\infty)} x\nu(\mathrm{d}x) < \infty$, in which case (bounded convergence) $\int_0^\infty \mathrm{e}^{-\alpha x}\nu(x,\infty)\,\mathrm{d}x$ vanishes, whence, $\varphi(\alpha)/\alpha \to d$ as $\alpha \to \infty$.

In all other cases (recalling that X is not a subordinator), that is, when there is a Brownian motion part or $\int_{(0,1]} x\nu(\mathrm{d}x) = \infty$ (that is, an unbounded variation component) we have that $\varphi(\alpha)/\alpha \to \infty$ as $\alpha \to \infty$, hence the right-hand side of (49) is zero. □

Remark 2. It is easy to check that for $\alpha \in [0, \alpha_\lambda)$ and with

$$g(\alpha) := \int_0^\alpha \mathrm{e}^{-\int_0^x \frac{\varphi(y)}{y(\lambda-\varphi(y))}\,\mathrm{d}y}\,\mathrm{d}x, \tag{51}$$

we have the following representation (when $U \sim \mathrm{Uniform}(0,1)$).

$$f(\alpha) = \mathrm{E}\mathrm{e}^{-\alpha Z^*} = \frac{1-\frac{g(\alpha)}{g(\alpha_\lambda)}}{g'(\alpha)\left(1-\frac{\varphi(\alpha)}{\lambda}\right)}. \tag{52}$$

https://doi.org/10.1017/jpr.2026.10073 Published online by Cambridge University Press

Consequently, from (24) and (26) it follows that

$$\mathrm{E}e^{-\alpha(Z^*U-L_\tau)^+} = \frac{1-\frac{g(\alpha)}{g(\alpha_\lambda)}}{g'(\alpha)\left(1-\frac{\alpha}{\alpha_\lambda}\right)}. \tag{53}$$

We complete this remark by emphasizing that $b = 1/g(\alpha_\lambda)$.

Remark 3. The proof of Theorem 1 can be easily modified to allow the more general case of $U \sim \mathrm{Beta}(\theta, 1)$. For b and the cases $\alpha \in [0, \alpha_\lambda)$ and $\alpha \in (\alpha_\lambda, \infty)$, the resulting expressions are the same as in (36) and (37), except that the integrals in the exponents are multiplied by θ. For the case $\alpha = \alpha_\lambda$, $1/\alpha_\lambda$ in the denominator is replaced by θ/α_λ. The expression $\mathrm{P}(Z^* = 0) = \lambda b/2d$ for the case of a subordinator with a negative drift is replaced by $\lambda b/((1+\theta)\mathrm{d})$. One may also apply the same ideas to handle a mixture of a $\mathrm{Beta}(\theta, 1)$ distribution and an atom at zero (clearing). For the sake of brevity we refrain from working out these exercises here.

Remark 4. Regarding Remark 3, we mention in passing that if we consider an on/off process with $\exp(\lambda)$ distributed on times and $\exp(\eta)$ distributed off times, where during on times the process behaves like a reflected Lévy process and during off times the process is a deterministic linear dam with release rate $r > 0$ (decreases at rate rx where x is a current state) then this process restricted to on times is the process Z and, hence, has the steady-state distribution of Z^* with $U \sim \mathrm{Beta}(r/\eta, 1)$. Furthermore, by PASTA, the steady-state distribution of the process restricted to off times is the distribution of Z^*U, the LST of which may be deduced from (30). Therefore, the steady-state distribution of the on/off process is a $(\eta/(\lambda+\eta), \lambda/(\lambda+\eta))$ mixture of the distributions of Z^* and Z^*U. Thus, our analysis covers this case as well.

6. Examples and Tail Behavior

In this section we work out the LST of Z^*, and hence of Z, for the two cases of reflected Brownian motion with collapse (Section 6.1) and the $M/M/1$ queue with collapse (Section 6.2). In both cases, collapses occur according to a Poisson process with rate λ, and instantaneously remove a Uniform(0,1) distributed fraction of the workload present. In these two examples we use the notation g and the expression for f from Remark 2. We conclude in Section 6.3 with a discussion about the tail behavior of Z^* for the compound Poisson case with a negative drift.

6.1. Reflected Brownian motion with collapse

The Laplace exponent of Brownian motion with drift c is given by (cf. (25))

$$\varphi(\alpha) = -c\alpha + \frac{\sigma^2}{2}\alpha^2. \tag{54}$$

In order to obtain the LST $f(\alpha)$ of the steady-state level of the reflected Brownian motion with collapses, we need to determine $g(\alpha)$ and its derivative, which feature in (52). First write

$$-\frac{\varphi(y)}{y(\lambda-\varphi(y))} = \frac{c-\frac{\sigma^2}{2}y}{\lambda+cy-\frac{\sigma^2}{2}y^2} = \frac{y-\frac{2c}{\sigma^2}}{(y-y_1)(y-y_2)}, \tag{55}$$

where

$$y_{1,2} = \frac{c}{\sigma^2} \pm \sqrt{\frac{c^2}{\sigma^4}+\frac{2\lambda}{\sigma^2}}. \tag{56}$$

https://doi.org/10.1017/jpr.2026.10073 Published online by Cambridge University Press

It is easily verified that $y_1 > 0$ and $y_2 < 0$, with $y_1 y_2 = -(2\lambda/\sigma^2)$ and $y_1 + y_2 = 2c/\sigma^2$; and of course $y_1 = \alpha_\lambda$, the unique positive zero of $\lambda - \varphi(\alpha)$.

Using a partial fraction expansion, we can rewrite (55) as follows:

$$-\frac{\varphi(y)}{y(\lambda - \varphi(y))} = \frac{D_1}{y - y_1} + \frac{D_2}{y - y_2}, \tag{57}$$

where

$$D_1 = \frac{-y_2}{y_1 - y_2} > 0, \quad D_2 = \frac{y_1}{y_1 - y_2} > 0. \tag{58}$$

It is now easily verified that

$$g(\alpha) = \int_0^\alpha \left(\frac{x - y_1}{-y_1}\right)^{D_1} \left(\frac{x - y_2}{-y_2}\right)^{D_2} \mathrm{d}x, \tag{59}$$

and, hence,

$$\frac{g(\alpha)}{g(\alpha_\lambda)} = \frac{g(\alpha)}{g(y_1)} = \frac{\int_0^\alpha (x - y_1)^{D_1}(x - y_2)^{D_2}\,\mathrm{d}x}{\int_0^{y_1} (x - y_1)^{D_1}(x - y_2)^{D_2}\,\mathrm{d}x}, \tag{60}$$

while

$$g'(\alpha) = \left(\frac{\alpha - y_1}{-y_1}\right)^{D_1} \left(\frac{\alpha - y_2}{-y_2}\right)^{D_2}. \tag{61}$$

Substitution in (52) shows that the LST of the steady-state level of our reflected Brownian motion with collapse is, for $\alpha \in [0, y_1)$, given by

$$\begin{aligned} f(\alpha) &= \frac{y_1 y_2}{(\alpha - y_1)(\alpha - y_2)} \left(\frac{\alpha - y_1}{-y_1}\right)^{-D_1} \left(\frac{\alpha - y_2}{-y_2}\right)^{-D_2} \\ &\quad \times \left(1 - \frac{\int_0^\alpha (x - y_1)^{D_1}(x - y_2)^{D_2}\,\mathrm{d}x}{\int_0^{y_1} (x - y_1)^{D_1}(x - y_2)^{D_2}\,\mathrm{d}x}\right) \\ &= \left(\frac{y_1}{y_1 - \alpha}\right)^{1+D_1} \left(\frac{-y_2}{-y_2 + \alpha}\right)^{1+D_2} \left(1 - \frac{\int_0^\alpha (x - y_1)^{D_1}(x - y_2)^{D_2}\,\mathrm{d}x}{\int_0^{y_1} (x - y_1)^{D_1}(x - y_2)^{D_2}\,\mathrm{d}x}\right). \end{aligned} \tag{62}$$

Note that the first two terms in the last line of (62) represent the LST of the difference of two independent Gamma distributed random variables.

Remark 5. It should be observed that $g(\alpha)$ can be expressed in incomplete beta functions. The incomplete beta function $B(z; a_1, a_2)$ is defined as (e.g. [1, p.263])

$$B(z; a_1, a_2) := \int_0^z t^{a_1 - 1}(1 - t)^{a_2 - 1}\,\mathrm{d}t; \tag{63}$$

it can also be expressed as a hypergeometric function via the identity

$$B(z; a_1, a_2) = \frac{z^{a_1}}{a_1} {}_2F_1(a_1, 1 - a_2; a_1 + 1; z). \tag{64}$$

https://doi.org/10.1017/jpr.2026.10073 Published online by Cambridge University Press

The transformation $t := (x-y_2)/(y_1-y_2)$ gives

$$\int_0^{\alpha} (x-y_1)^{D_1}(x-y_2)^{D_2}\,\mathrm{d}x = (-1)^{D_1}(y_1-y_2)^{D_1+D_2+1}\int_{\frac{-y_2}{y_1-y_2}}^{\frac{\alpha-y_2}{y_1-y_2}} t^{D_2}(1-t)^{D_1}\,\mathrm{d}t$$
$$=(-1)^{D_1}(y_1-y_2)^2\left[B\left(\frac{\alpha-y_2}{y_1-y_2};1+D_2,1+D_1\right)-B\left(\frac{-y_2}{y_1-y_2};1+D_2,1+D_1\right)\right]. \tag{65}$$

6.2. The $M/M/1$ queue with collapse

A second example in which one can explicitly determine the LST of the steady-state reflected spectrally positive Lévy process with collapse concerns the workload in an $M/M/1$ queue with collapse. Denote the arrival rate of customers by γ and the service rate by μ; the server works at speed d workload units per time unit. Collapses again occur according to a Poisson process with rate λ, and remove (and maintain) a Uniform(0,1) distributed part of the workload present.

The Laplace exponent of the workload process in an $M/M/1$ queue is given by

$$\varphi(\alpha) = d\alpha - \gamma\frac{\alpha}{\mu+\alpha}. \tag{66}$$

Very similar to the previous subsection, we can write

$$-\frac{\varphi(y)}{y(\lambda-\varphi(y))} = \frac{y+\mu-\frac{\gamma}{d}}{(y-z_1)(y-z_2)}, \tag{67}$$

where

$$z_{1,2} = \frac{1}{2}\left[\frac{\lambda+\gamma}{d}-\mu \pm \sqrt{\left(\frac{\lambda+\gamma}{d}-\mu\right)^2+4\frac{\lambda\mu}{d}}\right]. \tag{68}$$

It is easily verified that $z_1>0$ and $z_2<0$, with $z_1z_2=-(\lambda\mu/d)$ and $z_1+z_2=(\lambda+\gamma)/d-\mu$; and of course $z_1=\alpha_\lambda$, the unique positive zero of $\lambda-\varphi(\alpha)$.

Using a partial fraction expansion, we can rewrite (67) as follows:

$$-\frac{\varphi(y)}{y(\lambda-\varphi(y))} = \frac{F_1}{y-z_1}+\frac{F_2}{y-z_2}, \tag{69}$$

where

$$F_1 = \frac{\frac{\lambda}{d}-z_2}{z_1-z_2} > 0, \quad F_2 = \frac{z_1-\frac{\lambda}{d}}{z_1-z_2} > 0. \tag{70}$$

It is now easily verified that

$$g(\alpha) = \int_0^{\alpha}\left(\frac{x-z_1}{-z_1}\right)^{F_1}\left(\frac{x-z_2}{-z_2}\right)^{F_2}\mathrm{d}x, \tag{71}$$

and, hence,

$$\frac{g(\alpha)}{g(\alpha_\lambda)} = \frac{g(\alpha)}{g(z_1)} = \frac{\int_0^{\alpha}(x-z_1)^{F_1}(x-z_2)^{F_2}\,\mathrm{d}x}{\int_0^{z_1}(x-z_1)^{F_1}(x-z_2)^{F_2}\,\mathrm{d}x}, \tag{72}$$

https://doi.org/10.1017/jpr.2026.10073 Published online by Cambridge University Press

while

$$g'(\alpha) = \left(\frac{\alpha - z_1}{-z_1}\right)^{F_1} \left(\frac{\alpha - z_2}{-z_2}\right)^{F_2}. \tag{73}$$

Substitution in (52) shows that the LST of the steady-state workload in the $M/M/1$ queue with collapse is, for $\alpha \in [0, z_1)$, given by

$$\begin{aligned} f(\alpha) &= \frac{z_1 z_2}{(\alpha - z_1)(\alpha - z_2)} \left(\frac{\alpha - z_1}{-z_1}\right)^{-F_1} \left(\frac{\alpha - z_2}{-z_2}\right)^{-F_2} \\ &\quad \times \left(1 - \frac{\int_0^\alpha (x - z_1)^{F_1}(x - z_2)^{F_2}\,\mathrm{d}x}{\int_0^{z_1} (x - z_1)^{F_1}(x - z_2)^{F_2}\,\mathrm{d}x}\right) \\ &= \left(\frac{z_1}{z_1 - \alpha}\right)^{1+F_1} \left(\frac{-z_2}{-z_2 + \alpha}\right)^{1+F_2} \left(1 - \frac{\int_0^\alpha (x - z_1)^{F_1}(x - z_2)^{F_2}\,\mathrm{d}x}{\int_0^{z_1} (x - z_1)^{F_1}(x - z_2)^{F_2}\,\mathrm{d}x}\right). \end{aligned} \tag{74}$$

Just as in Remark 5, one can express this LST in terms of incomplete beta functions. Furthermore, the first two terms in the last line of (74) represent the LST of the difference of two independent Gamma distributed random variables.

Remark 6. We briefly indicate another approach that basically leads to the same first-order inhomogeneous differential equation as (38). The steady-state workload distribution has an atom p_0 at zero and conjecture that it is absolutely continuous on $(0, \infty)$ with density $h(\cdot)$, so that $\mathrm{P}(Z^* \le t) = p_0 + \int_0^t h(s)\,\mathrm{d}s$. Level crossing arguments lead to the following equation:

$$dh(x) + \lambda \int_x^\infty h(y)\frac{x}{y}\,\mathrm{d}y = \gamma \int_0^x \mathrm{e}^{-\mu(x-y)}\,\mathrm{dP}(Z^* < y). \tag{75}$$

Here the first term in the left-hand side corresponds to downcrossings of level x due to serving, and the second term in the left-hand side reflects downward jumps due to a uniform collapse with rate λ. The term in the right-hand side corresponds to a jump through level x due to a customer arrival: either a jump from zero or from some level $y \in (0, x)$. Multiplying both sides of (75) by $\mathrm{e}^{-\alpha x}$ and integrating from zero to infinity yields, with $f(\alpha) = \mathrm{E}\mathrm{e}^{-\alpha Z^*} = p_0 + \int_0^\infty \mathrm{e}^{-\alpha x} h(x)\,\mathrm{d}x$:

$$d(f(\alpha) - p_0) + \lambda \int_{y=0}^\infty \frac{h(y)}{y} \int_{x=0}^y x\mathrm{e}^{-\alpha x}\,\mathrm{d}x\,\mathrm{d}y = \frac{\gamma}{\mu + \alpha} f(\alpha). \tag{76}$$

The second term in the left-hand side can be rewritten as

$$\begin{aligned} \lambda \int_{y=0}^\infty h(y)\,\frac{1 - \mathrm{e}^{-\alpha y} - \alpha y \mathrm{e}^{-\alpha y}}{\alpha^2 y}\,\mathrm{d}y &= \frac{\lambda}{\alpha^2} \int_{u=0}^\alpha (f(u) - p_0)\,\mathrm{d}u - \lambda \frac{f(\alpha) - p_0}{\alpha} \\ &= \frac{\lambda}{\alpha^2} F(\alpha) - \frac{\lambda}{\alpha} f(\alpha), \end{aligned} \tag{77}$$

where $F(\alpha) = \int_0^\alpha f(x)\,\mathrm{d}x$, cf. above (38), and, hence, we have

$$F'(\alpha)\left(\frac{\lambda}{\alpha} - d + \frac{\gamma}{\mu + \alpha}\right) = \frac{\lambda}{\alpha^2} F(\alpha) - d p_0. \tag{78}$$

Dividing both sides by $\lambda/\alpha - d + \gamma/(\mu + \alpha)$ results in (38), with $b = 2dp_0/\lambda$; which is in agreement with Theorem 1.

https://doi.org/10.1017/jpr.2026.10073 Published online by Cambridge University Press

6.3. Tail behavior for compound Poisson with negative drift

Consider the special case in which the spectrally positive Lévy process consists of a compound Poisson process minus a positive drift. We denote the arrival rate of the Poisson jumps by γ, and a generic jump size by B, with distribution $B(\cdot)$ and LST $\beta(\cdot)$; the drift is denoted by d. The resulting model is an $M/G/1$ queue with arrival rate γ, generic service requirement B, service speed d, and uniformly distributed collapses occurring according to a Poisson process with rate λ. Z^* represents the steady-state workload of this queue; its LST is for $\alpha \in [0, \alpha_\lambda)$ given by (52), with $\varphi(\alpha) = d\alpha - \gamma(1 - \beta(\alpha))$.

We now demonstrate how (52) can be used to determine the tail behavior $\mathrm{P}(Z^* > t)$ for large t, in the case of regularly varying jump sizes of index $-\delta \in (-2, -1)$. In this case, the mean service requirement is finite but the variance is infinite. The Bingham–Doney lemma (cf. [5, Theorem 8.1.6]) states that the following two statements are equivalent:

(i) $\mathrm{P}(B > t) \approx (-1/\Gamma(1-\delta))t^{-\delta}l(t)$, as $t \to \infty$;

(ii) $\beta(\alpha) - 1 + \mathrm{E}B\alpha \approx \alpha^{\delta}l(1/\alpha)$, as $\alpha \downarrow 0$.

Here $l(\cdot)$ is a slowly varying function at infinity. If (i) holds, and hence so does (ii), then we have, for $\alpha \downarrow 0$,

$$\frac{\lambda}{\lambda - \varphi(\alpha)} - 1 - (d - \gamma \mathrm{E}B)\alpha \approx \frac{\gamma}{\lambda}\alpha^{\delta}l(1/\alpha), \tag{79}$$

$$\int_0^{\alpha} \frac{\varphi(y)}{y(\lambda - \varphi(y))}\,\mathrm{d}y - \frac{d - \gamma \mathrm{E}B}{\lambda}\alpha \approx \frac{\gamma}{\delta\lambda}\alpha^{\delta}l(1/\alpha), \tag{80}$$

and, hence, recalling Remark 2, for $\alpha \downarrow 0$,

$$g'(\alpha) - 1 + \frac{d - \gamma \mathrm{E}B}{\lambda}\alpha \approx -\frac{\gamma}{\delta\lambda}\alpha^{\delta}, \tag{81}$$

$$g(\alpha) - \alpha \approx -\frac{d - \gamma \mathrm{E}B}{2\lambda}\alpha^2. \tag{82}$$

Combining these asymptotic expansions with (52) shows that

$$\mathrm{E}\mathrm{e}^{-\alpha Z^*} - 1 + \left(\frac{1}{g(\alpha_\lambda)} - 2\frac{d - \gamma \mathrm{E}B}{\lambda}\right)\alpha \approx \frac{\gamma}{\lambda}\frac{\delta + 1}{\delta}\alpha^{\delta}l(1/\alpha), \quad \alpha \downarrow 0. \tag{83}$$

Now observe, using (34), that $g(\alpha_\lambda)^{-1} - 2(d - \gamma \mathrm{E}B)/\lambda$ equals $\mathrm{E}Z^*$. Indeed, $g(\alpha_\lambda)^{-1} = b$, and on the other hand the term $(1 - \mathrm{E}U)^{-1}(\mathrm{E}f(\alpha_\lambda U)/\alpha_\lambda)$ in (34) can be rewritten in the Uniform(0, 1) case as $(2/\alpha_\lambda^2)\int_0^{\alpha_\lambda} f(y)\,\mathrm{d}y = (2/\alpha_\lambda^2)F(\alpha_\lambda)$, which equals b (see the text preceding (38)). Hence, we have

$$\mathrm{E}\mathrm{e}^{-\alpha Z^*} - 1 + \mathrm{E}Z^*\alpha \approx \frac{\gamma}{\lambda}\frac{\delta + 1}{\delta}\alpha^{\delta}l(1/\alpha), \quad \alpha \downarrow 0. \tag{84}$$

Applying the Bingham–Doney lemma in the reverse direction reveals that

$$\mathrm{P}(Z^* > t) \approx \frac{-1}{\Gamma(1-\delta)}\frac{\gamma}{\lambda}\frac{\delta + 1}{\delta}t^{-\delta}l(t), \quad t \to \infty. \tag{85}$$

https://doi.org/10.1017/jpr.2026.10073 Published online by Cambridge University Press

In an ordinary $M/G/1$ queue with regularly varying service requirements, the workload tail is one degree heavier than the tail of the service requirement, whereas in this $M/G/1$ queue with uniform collapses, Z^* is regularly varying of index $-\delta$, just as $B : \mathrm{P}(Z^* > t) \approx (\gamma/\lambda)((\delta+1)/\delta)\mathrm{P}(B > t)$. In the $M/G/1$ shot-noise queue (an $M/G/1$ queue with service speed proportional to the workload) the same phenomenon was observed in [9]. This is not so surprising, as uniform collapses can also be seen as proportional (shot-noise) decrements during an exponentially distributed amount of time.

Funding information

OB was partially funded by the NWO Gravitation project Networks, grant number 024.002.003. OK was partially supported by ISF grant 3336/24 and the Vigevani Chair in Statistics. DP was partially supported by ISF grant 3336/24.

Competing interests

There were no competing interests to declare which arose during the preparation or publication process of this article.

References

[1] M. Abramowitz and I. A. Stegun (1970). *Handbook of Mathematical Functions*. Dover.
[2] E. Altman, K. Avrachenkov, C. Barakat and P. Dube (2005). Performance analysis of AIMD mechanisms over a multi-state Markovian path. *Computer Networks* **47**, 307–326.
[3] S. Asmussen (2003). *Applied Probability and Queues*, 2nd Ed. Springer.
[4] J. Bertoin (1996). *Lévy Processes*. Cambridge University Press.
[5] N. Bingham, C. Goldie and J. Teugels (1987). *Regular Variation*. Springer.
[6] O. Boxma, O. Kella and M. Mandjes (2023). On fluctuation-theoretic decompositions via Lindley-type recursions. *Stoch. Proc. Appl*. **165**, 316–336.
[7] O. Boxma, O. Kella and D. Perry (2011). On some tractable growth-collapse processes with renewal collapse epochs. *J. Appl. Probab.* **48A**, 217–234.
[8] O. Boxma, A. Löpker, M. Mandjes and Z. Palmowski (2021). A multiplicative version of the Lindley recursion. *Queueing Systems* **98**, 225–245.
[9] O. Boxma and M. Mandjes (2021). Shot-noise queueing models. *Queueing Systems* **99**, 121–159.
[10] O. Boxma, D. Perry and W. Stadje (2022). Peer-to-peer lending: a growth-collapse model and its steady-state analysis. *Math. Methods of Oper. Res.* **96**, 233–258.
[11] O. Boxma, D. Perry and W. Stadje (2024). Perishable inventories with random input: a unifying survey with extensions. *Annl. Oper. Res.* **332**, 1069–1105.
[12] O. Boxma, D. Perry, W. Stadje and S. Zacks (2006). A Markovian growth-collapse model. *Adv. Appl. Prob.* **38**, 221–243.
[13] K. Debicki and M. Mandjes (2015). *Queues and Lévy Fluctuation Theory*. Springer.
[14] V. Dumas, F. Guillemin and P. Robert (2002). A Markovian analysis of additive-increase multiplicative-decrease algorithms. *Adv. Appl. Prob.* **34**, 85–111.
[15] F. Guillemin, P. Robert and B. Zwart (2004). AIMD algorithms and exponential functionals. *Ann. Appl. Probab.* **14**, 90–117.
[16] R. van der Hofstad, S. Kapodistria, Z. Palmowski and S. Shneer (2023). Unified approach for solving exit problems for additive-increase and multiplicative-decrease processes. *J. Appl. Probab.* **60**, 85–105.
[17] O. Kella (2006). Reflecting thoughts. *Stat. Prob. Letters* **76**, 1808–1811.
[18] O. Kella (2009). On growth-collapse processes with stationary structure and their shot-noise counterparts. *J. Appl. Probab.* **46**, 363–371.
[19] O. Kella and O. Boxma (2013). Useful martingales for stochastic storage processes with Lévy-type input. *J. Appl. Probab.* **50**, 439–449.
[20] O. Kella and A. Löpker (2010). A Markov-modulated growth collapse model. *Prob. Eng. Inf. Sci.* **24**, 99–107.
[21] O. Kella, D. Perry and W. Stadje (2003). A stochastic clearing model with Brownian and compound Poisson components. *Prob. Eng. Inf. Sci.* **17**(1), 1–22.

https://doi.org/10.1017/jpr.2026.10073 Published online by Cambridge University Press

[22] O. KELLA AND W. WHITT (1992). Useful martingales for stochastic storage processes with Lévy input. *J. Appl. Probab.* **29**, 396–403.
[23] O. KELLA AND W. WHITT (1996). Stability and structural properties of stochastic storage networks. *J. Appl. Probab.* **33**, 1169–1180.
[24] O. KELLA AND M. YOR (2017). Unifying the Dynkin and Lebesgue-Stieltjes formulae. *J. Appl. Probab.* **54**, 252–266.
[25] A.E. KYPRIANOU (2006). *Introductory Lectures on Fluctuations of Lévy Processes with Applications*. Springer.
[26] A. LÖPKER AND J. VAN LEEUWAARDEN (2008). Transient moments of the TCP window size process. *J. Appl. Probab.* **45**, 163–175.
[27] A. LÖPKER AND W. STADJE (2011). Hitting times and the running maximum of Markovian growth-collapse processes. *J. Appl. Probab.* **48**, 295–312.
[28] Z. PALMOWSKI AND M. VLASIOU (2011). A Lévy input model with additional state-dependent services. *Stoch. Proc. Appl*. **121**, 1546–1564.
[29] Z. PALMOWSKI, M. VLASIOU AND B. ZWART (2014). A Lévy input fluid queue with input and workload regulation. *Queueing Systems* **76**, 21–36.
[30] K. SATO (1999). *Lévy Processes and Infinitely Divisible Distributions*. Cambridge University Press.

https://doi.org/10.1017/jpr.2026.10073 Published online by Cambridge University Press