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Global attractors of analytic plane flows

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Abstract. In this paper the global attractors of analytic and polynomial plane flows are characterized up to homeomorphisms. Following on from previous results for continuous and differentiable dynamical systems, our theorem completes the characterization of the global attractors of plane flows.

1. *Introduction*

The literature on global (asymptotically stable) attractors of flows on topological spaces is rather extensive. Bhatia and Szegö [**2**, p. 81, Theorem 3.6], Garay [**4**], and Günther and Segal [**8**] characterized global attractors of continuous flows on $\mathbb{R}^n$.

THEOREM 1.1. *A compact set $\Omega \subset \mathbb{R}^n$ is a global attractor of a continuous flow on $\mathbb{R}^n$ if and only if $\mathbb{R}^n \setminus \Omega$ is homeomorphic to $\mathbb{R}^n \setminus \{\mathbf{0}\}$. When $n = 2$, the set Ω has the shape of a finite union of finite wedges of circles.*

This theorem was generalized to continuous flows on more general topological spaces by Günther and Segal [**8**], as well as by Giraldo and Sanjurjo [**6**], using the concepts from Borsuk's shape theory [**3**]; see also [**5**]. These results were found to be very useful for studying global attractors of dissipative evolution equations; see, e.g., Robinson [**13**].

Later, Günther [**7**] proved that Theorem 1.1 holds even if Φ is a C^k $(k < \infty)$ flow. In $\mathbb{R}^2$, this result can be extended to smooth (C^∞) flows by applying Gutiérrez's theorem [**9**]; therefore, up to homeomorphisms, there is no difference between global attractors of continuous and smooth plane flows.

The proofs of all these results make use of topological techniques and smoothing arguments, and thus are insufficient when one tries to extend Theorem 1.1 to analytic (C^ω) flows. In the present paper, we introduce new techniques which are adapted to the C^ω realm, and fill this gap by obtaining a complete characterization of global attractors for

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analytic and polynomial plane flows—see the 'Main Theorem' in §2. Since we do not use shape theory, our characterization is more explicit than the one stated in Theorem 1.1.

A spin-off from the Main Theorem is that not any global attractor of a smooth plane flow can be the global attractor of an analytic plane flow (up to homeomorphisms). For example, define a compact connected set $A \subset \mathbb{R}^2$ by

$$\begin{aligned} A = &\{(x, y) \in \mathbb{R}^2 \mid 0 < x \leq 1/\pi,\ y = \sin(1/x)\} \\ &\cup \{(x, y) \in \mathbb{R}^2 \mid x = 0,\ -1 \leq y \leq 1\} \\ &\cup \{\text{the arc joining } (0, -1) \text{ to } (1/\pi, 0)\}. \end{aligned}$$

This set is called the 'Warsaw circle', and it is well known [**8**] that the domain Ω that A encloses can be the global attractor of a C^k plane flow (where $0 \leq k \leq \infty$). On the contrary, as a consequence of the Main Theorem, no set that is homeomorphic to Ω can be the global attractor of some analytic plane flow.

The Warsaw circle is a 'pathological' set in the sense that it is not locally connected, but the reader should not think that any global attractor of an analytic plane flow must be locally or arc connected; in particular, it does not need to be (locally) homeomorphic to the zero-set of an analytic function. For instance, if (r, θ) are polar coordinates in $\mathbb{R}^2$, let us define the set

$$\begin{aligned} \Omega = &\{(x, y) \in \mathbb{R}^2 \mid r(\theta) = 2/(4 - 3\exp(-2\theta))^{1/2},\ \theta \in [0, \infty)\} \\ &\cup \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}, \end{aligned}$$

which is compact and connected but fails to be arc and locally connected. On the other hand, it is not difficult to check that Ω is the global attractor of the plane flow defined by the polynomial vector field

$$f(x, y) = ((x - 2)^2 + y^2)(y + x(1 - x^2 - y^2),\ -x + y(1 - x^2 - y^2)).$$

The set Ω is an example of what we call a 'generalized arc', i.e. one of the structures that can appear as global attractors (up to homeomorphisms) of analytic plane flows, according to the classification that we describe in §2.

The proofs of the direct and converse statements of the Main Theorem are presented in §§3 and 4, respectively. For the proof of the direct statement, we make extensive use of the good local properties of analytic sets and the local structure of the phase portraits of analytic flows at isolated singular points. For the proof of the converse, we construct a model of a smooth plane flow with a given set as the global attractor, and then deform this model via a homeomorphism to obtain a polynomial plane flow.

2. *Notation and statement of the Main Theorem*

In what follows, an *arc* (respectively, an *open arc*, a *circle*, and a *disk*) will be a topological space that is homeomorphic to $[0, 1]$ (respectively, to $\mathbb{R}$, to $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$, and to $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$).

Let X be a metrizable space. A *local flow* on X is a continuous map $\Phi : \Lambda \subset \mathbb{R} \times X \to X$ which satisfies the following properties.

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(i) Λ is open in $\mathbb{R} \times X$, and for every fixed $u \in X$, the set of numbers t for which $\Phi(t, u)$ is defined is an open interval $I_u = (a_u, b_u)$ where $-\infty \leq a_u < 0 < b_u \leq \infty$.

(ii) $\Phi(0, u) = u$ for every $u \in X$.

(iii) If $\Phi(t, u) = v$, then $I_v = \{r - t \mid r \in I_u\}$; moreover, $\Phi(s, v) = \Phi(s, \Phi(t, u)) = \Phi(s + t, u)$ for every $s \in I_v$.

If $\Lambda = \mathbb{R} \times X$, then we call Φ a *flow* on X. We write $\Phi_u(t) := \Phi(t, u)$ whenever it makes sense. We call $\Gamma_{\Phi,u} := \Gamma_u = \Phi_u(I_u)$ the *orbit* of u (for the local flow Φ). If J is a subinterval of I_u, then we call $\Phi_u(J)$ a *semi-orbit* of the orbit Γ_u. In particular, Γ_u^- and Γ_u^+ denote, respectively, the *negative* and *positive* semi-orbits $\Phi_u((a_u, 0])$ and $\Phi_u([0, b_u))$. If $\Gamma_u = \{u\}$, then we call u a *singular point*; otherwise we call u a *regular point* and Γ_u a *regular orbit*. In the particular case where Γ_u is a circle or, equivalently, there is a minimal $t > 0$ such that $\Phi_u(t) = u$, we also call Γ_u a *periodic orbit*. We say that a subset of X is *invariant* for Φ if it is the union of some orbits of Φ.

We define the *ω-limit set* of the orbit Γ_u (or the point u) as the set

$$\omega_\Phi(u) = \{v \in X \mid \exists t_n \to b_u \text{ with } \Phi_u(t_n) \to v\}.$$

The *α-limit set* $\alpha_\Phi(u)$ is defined analogously by taking $t_n \to a_u$ instead.

We say that a set $\Omega \subset X$ is a *global attractor* [**4**] for a local flow Φ provided that:

- Ω is non-empty, compact and invariant;
- $\omega_\Phi(u) \subset \Omega$ for every $u \in X$;
- for each $\epsilon > 0$ there is a $\delta > 0$ such that $d(\Phi_u(t), \Omega) < \epsilon$ for every $t \geq 0$ whenever $d(u, \Omega) < \delta$, where $d(\cdot, \cdot)$ stands for a fixed distance function defined on X.

This definition is equivalent [**13**] to the concept of 'globally asymptotically stable attractor' used by Bhatia and Szegö [**2**] and by Giraldo and Sanjurjo [**6**], as well as to the notion of global attractor introduced by Günther and Segal [**8**] and Günther [**7**].

We say that two sets $A \subset X$ and $B \subset Y$ are *topologically equivalent* if there is a homeomorphism $h : X \to Y$ mapping A onto B. Let Φ and Ψ be local flows on the spaces X and Y, respectively. We say that Φ and Ψ are *topologically equivalent* if there is a homeomorphism $h : X \to Y$ such that $h(\Gamma_{\Phi,u}) = \Gamma_{\Psi,h(u)}$ and which preserves the respective orientations of the flows. It is well known that if Φ is a local flow, then it is topologically equivalent to a flow (with, moreover, the identity being the conjugation homeomorphism h).

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a C^k vector field, where $k \in \{0, 1, \ldots, \infty, \omega\}$ (here C^0 means 'continuous' and C^ω means 'real analytic'). Assume that for every $u_0 \in \mathbb{R}^n$, there is a unique solution $\Phi_{u_0} = u : I_{u_0} \to \mathbb{R}^n$ of the equation $u' = f(u)$ satisfying $u(0) = u_0$ (which, of course, is true except possibly when $k = 0$). If we define $\Phi(t, u) := \Phi_u(t)$, then we get a C^k local flow (called the *local flow of* $u' = f(u)$ or simply the *local flow of* f). Conversely, if Φ is a C^{k+1} local flow on $\mathbb{R}^n$ (where we take $\infty + 1 = \infty$, $\omega + 1 = \omega$) and $f(u) := (\partial\Phi/\partial t)(0, u)$ is its C^k *vector field*, then the local flow of $u' = f(u)$ is precisely Φ.

We say that a compact topological space G is a *graph* if every $u \in G$ either is isolated in G or admits a neighbourhood in G that is homeomorphic to $\{z \in \mathbb{C} \mid z^r \in [0, 1]\}$ for some positive integer r, with the homeomorphism mapping u to 0. Notice that the number r is unambiguously defined. If $r \geq 3$, then we call u a *vertex* of G. If $r = 1$, then we call u

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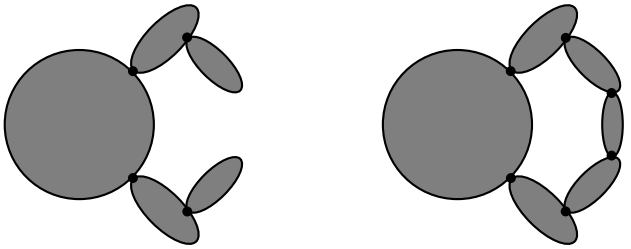


FIGURE 1. The set on the left is a cactus; the set on the right is not.

an *endpoint* of G. If V, P and I denote, respectively, the sets of vertices, endpoints and isolated points of G, then each of the components of $G \setminus (V \cup P \cup I)$ is called an *edge* of G. Note that a graph G consists of finitely many vertices, endpoints and edges. In fact, G is a graph if and only if it has finitely many connected components, each of which either is a single point or can be written as a connected union of finitely many arcs, with each pair of arcs intersecting at not more than one of their endpoints.

A set $A \subset \mathbb{R}^n$ is said to be *analytic* if it is the set of zeros of some analytic function $F : \mathbb{R}^n \to \mathbb{R}$. If $A \subset \mathbb{R}^2$ is a compact analytic set, then, as a consequence of the so-called Lojasiewicz theorem (see [**11**, p. 168, Theorem 6.3.3] and also [**10**]), A is a graph. Moreover, A has locally, at each non-isolated point z, an even number of branches [**15**], each of which admits an analytic parametrization (with non-vanishing derivative except possibly at z) and approaches z with a definite direction.

Recall that a *continuum* is a compact connected topological space. We say that a continuum T is a *cactus* if it is the simply connected union of finitely many disks. Equivalently, a cactus can be defined as a connected union of finitely many disks such that there are no circles in T other than those which are inside each disk (hence, in particular, each pair of disks have at most one common point); see Figure 1. Points and cactuses in $\mathbb{R}^2$ are also both called *dots* (with the term *degenerate dots* referring specifically to points). We say that the open arc $E \subset \mathbb{R}^2$ *ends at* the point $p \in \mathbb{R}^2$ if $E \cup \{p\}$ is homeomorphic to $[0, 1)$. We say that the open arc $E \subset \mathbb{R}^2$ *ends at* the cactus $T \subset \mathbb{R}^2$ if E spirals toward T, i.e. there is a homeomorphism $\varphi : \mathbb{R} \to E$ such that $d(\varphi(t), T) \to 0$ as $t \to \infty$, and there are continuous functions $\rho : \mathbb{R} \to [0, \infty)$, $\theta : \mathbb{R} \to \mathbb{R}$ together with a point $p \in T$ such that $\varphi(t) = p + \rho(t)e^{i\theta(t)}$ for every t and $|\theta(t)| \to \infty$ as $t \to \infty$. A disjoint union A of an open arc E and two dots P and Q with E ending at both P and Q is called a *generalized arc*; the dots P and Q are called the *endpoints* of A. See Figure 2. A disjoint union B of open arcs $\{E_i\}_{i=0}^{n-1}$ and dots $\{P_i\}_{i=0}^{n-1}$ such that $P_i \cup E_i \cup P_{i+1}$ is a generalized arc for every i (where we take $P_n = P_0$) is called a *generalized circle*; it encloses a compact set K called a *generalized disk*. See Figure 3.

Finally, we say that $\Omega \subset \mathbb{R}^2$ is a *flower* if it is either a dot or the connected union of finitely many generalized disks $\{K_i\}$ and generalized arcs $\{A_j\}$ and, moreover, contains no generalized circles other than those which are inside each generalized disk K_i. Note that this implies that each pair of sets in $\{K_i\} \cup \{A_j\}$ either are disjoint or intersect exactly at a dot; see Figure 4.

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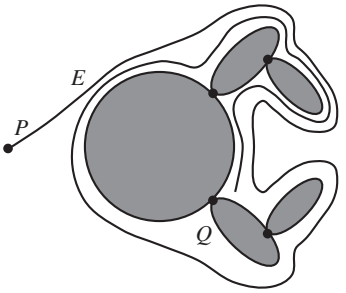


FIGURE 2. A generalized arc consisting of the degenerate dot P, the non-degenerate dot (cactus) Q and the open arc E which ends at both P and Q.

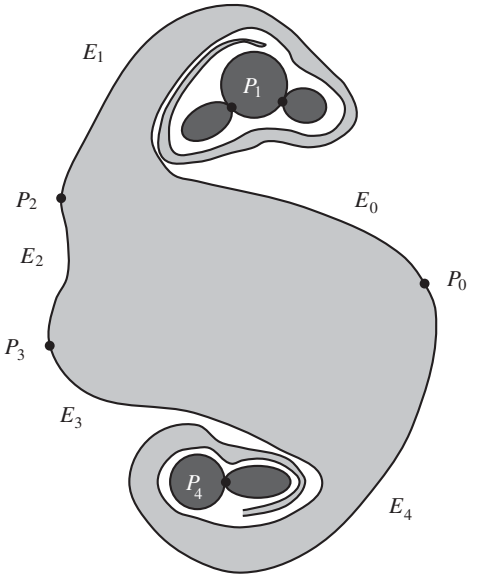


FIGURE 3. A generalized disk bounded by the generalized circle consisting of points P_0, P_2 and P_3, cactuses P_1 and P_4, and open arcs E_0, E_1, E_2, E_3 and E_4.

MAIN THEOREM. *If an analytic local flow on $\mathbb{R}^2$ has a global attractor, then it is a flower. Conversely, for every flower $\Omega \subset \mathbb{R}^2$ there is polynomial vector field whose local flow is topologically equivalent to a flow having Ω as its global attractor.*

3. *Proof of the direct statement of the Main Theorem*

First we recall some known facts about global attractors. Let Φ be a local flow on a metrizable space X. If Ω is a global attractor for Φ, then $u \in \Omega$ if and only if $\alpha_\Phi(u) \neq \emptyset$ [**2**, p. 90, Theorem 4.19]. Hence, if a local flow has a global attractor, then it is uniquely defined and contains all invariant compact subsets of X (and, in particular, all singular and periodic orbits of Φ). We emphasize that Bd Ω is invariant because Ω is invariant.

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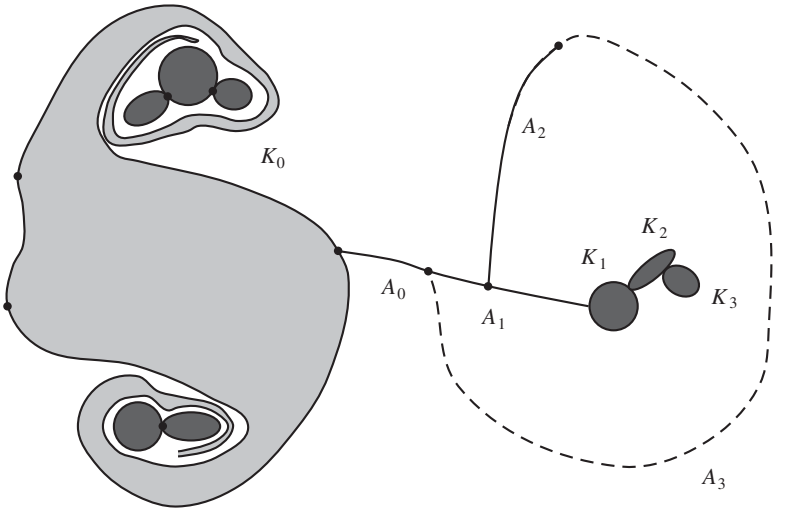


FIGURE 4. A flower consisting of the generalized disk K_0, the disks K_1, K_2 and K_3, and the arcs A_0, A_1 and A_2. Notice that if we add the dashed arc A_3, then the resultant set is not a flower.

If, in addition, X is connected, then we can easily see that Ω is connected as well. In the special case where X is the euclidean space $\mathbb{R}^n$, recall Theorem 1.1. If $X = \mathbb{R}^2$, then a further characterization of Ω can be provided, particularly when the local flow is analytic.

LEMMA 3.1. *Let Ω be a global attractor for a local flow Φ on $\mathbb{R}^2$. If $z \in \operatorname{Bd}\Omega$, then either Γ_z is periodic or $\alpha_\Phi(z)$ contains only singular points. In the latter case, if Φ is analytic, then $\alpha_\Phi(z)$ is a singular point.*

Proof. Assume that Γ_z is not periodic but $\alpha_\Phi(z)$ contains a regular point. Then Γ_z spirals towards $\alpha_\Phi(z)$ as time goes to $-\infty$, and so do the orbits of all points sufficiently close to z, which, in particular, includes the orbits of some points in $\mathbb{R}^2 \setminus \Omega$. This is, however, impossible because the α-limit set of every point of $\mathbb{R}^2 \setminus \Omega$ is empty. If, additionally, Φ is analytic and $\alpha_\Phi(z)$ consists of singular points (but is not just a single singular point), then [**10**, Lemma 4.6(iv)] applies and, for any fixed singular point $p \in \alpha_\Phi(z)$, we can find a homeomorphism $h : (-1, 1) \times [0, 1) \to A \subset \mathbb{R}^2$ such that $h(0, 0) = p$, every open arc $h((-1, 1) \times \{s\})$, $s \in (0, 1)$, is a regular semi-orbit of Φ, $h((-1, 1) \times \{0\}) \subset \alpha_\Phi(z)$, and $\Gamma_z \cap A$ accumulates at p. Thus Γ_z spirals towards $\alpha_\Phi(z)$ as before, and we arrive at the same contradiction. This implies the second statement of the lemma. □

In the next lemma, a set $B \subset \mathbb{R}^2$ is said to be *discrete* if its points do not accumulate at any point of $\mathbb{R}^2$.

LEMMA 3.2. (Non-oscillating property) *Let Φ be an analytic local flow on $\mathbb{R}^2$, let $p \in \mathbb{R}^2$, and assume that $\omega_\Phi(p) = \{q\}$ for an isolated singular point q which is not an attracting focus of Φ. Let $A \subset \mathbb{R}^2$ be an analytic set. Then either $\Gamma_p \subset A$ or $\Gamma_p^+ \cap A$ is a discrete set.*

Proof. Suppose that $\Gamma_p^+ \cap A$ is not a discrete set, and consider a sequence (p_n) of distinct points of $\Gamma_p^+ \cap A$ converging to a point u of A. According to the Lojasiewicz theorem,

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we may assume that u is the endpoint of an arc $B \subset A$, containing all the points p_n, which admits an analytic parametrization $\varphi : [0, 1] \to B$, $\varphi(s) = (\varphi_1(s), \varphi_2(s))$, such that $\varphi(0) = u$, $\varphi'(s) \neq (0, 0)$ for every $s \neq 0$, and $\varphi'(s)/\|\varphi'(s)\|$ converges as $s \to 0^+$. Let $f(x, y) = (P(x, y), Q(x, y))$ be the vector field of Φ, and let $z(t) = (x(t), y(t))$ be the solution of $z' = f(z)$ satisfying $z(0) = p$ (hence becoming a parametrization of Γ_p). Since q is not an attracting focus, we know that $z'(t)/\|z'(t)\|$ converges as $t \to \infty$, that is, the tangent to the orbit is well defined in the limit (see, e.g., [**1**, p. 331, Theorem 64]).

It suffices to show that $z(J) \subset A$ for some interval J, because $\Gamma \subset A$ then follows from the analyticity of z and A. This is easily seen to be the case when $u \in \Gamma_p^+$ (owing, again, to the analyticity of z and A), so we can assume that $u = q$. Now find sequences $s_n \to 0$ and $t_n \to \infty$ such that $p_n = \varphi(s_n) = z(t_n)$ for every n. Since neither Γ_p nor B spiral towards q, we can assume, after replacing (p_n) by an appropriately chosen subsequence if necessary, that $-y'(t_n)\varphi_1'(s_n) + x'(t_n)\varphi_2'(s_n) \geq 0$ (respectively, $-y'(t_n)\varphi_1'(s_n) + x'(t_n)\varphi_2'(s_n) \leq 0$) whenever n is odd (respectively, even). We then see that there is a sequence $r_n \to 0$ in $[0, 1]$ such that $\psi(r_n) = 0$ for every n for the analytic map $\psi(s) = -Q(\varphi(s))\varphi_1(s) + P(\varphi(s))\varphi_2(s)$; this further implies that $\psi(s) = 0$ for every $s \in [0, 1]$. Hence the vector field f is tangent to B at each of its points; in other words, B is the union of some semi-orbits of Φ and, in particular, $z(J) \subset A$ for some interval J as desired. □

LEMMA 3.3. *Let Ω be a global attractor for an analytic local flow Φ on $\mathbb{R}^2$. Then* Bd Ω *contains finitely many regular orbits.*

Proof. Let A be the set of singular points of Φ. Since A is compact (because it is a subset of Ω) and analytic, it is a graph; in particular, it has finitely many components. If a periodic orbit is contained in Bd Ω, then it encloses a disk that is included in Ω. Hence two such periodic orbits enclose disjoint disks and thus also different components of A. Therefore, Bd Ω must contain finitely many periodic orbits.

Next, to arrive at a contradiction, suppose that Bd Ω contains an infinite sequence (Γ_n) of non-periodic regular orbits. According to Lemma 3.1, we can assume that there is a sequence (q_n) of singular points, corresponding to α-limit sets of the orbits Γ_n, which converges to a singular point q. We can immediately discard the case where q is an isolated singular point (i.e. $q_n = q$ for all large enough n), because then the well-known local theory of analytic flows would apply, giving that either q is a repeller or q admits a small neighbourhood that is a union of finitely many hyperbolic, elliptic and parabolic sectors. Since the orbits Γ_n are contained in Bd Ω, we can easily find a point $w \in \mathbb{R}^2 \setminus \Omega$ such that $\alpha_\Phi(w) = \{q\}$, which is a contradiction.

Thus we can assume that there is an arc $B \subset A$ which contains all points q_n and ends at q (note that this does not rule out the possibility that $q_n = q$ for every n). Let $f = (f_1, f_2)$ be the vector field of Φ. Then there exist an open neighbourhood U of q and analytic functions $H, g_1, g_2 : U \to \mathbb{R}$ such that $f_1 = Hg_1$ and $f_2 = Hg_2$ in U, and such that $z' = g(z)$, where $g = (g_1, g_2)$, has no singular points in $U \setminus \{q\}$; see, e.g., [**10**]. We may assume that $B \subset U$, and then observe that all points of B (like those of A in U) belong to the set $\{(x, y) \in U \mid H(x, y) = 0\}$. By taking points $p_n \in \Gamma_n$ close enough to q, we can also assume that the semi-orbits $(\Gamma_n)^-_{p_n}$ are in U, and hence are semi-orbits of the

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local flow Ψ of g. By changing, if necessary, the sign of H, we may even assume that the orientations of the semi-orbits $(\Gamma_n)^-_{p_n}$ are the same in both flows.

Now, several possibilities arise. To begin with, q cannot be enclosed by periodic orbits of the new flow, for such orbits would become invariant sets for Φ and hence would enclose full subsets of Ω, in contradiction to $\Gamma_n \subset \operatorname{Bd}\Omega$. If q is a regular point of Ψ (when we obviously have $q_n \neq q$ for every n), then Ψ is locally rectifiable at q. Since the arc B crosses U to arrive at q, if a point $w \in \mathbb{R}^2 \setminus \Omega$ is close enough to a semi-orbit $(\Gamma_n)^-_{p_n}$ with n sufficiently large, then its orbit for Ψ must cross B as well. As the orientations coincide, the α-limit set of the orbit Γ_w of w for Φ must be a point from B, which is impossible. If q is an attracting or a repelling focus for Ψ, then every orbit of Ψ near q crosses B infinitely many times (because B approaches q from a definite direction), and again we can find a point in $\mathbb{R}^2 \setminus \Omega$ whose α-limit set for Φ is a point from B.

Finally, assume that q admits a small neighbourhood that is a union of finitely many hyperbolic, elliptic and parabolic sectors for Ψ. It is not restrictive to assume that U itself does the job. Again, we can easily discard the case where $q_n = q$ for every sufficiently large n, and assume that $q_n \neq q$ for every n. Now we resort to Lemma 3.2 to ensure (upon taking a smaller B if necessary) that either B is contained in one of the separatrices limiting the sectors or B does not intersect these separatrices except at q. The first possibility can be excluded owing to the condition $q_n \neq q$ for every n. Hence B and the semi-orbits $(\Gamma_n)^-_{p_n}$ must be contained in one of the sectors, and it is simple to check that, in each case, $\mathbb{R}^2 \setminus \Omega$ contains a point whose α-limit set is a point from B. Thus we have arrived at a contradiction. □

We are ready to prove the direct statement of the Main Theorem. Assume that an analytic local flow Φ on $\mathbb{R}^2$ has a global attractor Ω. Let $\Gamma = \Gamma_p$ be a regular orbit in $\operatorname{Bd}\Omega$; then the following possibilities exist: (i) Γ is a periodic orbit and all nearby exterior orbits spiral towards it in positive time, (ii) Γ spirals towards its ω-limit set, or (iii) $\omega_\Phi(p)$ is a single singular point. In the first two cases, $\omega_\Phi(p)$ is the boundary of a cactus C [**10**], which is invariant and contained in Ω. Consider the equivalence relation given by $x \sim y$ if $x = y$ or $x, y \in C$, for any $x, y \in \mathbb{R}^2$. If we collapse C to a point $\mathbf{q}$ by considering the quotient set $\mathbb{R}^2/\sim$, we obtain a topological space homeomorphic to $\mathbb{R}^2$ (use [**12**, p. 531, Fundamental Theorem 6]) which inherits, in the natural way, a (continuous) local flow that has $(\Omega \setminus C) \cup \{\mathbf{q}\}$ as its global attractor and for which the ω-limit set of p is $\mathbf{q}$. Notice that the described operation does not substantially damage the topological structure of the set of singular points, which will still be a graph.

Thus, after repeating the process finitely many times (recall that, by Lemma 3.3, $\operatorname{Bd}\Omega$ has a finite number of regular orbits), we arrive at a local flow Ψ whose set B of singular points is a graph and whose global attractor Σ has the property that both $\alpha_\Psi(z)$ and $\omega_\Psi(z)$ are single singular points for every $z \in \operatorname{Bd}\Sigma$ (we also use Lemma 3.1).

Hence $B \cup \operatorname{Bd}\Sigma$ is a graph. Since every subcontinuum of a graph is also a graph and $\operatorname{Bd}\Sigma$ is connected (because it is the boundary of a set homeomorphic to $\mathbb{R}^2 \setminus \{(0,0\}$), $\operatorname{Bd}\Sigma$ is a connected graph. Clearly, the continuum enclosed by $\operatorname{Bd}\Sigma$ (which is Σ itself) is either a single point or a flower with degenerate dots. By blowing up the collapsed cactuses, we see that Ω is either a dot or a flower, and we are done.

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4. *Proof of the converse statement of the Main Theorem*

The next lemma will be crucial in the proof. Here $\mathbb{S}^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$. A *polynomial vector field on* $\mathbb{S}^2$ is a tangent vector field to $\mathbb{S}^2$, each of whose coordinates is a polynomial in x, y and z. We say that an orbit is *positively* (respectively, *negatively*) *stable* if nearby orbits stay near it in positive (respectively, negative) time. More precisely, Γ is positively (respectively, negatively) stable if for any point $p \in \Gamma$ and any $\epsilon > 0$, there exists $\delta = \delta(p, \epsilon) > 0$ such that if $d(p, q) < \delta$, then Γ_q^+ (respectively, Γ_q^-) lies within ϵ of Γ_p^+ (respectively, Γ_p^-). Note that we are not requiring that $d(\Phi_p(t), \Phi_q(t))$ goes to zero as t goes to ∞. An orbit is said to be *unstable* if it is not positively stable or not negatively stable. An α- or ω-limit set which is neither a singular point nor a periodic orbit is called *non-trivial*.

LEMMA 4.1. *Let Φ be a flow on $\mathbb{S}^2$. Assume that Φ has the following properties:*

(i) *it is of finite type, that is, it has finitely many singular points and unstable orbits, and no open subset of $\mathbb{S}^2$ is the union of periodic orbits;*

(ii) *if p, A and Z are, respectively, a singular point, a non-trivial α-limit set and a non-trivial ω-limit set of Φ, then $p \in A \cap Z$ cannot happen;*

(iii) *the local phase portrait at every singular point belonging to a non-trivial limit set is the union of finitely many hyperbolic sectors;*

(iv) *every non-trivial limit set contains some saddle point.*

Then there is a polynomial vector field on $\mathbb{S}^2$ whose flow is topologically equivalent to Φ.

Proof. Since every flow on $\mathbb{S}^2$ is topologically equivalent to a C^∞ flow [**9**], there is no loss of generality in assuming that Φ is smooth. Property (i) implies that every singular point admits a local phase portrait that is the union of finitely many hyperbolic, elliptic and parabolic sectors [**1**, p. 315]. We can assume that local phase portraits corresponding to different singular points are disjoint, and hence that the semi-orbits limiting the sectors are disjoint as well. Let $\mathcal{R}$ be the family of all such separatrices. For any non-trivial limit set K, we denote by $\mathcal{C}_K$ the family of separatrices from $\mathcal{R}$ that are contained in K.

Let $f : \mathcal{R} \to (0, \infty)$ be defined by $f(\Gamma) = 2$ if Γ corresponds to a saddle point in some non-trivial ω-limit set, $f(\Gamma) = 1/2$ if Γ corresponds to a saddle point in some non-trivial α-limit set, or $f(\Gamma) = 1$ if Γ corresponds to any other singular point. By property (ii), this definition makes sense. Further, property (iv) implies that $\prod_{\Gamma \in \mathcal{C}_k} f(\Gamma) > 1$ (respectively, $\prod_{\Gamma \in \mathcal{C}_k} f(\Gamma) < 1$) for every ω-limit set (respectively, α-limit set) K. The last property, together with (i), (iii) and the smoothness of Φ, suffices to guarantee that Φ can be realized, up to topological equivalence, by a polynomial vector field [**14**, Theorem 1.1]. □

Let $\Omega \subset \mathbb{R}^2$ be a flower. We first assume that Ω is a tree, i.e. a graph containing no circles. The trivial case where Ω degenerates to one point can be discarded. Let $\{E_i\}$ and $\{p_j\}$ denote, respectively, the set of edges and the set of endpoints and vertices of Ω.

Since Ω contains no circles, there is a family of pairwise disjoint open arcs $\{\Gamma_i\}$ separating $\mathbb{R}^2$ such that every Γ_i intersects Ω at exactly one point $q_i \in E_i$. Now, for every component O of $\mathbb{R}^2 \setminus (\Omega \cup \bigcup_i \Gamma_i)$, one of the following two possibilities arises. If Bd O contains an endpoint p of Ω, then there is a homeomorphism $h : [0, \infty) \times \mathbb{R} \to \text{Cl } O$ that maps $\{0\} \times \mathbb{R}$ onto some Γ_i (with $h(0, 0) = q_i$), $(0, 1) \times \{0\}$ into E_i

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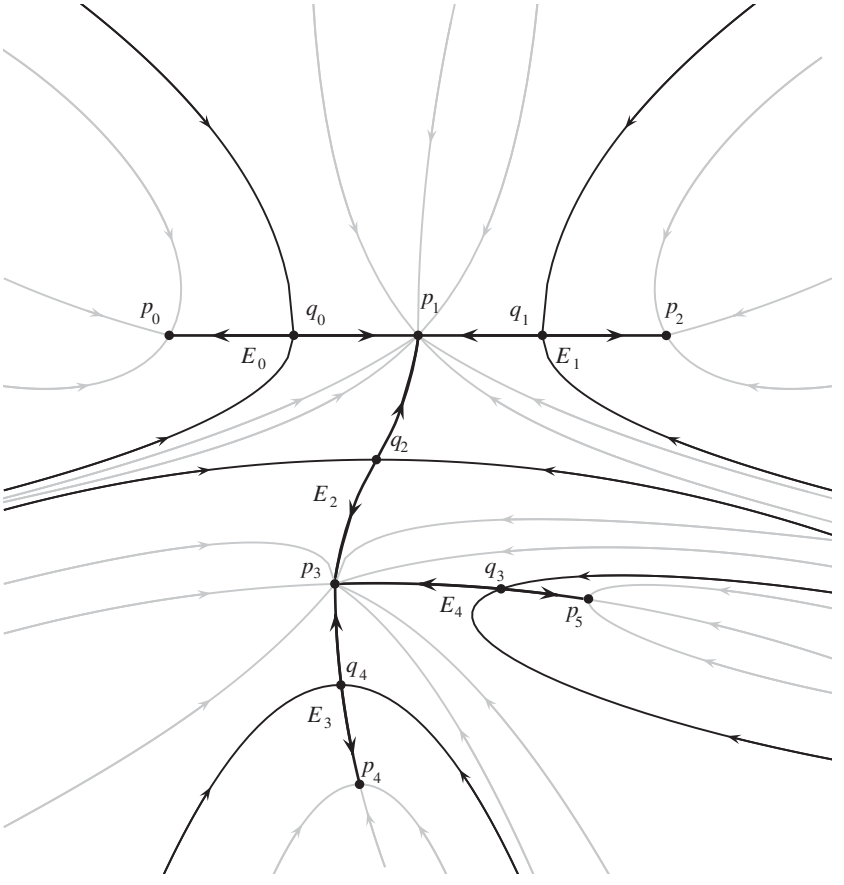


FIGURE 5. The tree consisting of the arcs E_0 (connecting p_0 and p_1), E_1 (connecting p_1 and p_2), E_2 (connecting p_1 and p_3), E_3 (connecting p_3 and p_4) and E_4 (connecting p_3 and p_5) is a global attractor for the depicted flow. The points p_j and q_i are the singular points of the flow.

(with $h(1, 0) = p$), and the remaining points onto O. If Bd O does not contain any endpoint of Ω, then there is a homeomorphism $h : [-1, 1] \times [0, \infty) \to \mathrm{Cl}\ O$ that maps $\{-1\} \times (0, \infty)$ and $\{1\} \times (0, \infty)$ into open arcs in Γ_i and Γ_j, respectively (with $h(-1, 0) = q_i$ and $h(1, 0) = q_j$), $(-1, 0) \times \{0\}$ and $(0, 1) \times \{0\}$ into E_i and E_j, respectively (with $h(0, 0)$ being the vertex of Ω at which both E_i and E_j end), and the rest of the points onto O. Now it is simple to induce a flow on every set Cl O, and hence on the whole plane, by using the flows of

$$P(x, y) = -x(x - 1),$$
$$Q(x, y) = -y$$

(if the first possibility holds) and

$$P(x, y) = x(x - 1)(x + 1),$$
$$Q(x, y) = -y$$

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(if the second possibility holds) together with the corresponding homeomorphisms h. In this way, we construct a flow that has $\{p_j\} \cup \{q_i\}$ as its set of singular points (with every p_j being an attracting node and every q_i being a saddle point) and which clearly has Ω as its global attractor. Notice that, at present, the obtained flow is just continuous. See Figure 5.

Next, we assume that Ω is a finite union of generalized arcs. Starting from a flow as above and blowing up some attracting nodes into cactuses with the property that all nearby orbits spiral towards them in positive time, there are many ways of devising a flow that has Ω as its global attractor. In particular, it is possible to construct the flow so that whenever D is a maximal disk contained in some of the cactuses, it contains as many singular points as intersection points with other disks, plus two additional singular points p_0 and p_2 in its boundary as well as two other points, p_1 and p_3 say, in its interior. Furthermore, there are heteroclinic orbits $\{\Gamma_i\}_{i=0}^3$ in Int D such that $\{p_i\}$ and $\{p_{i+1}\}$ are, respectively, the α-limit set and the ω-limit set of Γ_i (we identify p_4 with p_0). Reversing, if necessary, the orientations of the orbits Γ_i, it is possible to define the rest of the orbits inside D in such a way that they have either p_1 or p_3 as both their α- and their ω-limit sets. Notice that both p_0 and p_2 are then saddle points for the flow. See Figure 6.

In this manner, we are able to construct a continuous flow Φ on $\mathbb{R}^2$ which, besides having Ω as its global attractor, satisfies the following properties:

- it has finitely many singular points and no periodic orbits;
- it has finitely many unstable orbits;
- the only non-trivial α- or ω-limit sets for the flow are the boundaries of the maximal cactuses in Ω;
- the local phase portrait at a singular point belonging to the boundary of a maximal cactus in Ω is the union of finitely many hyperbolic sectors;
- the boundary of every maximal cactus in Ω contains some saddle point.

Upon viewing $\mathbb{R}^2 \cup \{\infty\}$ as the sphere $\mathbb{S}^2$ and ∞ as a repelling singular point, we have a flow on $\mathbb{S}^2$ that satisfies the requirements of Lemma 4.1. Let $f : \mathbb{S}^2 \subset \mathbb{R}^3 \to \mathbb{R}^3$ be a polynomial vector field whose flow is topologically equivalent to the former flow, say $f(x, y, z) = (P(x, y, z), Q(x, y, z), R(x, y, z))$. We can assume that the north pole is the repelling singular point of the new flow. Using the stereographic projection $u = x/(1 - z)$, $v = y/(1 - z)$, we transform f to the plane vector field

$$g(u, v) = (1 - z)^{-1}(P(x, y, z) + R(x, y, z)u, Q(x, y, z) + R(x, y, z)v),$$

with $x = 2u/(1 + u^2 + v^2)$, $y = 2v/(1 + u^2 + v^2)$ and $z = (u^2 + v^2 - 1)/(1 + u^2 + v^2)$. After multiplying g by a sufficiently high power of $1 + u^2 + v^2$, we get a polynomial vector field whose flow is topologically equivalent to Φ. Therefore we have shown that every flower containing no generalized disks is, up to topological equivalence, the global attractor of a polynomial vector field.

To finish the proof of the converse statement of the Main Theorem, it remains to tackle the case in which the set Ω is a flower containing generalized disks. To achieve this, let us consider finitely many arcs $\{A_i\}_{i=1}^N \subset \text{Bd } \Omega$. It is convenient to introduce embedding maps $F_i : [0, 1] \to \mathbb{R}^2$ such that $F_i([0, 1]) = A_i$ and write $F_i((0, 1)) = L_i$. These arcs are assumed to satisfy the following properties.

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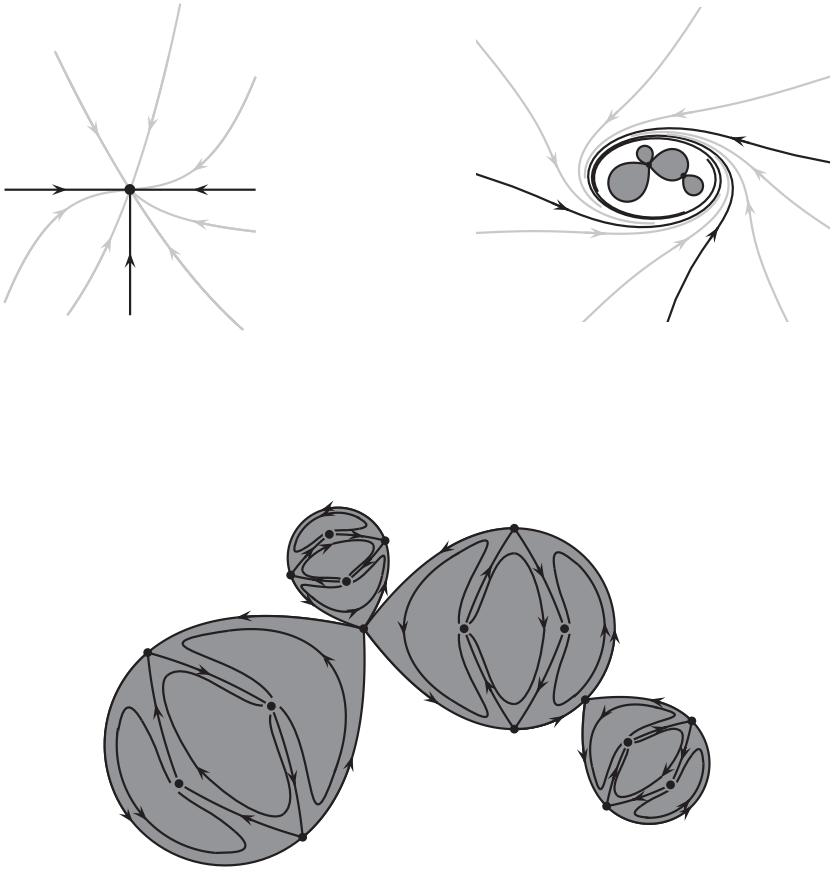


FIGURE 6. The singular point p_1 (upper left diagram) of the flow from Figure 5 is blown-up to a cactus (upper right diagram), inside of which a new flow is defined (lower diagram) so as to inherit the structure of the global attractor of the former flow.

(1) $A_i \cap A_j = \emptyset$ if $i \neq j$ and $A_i \cap P_j = \emptyset$ for any $i = 1, \ldots, N$ and $j = 1, \ldots, n$; here $\{P_j\}_{j=1}^n \subset \Omega$ is the set of dots contained in the flower Ω.

(2) Bd $\Omega \setminus \bigcup_{i=1}^N L_i$ is connected and does not contain any generalized circles.

It is clear from these two properties that Bd $\Omega \setminus \bigcup_{i=1}^N L_i$ is the boundary of a flower Ω' which is a finite union of generalized arcs; in particular, it does not contain any generalized disks. Roughly speaking, the set Ω' is obtained by taking off finitely many open arcs from the boundary of the constituent generalized disks of Ω (and also the inner parts of the generalized disks).

We apply the previous construction to the set Ω' to generate a polynomial vector field f having Λ' (a set topologically equivalent to Ω') as its global attractor. This homeomorphism transforms the 'loose ends' $F_i(0)$ and $F_i(1)$ of Ω' ($i = 1, \ldots, N$) into points, say a_i and b_i, of Λ'. Recall that, owing to the way the set Λ' is defined, the arcs in Λ' ending at these points approach the points along definite directions and, in particular, do not spiral around them. It is then easy to connect a_i and b_i with polygonal arcs R_i which

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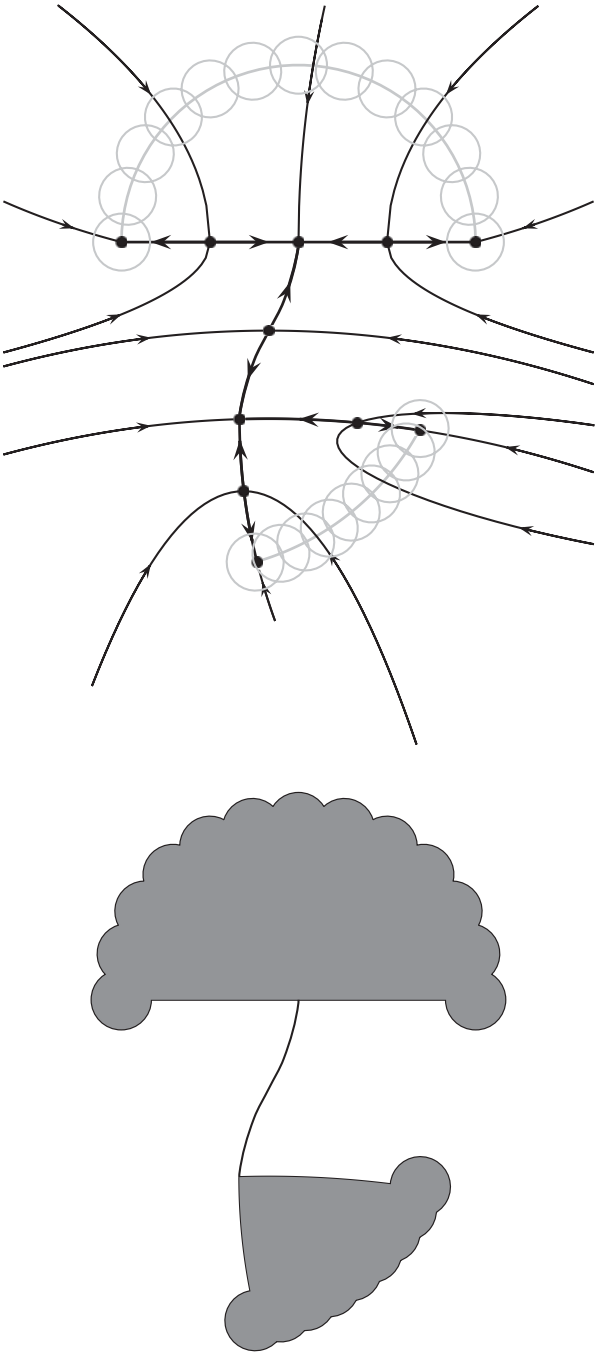


FIGURE 7. After taking off the grey arcs, the flower bounded by the thick black and grey lines becomes a tree which is the global attractor of a flow (upper picture); after adding the disks, the resultant flower (lower picture) is also a global attractor.

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intersect Λ' exactly at a_i and b_i, in such a way that the resultant set encloses a flower Λ'' topologically equivalent to Ω. The last step involves covering each arc R_i with finitely many disks $D_{ij} = \{z : \|z - p_{ij}\|_2 \leq r_{ij}\}$, $j = 1, \ldots, J$, satisfying $p_{ij} \in R_i$ and $p_{i1} = a_i$, $p_{iJ} = b_i$, so that $\Lambda = \Lambda'' \cup \bigcup_{i,j=1}^{N,J} D_{ij}$ is still a flower that is topologically equivalent to Ω. See Figure 7.

Now, define the polynomial vector field

$$g(z) = \left[\prod_{i,j}(\|z - p_{ij}\|_2^2 - r_{ij}^2)\right] f(z),$$

and observe that the local flow of g satisfies:

- Λ is invariant;
- the ω-limit of any point $z \in \mathbb{R}^2$ is contained in Λ, because $\Lambda' \subset \Lambda$ and $\mathrm{Bd}\,\Lambda \backslash \Lambda'$ consists of singular points.

This immediately yields that the local flow of g has Λ as its (asymptotically stable) global attractor; hence the proof is complete.

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