THE S-MATRIX AND RADIATION DAMPING

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Using the invariant perturbation theory, developed in recent years, Dyson (1) has obtained for the S-matrix

$$S = 1 + \sum_{n=1}^{\infty} \frac{(-i)^n}{n!} P_n, \tag{1}$$

where
$$P_n \equiv \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dx'' \ldots \int_{-\infty}^{\infty} dx^{(n)} P(H(x'), H(x''), \ldots, H(x^{(n)})). \tag{2}$$

In this form of the S-matrix the damping effects in collision processes are not exhibited in an obvious form. But we shall show that one can obtain another form of the S-matrix, which gives the damping effect in accordance with the ideas of Heitler (2) and others.

For this, we start from the S-matrix in the Cayley form:

$$S = \frac{1 - \frac{1}{2}iK}{1 + \frac{1}{2}iK}, \tag{3}$$

where, as shown by Schwinger (3), K is given by

$$K = \int_{-\infty}^{\infty} H(x)\, V[\sigma]\, dx, \tag{4}$$

with
$$V[\sigma] = 1 - \frac{i}{2} \int_{-\infty}^{\infty} \epsilon[\sigma, \sigma']\, H(x')\, V[\sigma']\, dx'. \tag{5}$$

According to (5), we can expand $V[\sigma]$ as an infinite series by repeated substitutions. Then, we may write (4) as

$$\begin{aligned} K &= \int_{-\infty}^{\infty} dx' H(x') + \left(\frac{-i}{2}\right) \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dx'' \epsilon[\sigma', \sigma'']\, H(x')\, H(x'') \\ &\quad + \left(\frac{-i}{2}\right)^2 \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dx'' \int_{-\infty}^{\infty} dx''' \epsilon[\sigma', \sigma'']\, \epsilon[\sigma'', \sigma''']\, H(x')\, H(x'')\, H(x''') + \ldots \\ &= \sum_{n=1}^{\infty} K_n, \end{aligned} \tag{6}$$

where
$$K_n = \left(\frac{-i}{2}\right)^{n-1} \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dx'' \ldots \int_{-\infty}^{\infty} dx^{(n)}\, \epsilon[\sigma', \sigma'']\, \epsilon[\sigma'', \sigma'''] \ldots \times \epsilon[\sigma^{(n-1)}, \sigma^{(n)}]\, H(x')\, H(x'') \ldots H(x^{(n)}). \tag{7}$$

Now, from Dyson's definition of the P-bracket, it follows that

$$P(H(x'), H(x''), \ldots, H(x^{(n)})) = \sum \frac{\epsilon[\sigma', \sigma''] + 1}{2} \frac{\epsilon[\sigma'', \sigma'''] + 1}{2} \cdots \frac{\epsilon[\sigma^{(n-1)}, \sigma^{(n)}] + 1}{2} H(x')\, H(x'') \ldots H(x^{(n)}), \tag{8}$$

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where the summation Σ extends over the $n!$ permutations obtained by interchanging the dashes in all possible ways. Integrating both sides of (8) over $x', x'', \ldots, x^{(n)}$, and noting that each permutation on the right-hand side of (8) gives rise to the same integral, we obtain

$$P_n = \frac{n!}{2^{n-1}} \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dx'' \ldots \int_{-\infty}^{\infty} dx^{(n)} (\epsilon[\sigma', \sigma''] + 1)(\epsilon[\sigma'', \sigma'''] + 1) \ldots \times (\epsilon[\sigma^{(n-1)}, \sigma^{(n)}] + 1) H(x') H(x'') \ldots H(x^{(n)}). \quad (9)$$

With the help of (9) we can easily convert the ϵ's occurring in (7) into the P-brackets, getting

$$K_1 = P_1, \quad (10a)$$

$$K_2 = \frac{(-i)}{2!} P_2 - \left(\frac{-i}{2}\right) P_1^2, \quad (10b)$$

$$K_3 = \frac{(-i)^2}{3!} P_3 - \left(\frac{-i}{2}\right)^2 (P_1 P_2 + P_2 P_1) + \left(\frac{-i}{2}\right)^2 P_1^3, \quad (10c)$$

$$K_4 = \frac{(-i)^3}{4!} P_4 - \left(\frac{-i}{2}\right)^3 P_2^2 - \left(\frac{-i}{2}\right) \frac{(-i)^2}{3!} (P_1 P_3 + P_3 P_1) + \left(\frac{-i}{2}\right)^3 (P_1^2 P_2 + P_1 P_2 P_1 + P_2 P_1^2) - \left(\frac{-i}{2}\right)^3 P_1^4, \quad (10d)$$

and so on. Thus, the equation (3), together with (7) or (10), provides an alternative form of the S-matrix.

In order to use the above S-matrix for collision processes, we write (3) in a more familiar form as

$$S = 1 - iU, \quad (11)$$

where, according to (3) and (11), U is given by

$$U = K - \frac{i}{2} KU, \quad (12)$$

which is simply a covariant form of Heitler's well-known integral equation. Now, for any process under consideration, we can calculate K up to the required order of approximation, and solve the integral equation (12) to obtain U. This then gives the scattering cross-section with damping, which is proportional to $|U|^2$. Similar results have also been obtained by Miyazima and collaborators (4), but their treatment is not quite clear.

We have seen that the S-matrix can be expanded in two alternative forms. The expansion (1) is a direct expansion of the S-matrix in powers of the coupling constant, and it may be called the *Born expansion*. On the other hand, the expansion given by (3) and (6) gives rise to the damping, and we may call it the *Heitler expansion*. Since, according to Bhabha (5) and Heitler (2), the damping plays a vital role in the meson theory, it is very important to decide which of these expansions is to be preferred. Now, it is well known that in order that the normalization of the final state in a collision process be the same as that of the initial state, the S-matrix must be unitary. We therefore examine the unitary property of the two expansions. Though the Born expansion (1) is unitary *as a whole*, in practice we have to confine our attention to

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a finite number of terms in this expansion, and in that case it is not unitary. But the Heitler expansion, given by (3) and (6), is unitary *in any order of approximation*, and therefore it is more exact.

The significance of the relations (10) expressing the K_n in terms of the P_n is as follows: The first term on the right-hand side of (10*d*), for instance, contains the contributions of all possible fourth-order transitions, while the remaining terms simply cancel the contributions of those transitions, which are made up of two or more real transitions. Similarly, K_n will contain the contributions of all those nth-order transitions which are not made up of two or more real transitions. Thus, according to the Heitler expansion, a transition, which is made up of two or more real transitions, is not regarded as a new transition, but its effect is taken into account through the integral equation (12).

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