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Shapes of polynomial Julia sets

KATHRYN A. LINDSEY

Department of Mathematics, Cornell University, Ithaca, NY 14853, USA
(e-mail: klindsey@math.cornell.edu)

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Abstract. Any Jordan curve in the complex plane can be approximated arbitrarily well in the Hausdorff topology by Julia sets of polynomials. Finite collections of disjoint Jordan domains can be approximated by the basins of attraction of rational maps.

1. *Introduction*

For any complex-valued rational function f on the Riemann sphere $\hat{\mathbb{C}}$ such that the point ∞ is an attracting fixed point of f, define the following notation:

$$\begin{aligned}\mathcal{L}(f) &= \text{the basin of attraction of } \infty \text{ for the map } f,\\ \mathcal{K}(f) &= \hat{\mathbb{C}} \setminus \mathcal{L}(f),\\ \mathcal{J}(f) &= \partial(\mathcal{K}(f)),\end{aligned}$$

where $\partial(X)$ denotes the boundary of a set X. If f is a a polynomial, then $\mathcal{J}(f)$ is the *Julia set* of f and $\mathcal{K}(f)$ is the *filled Julia set* of f.

Theorem 2.6 gives a constructive proof that for any closed, bounded annulus $A \subset \hat{\mathbb{C}}$ whose boundary consists of two disjoint Jordan curves, there exists a polynomial P such that the bounded connected component of $\hat{\mathbb{C}} \setminus A$ is contained in $\mathcal{K}(P)$, the unbounded connected component of $\hat{\mathbb{C}} \setminus A$ is contained in $\mathcal{L}(P)$, $\mathcal{J}(P) \subset A$, and any path connecting the two components of $\hat{\mathbb{C}} \setminus A$ contains a point in $\mathcal{J}(P)$. Theorem 3.2 reformulates this result in terms of the Hausdorff metric, showing that any Jordan curve in the complex plane can be approximated arbitrarily well in the Hausdorff topology by Julia sets of polynomials. Theorems 4.2, 4.3 and 4.4 combine these polynomials to form rational maps whose basins of attraction approximate more complicated sets.

2. *Annular decompositions and polynomial Julia sets*

Define $\mathcal{A}$ to be the collection of all closed, bounded annuli in $\hat{\mathbb{C}}$ whose boundary consists of two disjoint Jordan curves, i.e.,

$$\mathcal{A} = \{A \subset \hat{\mathbb{C}} \mid \exists \text{ a homeomorphism } f : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \text{ s.t. } f(R) = A \text{ and } \infty \notin A\},$$

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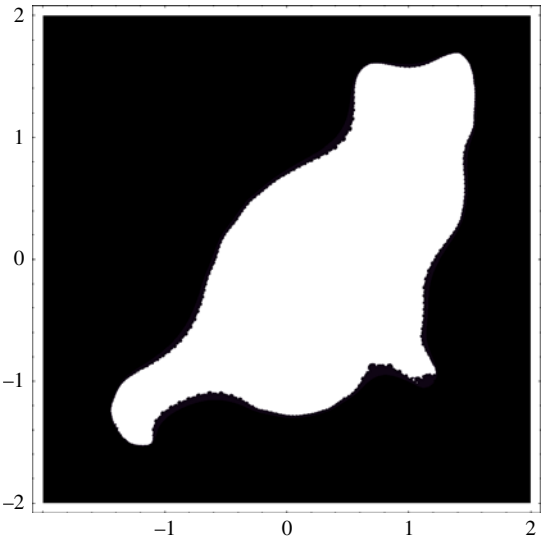


FIGURE 1. A filled Julia set in the shape of a cat. The polynomial that generates this Julia set has degree 301.

where $R = \{z \in \hat{\mathbb{C}} \mid 1 \leq |z| \leq 2\}$. For any $A \in \mathcal{A}$, $\hat{\mathbb{C}} \setminus A$ consists of two topological disks; let A_∞ denote the unbounded connected component of $\hat{\mathbb{C}} \setminus A$, and let A_0 denote the bounded connected component of $\hat{\mathbb{C}} \setminus A$. Thus, for each $A \in \mathcal{A}$, $\hat{\mathbb{C}} = A_\infty \sqcup A \sqcup A_0$.

For each $A \in \mathcal{A}$, let γ_A be a fixed Jordan curve that is a core curve of A and is contained in the interior of A. Define U_A to be the unbounded connected component of $\hat{\mathbb{C}} \setminus \gamma_A$, which is a topological disk. Let $\widetilde{\varphi_A}$ denote a fixed inverse Riemann map

$$\widetilde{\varphi_A} : \{z \in \hat{\mathbb{C}} \mid 1 < |z|\} \to U_A,$$

such that the points at infinity correspond. For some sufficiently small $\epsilon = \epsilon(A) > 0$, the set $\widetilde{\varphi_A}(\{z \in \hat{\mathbb{C}} \mid |z| = 1 + \epsilon\})$ is contained in the interior of A. Define C_A to be this set, i.e.,

$$C_A = \widetilde{\varphi_A}(\{z \in \hat{\mathbb{C}} \mid |z| = 1 + \epsilon\}).$$

We will now rescale the conformal map $\widetilde{\varphi_A}$ so that the unit circle maps to C_A. Specifically, define the map $\varphi_A : \{z \in \hat{\mathbb{C}} \mid 1/(1+\epsilon) < |z|\} \to \hat{\mathbb{C}}$ by

$$\varphi_A(z) = \widetilde{\varphi_A}((1+\epsilon)(z)).$$

LEMMA 2.1. *For each $A \in \mathcal{A}$:*

(i) *the set C_A is a Jordan curve contained in the interior of A and is a core curve of A;*
(ii) *C_A is rectifiable with respect to the Euclidean metric on $\mathbb{C}$; and*
(iii) *φ_A' is non-vanishing and of bounded variation on the unit circle.*

Proof. The set C_A is a conformal image of S^1, and hence a Jordan curve. It is homotopic to γ_A, which, by assumption, is a core curve of A, proving (i). Since φ_A is a conformal map, φ' is everywhere non-zero. By the maximum modulus principle, $|\varphi_A'|$ is bounded on S^1, which is itself of finite Euclidean arc length, implying (ii) and (iii). □

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For each $A \in \mathcal{A}$, the set $\hat{\mathbb{C}} \setminus C_A$ consists of two topological disks; let O_A denote the unbounded connected component of $\hat{\mathbb{C}} \setminus C_A$ and let I_A denote the bounded connected component of $\hat{\mathbb{C}} \setminus C_A$. Thus, for each $A \in \mathcal{A}$, we have a decomposition $\hat{\mathbb{C}} = O_A \sqcup C_A \sqcup I_A$.

Because φ_A is injective, the coefficients for all the terms of degree greater than 1 in the Laurent series centered at 0 for φ_A must be zero. Hence, any such Laurent series has the form

$$\varphi_A(z) = c_A z + c_{(A,0)} + \frac{c_{(A,1)}}{z} + \frac{c_{(A,2)}}{z^2} + \cdots, \quad |z| \geq 1,$$

with $c_A, c_{(A,0)}, c_{(A,1)}, \ldots \in \mathbb{C}$. For each pair of natural numbers (k, n) with $k \leq n$, and for each $A \in \mathcal{A}$, set

$$r_A^{(k,n)} = \varphi_A(e^{k2\pi i/n}).$$

For each $n \in \mathbb{N}$ and $A \in \mathcal{A}$, define $\omega_{A,n} : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ by

$$\omega_{A,n}(z) = c_A^{-n} \prod_{j=1}^{n} (z - r_A^{(k,n)}).$$

For each $n \in \mathbb{N}$ and $A \in \mathcal{A}$, define the polynomial $P_{A,n} : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ by

$$P_{A,n}(z) = z(\omega_{A,n}(z) + 1).$$

The definitions of the map φ and the polynomials ω_n use only the Jordan curve C_A, and not the fact that C_A came from the annulus A. Thus, any Jordan curve C in $\mathbb{C}$ determines both a conformal map φ_C and a family of polynomials $\omega_{C,n}$, $n \in \mathbb{N}$; these polynomials are called *fundamental polynomials of Lagrange interpolation*. Curtiss proved the following two theorems about the uniform convergence of such polynomials in [**1**] (we transcribe them here using our notation).

THEOREM 2.2. [**1**] *If C is a rectifiable Jordan curve in $\mathbb{C}$, then*

$$\lim_{n\to\infty} \omega_{C,n}(z) = -1 \tag{1}$$

uniformly for z on any closed subset of the region interior to C, and

$$\lim_{n\to\infty} \frac{\omega_{C,n}(z)}{w^n - 1} = 1 \quad \text{where } z = \varphi_C(w) \tag{2}$$

uniformly for z on any closed subset of the region exterior to C.

THEOREM 2.3. [**1**] *If $\varphi_C'(w)$ is non-vanishing and of bounded variation for $|w| = 1$, then limit (2) in Theorem 2.2 converges uniformly for z in the closed region $C \sqcup C_\infty$, where C_∞ denotes the unbounded connected component of $\hat{\mathbb{C}} \setminus C$.*

The reason for choosing the curve C_A as we did was to use Curtiss's theorems to guarantee that the polynomials $\omega_{A,n}$ converge uniformly not only on bounded subsets of O_A (as in Theorem 2.2), but also on unbounded compact subsets of $O_A \subset \hat{\mathbb{C}}$. Lemma 2.1 together with Theorems 2.2 and 2.3 immediately yield the following.

COROLLARY 2.4. *For any $A \in \mathcal{A}$ and any compact set $I' \subset I_A$,*

$$\lim_{n\to\infty} \omega_{A,n}(z) = -1$$

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uniformly for z on I', and

$$\lim_{n\to\infty} \frac{\omega_{A,n}(z)}{w^n - 1} = 1 \quad \textit{where } z = \varphi_A(w)$$

uniformly for z on $C_A \sqcup O_A$.

LEMMA 2.5. *Fix a real number $r > 0$ and any $A \in \mathcal{A}$ such that $0 \in A_0$. Then there exists a real number $K \geq 1$ such that, for any real number $\kappa > K$, there exists $N \in \mathbb{N}$ such that $n \geq N$ implies:*

(i) $|P_{A,n}(z)| < r$ *for all* $z \in A_0$;

(ii) $|P_{A,n}(z)| > \kappa \cdot |z|$ *for all* $z \in A_\infty$; *and*

(iii) $\kappa \cdot |z| > \sup\{|x| : x \in \hat{\mathbb{C}} \setminus A_\infty\}$ *for all* $z \in A_\infty$.

Proof. Without loss of generality, we will assume that r is small enough that $B_r(0) \subset A_0$, where $B_r(0)$ denotes the open ball of radius r about 0. Set

$$\begin{aligned} \alpha &= \sup\{|x| : x \in A_0\}, \\ \beta &= \sup\{|x| : x \in A_0 \cup A\}, \\ \gamma &= \inf\{|x| : x \in A_\infty\}, \\ \delta &= \inf\{|x| : x \in \varphi_A^{-1}(A_\infty)\}. \end{aligned}$$

Note that α, β, and γ are all at least r, and, in particular, are non-zero. Note also that $\delta > 1$. This is because the distance function from a point in the unit circle to the set $\varphi_A^{-1}(\partial A_\infty) \subset \{w : |w| > 1\}$ is continuous on the unit circle, and hence attains its minimum, which is non-zero. Hence, $\inf\{|x| : x \in \varphi_A^{-1}(A_\infty)\} > 1$.

Let $K = \beta/\gamma$. Since $\beta \geq \gamma$ by construction, $K \geq 1$. Fix $\kappa > K$; then $\kappa > 1$ and $\kappa \cdot \gamma > \beta$. Pick $\epsilon_2 > 0$ such that $\epsilon_2 \alpha < r$. Since

$$\lim_{n\to\infty} \frac{\omega_{A,n}(z)}{(\varphi_A^{-1}(z))^n - 1} = 1$$

uniformly for $z \in A_\infty$, by Corollary 2.4, and $|\varphi_A^{-1}(z)| > 1$ on A_∞, we also have that

$$\lim_{n\to\infty} \frac{\omega_{A,n}(z) + 1}{(\varphi_A^{-1}(z))^n - 1} = 1$$

uniformly on A_∞. Hence, there exists $N_1 \in \mathbb{N}$ such that $n \geq N_1$ implies

$$|\omega_{A,n}(z) + 1| > (1 - \epsilon_2) \cdot |w^n - 1|,$$

for all $z \in A_\infty$, with $z = \varphi_A(w)$. Pick $N_2 \in \mathbb{N}$ such that $n \geq N_2$ implies

$$\delta^n > 1 + \frac{\kappa}{1 - \epsilon_2}.$$

Since $\omega_{A,n}(z)$ converges uniformly to -1 on A_0, by Corollary 2.4, there exists $N_3 \in \mathbb{N}$ such that $n \geq N_3$ implies $|\omega_{A,n}(z) + 1| < \epsilon_2$, for all $z \in A_0$. Let $N = \max\{N_1, N_2, N_3\}$.

Now consider $P_{A,n}$ for $n \geq N$. For $z \in A_0$,

$$|P_{A,n}(z)| = |z| \cdot |\omega_{A,n}(z) + 1| \leq \alpha \cdot \epsilon_2 < r.$$

For $z \in A_\infty$,

$$\begin{aligned} |P_{A,n}(z)| &= |z| \cdot |\omega_{A,n}(z) + 1| \geq |z| \cdot (1 - \epsilon_2) \cdot |w^n + 1| \geq |z| \cdot (1 - \epsilon_2) \cdot ||w|^n - 1| \\ &\geq |z| \cdot (1 - \epsilon_2) \cdot (\delta^n - 1) > |z| \cdot (1 - \epsilon_2)\left(\frac{\kappa}{1 - \epsilon_2}\right) = |z| \cdot \kappa. \end{aligned}$$

□

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THEOREM 2.6. *Fix $A \in \mathcal{A}$ and any point $t \in A_0$. Let $\tau : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ denote the translation $z \mapsto z - t$. Then there exists $N \in \mathbb{N}$ such that $n \geq N$ implies:*

(i) $A_0 \subset \mathcal{K}(\tau^{-1} \circ P_{\tau(A),n} \circ \tau)$;

(ii) $A_\infty \subset \mathcal{L}(\tau^{-1} \circ P_{\tau(A),n} \circ \tau)$;

(iii) $\mathcal{J}(\tau^{-1} \circ P_{\tau(A),n} \circ \tau) \subset A$; *and*

(iv) *any path from a point in ∂A_∞ to a point in ∂A_0 has a non-empty intersection with $\mathcal{J}(\tau^{-1} \circ P_{\tau(A),n} \circ \tau)$.*

(Note: the annulus $\tau(A)$ is an element of $\mathcal{A}$, and the polynomials $P_{\tau(A),n}$ are well defined and are constructed using the Jordan curve, Riemann map, etc., associated with $\tau(A)$.)

Proof. To start, we will consider only the case that $0 \in A_0$ and τ is the identity map. So, pick $r > 0$ such that $B_r(0) \subset A_0$. Pick a real number $\kappa > 1$ and $N \in \mathbb{N}$, as in the statement of Lemma 2.5. Let $n \geq N$, $n \in \mathbb{N}$.

Since A_0 contains the disk $B_r(0)$, the property $|P_{A,n}(z)| < r$, for all $z \in A_0$, implies $A_0 \subset \mathcal{K}(P_{A,n})$. The property

$$|P_{A,n}(z)| > \kappa \cdot |z| > \sup\{|w| : w \in \hat{\mathbb{C}} \setminus A_\infty\},$$

for all $z \in A_\infty$, implies that $P_{A,n}(A_\infty) \subset A_\infty$, and thus that $|P_{A,n}^m(z)| > \kappa^m \cdot |z|$, for all $m \in \mathbb{N}$ and $z \in A_\infty$. Hence, $A_\infty \subset \mathcal{L}(P_{A,n})$.

The partition of the Riemann sphere as $\hat{\mathbb{C}} = A_\infty \sqcup A \sqcup A_0$ together with the facts $A_0 \subset \mathcal{K}(P_{A,n})$ and $A_\infty \subset \mathcal{L}(P_{A,n})$ imply that $\partial\mathcal{K}(P_{A,n}) \subset A$. Any path connecting a point in $\mathcal{K}(P_{A,n})$ and a point in $(\mathcal{K}(P_{A,n}))^c = \mathcal{L}(P_{A,n})$ must have a non-empty intersection with $\partial\mathcal{K}(P_{A,n})$. This proves the theorem in the case that $0 \in A_0$ and τ is the identity.

Now consider the general case, in which τ is any fixed map $z \mapsto z - t$ for some $t \in A_0$. The translated annulus $\tau(A)$ is an element of $\mathcal{A}$ and its interior region, $\tau(A)_0$, contains the point 0. Hence, the conclusion of the theorem holds for the annulus $\tau(A)$ and the identity translation. The composition $\tau^{-1} \circ P_{\tau(A),n} \circ \tau$ is a polynomial and $\mathcal{K}(\tau^{-1} \circ P_{\tau(A),n} \circ \tau) = \tau^{-1}(\mathcal{K}(P_{\tau(A),n}))$. Therefore,

$$\mathcal{K}(\tau^{-1} \circ P_{\tau(A),n} \circ \tau) \supset \tau^{-1}(\tau(A)_0) = A_0.$$

Statements (ii), (iii) and (iv) follow similarly. □

3. *Hausdorff distance and annular approximations*

The purpose of this section is to carry out the technical arguments necessary to reformulate the conclusion of Theorem 2.6 in terms of the Hausdorff topology and Jordan curves (Theorem 3.2).

For any two non-empty subsets X and Y of $\mathbb{C}$, we denote by $d_H(X, Y)$ the Hausdorff distance between X and Y, defined by

$$d_H(X, Y) = \inf\{\epsilon > 0 \mid X \subseteq N_\epsilon(Y) \text{ and } Y \subseteq N_\epsilon(X)\},$$

where

$$N_\epsilon(X) = \bigcup_{x \in X} \{z \in \hat{\mathbb{C}} \mid d(z, x) \leq \epsilon\}.$$

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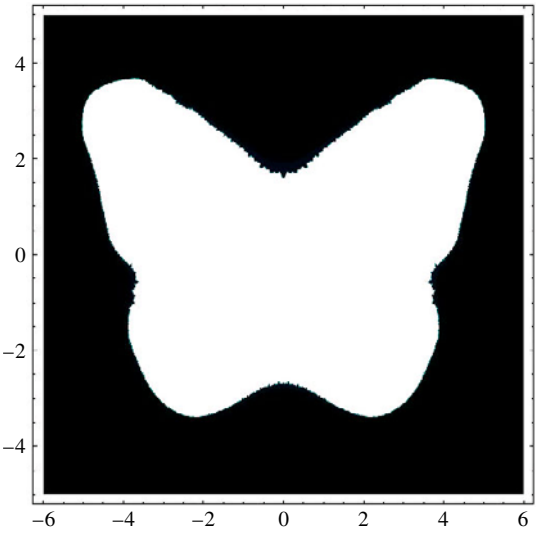


FIGURE 2. A filled Julia set in the shape of a butterfly. The polynomial that generates this Julia set has degree 301.

We will also consider the Hausdorff distance $d_H(X, Y)$ in the case where $X, Y \subset \hat{\mathbb{C}}$, and both X and Y contain an open ball, say B, about the point ∞. In this case, we define $d_H(X, Y)$ to be the Hausdorff distance $d_H(X \setminus B, Y \setminus B)$, which is well defined and finite.

We will denote by $\mathcal{C}$ the collection of Jordan curves in the complex plane $\mathbb{C} \subset \hat{\mathbb{C}}$. The Jordan–Schoenflies theorem says that for each Jordan curve $J \in \mathcal{C}$, there exists a homeomorphism $f_J : \mathbb{C} \to \mathbb{C}$ such that $f_J(J) = S^1$, and such that f maps the bounded component of J^c to the unit disk. For each real number ϵ with $0 < \epsilon < 1$, define

$$J^\epsilon = f_J^{-1}(\{z \in \mathbb{C} \mid 1 - \epsilon \leq |z| \leq 1 + \epsilon\}).$$

The set $\hat{\mathbb{C}} \setminus J^\epsilon$ consists of two connected components; let J_0^ϵ denote the bounded component and let J_∞^ϵ denote the unbounded component. Thus, for each $J \in \mathcal{C}$ and $0 < \epsilon < 1$, we have a decomposition $\hat{\mathbb{C}} = J_0^\epsilon \sqcup J^\epsilon \sqcup J_\infty^\epsilon$.

LEMMA 3.1. *Fix a Jordan curve $J \in \mathcal{C}$ and a real number $\delta > 0$. Then there exists a real number $\rho > 0$ such that $\rho > \epsilon > 0$ implies:*

(i) $J^\epsilon \subset N_\delta(J)$*; and*

(ii) *for every point $j \in J$, $B_\delta(j) \cap J_0^\epsilon \neq \emptyset$ and $B_\delta(j) \cap J_\infty^\epsilon \neq \emptyset$.*

Proof. Define a function $w_\delta : S^1 \to \mathbb{R}$ by

$$w_\delta(s) = \sup\{r > 0 \mid B_r(s) \subset f_J(N_\delta(J))\}.$$

Since $f_J(N_\delta(J))$ is an open set, which contains S^1, $w_\delta(s)$ is well defined and positive for each $s \in S^1$. Since w_δ is continuous on S^1 and S^1 is compact, there exists a real number $\rho_1 > 0$ such that $w_\delta(s) \geq \rho_1$, for all $s \in S^1$. Then, for any real number ϵ with $0 < \epsilon < \rho_1$, and for any $s \in S^1$, we have $B_\epsilon(s) \subset f_J(N_\delta(J))$, and thus

$$N_\epsilon(S^1) = \bigcup_{s \in S^1} B_\epsilon(s) \subset f_J(N_\delta(J)).$$

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Therefore, $0 < \epsilon < \rho_1$ implies $J^\epsilon \subset N_\delta(J)$.

Define a function $y_\delta : J \to \mathbb{R}$ by

$$y_\delta(j) = \sup\{\epsilon > 0 \mid B_\delta(j) \cap J^\epsilon_\infty \neq \emptyset \text{ and } B_\delta(j) \cap J^\epsilon_0 \neq \emptyset\}.$$

Note that $y_\delta(j)$ is well defined and positive for each point $j \in J$. Since y_δ is continuous on J, and J is compact (J is the image of S^1 under f^{-1}), there exists a real number $\rho_2 > 0$ such that $y_\delta(j) > \rho_2$, for all $j \in J$. Then $\rho_2 > \epsilon > 0$ implies $B_\delta(j) \cap J^\epsilon_\infty \neq \emptyset$ and $B_\delta(j) \cap J^\epsilon_0 \neq \emptyset$ for all $j \in J$.

Let $\rho = \min\{\rho_1, \rho_2\}$. Then $\rho > \epsilon > 0$ implies properties (i) and (ii). □

THEOREM 3.2. *Let J be a Jordan curve in $\mathbb{C}$ and fix a real number $\delta > 0$. Denote by I the bounded component of $\hat{\mathbb{C}} \setminus J$, and denote by O the unbounded component of $\hat{\mathbb{C}} \setminus J$. There exists a polynomial $P : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that:*

(i) $d_H(I, \mathcal{K}(P)) < \delta$;

(ii) $d_H(J, \mathcal{J}(P)) < \delta$; *and*

(iii) $d_H(O, \mathcal{L}(P)) < \delta$.

Proof. Pick a real number δ_1 such that $0 < \delta_1 < \delta/3$. Apply Lemma 3.1, using the parameter δ_1, to obtain $\rho > 0$ such that $\rho > \epsilon > 0$ implies:

(a) $J^\epsilon \subseteq N_{\delta_1}(J)$; and

(b) $B_{\delta_1}(j) \cap J^\epsilon_0 \neq \emptyset$ and $B_{\delta_1}(j) \cap J^\epsilon_\infty \neq \emptyset$, for all $j \in J$.

Fix a real number ϵ with $\rho > \epsilon > 0$. Apply Theorem 2.6 to the annulus $J^\epsilon \in \mathcal{A}$ to obtain a polynomial P_n such that:

(i) $J^\epsilon_0 \subset \mathcal{K}(P_n)$;

(ii) $J^\epsilon_\infty \subset \mathcal{L}(P_n)$;

(iii) $\mathcal{J}(P_n) \subset J^\epsilon$; and

(iv) any path from a point in $\partial J^\epsilon_\infty$ to a point in ∂J^ϵ_0 has a non-empty intersection with $\mathcal{J}(P_n)$.

We know $J^\epsilon_0 \subseteq \mathcal{K}(P_n) \subseteq J^\epsilon \sqcup J^\epsilon_0$ and $J^\epsilon_0 \subseteq I \subseteq J^\epsilon \sqcup J^\epsilon_0$. Consequently, the symmetric difference $\mathcal{K}(P_n) \triangle I$ is contained in J^ϵ. By (a), any point $x \in J^\epsilon$ is within distance at most δ_1 of some point $j \in J$, and by (b), the point j is within distance at most δ_1 of a point in $J^\epsilon_0 \subset I \cap \mathcal{K}(P_n)$. Hence, $d_H(I, \mathcal{K}(P_n)) \leq 2\delta_1 < \delta$.

Similarly, $J^\epsilon_\infty \subseteq \mathcal{L}(P_n) \subseteq J^\epsilon \sqcup J^\epsilon_\infty$ and $J^\epsilon_\infty \subseteq O \subseteq J^\epsilon \sqcup J^\epsilon_\infty$, implying $\mathcal{L}(P_n) \triangle O \subset J^\epsilon$, and thus $d_H(\mathcal{L}(P_n), O) < \delta$.

Let j be a point in J. By (b), pick points $p_1 \in B_{\delta_1}(j) \cap J^\epsilon_0$ and $p_2 \in B_{\delta_1}(j) \cap J^\epsilon_\infty$, and let γ be the path consisting of the two line segments $\overline{p_1 j}$ and $\overline{p_2 j}$. By construction, the length of γ is at most $2\delta_1$. Since γ connects $\mathcal{K}(P_n)$ with $\mathcal{L}(P_n) = (K(P_n))^c$, γ contains a point in $\partial(\mathcal{K}(P_n)) = \mathcal{J}(P_n)$. Hence, $d_H(j, \mathcal{J}(P_n)) \leq 2\delta_1$. The containments

$$\mathcal{J}(P_n) \subset J^\epsilon \subset N_{\delta_1}(J)$$

imply $d_H(a, J) \leq \delta_1$, for any point $a \in \mathcal{J}(P_n)$. Therefore,

$$d_H(\mathcal{J}(P_n), J) < \delta.$$

□

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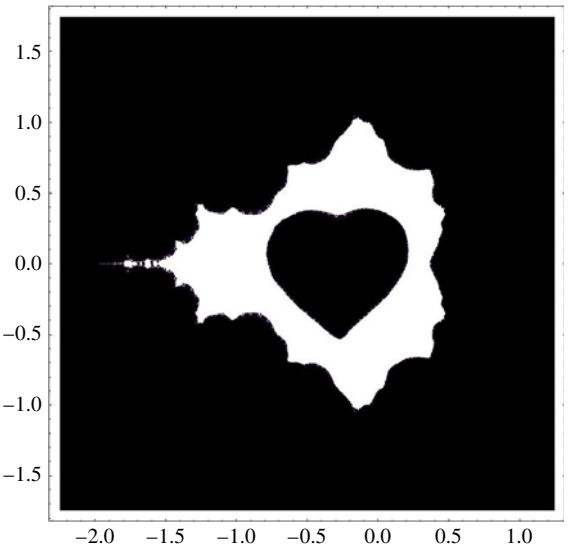


FIGURE 3. $\mathcal{K}(\rho)$ for a rational map ρ constructed as in Theorem 4.4.

4. *Rational maps*

Let $\mathfrak{A}$ denote a finite collection $\{A_1, \ldots, A_m\}$ of annuli in $\mathcal{A}$. For each i, let $A_{i,0}$ and $A_{i,\infty}$ denote the bounded component and unbounded component, respectively, of $\hat{\mathbb{C}} \setminus A_i$. Let $|\mathfrak{A}|$ denote the number of elements of $\mathfrak{A}$. We will say that the elements of $\mathfrak{A}$ are *mutually exterior* if $A_j \cap (A_i \sqcup A_{i,0}) = \emptyset$ for all $i \neq j$.

For any finite collection $\mathfrak{A}$ of mutually exterior annuli in $\mathcal{A}$ and any $n \in \mathbb{N}$, define a rational function $\Omega_{\mathfrak{A},n} : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ by

$$\Omega_{\mathfrak{A},n}(z) = \left(\sum_{j=1}^{|\mathfrak{A}|} (\omega_{A_j,n}(z) + 1)^{-1} \right)^{-1}. \tag{3}$$

Also, define for each $n \in \mathbb{N}$ the rational function $R_{\mathfrak{A},n} : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ by

$$R_{\mathfrak{A},n}(z) = z \cdot \Omega_{\mathfrak{A},n}(z).$$

Define also

$$\begin{aligned} \mathfrak{A}_+ &= \bigcup_{j=1}^{|\mathfrak{A}|} A_j, \\ \mathfrak{A}_0 &= \bigcup_{j=1}^{|\mathfrak{A}|} A_{j,0}, \\ \mathfrak{A}_\infty &= \bigcap_{j=1}^{|\mathfrak{A}|} A_{j,\infty}. \end{aligned}$$

Thus, we have a decomposition $\hat{\mathbb{C}} = \mathfrak{A}_0 \sqcup \mathfrak{A}_+ \sqcup \mathfrak{A}_\infty$.

LEMMA 4.1. *Let $\mathfrak{A}$ be a finite collection of mutually exterior annuli in $\mathcal{A}$, and fix real numbers $B > b > 0$. Then there exists $N \in \mathbb{N}$ such that, for all integers $n \geq N$:*

(i) $|\Omega_{\mathfrak{A},n}(z)| < b$ *for all* $z \in \mathfrak{A}_0$*; and*

(ii) $|\Omega_{\mathfrak{A},n}(z)| > B$ *for all* $z \in \mathfrak{A}_\infty$.

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Proof. Pick positive real numbers d and D, with d small enough and D large enough that

$$\left|\frac{1}{d} - \frac{|\mathfrak{A}|-1}{D}\right|^{-1} < b \quad \text{and} \quad \frac{D}{|\mathfrak{A}|} > B.$$

As a consequence of Corollary 2.4, there exists $N \in \mathbb{N}$ such that, for all $n \geq N$ and all $1 \leq j \leq |\mathfrak{A}|$, $|\omega_{A_j,n}(z)+1| < d$ for all $z \in A_{j,0}$, and $|\omega_{A_j,n}(z)+1| > D$ for all $z \in A_{j,\infty}$. Fix $n \geq N$. For $z \in A_{i,0}$,

$$\left|\sum_{j=1}^{|\mathfrak{A}|} \frac{1}{\omega_{A_j,n}(z)+1}\right| \geq \left|\left|\frac{1}{\omega_{A_i,n}(z)+1}\right| - \sum_{j=1, j\neq i}^{|\mathfrak{A}|}\left|\frac{1}{\omega_{j,n}(z)+1}\right|\right| \geq \left|\frac{1}{d} - \frac{|\mathfrak{A}|-1}{D}\right|.$$

Hence, $|\Omega_{\mathfrak{A},n}(z)| < b$, for all $z \in \mathfrak{A}_0$. For $z \in \mathfrak{A}_\infty$,

$$\left|\sum_{j=1}^{|\mathfrak{A}|} \frac{1}{\omega_{A_j,n}(z)+1}\right| \leq \sum_{j=1}^{|\mathfrak{A}|}\left|\frac{1}{\omega_{A_j,n}(z)+1}\right| \leq \frac{|\mathfrak{A}|}{D}.$$

Hence, $|\Omega_{\mathfrak{A},n}(z)| > B$, for all $z \in \mathfrak{A}_\infty$. □

THEOREM 4.2. *Let $\mathfrak{A} = \{A_1, \ldots, A_m\}$ be a finite collection of mutually exterior annuli in $\mathcal{A}$. Fix any point $t \in \mathfrak{A}_0$ and let $\tau : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be the translation $z \mapsto z - t$. Then there exists $N \in \mathbb{N}$ such that, for any integer $n \geq N$:*

(i) $\mathfrak{A}_0 \subset \mathcal{K}(\tau^{-1} \circ R_{\tau(\mathfrak{A}),n} \circ \tau)$;

(ii) $\mathfrak{A}_\infty \subset \mathcal{L}(\tau^{-1} \circ R_{\tau(\mathfrak{A}),n} \circ \tau)$;

(iii) $\mathcal{J}(\tau^{-1} \circ R_{\tau(\mathfrak{A}),n} \circ \tau) \subset \mathfrak{A}_+$; *and*

(iv) *for any $A_i \in \mathcal{A}$, any path connecting $A_{i,\infty}$ and $A_{i,0}$ has a non-empty intersection with $\mathcal{J}(\tau^{-1} \circ R_{\tau(\mathfrak{A}),n} \circ \tau)$.*

Proof. To begin, we will consider the case that $0 \in \mathfrak{A}_0$ and τ is the identity map. So, assume $0 \in \mathfrak{A}_0$, and let ρ be the radius of a small ball centered at 0 and contained in $\mathfrak{A}_0$. Fix a real number $b > 0$ small enough that

$$b \cdot \sup\{|z| \mid z \in \mathfrak{A}_0\} < \rho.$$

Fix a real number $B > 1$ large enough that

$$B \cdot \inf\{|z| \mid z \in \mathfrak{A}_\infty\} > \sup\{|z| \mid z \in \mathfrak{A}_+ \cup \mathfrak{A}_0\}.$$

Let N be the integer guaranteed by Lemma 4.1, and fix an integer $n \geq N$. Then, for any $z \in \mathfrak{A}_0$,

$$|R_{\mathfrak{A},n}(z)| = |z| \cdot |\Omega_{\mathfrak{A},n}(z)| \leq |z| \cdot b < \rho.$$

Hence, $R_{\mathfrak{A},n}(z) \in \mathfrak{A}_0$ for any $z \in \mathfrak{A}_0$. For any $z \in \mathfrak{A}_\infty$,

$$|R_{\mathfrak{A},n}(z)| = |z| \cdot |\Omega_{\mathfrak{A},n}(z)| \geq |z| \cdot B.$$

Hence, for any $z \in \mathfrak{A}_\infty$, we have $R_{\mathfrak{A},n}(z) \in \mathfrak{A}_\infty$, and so $|R^m_{\mathfrak{A},n}(z)| > B^m|z|$, for all $m \in \mathbb{N}$. Therefore, the basin of attraction of ∞ for the map $R_{\mathfrak{A},n}$ contains all of $\mathfrak{A}_\infty$ and has empty intersection with $\mathfrak{A}_0$. This proves the theorem in the case that $0 \in \mathfrak{A}_0$ and τ is the identity map.

For the general case, in which τ is any fixed map $z \mapsto z - t$ for some $t \in \mathfrak{A}_0$, it suffices to observe that $\mathcal{K}(\tau^{-1} \circ R_{\tau(\mathfrak{A}),n} \circ \tau) = \tau^{-1}(\mathcal{K}(R_{\mathfrak{A},n}))$. □

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For any finite collection $\mathfrak{J} = \{J_1, \ldots, J_m\}$ of mutually exterior bounded Jordan curves in $\hat{\mathbb{C}}$, define the sets $\mathfrak{J}_+$, $\mathfrak{J}_0$ and $\mathfrak{J}_\infty$ as follows. Let $\mathfrak{J}_+ = \cup_i J_i$. Let $\mathfrak{J}_0$ denote the union of the bounded connected components of $\hat{\mathbb{C}} \setminus J_i$. Let $\mathfrak{J}_\infty = \hat{\mathbb{C}} \setminus (\mathfrak{J}_+ \cup \mathfrak{J}_0)$. Thus, $\hat{\mathbb{C}} = \mathfrak{J}_0 \sqcup \mathfrak{J}_+ \sqcup \mathfrak{J}_\infty$.

THEOREM 4.3. *Let $\mathfrak{J} = \{J_1, \ldots, J_m\}$ be a finite collection of mutually exterior bounded Jordan curves in $\hat{\mathbb{C}}$, and fix a real number $\delta > 0$. Then there exists a rational map $Q : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ for which ∞ is an attracting fixed point and:*

(i) $d_H(\mathfrak{J}_0, \mathcal{K}(Q)) < \delta$;
(ii) $d_H(\mathfrak{J}_+, \mathcal{J}(Q)) < \delta$; *and*
(iii) $d_H(\mathfrak{J}_\infty, \mathcal{L}(Q)) < \delta$.

Proof. Let

$$\eta = \tfrac{1}{2} \inf\{d(x, y) \mid x \in J_{i,0},\ y \in J_{j,0},\ i \neq j\}.$$

Because the curves in $\mathfrak{J}$ are mutually exterior, $\eta > 0$. Let $\delta_1 = \min\{\delta/3, \eta\}$. For each i, let ρ_i be the positive real number guaranteed by Lemma 3.1 using the parameter δ_1. Let $\rho = \min\{\rho_0, \rho_1, \ldots, \rho_m\}$.

Fix a real number ϵ such that $\rho > \epsilon > 0$. Let $\mathfrak{A}$ denote the collection of annuli $J_1^\epsilon, \ldots, J_m^\epsilon$ (the notation J^ϵ is as defined at the beginning of §3). The condition that $\delta_1 < \eta$ implies that the annuli in $\mathfrak{A}$ are all pairwise disjoint and thus mutually exterior. Apply Theorem 4.3, using the finite collection of mutually exterior annuli $\mathfrak{A}$, to obtain a rational map $Q : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, such that:

(i) $\mathfrak{A}_0 \subset \mathcal{K}(Q)$;
(ii) $\mathfrak{A}_\infty \subset \mathcal{L}(Q)$;
(iii) $\mathcal{J}(Q) \subset \mathfrak{A}_+$; and
(iv) every path between a point in $\mathfrak{A}_0$ and a point in $\mathfrak{A}_\infty$ has a non-empty intersection with $\mathcal{J}(Q)$.

The Hausdorff distance estimates can then be proved in the same way as in the proof of Theorem 3.2. □

THEOREM 4.4. *Let A be an annulus in $\mathcal{A}$ and fix a real number $\delta > 0$. Then there exists a rational function $S : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ for which ∞ is an attracting fixed point, and:*

(i) $d_H(A, \mathcal{K}(S)) < \delta$;
(ii) $d_H(A_\infty \sqcup A_0, \mathcal{L}(S)) < \delta$; *and*
(iii) $d_H(\partial A, \mathcal{J}(S)) < \delta$.

Proof. The boundary of A is composed of two Jordan curves; let C_1 denote the 'outer' curve and C_2 denote the 'inner' curve. Set

$$\xi = \inf\{d(x, y) \mid x \in C_1,\ y \in C_2\}.$$

Note the compactness of the Jordan curves C_1 and C_2 implies $\xi > 0$. Let E and F be the annuli guaranteed by Lemma 3.1 using C_1 and C_2, respectively, and the parameter $\delta_1 = 1/3 \min\{\delta, \xi\}$. By construction, E and F are disjoint. Thus, $\hat{\mathbb{C}} \setminus (E \cup F)$ consists of three connected components; denote the 'outside' component (which contains ∞) by O, denote the 'middle' component by M, and denote the 'inside' component by I. Thus, $\hat{\mathbb{C}} = O \sqcup E \sqcup M \sqcup F \sqcup I$.

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Assume for now that $0 \in M$. Fix real numbers $R > r > 0$ such that $B_r(0) \subset M$ and $(\hat{\mathbb{C}} \setminus O) \subset B_R(0)$. Set

$$\eta = \frac{1}{\inf\{|z| : z \in O\}}.$$

Fix a real number κ, such that

$$\kappa > \max\left\{1, 2 + \frac{r\eta}{2}, \eta\left(R + \frac{r}{2}\right)\right\}.$$

By Lemma 2.5, there exists $N_E \in \mathbb{N}$ such that $n > N_E$ implies

$$|P_{E,n}(z)| < \frac{r}{2} \quad \text{for all } z \in M \sqcup F \sqcup I$$

and

$$|P_{E,n}(z)| > \kappa|z| \quad \text{for all } z \in O.$$

Note that

$$\kappa|z| > \eta\left(R + \frac{r}{2}\right)|z| \geq R + \frac{r}{2},$$

for all $z \in O$. Note also that

$$\kappa|z| - \frac{r}{2} > \left(2 + \frac{r\eta}{2}\right)|z| - \frac{r}{2} = 2|z| + \frac{r}{2}(\eta|z| - 1) \geq 2|z|, \tag{4}$$

for all $z \in O$.

By Corollary 2.4, there exists $N_F \in \mathbb{N}$ such that, for any integer $m > N_F$,

$$\left|\frac{1}{\omega_{F,m}(z) + 1}\right| > R + \frac{r}{2} \quad \text{for all } z \in I$$

and

$$\left|\frac{1}{\omega_{F,m}(z) + 1}\right| < \frac{r}{2} \quad \text{for all } z \in M \sqcup E \sqcup O.$$

For each $n \in \mathbb{N}$, define a rational map $S_n : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ by

$$S_n(z) = P_{E,n}(z) + \frac{1}{\omega_{F,n}(z) + 1}.$$

Set $N = \max\{N_E, N_F\}$. Fix any integer $n > N$. The inequalities above imply that $|S_n(z)| > R$, for all $z \in O \sqcup I$, and $|S_n(z)| < r$, for all $z \in M$. Consequently, $S_n(z) \in O$, for all $z \in O \sqcup I$, and $S_n(z) \in M$, for all $z \in M$. Furthermore, for any $z \in O$, the inequality (4) implies

$$|S_n(z)| \geq \left||P_{E,n}(z)| - \left|\frac{1}{\omega_{F,n}(z) + 1}\right|\right| \geq \kappa|z| - \frac{r}{2} > 2|z|.$$

Hence, $|S_n^t(z)| > 2^t|z|$, for all $z \in O$ and $t \in \mathbb{N}$. Consequently, $O \sqcup I \subset \mathcal{L}(S_n)$ and $M \subset \mathcal{K}(S_n)$. The Hausdorff distance estimates may then be obtained in a way similar to that in the proof of Theorem 3.2. □

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5. *Discussion*

Theorem 3.2 extends the analogy between the dynamics of rational maps and Kleinian groups. The discussion of inversive two-manifolds in [**3**] examines Kleinian groups generated by inversions through circles arranged in a chain along a Jordan curve. The approach in [**3**] leads to the following result.

THEOREM 5.1. *Any simple closed curve C can be approximated in the Hausdorff topology by limits sets of finitely generated quasi-Fuchsian groups.*

The result above is stated in [**2**]. The maps constructed in this paper and in Theorem 5.1 all use a set of special points strung along the curve—in the first case it is the roots of the polynomials, and in the second case it is the centers of the circles that define the inversions.

Question. Characterize sets $S \subset \mathbb{C}$ such that, for any $\epsilon > 0$, there exists a polynomial (or rational map) P such that $d_{\text{Haus}}(S, \mathcal{K}(P)) < \epsilon$ and $d_{\text{Haus}}(\hat{\mathbb{C}} - S, \hat{\mathbb{C}} - \mathcal{K}(P)) < \epsilon$.

The theme of this paper has been first to fix the 'shape' we wish to approximate, and then vary the degree of the polynomials whose Julia sets approximate the shape. However, the reverse approach—first fix a degree d, and then analyze the set of Julia sets for polynomials of degree d—also warrants investigation.

Question. Find a method to determine

$$\{P : P \text{ is a polynomial of degree } d \text{ and } d_{\text{Haus}}(\mathcal{J}(P), S) \leq \epsilon\}$$

for any given $d \in \mathbb{N}$, $\epsilon \geq 0$, and $S \subset \mathbb{C}$.

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