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How to play the triangle game

TONY CRILLY

> We are given n sticks of lengths 1, 2, 3, …, n and three are selected at random. Which selections enable a triangle to be formed?

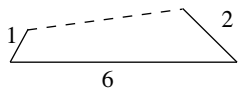
This question can be written in the form of a game: you win if a triangle can be formed from the three numbers interpreted as side lengths, you lose if they do not.

We let $\mathbb{N}_n = \{1, 2, 3, \ldots, n\}$, and see there are two variants of the game:

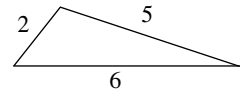
- From $\mathbb{N}_n$ draw at random three numbers sequentially. At each stage do not replace the drawn number.
- From $\mathbb{N}_n$ a number is drawn at random and is recorded. In three stages three numbers are drawn. Replace the number at each stage.

Which game would you prefer to play?

When is a triangle formed? For example, if $n = 6$ (say) and with selections such as 126 or 246 a failed triangle would result, but for the string 256, a triangle would be possible (see Figure 1 below).



A failed triangle



A (genuine) triangle

FIGURE 1

When a triple does not represent a triangle, or is a straight line (resulting from a triple such as 246), a triple gives us a ‘failed triangle’.

1. *Sampling without replacement*

This is the first variant of the game. We define $X(n)$ to be the set of all triples xyz, where each of x, y, z is a member of $\mathbb{N}_n$. The subset $R(n)$

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consists of those triples that do form triangles and $H(n)$ the triples that do not. In this variant of the game only scalene triangles are possible since equality between x, y, z does not arise.

The subset $W(n)$, is the set of all *ordered* triples xyz where $(x < y < z)$, $Q(n)$ the subset of *ordered* triples which form a triangle and $D(n)$ the subset of *ordered* triples that do not. The sets $R(n)$ and $H(n)$ are mutually exclusive, as are $Q(n)$ and $D(n)$, while also $X(n) = R(n) \cup H(n)$ and $W(n) = Q(n) \cup D(n)$, and

$$Q(n) \subset R(n) \subset X(n),$$
$$D(n) \subset H(n) \subset X(n).$$

From the definitions it follows that the number of triples in each set are related by:

$$|X(n)| = |R(n)| + |H(n)|,$$
$$|W(n)| = |Q(n)| + |D(n)|.$$

The number of subsets of three elements selected from $\mathbb{N}_n$ is the binomial coefficient $\binom{n}{3}$, and, as any subset has a unique ordered representative, we have $|W(n)| = \binom{n}{3}$. Furthermore, $|X(n)| = |W(n)| \times 3!$, $|R(n)| = |Q(n)| \times 3!$ and $|H(n)| = |D(n)| \times 3!$.

In terms of probability, assuming each number selected from $\mathbb{N}_n$ has the same chance of being selected, the probability of a genuine triangle is

$$\frac{|R(n)|}{|X(n)|} = \frac{|Q(n)|}{|W(n)|}.$$

Our problem is to find $|Q(n)|$, or equivalently $|D(n)|$, the number of triples in $D(n) = \{xyz : x < y < z, x + y \leqslant z\}$. For example, for $n = 6$,

$$D(6) = \{123, 124, 125, 126, 134, 135, 136, 145, 146, 156, 235, 236, 246\},$$

and so $|D(6)| = 13$, while

$$Q(6) = \{234, 245, 256, 345, 346, 356, 456\}$$

and we have $|Q(6)| = 7$, noting that $|X(6)| = 20 = \binom{6}{3}$.

A useful observation is the limitation on the numerical values x, y, z for triples $xyz \in D(n)$:

Proposition 1.1: For $d = xyz \in D(n)$:

(i) $1 \leqslant x \leqslant \lfloor \frac{1}{2}(n-1) \rfloor = u_n$

(ii) $2 \leqslant y \leqslant n - 1$

(iii) $3 \leqslant z \leqslant n$

where $\lfloor\ \rfloor$ is the floor function.

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Proof: For (i) suppose to the contrary, that $x > \frac{1}{2}(n-1)$. Since

$$y \geqslant \frac{n-1}{2} + 1 = \frac{n+1}{2}$$

and therefore $x + y > \dfrac{n-1}{2} + \dfrac{n+1}{2} = n$, which shows $d = xyz \notin D(n)$. This proves (i).

For (ii) suppose $y > n-1$, then as $x \geqslant 1$ we again have $x + y > n$. Part (iii) follows from the requirement that components of strings are required to be in $\mathbb{N}_n$.

Schemes

We denote by $S_i \subset D(n)$ the set of ordered strings xyz with $x = i$ and consider them arranged in a triangular *scheme*:

$$\begin{array}{llll} i(i+1)(2i+1) & i(i+1)(2i+2) & \ldots\ i(i+1)(n-1) & i(i+1)(n) \\ & i(i+2)(2i+2) & \ldots\ i(i+2)(n-1) & i(i+2)(n) \\ & & & \vdots \\ & & & i(i+(n-2i))(n) \end{array}$$

Every triple in $D(n)$ is in one such scheme. The triangular shape of a scheme indicates the close connection with triangular numbers, and we see that $|S_i(n)| = t_{n-2i}$.

Since, by Proposition 1.1,

$$D(n) = S_1(n) \cup S_2(n) \cup \ldots \cup S_{u_n}(n)$$

we have

$$|D(n)| = \sum_{i=1}^{u_n} |S_i(n)| = \sum_{i=1}^{u_n} t_{n-2i} = \sum_{i=1}^{u_n} \binom{n-2i+1}{2}$$

where $u_n = \lfloor \frac{1}{2}(n-1) \rfloor$.

For example, with $n = 6$, $u_n = \lfloor \frac{1}{2}(6-1) \rfloor = 2$ and

$$|D(6)| = \sum_{i=1}^{2} |S_i(n)| = \binom{5}{2} + \binom{3}{2} = 10 + 3 = 13.$$

We now define

$$|G(n)| = |D(n)| + |D(n+1)|,$$

a useful function in what follows.

Proposition 1.2:

$$|G(n)| = \sum_{i=1}^{n-1} t_i = \binom{n+1}{3}.$$

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Proof: Suppose n is even. With $n = 2k$, $u_n = k - 1$ and $u_{n+1} = k$. So

$$|G(n)| = \sum_{i=1}^{k-1} t_{n-2i} + \sum_{i=1}^{k} t_{n+1-2i}$$

$$= \sum_{i=1}^{k-1} (t_{n-2i} + t_{n+1-2i}) + t_1,$$

$$|G(n)| = (t_{n-2} + t_{n-1}) + \ldots + (t_2 + t_3) + t_1.$$

If n is odd, then $u_n = k$ and $u_{n+1} = k$, then we deduce that

$$|G(n)| = (t_{n-2} + t_{n-1}) + \ldots + (t_1 + t_2).$$

In both cases $|G(n)|$ is the sum of the first $n - 1$ triangular numbers and by their well-known relationship with binomial coefficients (as illustrated in [1]), the result follows.

Proposition 1.3: The number of failed triangles formed from $\mathbb{N}_n$ is equal to the number of genuine triangles formed from $\mathbb{N}_{n+1}$:

$$|D(n)| = |Q(n+1)|.$$

Proof: Since $|W(n)| = \binom{n}{3}$, as initially observed,

$$|W(n+1)| = |Q(n+1)| + |D(n+1)| = \binom{n+1}{3}$$

and combining this with the result of Proposition 1.2, that is

$$|D(n)| + |D(n+1)| = \binom{n+1}{3},$$

we have the result.

We can probe a little further and define a *geometric* way of turning a failed triangle into a complete triangle, and thus a transformation $D(n) \to Q(n+1)$:

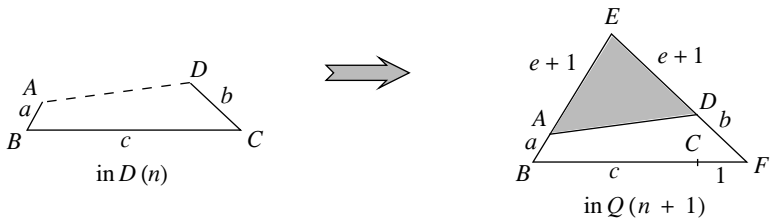


FIGURE 2

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In Figure 2, the base BC of a failed triangle of length c is extended by unity (to give BF) and an isosceles triangle ADE (shaded) is constructed on the base AD with $AE = ED = e + 1$. In this transformation, $e = c - (a + b)$ is the excess of the failed triangle, and $e \geqslant 0$.

This mapping is equivalent to a transformation of triples:

$$abc \rightarrow (c - b + 1)(c - a + 1)(c + 1).$$

This is a bijection, the main thing being to check is that it is well-defined.

Since $a < b < c$, we have $c + 1 - b < c + 1 - a < c + 1$, so that $(c - b + 1)(c - a + 1)(c + 1)$ is an ordered triple in $X(n + 1)$.

We further observe that

$$(c - b + 1) + (c - a + 1) = 2c - (a + b) + 2$$

and because $e \geqslant 0$ $(abc \in D(n))$ we have

$$(c - b + 1) + (c - a + 1) = c + e + 2 \geqslant c + 2 > c + 1$$

and we see $(c - b + 1)(c - a + 1)(c + 1) \in Q(n + 1)$, as required.

Proposition 1.4: For any positive integer n, the number of triples representing failed triangles always exceeds the number of genuine triangles:

$$|H(n)| > |R(n)|.$$

Proof: By Proposition 1.3,

$$|D(n)| = |Q(n + 1)| > |Q(n)|$$

and because

$$|H(n)| = |D(n)| \times 3! \text{ and } |R(n)| = |Q(n)| \times 3!$$

the result follows.

Calculations

The numbers $|D(n)|$ can be calculated from the difference equation

$$|D(n + 1)| + |D(n)| = \binom{n + 1}{3}, \qquad |D(3)| = 1$$

or directly from

$$|D(n)| = \sum_{i=1}^{u_n} t_{n-2i}.$$

From the last formula, dividing into cases where n is odd or even, and using the standard formulae for $\sum i$ and $\sum i^2$ we have

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$$|D(n)| = \begin{cases} \dfrac{n(n-2)(2n+1)}{24} & (n \text{ even}), \\ \dfrac{(n-1)(n+1)(2n-3)}{24} & (n \text{ odd}). \end{cases}$$

We may now compare the number of failed triangles $|H(n)|$ with the number of genuine triangles $|R(n)|$.

n	$\lvert D(n)\rvert$	$\lvert Q(n)\rvert$	$\lvert W(n)\rvert$	$\lvert H(n)\rvert$	$\lvert R(n)\rvert$	$\lvert X(n)\rvert$
3	1	0	1	6	0	6
4	3	1	4	18	6	24
5	7	3	10	42	18	60
6	13	7	20	78	42	120
7	22	13	35	132	78	210
8	34	22	56	204	132	336
9	50	34	84	300	204	504
10	70	50	120	420	300	720

The Triangle Table (without replacement)

We observe that the sequence for $|Q(n)|$ is simply a displacement of the sequence for $|D(n)|$ and $|R(n)|$ a displacement of $|H(n)|$.

If we treat the problem as a game there will always be a bias in favour of failed triangles (by Proposition 1.4) but for large n the probability of gaining a successful triangle approaches 50%. This is the meaning of:

$$\lim_{n\to\infty} \frac{|Q(n)|}{\binom{n}{3}} = \frac{1}{2}.$$

2. *Sampling with replacement*

This is the second form of the game. In this problem, we select a stick, note the result, and replace the stick, in effect the classical sampling *with* replacement. In this form we have the possibility of equilateral and isosceles triangles in addition to scalene triangles.

We adapt the previous notation and let $X^+(n)$ be the set of strings xyz where $x, y, z \in \mathbb{N}_n$. In this case $|X^+(n)| = n^3$ (the use of + indicates with replacement). In the case of sampling without replacement $|X(6)| = 120$, but in this new case $|X^+(6)| = 6^3 = 216$.

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We define $B^+(n) = \{xyn : x + y \leqslant n\}$, representing it by points in the plane, shown here for $n = 6$:

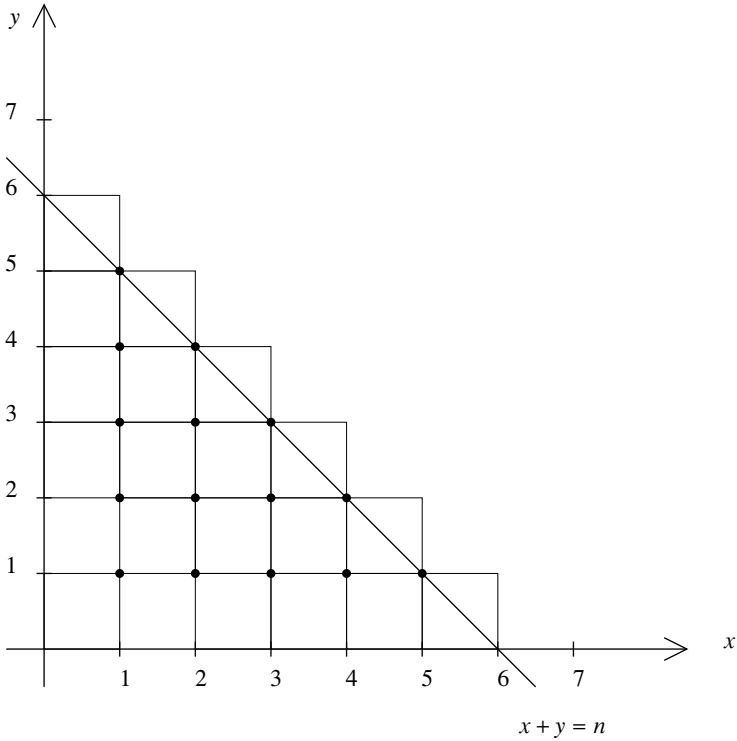


FIGURE 3

For example $B^+(1) = \varnothing$, $|B^+(1)| = 0$, and $B^+(2) = \{112\}$, $|B^+(2)| = 1$. In general

$$|B^+(n)| = t_{n-1} = \binom{n}{2}.$$

Considering sets of triples of the form *xny* and *nxy* we derive the set

$$A^+(n) = \{xyz : \max(x, y, z) = n \text{ and } x + y \leqslant n\}$$

and conclude

$$|A^+(n)| = 3\binom{n}{2}.$$

For example, $A^+(2) = \{112, 121, 211\}$ and $|A^+(2)| = 3$. From Figure 3, $|B^+(6)| = 15$ and $|A^+(6)| = 45$.

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Proposition 2.1: Sampling from $\mathbb{N}_n$, the totality of failed triangles is given by

$$\left|H^+(n)\right| = \sum_{i=1}^{n} \left|A^+(i)\right|.$$

Proof:

$$H^+(n) = \bigcup_{i=1}^{n} A^+(i)$$

and from this union of mutually exclusive sets the result follows.

Corollary 1:

$$\left|H^+(n)\right| = \sum_{i=1}^{n} 3\binom{i}{2} = 3\binom{n+1}{3}$$

hence $\left|H^+(n)\right| = \frac{1}{2}n(n^2-1)$.

Corollary 2: The number of genuine triangles is

$$\left|R^+(n)\right| = n^3 - \frac{n(n^2-1)}{2} = \frac{n(n^2+1)}{2}.$$

And in terms of triangular numbers

$$\left|R^+(n)\right| = \left(\frac{n^2+1}{n+1}\right) t_{n.}$$

3. *The two games*

Which game would you prefer to play? Without replacement there is always a greater chance of a failed triangle than a real triangle, but with replacement the reverse is true.

n	$\lvert R(n)\rvert$	$\lvert H(n)\rvert$	$\lvert R^+(n)\rvert$	$\lvert H^+(n)\rvert$
1	–	–	1	0
2	–	–	5	3
3	0	6	15	12
4	6	18	34	30
5	18	42	65	60
6	42	78	111	105
7	78	132	175	168
8	132	204	260	252
9	204	300	369	360
10	300	420	505	495

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Proposition 3:

$$|R^+(n)| > |H^+(n)|.$$

This is in contrast to the case when sampling is made without replacement, where $|R(n)| < |H(n)|$ (Proposition 1.4).

In terms of probability:

n	Without replacement Probability of a triangle	Without replacement Probability of a triangle
1	–	1
2	–	0.6250
3	0	0.5556
4	0.2500	0.5313
5	0.3000	0.5200
6	0.3500	0.5139
7	0.3714	0.5102
8	0.3929	0.5078
9	0.4048	0.5062
10	0.4167	0.5050

Acknowledgements

Superficially our triangle problem is related to the 'broken stick problem' [3, 4]. An integer variant of the 'broken stick problem' was tackled by Émile Lemoine in the 1870s. In it we are given a stick of length n and invited to break it at integer points a, b, and to count the triangles formed from the resulting three lengths (for example, a, $b - a$, $n - b$) which sum to n unlike the three numbers drawn from $\mathbb{N}_n$ here.

I would like to thank the referee for his comments on an early draft. Thanks also to Dr Alfred M. Bruckstein for sending me his paper on the triangle problem [2]. He discovered it while teaching a course in Discrete Mathematics. In studying the problem, he linked it to the problem of counting bipartite graphs and to 'Costas arrays' and to various formulae for $|Q(n)|$.

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4. Hans Schupp, The broken stick reconsidered again. *Mathematical Intelligencer*, **32** (March 2010) pp. 8-9.

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TONY CRILLY
10 Lemsford Road, St Albans AL1 3PB
e-mail: *t.crilly@btinternet.com*

Presidential Address

The Editor apologises to Peter Neumann for the fact that the first paragraph of the second section of his Presidential Address in the November 2016 issue of the *Gazette* was accidentally omitted.

Here is the missing section

2. *An inspiring teacher: Roger Wheeler* (1929–2014)

Roger Frank Wheeler was born in Sheffield in 1929. After primary school, he was educated at Bradford Grammar School (not then the 'independent' that it became in the mid-1970s) and Trinity College, Cambridge. His undergraduate studies were interrupted by service in the Royal Navy, after which he achieved First Class Honours in the Mathematical Tripos 1951. From Cambridge he moved to Hull, where he taught mathematics at Hymers College, 1952-1961. He joined the MA in 1954, and was an active member of the Yorkshire branch. There he met Dab, another enthusiastic member, and they married in 1960.

In 1961 Wheeler moved to a University Lectureship at Leicester, . . .

The complete article can be seen at:

http://www.m-a.org.uk/resources/Presidential-Address-2015-16-Peter-M-Neumann.pdf

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