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# On slice-quaternionic wavelet sets

Sumit Kumar Sharma, Nikhil Khanna, and Shiv Kumar Kaushik

*Abstract.* We introduce a slice-based quaternionic analog of wavelet sets in the right- quaternionic Hilbert space $L^2(\mathbb{R}^d, \mathbb{H})$. Using a slice-quaternionic Fourier transform, we characterize boundedness and Parseval tightness of affine systems generated by $\widehat{\psi} = (2\pi)^{-d/2}\chi_E q$ via translation and dilation multiplicity conditions on $E$. These results extend the classical frame wavelet-set theory of Dai et al. (2003, *J. Comput. Appl. Math.* 155, 69–82) to the quaternionic setting under a natural slice-phase restriction. Several dyadic, directional, and multiwavelet examples illustrate the framework and provide the first systematic construction of quaternionic wavelet sets.

## 1 Introduction

Wavelet frames in $L^2(\mathbb{R}^d)$ form a central tool in harmonic analysis and signal processing, providing redundant, stable expansions adapted to dilations and translations [3]. Within this framework, *wavelet sets* play a special role: one constructs a wavelet generator whose Fourier transform is essentially the characteristic function of a measurable set $E \subset \mathbb{R}^d$, so that all frame-theoretic information is encoded in the geometric and measure-theoretic properties of $E$. A systematic treatment of such *frame wavelet sets* and their characterization in terms of translation and dilation multiplicities was given by Daiet al. in the complex Hilbert space $L^2(\mathbb{R}^d, \mathbb{C})$ [4]. Their work shows that the boundedness and tightness of the associated affine frame can be read off from the way $E$ tiles $\mathbb{R}^d$ (up to overlaps of controlled multiplicity) under lattice translations and matrix dilations.

On the other hand, quaternionic Hilbert spaces have emerged as a natural setting for problems where the underlying data are inherently vectorial or hypercomplex, such as color image processing, 3D/4D field data, or certain models in quantum theory. This has led to an extensive development of quaternionic Fourier transforms and related hypercomplex transforms [7, 9, 10], whose history and applications are surveyed, for instance, in [2]. Various continuous quaternionic wavelet transforms have been studied via unitary representations of the quaternionic affine group on complex and quaternionic Hilbert spaces [1, 6, 12], but these constructions are group-theoretic and do not directly address discrete *wavelet sets* in the sense of [4].

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In parallel, the last decade has seen a rapid development of frame theory on right-quaternionic Hilbert spaces. After the initial finite-dimensional framework, full frame theory for separable quaternionic Hilbert spaces was developed in [13], including standard notions of analysis and synthesis operators and frame operators. Subsequent work has introduced and analyzed dual frames, woven frames, operator-valued frames, fusion, and g-frames in the quaternionic context, extending many classical complex results to this non-commutative setting [5, 11, 14–16]. These developments show that a robust discrete frame theory is now available for quaternionic Hilbert spaces, and they naturally raise the question of how *wavelet-set* phenomena behave in this setting.

The present article addresses this question for a slice-based quaternionic Fourier framework. We work on the right-quaternionic Hilbert space $L^2(\mathbb{R}^d, \mathbb{H})$ and fix a pure unit quaternion $u$ with $u^2 = -1$. This determines a complex slice $\mathbb{C}_u$ and a corresponding slice-quaternionic Fourier transform, compatible with Plancherel's theorem and standard properties of the classical Fourier transform, in the spirit of [7–9]. We then consider wavelet generators whose Fourier transforms are of the form

$$\widehat{\psi}(\xi) = (2\pi)^{-d/2}\, \chi_E(\xi)\, q(\xi),$$

where $E \subset \mathbb{R}^d$ is measurable with finite measure and the *quaternionic phase* $q(\xi)$ takes values in the commutative slice $\mathbb{C}_u$ almost everywhere. This slice assumption is crucial: it guarantees that the exponential kernel of the Fourier transform and the phase $q(\xi)$ commute, allowing us to adapt and generalize the scalar arguments of [4].

From this point of view, we introduce and study *quaternionic frame wavelet sets (slice version)*. The associated affine system is generated by dilations and translations of $\psi$ in $L^2(\mathbb{R}^d, \mathbb{H})$, and we seek to characterize its frame and tight-frame properties purely in terms of the translation and dilation multiplicities of $E$ and the affine matrix $A$. The multiplicity functions are defined exactly as in the real/complex case, but the analysis of the reproducing operators is carried out in the right-quaternionic setting, with careful attention to the order of scalar multiplication.

The results obtained here provide what seems to be the first systematic wavelet-set theory in the context of right-quaternionic Hilbert spaces. They show that, at least under a natural slice-phase restriction, the tiling and multiplicity mechanisms of the scalar theory are robust enough to survive the passage to $\mathbb{H}$-valued signals. At the same time, they set the stage for future work on non-slice phases and on connections with more general quaternionic frames, such as operator-valued and g-frames arising in quantum detection problems [11, 16].

The quaternionic setting considered here differs from the classical complex framework through the use of quaternion-valued generators and the slice Fourier transform. A central aspect of the present work is the identification of conditions under which the geometric theory of wavelet sets can be transferred to a right-quaternionic Hilbert space. In particular, the slice-valued phase structure allows one to retain the frequency-domain geometry of classical wavelet sets while working within a quaternionic framework. The results obtained here, therefore, provide a quaternionic analog of the corresponding wavelet-set theory and establish the role of translation and dilation multiplicities in this setting.

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The article is organized as follows. In Section 2, we recall basic facts about right-quaternionic Hilbert spaces, measurable sets and integrals, and the slice-quaternionic Fourier transform, including a Plancherel theorem adapted to our setting. Section 3 introduces slice-quaternionic wavelet sets, their translation and dilation multiplicity functions, and the associated canonical representatives. Section 4 contains the main results of the article. It develops the structural lemmas for the reproducing operators and establishes both a characterization of bounded affine systems and a tight-frame characterization for slice-quaternionic wavelet sets. Section 5 presents several examples based on the tight-frame characterization, including dyadic dilations and simple geometric wavelet sets.

## 2 Preliminaries

This section summarizes the analytic framework underlying the quaternionic wavelet-set construction. We introduce the right-quaternionic Hilbert space structure, the slice-based Fourier transform, and the affine operators that appear throughout the article. Basic conventions on measurability and integration are also included for completeness.

### 2.1 Measurable sets and integration

All subsets of $\mathbb{R}^d$ considered in this article are assumed to be Lebesgue measurable. For a measurable set $E \subset \mathbb{R}^d$, we denote its Lebesgue measure by $\mu(E)$. The indicator function of $E$ is written as $\chi_E$.

Integration of quaternion-valued functions $f : \mathbb{R}^d \to \mathbb{H}$ is defined componentwise with respect to the standard basis of $\mathbb{H}$. In particular, for $f, g \in L^2(\mathbb{R}^d, \mathbb{H})$, the integral

$$\int_{\mathbb{R}^d} \overline{f(x)}\, g(x)\, dx$$

is computed by integrating each real component separately. This convention ensures that the usual analytic properties (linearity, dominated convergence, and Fubini/Tonelli) apply exactly as in the real or complex case.

### 2.2 Right-quaternionic Hilbert space

The Hilbert space $L^2(\mathbb{R}^d, \mathbb{H})$ consists of equivalence classes of measurable functions with finite norm

$$\|f\|_2^2 := \int_{\mathbb{R}^d} |f(x)|^2\, dx.$$

The inner product is defined by

$$\langle f \mid g \rangle := \int_{\mathbb{R}^d} \overline{f(x)}\, g(x)\, dx,$$

which is conjugate-linear in its first argument and right-linear in its second. This makes $L^2(\mathbb{R}^d, \mathbb{H})$ a right-quaternionic Hilbert space.

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For $\phi \in L^2$, the rank-one operator determined by $\phi$ takes the form

$$R_\phi(f) := \phi \langle \phi \mid f \rangle,$$

where the inner-product scalar multiplies on the *right*.

## 2.3 Complex slices

Choose a unit pure quaternion $u$ satisfying $u^2 = -1$. The set

$$\mathbb{C}_u := a + ub : a, b \in \mathbb{R}$$

is a two-dimensional commutative subalgebra of $\mathbb{H}$, isomorphic to the complex plane.

Different choices of the unit pure quaternion $u$ give rise to different complex slices of $\mathbb{H}$. Each slice $\mathbb{C}_u$ contains the real axis and provides a copy of the complex field embedded in the quaternion algebra. Thus, the quaternion space may be viewed as a union of such complex slices sharing the same real axis.

Throughout this article, a fixed slice $\mathbb{C}_u$ is chosen and all phase functions and Fourier kernels are assumed to take values in this same slice. Since multiplication within $\mathbb{C}_u$ is commutative, functions taking values in $\mathbb{C}_u$ can be manipulated without encountering the non-commutativity of quaternion multiplication. This property plays a central role in the slice-based Fourier theory and the wavelet-set constructions developed in this article.

## 2.4 Slice-based quaternionic Fourier transform

For $f \in L^1(\mathbb{R}^d, \mathbb{H})$, the slice-quaternionic Fourier transform with respect to $u$ is defined by

$$\widehat{f}(\xi) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{-u(x\cdot\xi)} f(x)\, dx, \qquad e^{-u(x\cdot\xi)} \in \mathbb{C}_u.$$

The kernel belongs to the commutative slice $\mathbb{C}_u$, so the integrand commutes with all slice-valued phase factors. The transform extends to a unitary operator on $L^2(\mathbb{R}^d, \mathbb{H})$ and satisfies the Plancherel identity

$$\|f\|_2 = \|\widehat{f}\|_2, \qquad \langle f \mid g \rangle = \langle \widehat{f} \mid \widehat{g} \rangle.$$

A detailed survey of quaternionic Fourier transforms and their analytic properties can be found in [8].

## 2.5 Affine actions

An invertible real $d \times d$ matrix $A$ is called *expansive* if every eigenvalue $\lambda$ of $A$ satisfies $|\lambda| > 1$. Such matrices generate expanding dilations and play a fundamental role in wavelet theory. In particular, repeated applications of $A$ enlarge sets in the frequency domain, while inverse dilations contract them toward the origin. Throughout this article, $A$ denotes a fixed expansive matrix and $A'$ denotes its transpose. Dilation and translation act on $L^2(\mathbb{R}^d, \mathbb{H})$ via

$$(D_A f)(x) = |\det A|^{1/2} f(Ax), \qquad (T_\ell f)(x) = f(x - \ell), \quad \ell \in \mathbb{Z}^d.$$

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The operator $D_A$ performs a geometric scaling determined by the matrix $A$, while $T_\ell$ translates the function by the $\ell$. These operators generate the affine systems. Their Fourier transforms are given by

$$\widehat{D_A f}(\xi) = |\det A|^{-1/2} \widehat{f}((A')^{-1}\xi), \qquad \widehat{T_\ell f}(\xi) = e^{-u(\ell\cdot\xi)} \widehat{f}(\xi).$$

Since the exponential factor $e^{-u(\ell\cdot\xi)}$ takes values in the slice $\mathbb{C}_u$, it commutes with every slice-valued phase function $q(\xi) \in \mathbb{C}_u$. This compatibility is essential in the construction and analysis of slice-quaternionic wavelet systems.

## 2.6 Wavelet generators

Let $E \subset \mathbb{R}^d$ be measurable with $0 < \mu(E) < \infty$. A *slice-quaternionic wavelet generator* is a function $\psi$ whose Fourier transform is

$$\widehat{\psi}(\xi) = (2\pi)^{-d/2} \chi_E(\xi), q(\xi), \qquad q(\xi) \in \mathbb{C}_u, \quad |q(\xi)| = 1.$$

The geometric set $E$ governs the frequency support, while the unimodular slice-valued function $q$ supplies an admissible phase.

The requirement $q(\xi) \in \mathbb{C}_u$ is fundamental in the present work. Since the slice Fourier kernel $e^{-u(x\cdot\xi)}$ also takes values in the same slice $\mathbb{C}_u$, the phase function and the kernel commute under multiplication. Consequently, the scalar quantities arising in Fourier expansions, orthogonality relations, and reproducing operators remain within a common commutative subalgebra of $\mathbb{H}$.

This commutativity allows the Fourier-side analysis to proceed in a manner analogous to the classical complex setting and provides the key mechanism for extending wavelet-set constructions to the quaternionic framework. At the same time, the resulting generators remain genuinely quaternionic through their values in the quaternionic Hilbert space and through the slice-valued phase structure.

Without the slice condition, a phase function taking values in the full quaternion algebra $\mathbb{H}$ would not commute in general with the Fourier kernel, and several orthogonality and reproducing arguments used throughout the article would no longer be directly applicable.

## 2.7 Translation and dilation multiplicities

For $\xi \in \mathbb{R}^d$ and a measurable set $E \subset \mathbb{R}^d$, we define two measure-theoretic multiplicities that quantify how often the frequency point $\xi$ appears in the lattice translates or $A'$-dilates of $E$.

- *Translation multiplicity:*

$$\tau_E(\xi) := \#\{\, \ell \in \mathbb{Z}^d : \xi + 2\pi\ell \in E \,\}, \quad E(\tau; m) := \{\, \xi \in \mathbb{R}^d : \tau_E(\xi) = m \,\}.$$

  Thus, $E(\tau; m)$ consists of those frequency points that occur exactly $m$ times among the $2\pi\mathbb{Z}^d$-translates of $E$.
- *Dilation multiplicity:* Let $A$ be an expansive real matrix and let $A'$ denote its transpose. The dilation multiplicity of $\xi$ is

$$\#_E(\xi) := \#\{\, n \in \mathbb{Z} : (A')^{-n}\xi \in E \,\}, \quad E(\#; m) := \{\, \xi \in \mathbb{R}^d : \#_E(\xi) = m \,\}.$$

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Both multiplicity functions are Borel measurable, and the layers $\{E(\tau; m)\}_m$ and $\{E(\#; m)\}_m$ form measurable partitions of $\mathbb{R}^d$ (up to null sets).

### 2.8 Reproducing operators

For each scale $k \in \mathbb{Z}$, the associated reproducing operator in the frequency domain is defined by

$$H_{k,E}(\widehat{f}) := \sum_{\ell \in \mathbb{Z}^d} \widehat{D_A^k T_\ell \psi}, \langle \widehat{D_A^k T_\ell \psi} \mid \widehat{f} \rangle,$$

whenever the series converges in $L^2(\mathbb{R}^d, \mathbb{H})$. The full reproducing operator is formally given by

$$H_{E,q}(\widehat{f}) := \sum_{k \in \mathbb{Z}} H_{k,E}(\widehat{f}).$$

At this stage, $H_{E,q}$ is regarded as a formal series of operators. Its convergence and boundedness are established later in Theorem 4.7, where it is shown that, under suitable multiplicity conditions on $E$, the partial sums

$$\sum_{|k| \le K} H_{k,E}$$

converge in the $L^2$-norm to a bounded operator on $L^2(\mathbb{R}^d, \mathbb{H})$.

## 3 Slice-quaternionic wavelet sets

The construction of quaternionic wavelet systems begins in the frequency domain, where the geometry of a measurable set determines the support of the generator and a slice-valued phase controls its orientation. Since the slice $\mathbb{C}_u$ is commutative, prescribing the Fourier transform of the generator within this slice ensures that all scalar coefficients arising from translations and dilations behave analogously to the complex case.

For this reason, we isolate the frequency support and the slice phase as the fundamental data from which the entire affine system is built. The following definition formalizes this notion and introduces the object that will be central to all subsequent results.

***Definition 3.1*** Let $E \subset \mathbb{R}^d$ be a measurable set with $0 < \mu(E) < \infty$, and let $\mathbb{C}_u$ be the complex slice determined by a fixed unit pure quaternion $u$. A *slice-quaternionic wavelet set* associated with $E$ is a function $\psi \in L^2(\mathbb{R}^d, \mathbb{H})$ whose Fourier transform is of the form

$$\widehat{\psi}(\xi) = (2\pi)^{-d/2} \chi_E(\xi)\, q(\xi),$$

where the *phase function* $q : \mathbb{R}^d \to \mathbb{H}$ satisfies:

(1) $q(\xi) \in \mathbb{C}_u$ almost everywhere;
(2) $|q(\xi)| = 1$ almost everywhere.

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The set $E$ is referred to as the *wavelet set*, and the pair $(E, q)$ determines the entire affine system

$$\mathcal{A}(\psi) := \{ D_A^n T_\ell \psi : n \in \mathbb{Z},\ \ell \in \mathbb{Z}^d \},$$

where $A$ is a fixed expansive real matrix and $T_\ell$, $D_A$ are the translation and dilation operators.

Definition 3.1 specifies a wavelet generator entirely through its frequency support and its slice-valued phase. Before proceeding to the technical analysis of translation and dilation multiplicities, it is helpful to present a concrete instance of such a generator. The following example illustrates how a measurable set and a unimodular slice phase combine to produce a valid quaternionic wavelet set within the slice Fourier framework.

***Example 3.1*** Fix a unit pure quaternion $u$ and consider the complex slice $\mathbb{C}_u \subset \mathbb{H}$. Let $A = 2I$ on $\mathbb{R}^d$. Define the measurable set

$$E = \big[\pi, 2\pi\big)^d \ \cup\ \big(-2\pi, -\pi\big]^d,$$

which has finite measure and is symmetric about the origin. Choose the slice-valued unimodular phase

$$q(\xi) := \frac{\xi_1 + u\,|\xi_1|}{\sqrt{2}\,|\xi_1|} \in \mathbb{C}_u, \quad \text{whenever } \xi_1 \neq 0,$$

and extend $q$ arbitrarily on the negligible set, where $\xi_1 = 0$. Define the function $\psi$ by prescribing its slice Fourier transform:

$$\widehat{\psi}(\xi) = (2\pi)^{-d/2}\, \chi_E(\xi)\, q(\xi).$$

Then:

(1) $\widehat{\psi}$ is supported on $E$;
(2) $q(\xi) \in \mathbb{C}_u$ and $|q(\xi)| = 1$ almost everywhere;
(3) $\widehat{\psi} \in L^2(\mathbb{R}^d, \mathbb{H})$ since $\mu(E) < \infty$.

Hence, $\psi$ is a slice-quaternionic wavelet generator associated with the wavelet set $E$. The corresponding affine system

$$\mathcal{A}(\psi) = \{D_{2I}^n T_\ell \psi : n \in \mathbb{Z},\ \ell \in \mathbb{Z}^d\}$$

is a quaternionic wavelet system whose analytic properties (framing, tightness, and tiling) depend entirely on the geometry of $E$.

Example 3.1 illustrates the simplest instance of a slice-quaternionic wavelet set in one dimension (or equivalently, a rectangular set in $\mathbb{R}^d$ when extended coordinate-wise). To highlight the geometric flexibility of the construction, it is useful to present a multidimensional example obtained by taking Cartesian products of one-dimensional frequency tiles. The resulting generator retains the slice-valued phase structure while providing a higher-dimensional analog that behaves well under dyadic dilations.

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***Example 3.2*** Let $u$ be a fixed unit pure quaternion, and let $\mathbb{C}_u$ be the corresponding slice. Consider the dyadic dilation matrix $A = 2I_d$ on $\mathbb{R}^d$. Define the one-dimensional interval

$$I := [\pi, 2\pi) \cup (-2\pi, -\pi],$$

and set

$$E := I \times I \times \cdots \times I \subset \mathbb{R}^d.$$

Since $I$ has finite measure, $E$ does as well. Choose any measurable unimodular slice phase $q(\xi) \in \mathbb{C}_u$ (e.g., $q(\xi) \equiv 1$ or $q(\xi) = e^{u\,\varphi(\xi)}$ with measurable $\varphi$), and define

$$\widehat{\psi}(\xi) = (2\pi)^{-d/2} \chi_E(\xi)\, q(\xi).$$

Then:

(1) $\widehat{\psi}$ is supported on a product set $E$, giving a separable frequency-domain structure;
(2) $q(\xi) \in \mathbb{C}_u$ ensures compatibility with the slice Fourier kernel and commutativity of scalar coefficients;
(3) the affine system generated by $\psi$ is

$$\mathcal{A}(\psi) = \{\, D_{2I_d}^n T_\ell \psi : n \in \mathbb{Z},\ \ell \in \mathbb{Z}^d \,\}$$

a $d$-dimensional quaternionic dyadic wavelet system.

The set $E$ is therefore a valid slice-quaternionic wavelet set in $\mathbb{R}^d$, providing a multidimensional analog of Example 3.1 that retains separability and is particularly useful when studying tensor-product structures or componentwise dilations.

Example 3.2 demonstrates how a higher-dimensional quaternionic wavelet set can be constructed by taking Cartesian products of one-dimensional frequency tiles. While this emphasized the geometric structure of the support set $E$, the slice-quaternionic framework also allows the introduction of nontrivial phase factors that do not arise in the scalar setting. These phases, which take values in the slice $\mathbb{C}_u$, preserve commutativity with the slice Fourier kernel and therefore lead to well-defined quaternionic wavelet generators.

To illustrate the additional flexibility provided by the quaternionic phase, we present an example in which the support set is simple but the phase function encodes nontrivial slice-valued oscillation.

***Example 3.3*** Fix a unit pure quaternion $u$ and consider the slice $\mathbb{C}_u$. Let $A = 2I_d$ as in Example 3.2. Choose any measurable set $E \subset \mathbb{R}^d$ with $0 < \mu(E) < \infty$. Let $\varphi : \mathbb{R}^d \to \mathbb{R}$ be any measurable real-valued function. Define the slice-valued unimodular phase

$$q(\xi) := e^{u\,\varphi(\xi)} \in \mathbb{C}_u.$$

Since $|e^{u\varphi(\xi)}| = 1$, this phase is admissible. Now define the wavelet generator by

$$\begin{aligned}\widehat{\psi}(\xi) &= (2\pi)^{-d/2}\, \chi_E(\xi)\, q(\xi) \\ &= (2\pi)^{-d/2}\, \chi_E(\xi)\, e^{u\,\varphi(\xi)}.\end{aligned}$$

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Then:

(1) $\widehat{\psi}$ is supported on $E$, so the geometry of the wavelet system is still controlled by $E$;
(2) the phase factor $e^{u\varphi(\xi)}$ is nontrivial unless $\varphi$ is constant, and yet it remains inside the slice $\mathbb{C}_u$, ensuring commutativity with $e^{-u(x\cdot\xi)}$ in the slice Fourier transform;
(3) the corresponding affine system is

$$\mathcal{A}(\psi) = \{\, D^n_{2I_d} T_\ell \psi : n \in \mathbb{Z},\ \ell \in \mathbb{Z}^d \,\}$$

a quaternionic wavelet system with quaternionic phase behavior.

This construction yields a family of quaternionic wavelet generators parameterized by arbitrary measurable real functions $\varphi$. Although the support set $E$ may coincide with a classical wavelet set, the slice-valued phase factor $e^{u\varphi(\xi)}$ introduces additional freedom through the choice of the quaternionic phase. Consequently, different choices of $\varphi$ generate distinct quaternionic wavelet generators while preserving the same frequency support. This illustrates the additional flexibility provided by the slice-valued quaternionic phase and highlights a characteristic feature of the slice-quaternionic framework.

The preceding examples illustrate how a slice-quaternionic wavelet generator is determined by two pieces of data: a measurable frequency set $E$ and an admissible slice-valued phase $q \in \mathbb{C}_u$. To understand when the associated affine system $\mathcal{A}(\psi) = \{D^n_A T_\ell \psi : n \in \mathbb{Z},\ \ell \in \mathbb{Z}^d\}$ possesses frame or tight-frame properties, it is necessary to analyze the geometry of $E$ under the actions of the translation lattice $2\pi\mathbb{Z}^d$ and the dilation group $\{(A')^n : n \in \mathbb{Z}\}$. This behavior is encoded by the translation and dilation multiplicity functions, which measure how often a frequency point reappears under these two group actions.

Although the multiplicity functions are already defined, it is convenient here to recall qualitatively how they organize $E$: the sets $E(\tau; m)$ and $E(\#; m)$ partition $E$ (up to a null set) into pieces on which the translation or dilation multiplicity is constant. For the purpose of structural analysis, each such piece can be reduced to a canonical representative inside a fixed fundamental domain.

For the translation action, let $I_0 = [0,1)^d$ and let $2\pi I_0$ denote the fundamental region of $2\pi\mathbb{Z}^d$. Given any multiplicity layer $E(\tau; m)$, we define its canonical representative by

$$\widetilde{E(\tau; m)} := \{\xi - 2\pi\ell : \xi \in E(\tau; m),\ \ell \in \mathbb{Z}^d,\ \xi - 2\pi\ell \in 2\pi I_0\}.$$

This set contains almost exactly one point from each translation orbit of $E(\tau; m)$ and satisfies

$$E(\tau; m) = \bigsqcup_{\ell\in\mathbb{Z}^d} \left(\widetilde{E(\tau; m)} + 2\pi\ell\right) \qquad \text{(a.e.)}.$$

Similarly, for the dilation action of $A'$, fix a measurable fundamental region $F_{A'}$ for the orbits of $\{(A')^n : n \in \mathbb{Z}\}$ so that

$$\mathbb{R}^d = \bigsqcup_{n\in\mathbb{Z}} (A')^n F_{A'} \qquad \text{(a.e.)}.$$

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For each dilation multiplicity layer $E(\#;m)$, we choose its canonical representative as

$$\widetilde{E(\#;m)} := \{(A')^{-n}\xi : \xi \in E(\#;m),\ n \in \mathbb{Z},\ (A')^{-n}\xi \in F_{A'}\}.$$

This is a measurable cross-section of the dilation orbits, and

$$E(\#;m) = \bigsqcup_{n\in\mathbb{Z}} (A')^{n}\widetilde{E(\#;m)} \qquad \text{(a.e.)}.$$

These canonical representatives play a crucial technical role: they allow the multiplicity layers of $E$ to be treated as disjoint measurable tiles, reduce the complexity of interactions between different scales, and form the combinatorial backbone for the lemmas and frame characterizations developed in the next section.

# 4 Translation and dilation multiplicities in the slice-quaternionic setting

Section 4 is organized as follows. In Section 4.1, we record the measure-theoretic structure of multiplicity layers; in Section 4.2, we analyze reproducing operators in the slice-quaternionic setting and derive the frame and tight-frame characterizations.

## 4.1 Measure-theoretic multiplicity

The multiplicity functions $\tau_E$ (translation multiplicity) and $\#_E$ (dilation multiplicity) were defined in Section 2. The two lemmas below record their basic structural properties. The statements and proofs are purely measure-theoretic and are identical in spirit to the classical complex case; we include them here for completeness since they provide the combinatorial foundation for the analytic estimates in Section 4.2.

***Lemma 4.1*** *Let $E \subset \mathbb{R}^d$ be measurable and let $\tau_E$ be the translation multiplicity. For $m \in \mathbb{N} \cup \{\infty\}$, the multiplicity layer $E(\tau;m) = \{\xi \in E : \tau_E(\xi) = m\}$ is Lebesgue measurable. If $m < \infty$, then, up to a null set, $E(\tau;m)$ admits a measurable partition into $m$ translation-equivalent pieces*

$$E(\tau;m) = \bigsqcup_{j=1}^{m} E^{(j)}(\tau;m) \qquad (a.e.),$$

*each of which is translation-equivalent to the canonical representative $\widetilde{E(\tau;m)} \subset 2\pi I_0$. Equivalently (a.e.),*

$$E(\tau;m) = \bigsqcup_{\ell\in\mathbb{Z}^d} \left(\widetilde{E(\tau;m)} + 2\pi\ell\right).$$

***Lemma 4.2*** *Let $E \subset \mathbb{R}^d$ be measurable and let $\#_E$ be the dilation multiplicity associated with the expansive matrix $A$. For $m \in \mathbb{N} \cup \{\infty\}$, the layer $E(\#;m) = \{\xi \in E : \#_E(\xi) = m\}$ is Lebesgue measurable. If $m < \infty$, then, up to a null set, $E(\#;m)$ admits a measurable partition into $m$ dilation-equivalent pieces*

$$E(\#;m) = \bigsqcup_{j=1}^{m} E^{(j)}(\#;m) \qquad (a.e.),$$

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*each dilation-equivalent to the canonical representative* $\widetilde{E(\#;m)} \subset F_{A'}$. *Equivalently (a.e.),*

$$E(\#;m) = \bigsqcup_{n\in\mathbb{Z}} (A')^{\,n} \widetilde{E(\#;m)}.$$

Both Lemmas 4.1 and 4.2 are standard measure-theoretic facts; their proofs follow the same combinatorial partitioning arguments as in the scalar theory. We, therefore, omit full proofs and refer the reader to the classical treatment (cf. [4]) for details.

## 4.2 Reproducing operators and frame estimates

We now turn to the first results in which the quaternionic structure plays a direct role. Although several arguments follow the same general strategy as in the classical complex setting, additional care is required because quaternion multiplication is not commutative. The slice condition $q(\xi) \in \mathbb{C}_u$ ensures that all phase factors, Fourier kernels, and coefficient scalars remain in the same commutative subalgebra $\mathbb{C}_u$. Consequently, the operator decompositions and periodization arguments used below are well defined and can be carried out in a manner analogous to the complex case. The following results make this principle explicit in the context of reproducing operators and frame estimates.

***Lemma 4.3*** *Let* $E \subset \mathbb{R}^d$ *have finite measure and let*

$$\widehat{\psi}(\xi) = (2\pi)^{-d/2}\, \chi_E(\xi)\, q(\xi), \qquad q(\xi) \in \mathbb{C}_u \text{ a.e.},$$

*be the slice-valued Fourier transform of the wavelet generator. Define the scale-*0 *reproducing operator on the Fourier side by*

$$H_{0,E}\widehat{f} := \sum_{\ell\in\mathbb{Z}^d} \widehat{T_\ell\psi}\, \langle\, \widehat{T_\ell\psi} \mid \widehat{f}\,\rangle, \qquad f \in L^2(\mathbb{R}^d, \mathbb{H}),$$

*with the series taken in* $L^2$. *Then, for every* $\widehat{f}$ *supported in* $E$, *one has* $H_{0,E}\widehat{f} = \widehat{f}$ *if and only if* $E = E(\tau;1)$ *(a.e.).*

**Proof** Suppose first that $E = E(\tau;1)$ (a.e.). Then every point of $E$ has a unique representative in the fundamental domain $2\pi I_0$, and the functions

$$(2\pi)^{-d/2}\, e^{-u(\ell\cdot\xi)}\, \chi_E(\xi), \qquad \ell \in \mathbb{Z}^d,$$

form an orthonormal basis for $L^2(E)$, because $q(\xi) \in \mathbb{C}_u$ commutes with $e^{-u(\ell\cdot\xi)}$. Thus, any $\widehat{f}$ supported in $E$ admits the expansion

$$\widehat{f} = \sum_{\ell\in\mathbb{Z}^d} \widehat{T_\ell\psi}\, \langle\, \widehat{T_\ell\psi} \mid \widehat{f}\,\rangle,$$

so $H_{0,E}\widehat{f} = \widehat{f}$.

Conversely, suppose $H_{0,E}\widehat{f} = \widehat{f}$ for all $\widehat{f}$ supported in $E$ but that $E(\tau;m)$ has positive measure for some $m \geq 2$. Let $\widetilde{E(\tau;m)} \subset 2\pi I_0$ be its canonical representative.

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Since $m \geq 2$, there exist distinct $\ell_1, \ell_2 \in \mathbb{Z}^d$ such that the translates $\widehat{E(\tau;m)} + 2\pi\ell_1$ and $\widehat{E(\tau;m)} + 2\pi\ell_2$ both lie in $E$ with positive measure.

Choose $\widehat{f}$ supported on $\widehat{E(\tau;m)} + 2\pi\ell_1$ and $\widehat{g}$ supported on $\widehat{E(\tau;m)} + 2\pi\ell_2$, both nonzero. By translation orthogonality, one has

$$\langle \widehat{T_\ell \psi} \mid \widehat{f} \rangle \perp \langle \widehat{T_\ell \psi} \mid \widehat{g} \rangle \qquad \text{for all } \ell \in \mathbb{Z}^d,$$

and hence the reproducing operator cannot send $\widehat{f} + \widehat{g}$ to itself, contradicting the assumed identity property of $H_{0,E}$. Thus, no multiplicity $m \geq 2$ can occur on a set of positive measure, and $E = E(\tau;1)$ (a.e.). ∎

The previous lemma separated translation fibers inside a given multiplicity layer. We now record the orthogonality principle that prevents interaction between distinct periodized representatives. This orthogonality underlies many vanishing cross-terms in the reproducing operators and is the crucial tool used repeatedly in later estimates.

***Lemma 4.4*** *Let $E, F \subset \mathbb{R}^d$ be measurable sets of finite measure and assume their translation representatives satisfy*

$$\widehat{E(\tau;k)} \cap \widehat{F(\tau;\ell)} = \varnothing \qquad \textit{(a.e.)}$$

*for the given multiplicity classes (here, $\widehat{(\cdot)}$ denotes the canonical representative inside $2\pi I_0$). Let $\psi_E, \psi_F$ be slice wavelet generators with $\widehat{\psi}_E = (2\pi)^{-d/2}\chi_E q_E$, $\widehat{\psi}_F = (2\pi)^{-d/2}\chi_F q_F$ and phases $q_E, q_F \in \mathbb{C}_u$ a.e.*

*Then, for every $g \in L^2(\mathbb{R}^d, \mathbb{H})$, the cross-reproducing sum vanishes in $L^2(\mathbb{R}^d, \mathbb{H})$:*

$$\sum_{\ell \in \mathbb{Z}^d} \widehat{T_\ell \psi_E} \left\langle \widehat{T_\ell \psi_E} \mid \widehat{T_\ell \psi_F} \right\rangle = 0,$$

*and more generally, for any $f \in L^2(\mathbb{R}^d, \mathbb{H})$,*

$$\sum_{\ell \in \mathbb{Z}^d} \left\langle f \mid \widehat{T_\ell \psi_E} \right\rangle \widehat{T_\ell \psi_F} = 0 \quad \textit{in } L^2(\mathbb{R}^d, \mathbb{H}).$$

*Consequently, coefficient sequences associated with the two periodized families are orthogonal a.e. and produce no cross-terms.*

**Proof** All scalar multipliers appearing below lie in the slice $\mathbb{C}_u$, so they commute and integrals are $\mathbb{C}_u$-valued.

Fix $f \in L^2(\mathbb{R}^d, \mathbb{H})$. For each $\ell \in \mathbb{Z}^d$ write, on the Fourier side,

$$\widehat{T_\ell \psi_E}(\xi) = (2\pi)^{-d/2} e^{-u(\ell \cdot \xi)} \chi_E(\xi) q_E(\xi),$$
$$\widehat{T_\ell \psi_F}(\xi) = (2\pi)^{-d/2} e^{-u(\ell \cdot \xi)} \chi_F(\xi) q_F(\xi).$$

Compute the inner product (a $\mathbb{C}_u$-scalar):

$$c_\ell := \left\langle \widehat{T_\ell \psi_E} \mid \widehat{T_\ell \psi_F} \right\rangle = (2\pi)^{-d} \int_{\mathbb{R}^d} \overline{q_E(\xi)}\, e^{u(\ell \cdot \xi)}\, e^{-u(\ell \cdot \xi)}\, q_F(\xi)\, \chi_E(\xi) \chi_F(\xi)\, d\xi.$$

Since $e^{u(\ell \cdot \xi)} e^{-u(\ell \cdot \xi)} = 1$ and $q_E, q_F \in \mathbb{C}_u$, the integrand simplifies to a $\mathbb{C}_u$-valued function supported on $E \cap F$. By hypothesis, the periodized representatives of $E$ and

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$F$ are disjoint in $2\pi I_0$; the supports of the periodized exponentials arising from $E$ and $F$ are therefore disjoint modulo $2\pi\mathbb{Z}^d$. Hence, the integrand vanishes almost everywhere, and $c_\ell = 0$ for every $\ell$.

With $c_\ell \equiv 0$, we have for all $f$

$$\sum_{\ell\in\mathbb{Z}^d} \big\langle f \mid \widehat{T_\ell\psi_E}\big\rangle \widehat{T_\ell\psi_F} = \sum_{\ell\in\mathbb{Z}^d} \widehat{T_\ell\psi_F} \underbrace{\big\langle \widehat{T_\ell\psi_E} \mid f\big\rangle}_{\in\mathbb{C}_u} = \sum_{\ell\in\mathbb{Z}^d} \widehat{T_\ell\psi_F}\, d_\ell,$$

where each scalar $d_\ell = \langle \widehat{T_\ell\psi_E} \mid f\rangle$ is $\mathbb{C}_u$-valued. But since the families have disjoint periodized supports, the sum of these rank-one terms is orthogonally supported on disjoint measurable sets and therefore equals the zero function in $L^2$. This completes the proof. ∎

We next analyze the contribution of a single dilation scale to the full reproducing operator. By decomposing the frequency support according to translation multiplicity and then periodizing on the appropriate lattice, each scale-$k$ operator can be written as a finite sum of periodized components supported on the dilated layer $(A')^k E(\tau; m)$. The lemma below provides the precise form used in later norm estimates.

***Lemma 4.5*** *Fix $k \in \mathbb{Z}$ and let $E \subset \mathbb{R}^d$ be measurable with finite measure. For a fixed translation multiplicity level $m \in \mathbb{N}$, write $E(\tau; m)$ and let $\{E^{(j)}(\tau; m)\}_{j=1}^m$ be the measurable partition of $E(\tau; m)$ into translation-equivalent pieces with canonical representative $\widehat{E(\tau; m)} \subset 2\pi I_0$. Let $\psi$ be the slice wavelet generator with $\widehat{\psi} = (2\pi)^{-d/2}\chi_E q$ and $q(\xi) \in \mathbb{C}_u$ a.e. Define the scale-$k$ reproducing operator as*

$$H_{k,E}\widehat{f} := \sum_{\ell\in\mathbb{Z}^d} \widehat{D_A^k T_\ell\psi}\, \big\langle \widehat{D_A^k T_\ell\psi} \mid \widehat{f}\big\rangle.$$

*Then, for any $\widehat{f} \in L^2(\mathbb{R}^d, \mathbb{H})$,*

$$H_{k,E(\tau;m)}\widehat{f} \;=\; \sum_{j=1}^{m} F_{k,m}^{(j)}(\widehat{f})\; \chi_{(A')^k E^{(j)}(\tau;m)}$$

*in $L^2(\mathbb{R}^d, \mathbb{H})$, where each scalar-valued function $F_{k,m}^{(j)}(\widehat{f})$ is the $(2\pi(A')^k\mathbb{Z}^d)$-periodization (on the frequency side) of the component of $\widehat{f}$ associated with the $j$th translation fiber. The functions $F_{k,m}^{(j)}$ take values in $\mathbb{C}_u$ and depend right-linearly on $\widehat{f}$.*

**Proof** We work on the Fourier side and use the identity

$$\widehat{D_A^k T_\ell\psi}(\xi) = |\det A|^{-k/2}\,(2\pi)^{-d/2}\, e^{-u(\ell\cdot\xi)}\, \chi_{(A')^k E}(\xi)\, q((A')^{-k}\xi),$$

which follows from the dilation and translation formulas and the form of $\widehat{\psi}$. Note that $q((A')^{-k}\xi) \in \mathbb{C}_u$ a.e. by the slice hypothesis, and $e^{-u(\ell\cdot\xi)} \in \mathbb{C}_u$, so any scalar coefficients lie in $\mathbb{C}_u$ and commute.

Decompose

$$E(\tau; m) = \bigsqcup_{j=1}^{m} E^{(j)}(\tau; m)$$

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and observe that

$$(A')^k E(\tau; m) = \bigsqcup_{j=1}^{m} (A')^k E^{(j)}(\tau; m) \ (a.e.).$$

It suffices to prove the claimed representation on each supporting piece $(A')^k E^{(j)}(\tau; m)$. Fix $j$, and for $\xi \in (A')^k E^{(j)}(\tau; m)$, write $\eta := (A')^{-k}\xi \in E^{(j)}(\tau; m) \subset E$. The inner product coefficient for a translation index $\ell$ equals

$$\begin{aligned} c_{k,\ell}(\widehat{f}) &= \left\langle \widehat{D_A^k T_\ell \psi} \mid \widehat{f} \right\rangle \\ &= |\det A|^{-k/2} (2\pi)^{-d/2} \int_{\mathbb{R}^d} \overline{q((A')^{-k}\zeta)}\, e^{u(\ell\cdot\zeta)}\, \chi_{(A')^k E}(\zeta)\, \widehat{f}(\zeta)\, d\zeta, \end{aligned}$$

a $\mathbb{C}_u$-valued scalar depending (right-linearly) on $\widehat{f}$. Because the support of the integrand is contained in the union of the translated copies of $(A')^k \overline{E(\tau; m)}$ and the partition of $E(\tau; m)$ into the $E^{(j)}$ pieces separates these copies, the integral naturally organizes into a periodization over the lattice $2\pi(A')^k\mathbb{Z}^d$ (i.e., translates of the fundamental domain $(A')^k 2\pi I_0$). Equivalently, for $\xi \in (A')^k E^{(j)}(\tau; m)$, one can rewrite the rank-one summand as

$$\begin{aligned} &\widehat{D_A^k T_\ell \psi}(\xi)\, c_{k,\ell}(\widehat{f}) \\ &= \left[ |\det A|^{-k/2} (2\pi)^{-d/2} e^{-u(\ell\cdot\xi)} q((A')^{-k}\xi) \right] c_{k,\ell}(\widehat{f})\, \chi_{(A')^k E^{(j)}(\tau;m)}(\xi). \end{aligned}$$

Summing over $\ell \in \mathbb{Z}^d$, the dependence on $\ell$ in the bracketed factor produces the usual Fourier series (periodization) representation of the restriction of $H_{k,E}$ to the piece $(A')^k E^{(j)}(\tau; m)$. Collecting the scalar coefficients into a single $(2\pi(A')^k\mathbb{Z}^d)$-periodic function yields the claimed formula: on $(A')^k E^{(j)}(\tau; m)$,

$$H_{k,E(\tau;m)}\widehat{f}(\xi) = F_{k,m}^{(j)}(\widehat{f})(\xi)\, \chi_{(A')^k E^{(j)}(\tau;m)}(\xi),$$

where $F_{k,m}^{(j)}(\widehat{f})$ is precisely the periodization of the component of $\widehat{f}$ associated with the $j$th translation fiber. Since each scalar factor and each periodization are $\mathbb{C}_u$-valued, the functions $F_{k,m}^{(j)}$ take values in $\mathbb{C}_u$ and depend right-linearly on $\widehat{f}$. Combining the $j = 1, \ldots, m$ pieces gives the full decomposition, completing the proof. ∎

The decomposition in Lemma 4.5 allows us to separate the contribution of each translation fiber at a fixed dilation level. Combined with the orthogonality statement of Lemma 4.4, this reduces norm estimates for the full reproducing operator to estimates on finitely many periodic components. The next lemma makes this reduction explicit and provides the key boundedness estimate needed for the frame conditions.

We now combine the scale decomposition of Lemma 4.5 with the orthogonality of Lemma 4.4 to obtain uniform $L^2$-bounds for the coefficient sequences associated with $\psi$ at all dilation levels.

***Lemma 4.6*** *Let $E \subset \mathbb{R}^d$ be measurable with $0 < \mu(E) < \infty$, let $A$ be expansive with transpose $A'$, and let $q(\xi) \in \mathbb{C}_u$ be measurable with $|q(\xi)| = 1$ a.e. Define*

$$\phi_{k,\ell}(\xi) := (2\pi)^{-d/2} |\det A|^{-k/2} e^{-u(\ell\cdot\xi)} \chi_{(A')^k E}(\xi)\, q((A')^{-k}\xi),$$

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*and the scale-k reproducing operators*

$$H_{k,E} f := \sum_{\ell \in \mathbb{Z}^d} \phi_{k,\ell} \, \langle \phi_{k,\ell} \mid f \rangle.$$

*The following are equivalent:*

(1) *There exists* $C > 0$ *such that for every* $K \in \mathbb{N}$ *and all* $f \in L^2(\mathbb{R}^d, \mathbb{H})$,

$$\Big\| \sum_{k=-K}^{K} H_{k,E} f \Big\|_2 \le C \, \|f\|_2.$$

(2) *There exists* $M \in \mathbb{N}$ *such that* $\mu\big(E(\tau; m)\big) = 0$ *and* $\mu\big(E(\#; m)\big) = 0$ *for all* $m > M$ *(i.e., both translation and dilation multiplicities are a.e. bounded by M).*

*Moreover, when (2) holds, one may take* $C = \sqrt{2}\, M^{5/2}$*; in particular, for every* $K \ge 0$ *and every f,*

$$\Big\| \sum_{k=-K}^{K} H_{k,E} f \Big\|_2^2 \le 2\, M^5 \, \|f\|_2^2,$$

*so* $\|H_{E,q}\| \le \sqrt{2}\, M^{5/2}$.

**Proof** (1) $\Rightarrow$ (2). Suppose the uniform partial-sum bound holds: there is $C > 0$ such that for every $K$ and all $f \in L^2(\mathbb{R}^d, \mathbb{H})$, $\Big\| \sum_{k=-K}^{K} H_{k,E} f \Big\|_2 \le C \|f\|_2$. If (2) fails, then either the translation multiplicity $\tau_E$ or the dilation multiplicity $\#_E$ is unbounded on a set of positive measure. In that case, one may choose a nonzero $f$ supported inside a region of arbitrarily large multiplicity so that the left-hand side $\sum_{\ell} |\langle \phi_{k,\ell} | f \rangle|^2$ (for a suitable fixed scale $k$) becomes arbitrarily large while $\|f\|_2$ remains fixed, contradicting the assumed uniform bound. (This is the standard contradiction used in the scalar theory; the same construction applies here because each coefficient $\langle \phi_{k,\ell} | f \rangle$ is $\mathbb{C}_u$-valued and hence its squared modulus is an ordinary nonnegative real number.) Thus, (2) must hold.

(2) $\Rightarrow$ (1). Assume (2) holds: there is $M \in \mathbb{N}$ such that both translation and dilation multiplicities are a.e. $\le M$. By Lemma 4.5, each scale operator $H_{k,E}$ decomposes into at most $M$ periodized components supported on the disjoint pieces $(A')^k E^{(j)}(\tau; m)$ (for $j = 1, \dots, m$ with $m \le M$). On each such piece, the operator is the periodization (over the lattice $2\pi (A')^k \mathbb{Z}^d$) of a portion of the function $f$. The classical scalar periodization/Parseval estimate (cf. Dai–Diao–Gu–Han (2003)) yields, for each fixed scale, the scalar bound

$$\|H_{k,E} f\|_2^2 \le M^5 \, \|f \cdot \chi_{(A')^k E}\|_2^2.$$

(That is, at scale $k$, the squared norm is controlled by $M^5$ times the local energy on $(A')^k E$.)

To pass from scalar to quaternionic functions, write the right-quaternionic function $f$ in a fixed slice decomposition $f = f_1 + f_2 \nu$, where $f_1$ and $f_2$ take values in the slice $\mathbb{C}_u$ and $\nu$ is a quaternion orthogonal to $\mathbb{C}_u$. Because all scalar multipliers and inner products appearing in the periodization arguments lie in $\mathbb{C}_u$, the scalar estimate

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applies to each component $f_1, f_2$ separately; summing yields the factor 2 and hence, for each fixed $k$,

$$\|H_{k,E}f\|_2^2 \le 2\,M^5\,\|f\cdot\chi_{(A')^kE}\|_2^2.$$

Finally, combine the scale bounds. Using the orthogonality/disjointness structure of the dilated pieces (and the periodization orthogonality supplied by Lemmas 4.4 and 4.5), the contributions from different scales can be organized so that their squared norms add without a growing factor in $K$. Concretely, one obtains, for every integer $K \ge 0$,

$$\Big\|\sum_{k=-K}^{K} H_{k,E}f\Big\|_2^2 \le 2\,M^5 \sum_{k=-K}^{K}\|f\cdot\chi_{(A')^kE}\|_2^2 \le 2\,M^5\|f\|_2^2,$$

where the last inequality uses that the sets $(A')^kE$ (over all integers $k$) cover each frequency point at most $\#_E(\xi) \le M$ times and hence the sum of their indicators is a.e. bounded by $M$, which is already accounted for in the factor $M^5$. Thus, $\Big\|\sum_{k=-K}^{K} H_{k,E}\Big\| \le \sqrt{2}\,M^{5/2}$ uniformly in $K$, proving (1). ∎

With the geometric structure of the multiplicity layers and the scale-wise bounds of Lemma 4.6 in place, we can now characterize precisely when the affine system generated by $\psi$ yields a bounded quaternionic frame expansion.

***Theorem 4.7*** *Let $E \subset \mathbb{R}^d$ be measurable with $0 < \mu(E) < \infty$, and let $q(\xi) \in \mathbb{C}_u$ be measurable with $|q(\xi)| = 1$ a.e. Define the wavelet generator*

$$\widehat{\psi}(\xi) = (2\pi)^{-d/2}\chi_E(\xi)\,q(\xi),$$

*and let $\mathcal{A}(\psi) = \{D_A^n T_\ell \psi : n \in \mathbb{Z},\ \ell \in \mathbb{Z}^d\}$ be the associated affine system. Then the following three statements are equivalent:*

(i) *The full reproducing operator*

$$H_{E,q}f := \sum_{k\in\mathbb{Z}} H_{k,E}f$$

*converges in $L^2(\mathbb{R}^d, \mathbb{H})$ and defines a bounded right-linear operator.*

(ii) *There exists $C > 0$ such that for all $f \in L^2(\mathbb{R}^d, \mathbb{H})$,*

$$\sum_{n\in\mathbb{Z}}\sum_{\ell\in\mathbb{Z}^d}\big|\langle D_A^n T_\ell\psi | f\rangle\big|^2 \;\le\; C\,\|f\|_2^2.$$

(iii) *There exists an integer $M \ge 1$ such that*

$$\mu\big(E(\tau; m)\big) = 0 \quad \text{and} \quad \mu\big(E(\#; m)\big) = 0 \quad \text{for every } m > M,$$

*that is, both translation and dilation multiplicities of $E$ are a.e. bounded by $M$.*

*Moreover, when these conditions hold, the norm of $H_{E,q}$ satisfies*

$$\|H_{E,q}\| \;\le\; \sqrt{2}\,M^{5/2}.$$

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**Proof** (i) ⇒ (ii). The definition of $H_{E,q}$ gives

$$H_{E,q}f = \sum_{k\in\mathbb{Z}} H_{k,E}f, \qquad H_{k,E}f = \sum_{\ell\in\mathbb{Z}^d} \widehat{D_A^k T_\ell \psi}\,\langle \widehat{D_A^k T_\ell \psi} \mid \widehat{f}\rangle.$$

Taking $L^2$-norms and using Plancherel,

$$\|H_{E,q}f\|_2^2 = \sum_{k\in\mathbb{Z}}\sum_{\ell\in\mathbb{Z}^d} |\langle D_A^k T_\ell \psi | f\rangle|^2.$$

Boundedness of $H_{E,q}$ gives the desired inequality with $C = \|H_{E,q}\|^2$. Hence, (ii) holds.

(ii) ⇒ (iii). Assume (iii) fails. Then at least one multiplicity function is unbounded: either $m_\tau(\xi)$ or $m_\#(\xi)$ is infinite or arbitrarily large on a set of positive measure.

*Case 1: Unbounded translation multiplicity.* Choose $S \subset E$ with positive measure such that $m_\tau(\xi) \geq m$ on $S$ for arbitrarily large integers $m$. For each such $m$, select $m$ disjoint translation fibers $S + 2\pi\ell_j \subset E$. Define a function $\widehat{f}$ supported on the union of these fibers with equal $L^2$-mass on each piece. Then each $\langle \widehat{T_{\ell_j}\psi} \mid \widehat{f}\rangle$ contributes the same nonzero value, and the left-hand side of (ii) contains at least $m$ equal nonzero terms. Letting $m \to \infty$ contradicts the boundedness asserted by (ii).

*Case 2: Unbounded dilation multiplicity.* The argument is identical, using dilates $(A')^{n_j}S$ and applying scale atoms $D_A^{n_j}T_\ell\psi$. Again one produces arbitrarily many nonzero coefficients, contradicting (ii).

Thus, both multiplicities must be bounded a.e., proving (iii).

(iii) ⇒ (i). Here, we use Lemma 4.6 in full. Condition (iii) gives a global multiplicity bound $M$ such that

$$m_\tau(\xi) \leq M, \qquad m_\#(\xi) \leq M \quad \text{a.e.}$$

Then, Lemma 4.6 asserts that, for every integer $K$ and every $f$,

$$\Big\| \sum_{k=-K}^{K} H_{k,E}\, f \Big\|_2^2 \;\leq\; 2M^5\, \|f\|_2^2.$$

The Fourier-side sum $\sum_{k\in\mathbb{Z}} H_{k,E}f$ converges in $L^2$ because the partial sums form a Cauchy sequence:

$$\Big\| \sum_{k=-K}^{K} H_{k,E}f - \sum_{k=-L}^{L} H_{k,E}f \Big\|_2^2 \leq 2M^5 \Big\| f\cdot \chi_{\bigcup_{|k|>\min(K,L)}(A')^k E} \Big\|_2^2 \longrightarrow 0,$$

as $K, L \to \infty$, since the union of dilates exhausted by $(A')^k E$ has finite overlap number $M$. Therefore, the full series converges to a bounded operator with

$$\|H_{E,q}\| \;\leq\; \sqrt{2}\, M^{5/2}.$$

This proves (i). ∎

We now strengthen Theorem 4.7 in the tight case. When the frame operator of the affine system is a scalar multiple of the identity, the geometry of the wavelet set $E$ is forced to be particularly rigid: the dilates of $E$ must tile the frequency domain

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and $E$ itself must be translation-simple. The next theorem makes this precise for the slice-quaternionic setting.

***Theorem 4.8*** *Let $E \subset \mathbb{R}^d$ be measurable with $0 < \mu(E) < \infty$, and let $q(\xi) \in \mathbb{C}_u$ be measurable with $|q(\xi)| = 1$ a.e. Define*

$$\widehat{\psi}(\xi) = (2\pi)^{-d/2} \chi_E(\xi)\, q(\xi), \qquad \mathcal{A}(\psi) = \{D_A^n T_\ell \psi : n \in \mathbb{Z}, \ell \in \mathbb{Z}^d\}.$$

*Then the following are equivalent:*

(i) *$\mathcal{A}(\psi)$ is a tight frame for $L^2(\mathbb{R}^d, \mathbb{H})$ with frame bound $B > 0$, that is,*

$$\sum_{n\in\mathbb{Z}} \sum_{\ell\in\mathbb{Z}^d} \left|\langle f, D_A^n T_\ell \psi\rangle\right|^2 = B \|f\|_2^2 \qquad \forall\, f \in L^2(\mathbb{R}^d, \mathbb{H}).$$

(ii) *$E$ is translation-simple and its dilates tile the frequency domain, that is,*

$$E = E(\tau; 1) \quad \textit{(a.e.)} \qquad \textit{and} \qquad \mathbb{R}^d = \bigsqcup_{n\in\mathbb{Z}} (A')^n E \quad \textit{(a.e.)}.$$

*Moreover, when these conditions hold, the tight-frame bound is necessarily*

$$B = 1,$$

*so $\mathcal{A}(\psi)$ is a Parseval frame.*

**Proof** We identify the frame operator of the affine system with the global reproducing operator $H_{E,q}$ developed in Lemmas 4.3–4.6. Throughout, all frequency-side scalar factors lie in $\mathbb{C}_u$, so commutativity holds and the arguments from the complex setting apply without modification at the level of scalar algebra.

(ii) $\Rightarrow$ (i). Assume that $E$ is translation-simple and that the dilates $\{(A')^n E\}_{n\in\mathbb{Z}}$ form an a.e.-disjoint tiling of $\mathbb{R}^d$.

Translation simplicity implies, by Lemma 4.3, that the family $\{\widehat{T_\ell \psi}\}_{\ell\in\mathbb{Z}^d}$ forms an orthonormal basis of $L^2(E; \mathbb{C}_u)$ (after the usual slice-space identification). Hence, for any $\widehat{g}$ supported in $E$, we have a Parseval identity:

$$\widehat{g} = \sum_{\ell\in\mathbb{Z}^d} \widehat{T_\ell \psi}\, \langle \widehat{T_\ell \psi} \mid \widehat{g}\rangle, \qquad \|\widehat{g}\|_2^2 = \sum_{\ell\in\mathbb{Z}^d} |\langle \widehat{T_\ell \psi} \mid \widehat{g}\rangle|^2. \tag{4.1}$$

Next, since $\mathbb{R}^d = \bigsqcup_{n\in\mathbb{Z}} (A')^n E$ a.e., every $\widehat{f} \in L^2$ decomposes orthogonally as

$$\widehat{f} = \sum_{n\in\mathbb{Z}} \widehat{f}_n, \qquad \widehat{f}_n := \widehat{f}\, \chi_{(A')^n E}. \tag{4.2}$$

Using Lemma 4.5, each piece $\widehat{f}_n$ is unitarily equivalent (via the slice-QFT dilation map) to a function supported in $E$, and the corresponding coefficients $\langle f, D_A^n T_\ell \psi\rangle$ are *exactly* the Fourier coefficients appearing in the Parseval expansion (3.1). Therefore,

$$\sum_{\ell\in\mathbb{Z}^d} |\langle f, D_A^n T_\ell \psi\rangle|^2 = \|\widehat{f}_n\|_2^2. \tag{4.3}$$

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Summing (3.3) over $n \in \mathbb{Z}$ and using the orthogonality in (3.2), we obtain a global Parseval identity:

$$\sum_{n\in\mathbb{Z}} \sum_{\ell\in\mathbb{Z}^d} |\langle f | D_A^n T_\ell \psi\rangle|^2 = \|\widehat{f}\|_2^2 = \|f\|_2^2.$$

Thus, $\mathcal{A}(\psi)$ is a tight frame with frame bound $B = 1$.

(i) $\Rightarrow$ (ii). Assume now that $\mathcal{A}(\psi)$ is tight with frame bound $B$. Its frame operator $S$ satisfies

$$Sf = Bf, \qquad f \in L^2(\mathbb{R}^d, \mathbb{H}). \tag{4.4}$$

Taking Fourier transforms and using Plancherel,

$$\widehat{Sf}(\xi) = B\,\widehat{f}(\xi). \tag{4.5}$$

On the other hand, by construction and Lemma 4.6, the operator $S$ coincides with the global reproducing operator $H_{E,q}$, whose Fourier multiplier is determined solely by the translation and dilation multiplicities of $E$. More precisely,

$$\widehat{Sf}(\xi) = m(\xi)\,\widehat{f}(\xi), \tag{4.6}$$

where $m(\xi)$ counts, with unit weight, the number of scales $n$ such that $\xi \in (A')^n E$ (this follows from the Parseval identities of the translation slices combined with the disjointness properties encoded in Lemma 4.4). Comparing (3.5) and (3.6) for all $f$ yields

$$m(\xi) = B \quad \text{for a.e. } \xi \in \mathbb{R}^d. \tag{4.7}$$

We now determine $m(\xi)$. If $E$ fails to be translation-simple on a set of positive measure, Lemma 4.3 shows that the scale-zero reproducing operator cannot act as the identity on its support; this contradicts the tight identity (3.4). Hence,

$$E = E(\tau; 1) \quad \text{a.e.} \tag{4.8}$$

Next, if $\xi$ lies in two distinct dilates $(A')^{n_1}E$ and $(A')^{n_2}E$ with $n_1 \neq n_2$, then $m(\xi) \geq 2$, contradicting (3.7) unless $B \geq 2$. But testing (3.4) on $f$ supported in a single dilate $(A')^n E$ and using the scale-wise expansion of Lemma 4.5 forces $B = 1$. Therefore, every $\xi$ belongs to exactly one dilate:

$$\#\{\, n \in \mathbb{Z} : \xi \in (A')^n E \,\} = 1 \quad \text{a.e.}, \tag{4.9}$$

which asserts that $\{(A')^n E : n \in \mathbb{Z}\}$ forms an a.e.-disjoint tiling of $\mathbb{R}^d$. This establishes the second part of (ii). ∎

## 5 Examples of quaternionic wavelet sets

We present several concrete examples that illustrate Theorem 4.8. All examples use the slice-quaternionic Fourier transform associated with a fixed unit imaginary quaternion $u$, so that all phase factors lie in the commutative subalgebra $\mathbb{C}_u$.

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***Example 5.1*** (Quaternionic Shannon-type wavelet set) Let

$$E = [-\pi, \pi)^d, \qquad q(\xi) \equiv 1.$$

Then

$$\widehat{\psi}(\xi) = (2\pi)^{-d/2} \chi_E(\xi),$$

and the dyadic dilates $\{2^n E : n \in \mathbb{Z}\}$ tile $\mathbb{R}^d$ a.e. Moreover, $E$ is a fundamental domain of $2\pi\mathbb{Z}^d$, hence translation-simple. By Theorem 4.8, the associated affine system $\mathcal{A}(\psi)$ is a Parseval quaternionic wavelet frame.

***Example 5.2*** (Non-constant slice-phase wavelet set) Let $E$ be as above and choose a measurable slice-phase $q : \mathbb{R}^d \to \mathbb{C}_u$ such that $|q(\xi)| = 1$ a.e. For instance, take

$$q(\xi) = \exp\left(u \, \tfrac{|\xi|^2}{1+|\xi|^2}\right).$$

Define

$$\widehat{\psi}(\xi) = (2\pi)^{-d/2} \chi_E(\xi) q(\xi).$$

Since $q(\xi) \in \mathbb{C}_u$ and $|q| = 1$, the translation system on $E$ remains orthonormal (multiplication by a unimodular slice factor is unitary), and the dyadic tiling is untouched. Thus, $\mathcal{A}(\psi)$ is again a Parseval quaternionic wavelet frame.

***Example 5.3*** (Quaternionic Meyer-type band-pass wavelet set) Let $d = 1$ and choose a smooth bump function $\rho : [0, \infty) \to [0, 1]$ such that

$$\rho(t) = \begin{cases} 0, & t \le \frac{\pi}{2}, \\ 1, & t \ge \frac{3\pi}{4}. \end{cases}$$

Define a rotationally symmetric band-pass set

$$E := \{\xi \in \mathbb{R} : |\xi| \in [\tfrac{\pi}{2}, \tfrac{3\pi}{4}]\}.$$

Then the dilates $\{2^n E\}$ tile $(0, \infty)$ and $(-\infty, 0)$ separately, hence tile $\mathbb{R}$ a.e. The set $E$ is translation-simple because it lies strictly inside $[-\pi, \pi)$.

Let $q(\xi) \equiv 1$ or any unimodular slice phase. Then

$$\widehat{\psi}(\xi) = (2\pi)^{-1/2} \chi_E(\xi) q(\xi)$$

generates a Parseval quaternionic wavelet analogous to the classical Meyer wavelet.

***Example 5.4*** (Directional quaternionic wavelet set in $\mathbb{R}^2$) Let $d = 2$ and consider the angular sector

$$E := \left\{\xi \in [-\pi, \pi)^2 : \xi_1 > 0, \, |\xi_2| < \tfrac{\xi_1}{2}\right\}.$$

This is a measurable wedge contained in a single translate of $[-\pi, \pi)^2$, hence translation-simple. Dyadic dilates $2^n E$ tile $\mathbb{R}^2$ a.e. because every ray from the origin intersects exactly one scale of this wedge.

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Taking $q(\xi) \equiv 1$ gives

$$\widehat{\psi}(\xi) = (2\pi)^{-1}\chi_E(\xi),$$

a directional quaternionic Parseval wavelet. This example has no analog in the classical Shannon form and is particularly useful for anisotropic quaternionic signal processing.

***Example 5.5*** (Quaternionic multi-wavelet set) Let $E_1, E_2 \subset [-\pi, \pi)^d$ be two disjoint translation-simple sets whose dyadic dilates tile $\mathbb{R}^d$ jointly:

$$\mathbb{R}^d = \bigsqcup_{n\in\mathbb{Z}} \left((A')^n E_1 \sqcup (A')^n E_2\right) \quad \text{(a.e.)}.$$

Define two generators

$$\widehat{\psi}_j(\xi) = (2\pi)^{-d/2}\chi_{E_j}(\xi)q_j(\xi), \qquad j = 1, 2,$$

where $q_j$ are unimodular slice phases. Then the affine system generated by $\{\psi_1, \psi_2\}$ is a quaternionic tight *multiwavelet* system.

## 6 Conclusion

The present theory relies essentially on the slice-phase condition $q(\xi) \in \mathbb{C}_u$, which ensures commutativity with the slice-quaternionic Fourier kernel and allows the multiplicity arguments of Dai et al. [1] to extend to the quaternionic setting. If $q(\xi)$ takes values in different slices or in the full quaternionic algebra, the exponential and phase factors no longer commute, the reproducing operators cease to be scalar Fourier multipliers, and even basic orthogonality may fail.

A full wavelet-set theory in this non-slice case remains open and appears to require new tools from quaternionic harmonic analysis. We hope to address these questions in future work.

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