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DOI: 10.1017/jsl.2026.10274

THE COMPLEXITY OF CONNECTEDNESS RELATIONS ON POLISH SPACES

MICHAL HEVESSY, YUSUF UYAR, AND BENJAMIN VEJNAR

Abstract. We investigate the Borel complexity of some equivalence relations of connectedness on Polish spaces. In particular, we show that the arc-connection relation on Polish subspaces of the plane is Borel reducible to the Vitali equivalence relation and thus of a very low complexity. We also provide optimal upper bounds for the complexity of the continuum-connection and the chain continuum-connection relations on locally compact Polish spaces.

1. Introduction

Given a metric space X, we consider the equivalence relations

$$E_X^A = \{(x,y) \in X \times X : \text{ there is an arc in } X \text{ containing } x \text{ and } y\}$$
$$E_X^C = \{(x,y) \in X \times X : \text{ there is a continuum in } X \text{ containing } x \text{ and } y\}$$
$$E_X^{CC} = \{(x,y) \in X \times X : \text{ for all } \varepsilon > 0, \text{ there are continua } K_0, ..., K_k \subseteq X, k \in \mathbb{N},$$
$$\text{of diameter } < \varepsilon \text{ such that } x \in K_0, y \in K_k \text{ and } K_i \cap K_{i+1} \neq \emptyset \text{ for } 0 \leq i < k\}$$

on X, which will be called *arc-connection relation*, *continuum-connection relation* and *chain continuum-connection relation* (or simply, chain-connection relation); respectively. For any metric space X, we have

$$E_X^A \subseteq E_X^{CC} \subseteq E_X^C$$

and both inclusions can be made strict at the same time by a simple example. All of these equivalence relations describe a notion of connectedness in a given space.

In this paper, we study the Borel complexity of these equivalence relations from the point of view of Borel reducibility. For this purpose, we restrict ourselves mainly to Polish spaces, i.e. completely metrizable separable topological spaces, although these are well-defined equivalence relations on any metric space. On a Polish space, all of the three equivalence relations can be easily seen to be analytic. The complexity of the arc-connection relation is the most studied one in the literature, and some of these works will be cited throughout the paper when appropriate.

Becker questioned in [1] how complex the arc-connection relation on compact subspaces of the plane could be. Debs and Saint Raymond recently showed that the arc-connection relation on any Polish subspace of the plane is Borel [6], which partially answers the question. Using this result, we prove that such a Borel equivalence relation is no more complex than the Vitali equivalence relation. This provides a conclusive answer to the question. We also show that the same holds for

2020 *Mathematics Subject Classification.* 54H05, 03E15, 54F16, 54E50.
Key words and phrases. Borel reduction, classification, connectedness.

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the chain-connection relation on locally compact subspaces of the plane, whereas the continuum-connection relation can be more complicated on such spaces.

In [22], Solecki studied the composant equivalence relation on indecomposable continua. The continuum-connection relation can be seen as a generalization of this equivalence relation on Polish spaces. It is also worth noting that for some continua such as the Knaster continuum (sometimes referred to as the buckethandle continuum) the composant equivalence relation coincides with the arc-connection relation [14, §48 V-VI].

In view of the properties of Peano continua, the chain-connection relation can be seen as a generalization of the arc-connection relation [19, Chapter VIII]. In fact, we will show that there are some natural assumptions to ensure that the two coincide (Lemmas 5.5 and 6.2).

The main reason why we focus primarily on planar spaces is that these equivalence relations can be very complicated even for Polish subspaces of $\mathbb{R}^3$ as we shall discuss (see also [1],[13] and [15]). A summary of the existing results together with the new ones is shown in Table 1 below.

TABLE 1. A Summary of Results

	Polish	locally compact Polish	$G_\delta \subseteq \mathbb{R}^2$	locally compact $\subseteq \mathbb{R}^2$
E_X^A	universal analytic Example 5.8 [1]	universal analytic Example 5.8 [1]	E_0 Theorem 3.2 Remark 3.3	E_0 Theorem 3.2 Remark 3.3
E_X^C	universal analytic Example 5.8	E_1 Proposition 4.1 Example 4.2	$\geq E_1$ Example 4.2 Questions 1 and 2	E_1 Proposition 4.1 Example 4.2
E_X^{CC}	universal analytic Example 5.8	$(E_{\ell_\infty})^{\mathbb{N}}$ Theorem 5.3 Example 5.4	$\geq E_0$ Remark 3.3 Questions 3 and 4	E_0 Theorem 5.6 Remark 3.3

Most of these complexity results are the optimal upper bounds on the corresponding spaces, in the sense that we construct spaces that attain these upper bounds. The two exceptions are the continuum-connection and the chain-connection relations considered on Polish subspaces of the plane. Although we construct some planar Polish spaces on which the continuum-connection and the chain-connection relations are non-trivial, we do not know if these have to be even Borel in general.

Another approach to study various notions of connectedness from the descriptive viewpoint is to consider the descriptive complexity of subsets satisfying these connectedness notions in some suitable hyperspaces. A great deal has been done concerning this approach previously (see [4], [5] and [16]). It is worth mentioning that the set of arc-connected subsets of the plane in the hyperspace of all compact subsets is of high descriptive complexity as shown in [4], which is in contrast with our result that the arc-connection relation on Polish subspas of the plane is of not complicated.

2. Preliminaries

In this section, we set the stage for studying the complexity of the connectedness relations. Firstly, we introduce basic definitions, examples and facts about the

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theory of Borel equivalence relations relevant to our study. We then present some properties of hyperspaces of Polish spaces. Lastly, we state the needed definitions and results from continuum theory.

2.1. **Borel Equivalence Relations.** We use the notion of Borel reducibility to investigate the complexity of the equivalence relations. All of the facts and theorems presented in this subsection are from [8] unless otherwise stated. We refer the reader to this reference for a detailed discussion of the theory of Borel equivalence relations and invariant descriptive set theory.

A *standard Borel space* is a measurable space which is isomorphic to the Borel σ-algebra of a Polish space, i.e. the σ-algebra generated by the open sets of the topology. We call measurable subsets of a standard Borel space *Borel*. It turns out that any Borel subset of a Polish space is standard Borel. An equivalence relation E on a standard Borel space X is called a *Borel equivalence relation* if E is a Borel subset of the product space $X \times X$. Likewise, a function between standard Borel spaces is called a *Borel function* if the inverse image of a Borel subset is Borel.

Let X, Y be standard Borel spaces and let E, F be equivalence relations on X, Y respectively. We say that E is *Borel reducible* to F, denoted by $E \leq_B F$, if there is a Borel function $f : X \to Y$, called a *Borel reduction*, such that

$$xEy \iff f(x)Ff(y)$$

for all $x, y \in X$. We say that E is *Borel bireducible* with F, denoted by $E \sim_B F$, if $E \leq_B F$ and $F \leq_B E$. We also write $E <_B F$ whenever $E \leq_B F$ and $F \not\leq_B E$.

Borel reductions compare in some sense the complexity of equivalence relations. If we have $E \leq_B F$, this intuitively means that the structure of E is captured in the structure of F, and thus E is no more complex than F. So it is a notion of relative complexity, sometimes referred to as *Borel complexity*, on equivalence relations.

Let X be a standard Borel space. A subset $A \subseteq X$ is said to be *analytic* if $A = \pi_1(B)$ for a Borel subset $B \subseteq X \times Y$ for some standard Borel space Y. For example, the connectedness equivalence relations we introduced are analytic since they can be obtained by projections over standard Borel hyperspaces of all arcs or continua, as their definitions suggest. It is clear that any Borel subset is analytic.

Our goal in general is to obtain Borel reductions between the connectedness equivalence relations and some well-understood Borel equivalence relations. Hence, we need to recall some examples of Borel equivalence relations. We start with the simplest class of equivalence relations in terms of Borel complexity.

A Borel equivalence relation on a standard Borel space is said to be *smooth* if it is Borel reducible to an identity equivalence relation on a standard Borel space. There are several characterizations of smooth Borel equivalence relations. Although it can be generalized in several ways, the following condition on Polish spaces for being smooth is sufficient for our purposes.

Theorem 2.1 ([8, Theorem 6.4.4])**.** *Let X be a Polish space and E be a Borel equivalence relation on X such that every equivalence class of E is closed in X. Then E is smooth.*

We will mainly use the following Borel equivalence relations, which follow smooth Borel equivalence relations in complexity.

- The equivalence relation E_0 is defined as

$$xE_0y \iff \exists m\, \forall n \geq m\; x(n) = y(n)$$

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for all $x, y \in 2^{\mathbb{N}}$.

- The equivalence relation E_1 is defined as

$$xE_1y \iff \exists m\ \forall n \geq m\ \forall k\ x(n,k) = y(n,k)$$

for all $x, y \in 2^{\mathbb{N}\times\mathbb{N}}$.

- The equivalence relation E_{ℓ_∞} is defined as

$$xE_{\ell_\infty}y \iff x - y \in \ell_\infty$$

for all $x, y \in \mathbb{R}^{\mathbb{N}}$.

These equivalence relations can be easily shown to be Borel. Moreover, they are of increasing complexity.

Theorem 2.2 ([8, Sections 8.1 and 15.2])**.** $E_0 <_B E_1 <_B E_{\ell_\infty}$

The equivalence relation E_0 is known to be Borel bireducible with the Vitali equivalence relation on $\mathbb{R}$, which is why it is sometimes called the *Vitali equivalence relation*. It is one of the simplest possible Borel equivalence relations in the following sense. Any Borel equivalence relation E on a Polish space is either smooth or satisfies $E_0 \leq_B E$ [9, Theorem 1.1].

An equivalence relation F in a class $\mathcal{C}$ of equivalence relations is called a *universal* $\mathcal{C}$ equivalence relation if $E \leq_B F$ for any $E \in \mathcal{C}$, i.e. if F is the most complicated equivalence relation in the class $\mathcal{C}$.

Remark 2.3. The significance of the equivalence relations defined above is in part due to the following universality properties that they satisfy.

- E_0 is a universal hyperfinite equivalence relation, where a Borel equivalence relation is said to be *hyperfinite* if it can be written as a countable increasing union of Borel equivalence relations with all equivalence classes finite.
- E_1 is a universal hypersmooth equivalence relation, where a Borel equivalence relation is said to be *hypersmooth* if it can be written as a countable increasing union of smooth Borel equivalence relations.
- E_{ℓ_∞} is a universal K_σ equivalence relation on Polish spaces.

We recall that the countable product of equivalence relations on the product of standard Borel spaces is defined as being component-wise equivalent. We will essentially use the infinite product $(E_{\ell_\infty})^{\mathbb{N}}$ together with E_{ℓ_∞} itself. Indeed, Gao showed that $E_{\ell_\infty} <_B (E_{\ell_\infty})^{\mathbb{N}}$ [7, Lemma 3.6 and the discussion after]. In light of Remark 2.3, any countable product of K_σ equivalence relations is no more complex than $(E_{\ell_\infty})^{\mathbb{N}}$.

For the convenience of the reader, we also include in Fig. 1 the Borel reducibility order among some equivalence relations that are used in this paper or that arise from them by natural operations. A comprehensive treatment of this reducibility order is given in Section 15.2 of our reference [8]. Although we will not go into the details, the symbol E^+ denotes the Friedman-Stanley jump of a Borel equivalence relation E, which yields a strictly more complex relation than E.

A Borel equivalence relation is said to be *countable* if all equivalence classes of the relation have at most countably many elements. We note that any equivalence relation satisfying the hypothesis of the following proposition is necessarily countable.

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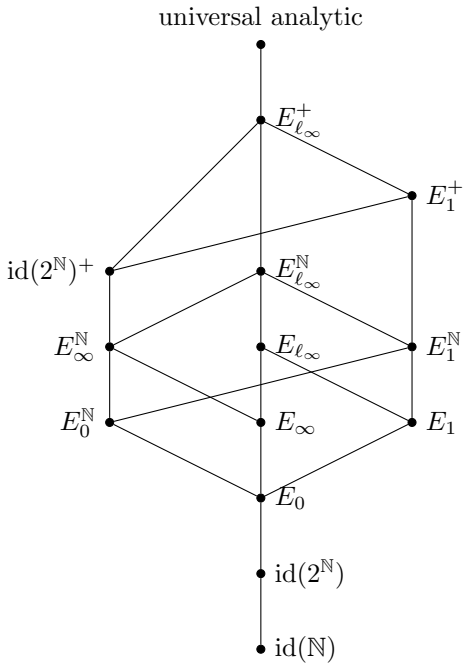


FIGURE 1. Borel Reducibility Order

Proposition 2.4 ([12, Remark 6.8])**.** *Let X be a standard Borel space and E be a Borel equivalence relation on X. Suppose that there exists a Borel graph of degree at most 2 on X whose connected components are the E-classes. Then $E \leq_B E_0$.*

The following result connects an intersection of equivalence relations to a product of equivalence relations. Since it can be easily shown by standard methods, the proof is omitted.

Lemma 2.5. *Let X, Y be standard Borel spaces and let E_n, F_n be Borel equivalence relations on X and Y for all $n \in \mathbb{N}$; respectively. Suppose that $E_n \leq_B F_n$ for all $n \in \mathbb{N}$. Then $\bigcap_{n\in\mathbb{N}} E_n \leq_B \prod_{n\in\mathbb{N}} F_n$.*

The following lemma helps us to consider different Borel parts of a hyperfinite Borel equivalence relation separately.

Lemma 2.6. *Let X be a standard Borel space and E be a Borel equivalence relation on X. Let $X = \bigcup_{n\in\mathbb{N}} X_n$ where each X_n is an E-invariant Borel subset of X. If $E|_{X_n} \leq_B E_0$ for all $n \in \mathbb{N}$, then $E \leq_B E_0$.*

Proof. Without loss of generality, we may assume that the X_n's are pairwise disjoint since otherwise we could replace X_n by $X_n \setminus \bigcup_{m<n} X_m$ for all $n \geq 1$. Then the lemma follows from the fact that the Borel equivalence relation $\sim$ on the space $\mathbb{N} \times 2^{\mathbb{N}}$ defined by

$$(m, x) \sim (n, y) \iff m = n \text{ and } xE_0y$$

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is Borel reducible to E_0. □

2.2. **Hyperspaces and Borel Structure.** In this subsection, we present earlier results on the hyperspaces of Polish spaces and the properties of Borel structure on these hyperspaces that will be used throughout the paper. We refer the reader to [11] for general descriptive properties of hyperspaces.

Let X be a Polish space. We denote the hyperspace of non-empty compact subsets of X and the hyperspace of continua in X by $\mathcal{K}(X)$ and $\mathcal{C}(X)$; respectively. We endow $\mathcal{K}(X)$ with the Vietoris topology, which can be induced by the Hausdorff metric on it. Recall that this makes $\mathcal{K}(X)$ a Polish space and that the space $\mathcal{C}(X)$ is a closed subset of $\mathcal{K}(X)$. Moreover, $\mathcal{K}(X)$ is a compact space whenever X is compact.

An *arc* is any topological space homeomorphic to $[0,1]$, and a *circle* is any topological space homeomorphic to S^1. We denote the set of all arcs and the set of all circles in X by $\mathcal{J}(X)$ and $\mathcal{S}(X)$; respectively. It turns out that both of these subsets are Borel. Moreover, there exists a Borel function $e : \mathcal{J}(X) \to \mathcal{K}(X)$ that assigns to every arc the set of its endpoints. This can be further improved by splitting the function e into two functions. That is, there are two Borel functions $e_1, e_2 : \mathcal{J}(X) \to X$ such that for every $I \in \mathcal{J}(X)$ we have $e(I) = \{e_1(I), e_2(I)\}$. We refer the reader to [2] and [6] for the proofs and details of these facts.

As mentioned earlier, an important result which sparked this investigation is that the arc-connection relation on a G_δ subspace of the plane is Borel [6], where we recall that G_δ subsets of a Polish space are exactly Polish subspaces of it.

Theorem 2.7 ([6, Theorem 1])**.** *E_X^A is Borel for any G_δ subspace $X \subseteq \mathbb{R}^2$.*

One of the most frequently used results throughout the paper is the Lusin-Novikov uniformization theorem.

Theorem 2.8 ([11, Theorem 18.10])**.** *Let X and Y be standard Borel spaces, and let $P \subseteq X \times Y$ be Borel. If every section $P_x = \{y \in Y : (x, y) \in P\}$ of P is countable, then P admits a Borel uniformization and therefore the projection $\pi_1(P)$ is Borel.*

2.3. **Continuum Theory and Triods.** Throughout the paper, we will extensively work with continua (compact connected metric spaces) for which we need several results from continuum theory. We refer the reader to [19] for the continuum-theoretic notions used in this paper.

Triod is a crucial notion that will come up very often in the study of all of the connectedness equivalence relations. We recall the definitions of several types of triods and briefly discuss the relations between them.

A continuum K is called a

- *weak triod* if it is a union of three intersecting subcontinua, none of which is contained in the union of the other two;
- *triod* if there is a subcontinuum $L \subseteq K$ such that $K \setminus L$ has at least three components;
- *simple triod* if it is a union of three arcs intersecting at a single point.

These notions of triods are listed in the increasing order of strength. Yet, it turns out that any weak triod contains a triod [23, Theorem 1.8]. The main reason why triods are of such importance for planar subsets is the following result of Moore.

Theorem 2.9 ([18, Theorem 1])**.** *Any collection of pairwise disjoint triods in the plane is countable.*

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This result is also the main reason why there are many essential differences between the behaviour of the connectedness equivalence relations in the plane and in spaces of higher dimensions. Recall that a continuum is said to be *atriodic* if it does not contain a triod. We will provide a sufficient condition for a continuum to contain a triod. To achieve this, we need the following results.

Lemma 2.10 ([20, Lemmas 3.3 and 3.5])**.** *Let K be an atriodic continuum, and let $A, B \in \mathcal{C}(K)$. Then*

- *both $B \setminus A$ and $A \cap B$ have at most two components;*
- *if $A \cap B$ is not connected, then $B \setminus A$ is connected.*

Theorem 2.11 ([19, Theorem 5.6] The Boundary Bumping Theorem)**.** *Let K be a continuum, and let $E \subseteq K$ be any non-empty proper subset. Then the closure of any component of E intersects ∂E.*

A continuum K is said to be *irreducible between* two points $x, y \in K$, if there is no proper subcontinuum of K containing x and y. Now, we are ready to give the sufficient condition.

Proposition 2.12. *Let K be a continuum. Assume that there are points $x, y \in K$ and three pairwise distinct continua $K_1, K_2, K_3 \subseteq K$ such that*

- *$K = K_1 \cup K_2 \cup K_3$, and*
- *K_i is irreducible between x and y for all $i \in \{1, 2, 3\}$.*

Then K contains a triod.

Proof. Assume towards a contradiction that K is atriodic. To arrive at a contradiction, it is enough to show that K contains a weak triod. Set $X = K_2 \setminus K_3$ and $Y = K_3 \setminus K_2$. Then X and Y are non-empty by the irreducibility assumption. We consider two distinct cases.

First, suppose that there are some points $a \in X \setminus K_1$ and $b \in Y \setminus K_1$. Then we can find open neighborhoods U_a of a and U_b of b in K such that $\overline{U_a} \subseteq K \setminus (K_1 \cup K_3)$ and $\overline{U_b} \subseteq K \setminus (K_1 \cup K_2)$. Let C_x^a and C_x^b be the components of x in $\overline{(K_2 \setminus \overline{U_a})}$ and $\overline{(K_3 \setminus \overline{U_b})}$, respectively. Then we have $C_x^a \cap \partial\overline{U_a} \neq \emptyset$ and $C_x^b \cap \partial\overline{U_b} \neq \emptyset$ by Theorem 2.11. Moreover, $K_1 \cap \partial\overline{U_a} = C_x^b \cap \partial\overline{U_a} = \emptyset$ and $K_1 \cap \partial\overline{U_b} = C_x^a \cap \partial\overline{U_b} = \emptyset$. We also have $y \notin C_x^a \cup C_x^b$ by the irreducibility assumption. But since x is in the intersection $K_1 \cap C_x^a \cap C_x^b$, we conclude that $K_1 \cup C_x^a \cup C_x^b$ is a weak triod.

Second, we assume without loss of generality that $X \subseteq K_1$. We observe that for any $1 \leq i < j \leq 3$, the intersection $K_i \cap K_j$ is disconnected by the irreducibility assumption and so has exactly two components by Lemma 2.10. Moreover, x and y cannot be contained in the same component of $K_i \cap K_j$ again by the irreducibility assumption. So we may let

A_1 and A_2 be the two components of $K_2 \cap K_3$ with $x \in A_1$ and $y \in A_2$;

B_1 and B_2 be the two components of $K_1 \cap K_2$ with $x \in B_1$ and $y \in B_2$;

C_1 and C_2 be the two components of $K_1 \cap K_3$ with $x \in C_1$ and $y \in C_2$.

We note that X is connected by Lemma 2.10. Since $K_2 = X \cup A_1 \cup A_2$ is connected and $A_1 \cap A_2 = \emptyset$, we have $A_1 \cap \overline{X} \neq \emptyset$, which implies that $X \cup A_1$ is connected. Since X is connected, we can without loss of generality assume that $X \subseteq B_2$. Since $X \subseteq B_2$ and $X \cup A_1$ is connected, we get that $A_1 \cup B_2$ is connected. Since A_1 and B_2 are closed in K, we get $A_1 \cap B_2 \neq \emptyset$. Let $c \in A_1 \cap B_2$. Then $c \in C_1$ or $c \in C_2$.

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Assume that $c \in C_1$. We will show that $A_1 \cup B_2 \cup C_1$ is a weak triod. If we had $C_1 \subseteq A_1 \cup B_2$, then $C_1 \cup B_2$ would be a subcontinuum of $K_1 \cap K_2$ containing x and y, which cannot be the case. Similarly, $A_1 \subseteq B_2 \cup C_1$ would imply that $A_1 \cup B_2$ is a subcontinuum of $K_1 \cap K_2$ containing x and y, which cannot be the case. We can argue that $B_2 \nsubseteq C_1 \cup A_1$ in the same way. Thus $A_1 \cup B_2 \cup C_1$ is a weak triod. We can use the same arguments to get that $A_1 \cup B_2 \cup C_2$ is a weak triod if $c \in C_2$. □

We recall that if K is a continuum and $x, y \in K$, then K contains a subcontinuum that is irreducible between x and y [19, Exercise 4.35]. This fact will be used a few times throughout the paper.

3. Arc-connection Relation

In this section, we explore the complexity of the arc-connection relation on Polish subspaces of the plane. We shall refer to an arc component of a metrizable space consisting of a singleton as a *trivial arc component*. It is well-known that any non-trivial arc component of a separable metrizable space either contains a triod, in which case we call it a *triodic arc component*, or it is an injective continuous image of an interval (for example, see [2]).

We will show that, in some sense, the complexity of the arc-connection relation on a planar Polish space is determined by the complexity of its non-triodic non-trivial arc components, whereas it is determined by the complexity of its triodic arc components on a general Polish space.

The main result of this section is that the arc-connection relation is Borel reducible to E_0 on a G_δ subspace of the plane. We start with a more general theorem, which will imply the main result.

Theorem 3.1. *Let X be a Polish space, and let*

$$X_T = \{x \in X : \text{ the arc component of } x \text{ is triodic}\}.$$

Suppose that $Y = X \setminus X_T$ is Borel. Then $E_Y^A \leq_B E_0$.

Proof. We may assume that $X \subseteq \mathbb{R}^{\mathbb{N}}$. We would first like to note that E_Y^A can be easily seen to be Borel by Theorem 2.8 for if $(x, y) \in E_Y^A$, then there can be at most two arcs in Y whose endpoints are x and y for any points $x, y \in Y$.

Since Y does not contain any simple triod, any arc component of Y that contains a circle is itself a circle. So we can partition the subspace Y into E_Y^A-invariant subsets as

$$\begin{aligned} Y_1 &= \{x \in Y : \text{ the arc component of } x \text{ is trivial}\} \\ Y_2 &= \{x \in Y : \text{ the arc component of } x \text{ is a circle}\} \\ Y_3 &= Y \setminus (Y_1 \cup Y_2). \end{aligned}$$

We know that Y_1 and $Y_2 \cup Y_3$ are Borel by [6, Proposition 10]. The subset

$$H = \{(x, S) \in Y \times \mathcal{S}(X) : x \in S\}$$

of $X \times \mathcal{K}(X)$ is Borel as the set $\mathcal{S}(X)$ of all circles in X is Borel as we discussed in Subsection 2.2. But since all sections of the set H are singletons, $Y_2 = \pi_1(H)$ is Borel by Theorem 2.8. The relations $E_{Y_1}^A$ and $E_{Y_2}^A$ are smooth, and hence Borel reducible to E_0, where the set H is a witness for the latter. We will show that $E_{Y_3}^A \leq_B E_0$ as well.

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Let e, e_1, e_2 be the Borel endpoint functions defined on $\mathcal{J}(X)$ as in Subsection 2.2. Then the set $\mathcal{J}(Y) = e^{-1}(Y \times Y)$ of all arcs in Y is Borel. We define an auxiliary Borel equivalence relation E_C on $\mathcal{J}(Y)$ of lying in the same arc component as

$$E_C = \{(I, J) \in \mathcal{J}(Y) \times \mathcal{J}(Y) : e_1(I) E_Y^A e_1(J)\}.$$

For any $n \in \mathbb{N}$ and $p \in \mathbb{R}$ consider the set

$$S(n, p) = \{(x_i)_{i \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N}} : x_n = p\},$$

which is a closed subset of $\mathbb{R}^{\mathbb{N}}$. For any $n \in \mathbb{N}$ and any distinct $p, q \in R$, define the Borel set $\mathcal{J}(n, p, q)$ as

$$\bigcup_{\{i,j\}=\{1,2\}} \{J \in \mathcal{J}(Y) : S(n, p) \cap J = \{e_i(J)\}, S(n, q) \cap J = \{e_j(J)\}, e_1(J) \in Y_3\}.$$

This is the set of all arcs in Y_3 that intersect the disjoint closed sets $S(n, p)$ and $S(n, q)$ only at their endpoints. We will use Proposition 2.4 to show that $E_C|_{\mathcal{J}(n,p,q)}$ is Borel reducible to E_0 for any $n \in \mathbb{N}$ and any distinct $p, q \in \mathbb{R}$. Fix $n \in \mathbb{N}$ and distinct $p, q \in \mathbb{R}$. Consider the Borel set $D = D_1 \cap D_2$ where

$$D_1 = \{(I, J, \Gamma) \in \mathcal{J}(n, p, q)^2 \times \mathcal{J}(Y) : I \neq J, e(\Gamma) \subseteq e(I) \cup e(J)\} \text{ and}$$

$$D_2 = \{(I, J, \Gamma) \in \mathcal{J}(n, p, q)^2 \times \mathcal{J}(X) : \Gamma \cap S(n, p) = \emptyset \text{ or } \Gamma \cap S(n, q) = \emptyset \text{ or } \Gamma = I \cup J\}.$$

Since the set $\mathcal{J}(n, p, q)$ contains only arcs from non-triodic arc components without circles, the section $D_{(I,J)}$ contains at most one arc for any pair (I, J) of arcs. Hence the projection $E = \pi_{1,2}(D)$ of D on the first two coordinates is Borel by Theorem 2.8. We will show that $G = (\mathcal{J}(n, p, q), E)$ is a Borel graph on $E_C|_{\mathcal{J}(n,p,q)}$ that satisfies the assumptions of Proposition 2.4.

The fact that G is a Borel graph follows from the construction. We need to show that the connected components of the graph G are exactly the equivalence classes of $E_C|_{\mathcal{J}(n,p,q)}$. Pick $(I, J) \in E_C|_{\mathcal{J}(n,p,q)}$. If $I \cap J \neq \emptyset$, then we have $(I, J, I \cup J) \in D$. So assume $I \cap J = \emptyset$. Let K be the smallest arc containing both I and J, which clearly exists as we are in a non-triodic arc component that is not a circle. Since K is compact, there are at most finitely many arcs from $\mathcal{J}(n, p, q)$ lying in K. Let

$$\{I = I_0, I_1, \ldots, I_k = J\} \subseteq \mathcal{J}(n, p, q)$$

be the set of all arcs lying in K in the listed order through a homeomorphism between K and $[0, 1]$. Let $0 \leq i < k$. If $I_i \cap I_{i+1} \neq \emptyset$, then $(I_i, I_{i+1}, I_i \cup I_{i+1}) \in D$. But if $I_i \cap I_{i+1} = \emptyset$, let K' be the smallest arc intersecting both I_i and I_{i+1}. Then $(I_i, I_{i+1}, K') \in D$. It follows that $(I_i, I_{i+1}) \in E$. This argument shows that I and J are in the same connected component of the graph G. Conversely, if $(I, J) \in E$, then it is clear from the definition of G that $(I, J) \in E_C|_{\mathcal{J}(n,p,q)}$.

Since we consider only non-triodic arc components, the degree of G is at most 2. It follows then that $E_C|_{\mathcal{J}(n,p,q)}$ is countable and that $E_C|_{\mathcal{J}(n,p,q)} \leq_B E_0$ by Proposition 2.4. Let

$$M(n, p, q) = \{(x, J) \in Y_3 \times \mathcal{J}(n, p, q) : x E_Y^A e_1(J)\},$$

which is a Borel set. Since $E_C|_{\mathcal{J}(n,p,q)}$ is countable, the section $M(n, p, q)_x$ is countable for any $x \in Y_3$. Then there is a Borel uniformization of $M(n, p, q)$ by Theorem 2.8. Moreover, this uniformization is a Borel reduction from $E^A_{\pi_1(M(n,p,q))}$

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to $E_C|_{\mathcal{J}(n,p,q)}$, implying that $E^A_{\pi_1(M(n,p,q))} \leq_B E_0$. Clearly, for any $n \in \mathbb{N}$ and distinct $p, q \in \mathbb{R}$, the set $\pi_1(M(n,p,q))$ is E^A_Y-invariant. It is easy to see that

$$Y_3 = \bigcup_{\substack{p \neq q \in \mathbb{Q} \\ n \in \mathbb{N}}} \pi_1\left(M(n,p,q)\right).$$

Combining all parts together, we get that

$$\{\pi_1(M(n,p,q)) : \ p,q \in \mathbb{Q},\ p \neq q \text{ and } n \in \mathbb{N}\} \cup \{Y_1, Y_2\}$$

is a collection as in Lemma 2.6 and so $E^A_Y \leq_B E_0$. □

Theorem 3.2. *$E^A_X \leq_B E_0$ for any Polish subspace $X \subseteq \mathbb{R}^2$.*

Proof. Let X_T and Y be as in the previous theorem. Since each arc component of X is Borel and X has only countably many triodic arc components by Theorems 2.7 and 2.9, X_T is Borel. So $E^A_Y \leq_B E_0$ by Theorem 3.1. Having countably many classes, $E^A_{X_T} \leq_B \mathrm{id}(\mathbb{N}) \leq_B E_0$. It follows that $E^A_X \leq_B E_0$. □

The following remark reveals that E_0 is the optimal upper bound for the complexity of the arc-connection relation on Polish subspaces of $\mathbb{R}^2$.

Remark 3.3. Let K be the Knaster continuum. The composant equivalence relation on K coincides with E^A_K and is known to be non-smooth [22]. We then have $E^{CC}_K = E^A_K \sim_B E_0$ by Theorem 3.2. This can also be shown directly as was done in [24]. Borel bireducibility of E^A_X with E_0 can in fact be shown for any generalized Knaster continuum X (see [25] for the definition).

4. Continuum-connection Relation

In this section, we give the optimal upper bound for the complexity of the continuum-connection relation on locally compact Polish spaces, as well as two supplementary results on the continuum-connection relation.

Proposition 4.1. *Let X be a locally compact Polish space. Then $E^C_X \leq_B E_1$.*

Proof. Since X is locally compact and separable, we can find an increasing sequence $(G_n)_{n\in\mathbb{N}}$ of open subsets such that each $\overline{G_n}$ is compact and $X = \bigcup_{n\in\mathbb{N}} G_n$. Define

$$E_n = E^C_{\overline{G_n}} \cup \mathrm{id}(X)$$

for all $n \in \mathbb{N}$. It is easy to see that E_n is a smooth Borel equivalence relation on X and that $E_n \subseteq E_{n+1}$ for every $n \in \mathbb{N}$.

We will show that $E^C_X = \bigcup_{n\in\mathbb{N}} E_n$. The inclusion $\bigcup_{n\in\mathbb{N}} E_n \subseteq E^C_X$ trivially holds. For the other inclusion, pick $(x,y) \in E^C_X$. Then there is $K \in \mathcal{C}(X)$ containing x and y. Since K is compact, $K \subseteq G_{n_0}$ for some $n_0 \in \mathbb{N}$. Thus $(x,y) \in E_{n_0} \subseteq \bigcup_{n\in\mathbb{N}} E_n$. This argument shows that E^C_X is hypersmooth and hence Borel reducible to E_1. □

Now, we show that this upper bound is optimal, which also indicates that in the plane the continuum-connection relation can be more complex than the arc-connection relation.

Example 4.2. *There exists a locally compact $X \subseteq \mathbb{R}^2$ for which $E^C_X \sim_B E_1$.*

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Proof. Let $\mathbb{P} \subseteq \mathbb{R}^2$ be the pseudo-arc. In [22], Solecki showed that the composant equivalence relation on $\mathbb{P}$, call F, is Borel bireducible with E_1. Let $x \in \mathbb{P}$ and consider $X = \mathbb{P} \setminus \{x\}$. By Proposition 4.1, we have $E_X^C \leq_B E_1$. Thus, it is enough to argue that $E_1 \leq_B E_X^C$. We will instead show that $F \leq_B E_X^C$. To this end, let C be the composant of $\mathbb{P}$ containing x. Now pick $y \in C \setminus \{x\}$ and consider the function $f : \mathbb{P} \to X$ defined by

$$f(p) = \begin{cases} p & \text{, if } p \in \mathbb{P} \setminus C \\ y & \text{, if } p \in C. \end{cases}$$

Then f is a Borel reduction from F to E_X^C, which finishes the proof. □

To prove that the arc-connection relation on a G_δ subset of the plane is Borel, Debs and Saint Raymond used the result by Becker and Pol that the arc-connection relation on such spaces is Borel if and only if all of the equivalence classes of the relation are Borel [2, Proposition 5.1]. Using a similar approach, we show that the same is true of the continuum-connection relation. However, note that this equivalence of conditions is not true in general for analytic equivalence relations.

Theorem 4.3. *Let $X \subseteq \mathbb{R}^2$ be a G_δ subspace. Then E_X^C is Borel if and only if every equivalence class of E_X^C is Borel.*

Proof. Each equivalence class of any Borel equivalence relation is clearly Borel. For the other implication, assume that every equivalence class of E_X^C is Borel. Let

$$A = \{(x, y, K) \in X^2 \times \mathcal{C}(X) : x, y \in K,\ K \text{ is irreducible between } x \text{ and } y\}.$$

We will show that this set is Borel. To this end, consider the set

$$B = \{(x, y, K, L) \in X^2 \times \mathcal{C}(X)^2 : x, y \in K \cap L \text{ and } L \subsetneq K\},$$

which can be easily seen to be Borel. For any $(x, y, K) \in X^2 \times \mathcal{C}(X)$, the section

$$B_{(x,y,K)} = \{L \in \mathcal{C}(K) : x, y \in L\} \setminus \{K\}.$$

is a K_σ set because $\mathcal{C}(K)$ is compact. It follows from the Arsenin-Kunugui Theorem [11, Theorem 35.46] that $\pi_{1,2,3}(B)$ is Borel. But since A is equal to

$$\{(x, y, K) \in X^2 \times \mathcal{C}(X) : x, y \in K\} \setminus \pi_{1,2,3}(B),$$

it is a Borel set. Now, let X_T be the union of all equivalence classes of E_X^C that contain a triod. There can be at most countably many such classes by Theorem 2.9. So since we assume that every equivalence class is Borel, then so are X_T and $E_{X_T}^C$. We set $X_N = X \setminus X_T$ and consider

$$C = A \cap (X_N \times X_N \times \mathcal{C}(X)).$$

For any $(x, y) \in X_N \times X_N$, the section $C_{(x,y)}$ contains at most two continua by Proposition 2.12. It follows that $\pi_{1,2}(C) = E_{X_N}^C$ is Borel by Theorem 2.8, where the equality is implied by the fact that any two points contained in a continuum are contained in a continuum that is irreducible between them. As a consequence, $E_{X_N}^C \cup E_{X_T}^C = E_X^C$ is Borel. □

It turns out that the continuum-connection relation is related to the arc-connection relation on the corresponding hyperspace as shown by the following theorem.

Theorem 4.4. *Let X be a Polish space. Then $E_X^C \sim_B E_{\mathcal{C}(X)}^A$.*

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Proof. ($\leq_B$) Consider the function $f : X \to \mathcal{C}(X)$ given by $x \mapsto \{x\}$. This function is continuous and hence Borel. We will show that it is the desired Borel reduction. Let $(x, y) \in E_X^C$. Then there exists $K \in \mathcal{C}(X)$ such that $x, y \in K$. We know then that the hyperspace $\mathcal{C}(K)$ is arc-connected [10, Theorem 14.9]. So there is an arc in $\mathcal{C}(K)$ from $\{x\}$ to $\{y\}$. Since $\mathcal{C}(K) \subseteq \mathcal{C}(X)$, we have that $\{x\}$ and $\{y\}$ are connected by an arc in $\mathcal{C}(X)$ as well, i.e. $f(x)E^A_{\mathcal{C}(X)}f(y)$. For the converse implication, suppose that $(\{x\}, \{y\}) \in E^A_{\mathcal{C}(X)}$. It follows that there is a continuous function $\varphi : [0, 1] \to \mathcal{C}(X)$ with $\varphi(0) = \{x\}$ and $\varphi(1) = \{y\}$. Since the union map on $\mathcal{K}(X)$ and the map φ are continuous, $\bigcup_{t\in[0,1]} \varphi(t)$ is a continuum that contains x and y. Thus, $(x, y) \in E_X^C$.

($\geq_B$) We know from [11, Theorem 12.13] that there exists a Borel selector function $S : \mathcal{C}(X) \to X$. We will show that S is the desired Borel reduction. Let $(K, L) \in E^A_{\mathcal{C}(X)}$. Since $\mathcal{C}(K)$ and $\mathcal{C}(L)$ are arc-connected, we can then find an arc in $\mathcal{C}(X)$ connecting $\{S(K)\}$, K, L and $\{S(L)\}$ consecutively. This implies, using a similar argument as before, that $S(K)$ and $S(L)$ are contained in a continuum in X. The converse implication also follows by the same reasoning as in the first Borel reduction above. □

Although there are parallels between the arc-connection and the continuum-connection relations, we do not know whether the continuum-connection relation on G_δ subspaces of the plane is always Borel.

5. Chain Continuum-connection Relation

In this section, we present some results concerning the complexity of the chain continuum-connection relation[1] on Polish metric spaces. Let X be a metric space, and let $\varepsilon > 0$. We define

$$\mathcal{C}_\varepsilon(X) = \{K \in \mathcal{C}(X) : \operatorname{diam}(K) < \varepsilon\} \text{ and}$$
$$\mathcal{C}_{\overline{\varepsilon}}(X) = \{K \in \mathcal{C}(X) : \operatorname{diam}(K) \leq \varepsilon\}.$$

A finite sequence $K_0, ..., K_k \in \mathcal{C}_\varepsilon(X)$ (respectively, in $\mathcal{C}_{\overline{\varepsilon}}(X)$) of consecutively intersecting continua such that $x \in K_0$ and $y \in K_k$ is called an *ε-chain* (respectively, an *$\overline{\varepsilon}$-chain*) in X from x to y. We consider the auxiliary equivalence relation

$$E_X^{CC,\varepsilon} = \{(x, y) \in X \times X : \text{there is an } \varepsilon\text{-chain in } X \text{ from } x \text{ to } y\}$$

on X to study the chain-connection relation. We fix a positive decreasing sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ with $\varepsilon_n \to 0$ throughout this section. So we have $E_X^{CC} = \bigcap_{n\in\mathbb{N}} E_X^{CC,\varepsilon_n}$.

We expressed the chain-connection relation as a countable intersection. In light of Lemma 2.5 this will prove useful. We continue our discussion by showing that there is an upper bound on the Borel complexity of $E_X^{CC,\varepsilon}$ on locally compact Polish metric spaces.

Lemma 5.1. *Let X be a locally compact Polish metric space and $\varepsilon > 0$. Then $E_X^{CC,\varepsilon}$ is a K_σ equivalence relation.*

[1]The property of being chain continuum-connected is closely related to continuum-chainability which was studied for example in [3].

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Proof. Since X is locally compact and separable, we can find an increasing sequence $(G_n)_{n\in\mathbb{N}}$ of open sets in X such that each $\overline{G_n}$ is compact and $X = \bigcup_{n\in\mathbb{N}} G_n$. For any $n \in \mathbb{N}$ and any $\varepsilon > 0$, the set

$$\{(x, y) \in \overline{G_n} \times \overline{G_n} : \text{there is an } \overline{\varepsilon}\text{-chain in } \overline{G_n} \text{ from } x \text{ to } y\}$$

is compact as it can be written as a projection of a compact set. Since the equivalence relation $E_{\overline{G_n}}^{CC,\varepsilon}$ can be written as a countable union of sets of the previous form, it is K_σ. We have $E_X^{CC,\varepsilon} = \bigcup_{n\in\mathbb{N}} E_{\overline{G_n}}^{CC,\varepsilon}$ by a similar argument as in the proof of Proposition 4.1. Hence $E_X^{CC,\varepsilon}$ is a K_σ equivalence relation. □

We can use this result together with the existence of a universal K_σ equivalence relation to bound the complexity of the chain continuum-connection relation on locally compact Polish spaces.

Remark 5.2. It is straightforward to check that $E_{X^\mathbb{N}}^A = (E_X^A)^\mathbb{N}$, $E_{X^\mathbb{N}}^C = (E_X^C)^\mathbb{N}$ and $E_{X^\mathbb{N}}^{CC} = (E_X^{CC})^\mathbb{N}$ for any Polish space X.

Theorem 5.3. *Let X be a locally compact Polish metric space. Then $E_X^{CC} \leq_B (E_{\ell_\infty})^\mathbb{N}$.*

Proof. We have $E_X^{CC} = \bigcap_{n\in\mathbb{N}} E_X^{CC,\varepsilon_n}$. Moreover, $E_X^{CC,\varepsilon_n} \leq_B E_{\ell_\infty}$ for all $n \in \mathbb{N}$ by Lemma 5.1 and Remark 2.3. Thus, $E_X^{CC} \leq_B (E_{\ell_\infty})^\mathbb{N}$ by Lemma 2.5 and Remark 5.2. □

This upper bound can actually be realized on a continuum. Recall that a metric space X is called a *metric graph* (respectively, *metric tree*) if there is a graph (respectively, tree) $G = (V, E)$ such that the following holds:

- X is obtained from G by replacing each edge $e \in E$ with a copy of the unit interval $[0, 1]$ and identifying its endpoints with the corresponding vertices;
- the metric on X is the induced path metric, so that the distance between two points is the infimum of the lengths of paths joining them.

Example 5.4. *There exists a continuum X for which $E_X^{CC} \sim_B (E_{\ell_\infty})^\mathbb{N}$.*

Proof. We will construct a continuum X for which $E_{\ell_\infty} \leq_B E_X^{CC}$. This will be enough since we will then have $(E_{\ell_\infty})^\mathbb{N} \sim_B E_{X^\mathbb{N}}^{CC}$ by Remark 5.2 and Theorem 5.3.

We shall construct the space as an inverse limit. Let $V_0 = \{0\}$ and set

$$V_{n+1} = V_n \times \{0, \dots, n + 1\}$$

for all $n \in \mathbb{N}$. Let X_n be a metric graph with vertices V_n and edges between any two distinct points $x, y \in V_n$ satisfying $\max_{0\leq i\leq n} |x(i) - y(i)| \leq 1$.

For $n \in \mathbb{N}$, let $f_n : X_{n+1} \to X_n$ be defined as follows. If $n = 0$, then $f_0 : X_1 \to X_0$ maps X_1 to the unique point of V_0. For $n \geq 1$, we define f_n on V_{n+1} as the natural projection onto V_n. On the edges, we have two possibilities. If the projections of the endpoints are different, f_n homeomorphically maps one edge onto the other. But if the projections of the endpoints are the same, then we pick any edge coming from the projection and let f_n map the edge onto the $[0, 1/2]$ part of the picked edge through $t \mapsto (1 - |2t - 1|)/2$ for all $t \in [0, 1]$. Now, let

$$X = \varprojlim (X_n, f_n).$$

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The equivalence relation E defined on $\prod_{n\in\mathbb{N}}\{0,\dots,n\}$ as

$$xEy \iff \exists N\in\mathbb{N} \text{ such that } \sup_{n\in\mathbb{N}}|x(n)-y(n)|\leq N$$

is Borel bireducible with E_{ℓ_∞} [21, Proposition 19]. Thus, it suffices to show that $E\leq_B E_X^{CC}$. Let

$$V=\lim_{\leftarrow}(V_n, f_n|_{V_{n+1}}).$$

Then there is a natural homeomorphism $V\approx\prod_{n\in\mathbb{N}}\{0,\dots,n\}$. We will show that this homeomorphism is a Borel reduction from E to E_X^{CC}. For any $x,y\in\prod_{n\in\mathbb{N}}\{0,\dots,n\}\approx V$, if $\sup_{n\in\mathbb{N}}|x(n)-y(n)|\leq 1$ then x and y are connected by an arc by the construction of X. So if $(x,y)\in E$ then $(x,y)\in E_X^A\subseteq E_X^{CC}$. On the other hand, if $(x,y)\notin E$ then any continuum in X that contains the points x,y is indecomposable and of diameter at least $1/2$. Furthermore, any continuum of sufficiently small diameter is arc-connected by the construction. Thus, $(x,y)\in E_X^{CC}$ would imply $(x,y)\in E_X^A$, which is not possible. □

It turns out that the complexity of the chain-connection relation decreases when we consider planar locally compact spaces. Let us first state a technical lemma showing that under some mild assumptions the chain-connection is the same as the arc-connection relation.

We recall that a continuum K is a Peano continuum if and only if it can be written as a finite union of elements of $\mathcal{C}_\varepsilon(K)$ for all $\varepsilon>0$ and that any Peano continuum is arc-connected [19, Chapter VIII].

Lemma 5.5. *Let X be a metric space that contains no triod. Then $E_X^{CC}=E_X^A$.*

Proof. It suffices to show that $E_X^{CC}\subseteq E_X^A$. Suppose tat $(x,y)\in E_X^{CC}$. Let $\mathcal{K}_n$ be an ε_n-chain in X from x to y for all $n\in\mathbb{N}$. Then $\bigcup\mathcal{K}_n$ contains a continuum L_n that is irreducible between x and y for any $n\in\mathbb{N}$. But since X does not contain a triod, the sequence $(L_n)_{n\in\mathbb{N}}$ consists of at most two elements by Proposition 2.12. In particular, it has a constant subsequence, say $(L=L_{k_n})_{n\in\mathbb{N}}$. We shall show that L is a Peano continuum. Let $\varepsilon>0$. Choose $n_0\in\mathbb{N}$ such that $\varepsilon_{k_{n_0}}<\varepsilon$. Say $\mathcal{K}_{k_{n_0}}=\{K_0,K_1,\dots,K_k\}$ and define $K=\bigcup\mathcal{K}_{k_{n_0}}$. Applying Lemma 2.10 to $L,K_i\in\mathcal{C}(K)$, we get that $L\cap K_i$ has at most two components for all $0\leq i\leq k$. Then let $M_i^0,M_i^1\in\mathcal{C}_{\varepsilon_{k_{n_0}}}(X)$ be such that $L\cap K_i=M_i^0\cup M_i^1$ for all $0\leq i\leq k$. We have

$$L=\bigcup_{i=0}^{k}M_i^0\cup M_i^1.$$

Since ε was arbitrary, L is a Peano continuum. In particular, it is arc-connected, which shows that $(x,y)\in E_X^A$. □

Theorem 5.6. *Let X be a locally compact subset of the plane. Then $E_X^{CC}\leq_B E_0$.*

Proof. The relation $E_X^{CC,\varepsilon}$ is Borel for every $\varepsilon>0$ by Lemma 5.1. For any $n\in\mathbb{N}$, let T_n be the union of all equivalence classes of E_X^{CC,ε_n} that contain a triod. We note that $T_{n+1}\subseteq T_n$ for all $n\in\mathbb{N}$. Since E_X^{CC,ε_n} is Borel, then so is T_n by Theorem 2.9. Let $N_n=X\setminus T_n$ for all $n\in\mathbb{N}$. Define $N=\bigcup_{n\in\mathbb{N}}N_n$ and $T=\bigcap_{n\in\mathbb{N}}T_n$. Since each T_n is E_X^{CC}-invariant, T and N are also E_X^{CC}-invariant. Then $E_N^{CC}=E_N^A\leq_B E_0$ by Lemma 5.5 and Theorem 3.2.

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We show that E_T^{CC} is smooth, which suffices to conclude that $E_X^{CC} \leq_B E_0$. Without loss of generality, we may assume that each T_n is non-empty and has countably infinite E_X^{CC,ε_n}-classes. Let $R_n = \{x_n^i : i \in \mathbb{N}\}$ be a set that contains exactly one point from every equivalence class of T_n in E_X^{CC,ε_n} for all $n \in \mathbb{N}$. For every $n \in \mathbb{N}$, the set R_n is countable and thus Borel. Given $x \in T$, let $f(x)$ be the unique point in $\prod_{n\in\mathbb{N}} R_n$ such that $(x, f(x)) \in E_X^{CC,\varepsilon_n}$ for every $n \in \mathbb{N}$. Then $f : T \to \prod_{n\in\mathbb{N}} R_n$ is a Borel function because the inverse image of a basic open set is either empty or the whole set T_k for some $k \in \mathbb{N}$. It is straightforward to check that f Borel reduces T to the identity relation on the product. □

We do not know whether Theorem 5.6 holds for a general planar G_δ set. However, we note that in the proof above, we essentially used nothing but the fact that $E_X^{CC,\varepsilon}$ is Borel for every $\varepsilon > 0$. Thus, for E_X^{CC} on a G_δ subset $X \subseteq \mathbb{R}^2$ to be Borel reducible to E_0, it would be enough to have that E_X^{CC,ε_n} is Borel for all $n \in \mathbb{N}$.

Proposition 5.7. *Let X be a Polish metric space. Then $E_X^{CC} \sim_B \bigcap_{n\in\mathbb{N}} E^A_{\mathcal{C}_{\varepsilon_n}(X)}$.*

Proof. ($\leq_B$) Consider the continuous function $x \mapsto \{x\}$ defined from X to $\mathcal{C}(X)$. Since $E_X^{CC} = \bigcap_{n\in N} E_X^{CC,\varepsilon_n}$, it is enough to show that for a fixed $\varepsilon > 0$ and any $x, y \in X$ we have

$$(x, y) \in E_X^{CC,\varepsilon} \iff (\{x\}, \{y\}) \in E^A_{\mathcal{C}_\varepsilon(X)}.$$

Fix $\varepsilon > 0$ and let $x, y \in X$. If $(x, y) \in E_X^{CC,\varepsilon}$, then the fact that $(\{x\}, \{y\}) \in E^A_{\mathcal{C}_\varepsilon(X)}$ can be shown in a similar way as in the proof of Theorem 4.4.

Now, suppose that there is an arc $\varphi : [0, 1] \to \mathcal{C}_\varepsilon(X)$ such that $\varphi(0) = \{x\}$ and $\varphi(1) = \{y\}$. Let d be a compatible metric with the topology of X. We have $\sup_{r\in[0,1]} \operatorname{diam} \varphi(r) < \varepsilon$ by compactness of $[0, 1]$. Thus, there exists $\eta > 0$ such that $\sup_{r\in[0,1]} \operatorname{diam} \varphi(r) = \varepsilon - \eta$. Since φ is uniformly continuous, there is $\delta > 0$ such that $|r - s| < \delta$ implies $d_H(\varphi(r), \varphi(s)) < \eta/2$. Let $I \subseteq [0, 1]$ be a closed interval of length less than δ. Let $x_0, y_0 \in \bigcup \varphi(I)$ such that $x_0 \in \varphi(r_0)$ and $y_0 \in \varphi(s_0)$ where $r_0, s_0 \in I$. Then since $d_H(\varphi(r_0), \varphi(s_0)) < \eta/2$, there is $x' \in \varphi(r_0)$ such that $d(x', y_0) < \eta/2$. It follows that

$$d(x_0, y_0) < d(x_0, x') + d(x', y_0) < (\varepsilon - \eta) + \eta/2 < \varepsilon$$

which shows that the diameter of the compact set $\bigcup \varphi(I)$ is less than ε. We can cover the interval $[0, 1]$ by finitely many closed intervals $I_0, \ldots, I_n$ of length $< \delta$. Then $\bigcup \varphi(I_0), \ldots, \bigcup \varphi(I_n)$ is an ε-chain in X between x and y, where the argument that $\bigcup \varphi(I_i)$ is a continuum is similar to that of the proof of Theorem 4.4.

($\geq_B$) It is routine to check that a Borel selector function on $\mathcal{C}(X)$ works. □

The following two examples show that the three equivalence relations of connectedness can be very complicated, unless we consider planar or Polish spaces.

Example 5.8. *There exists a Polish space $X \subseteq \mathbb{R}^3$ for which $E_X^A = E_X^{CC} = E_X^C$ and these equivalence relations are analytic non-Borel subsets of $X \times X$.*

Proof. Becker states that for any analytic equivalence relation E there is a compact set $K \subseteq \mathbb{R}^3$ such that $E \sim_B E_K^A$ [1, Theorem 4.1]. He does not include a complete proof; however, it can be found in [26, Theorem 3.2]. The construction can in fact

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be easily modified to yield a Polish space on which the arc-connection, continuum-connection and chain-connection relations coincide, where the Borel complexity remains the same. □

Example 5.9. *There exists an F_σ subspace $X \subseteq \mathbb{R}^2$ for which $E_X^A = E_X^{CC} = E_X^C$ and these equivalence relations are not analytic subsets of $X \times X$.*

Proof. Let $\mathcal{C}$ be the Cantor middle-third set in $[0,1]$. Let $B \subseteq \mathcal{C} \times \mathcal{C}$ be an F_σ set such that

$$A = \{x \in \mathcal{C} : \forall y \in \mathcal{C} \ \ (x,y) \in B\}$$

is coanalytic non-Borel. In particular, A is not analytic. Now, set

$$X_0 = \mathcal{C} \times ([-1,2] \setminus \mathcal{C}) \text{ and } X_1 = [0,1] \times \{-1\}.$$

It is straightforward to check that X_0 and X_1 are F_σ subsets of the plane. Define

$$X = X_0 \cup X_1 \cup B.$$

Then X is an F_σ subset of the plane. Let $x = (0,-1)$. It is easy to see that

$$A = [x]_{E_X^A} \cap (\mathcal{C} \times \{2\}).$$

If E_X^A were analytic, then $[x]_{E_X^A}$, and hence A, would be analytic as well. Thus, E_X^A is not analytic. To see $E_X^A = E_X^C$, it suffices to note that any subcontinuum of X is arc-connected. This implies that $E_X^{CC} = E_X^A$. □

The main reason why the equivalence relations are not even analytic for some planar F_σ sets is that for space that is not Polish, the families of arcs and subcontinua need not be Borel in the corresponding hyperspace.

6. Special classes of Continua

In this section, we obtain some results on the complexity of the equivalence relations of connectedness on some classes of continua. These results are obtained by using the properties of these classes, which are taken from [19] and will be used without explicit reference in this section.

A continuum is said to be

- *arc-like* if it is an inverse limit of arcs with surjective bonding maps;
- *circle-like* if it is homeomorphic to an inverse limits of circles;
- *tree-like* if it is homeomorphic to an inverse limit of metric trees.

We would like to note that an arc-like continuum can be embedded into the plane but the same is not necessarily true for a circle-like or a tree-like continuum.

We also recall that a continuum K is *unicoherent* if $C_1 \cap C_2$ is connected for any subcontinua C_1 and C_2 that cover K; *hereditarily unicoherent* if any subcontinua of K is unicoherent.

Corollary 6.1. *Let X be a circle-like continuum. Then $E_X^A \leq_B E_0$.*

Proof. Since X is circle-like, it does not contain any triod. So the result follows directly from Theorem 3.1. □

We show that under a quite natural condition which is satisfied by hereditarily unicoherent, circle-like and arc-like continua, the arc-connection relation is equal to the chain-connection relation.

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Lemma 6.2. *Let X be a metric space such that that $C_1 \cap C_2$ is connected for any continua $C_1, C_2 \subseteq X$. Then $E_X^{CC} = E_X^A$.*

Proof. Let $(x, y) \in E_X^{CC}$. Since x and y are connected by a continuum in X, we can find a continuum C in X that is irreducible between x and y. By the assumption of the lemma, C is the unique such continuum.

We will show that C is a Peano continuum and hence arc-connected. Let $\varepsilon > 0$. We can find an ε-chain $K_0, \dots, K_k$ in X between x and y. From the uniqueness of C, we get $C \subseteq \bigcup_{i=0}^{k} K_i$. By the assumption of the lemma, we have $C \cap K_i \in \mathcal{C}_\varepsilon(X)$ for every $0 \leq i < k$. Thus C is covered by a finite number of continua of diameter smaller than ε. Since ε was arbitrary, we conclude that C is a Peano continuum and thus arc-connected. Thus $(x, y) \in E_X^A$. So we obtain $E_X^{CC} = E_X^A$. □

Remark 6.3. If a Polish space satisfies the assumption of Lemma 5.5 or Lemma 6.2, the arc-connection and chain-connection relations on the space are not just equal to each other but they are also Borel.

Corollary 6.4. *Let X be a tree-like continuum. Then $E_X^A \leq_B (E_{\ell_\infty})^{\mathbb{N}}$.*

Proof. Since X is a tree-like continuum, it is hereditarily unicoherent. We have $E_X^A = E_X^{CC}$ by Lemma 6.2. So we get $E_X^A \leq_B (E_{\ell_\infty})^{\mathbb{N}}$ by Theorem 5.3. □

Existence of a universal tree-like continuum (see [17]) yields the following.

Proposition 6.5. *There exists a Borel equivalence relation E such that $E_X^A \leq_B E$ for every tree-like continuum X and $E \leq_B E_T^A$ for some tree-like continuum T.*

Proof. Let T be a universal tree-like continuum. We will show that E_T^A is the desired equivalence relation. Let X be any tree-like continuum and $\varphi : X \to T$ be an embedding. Clearly, $(x, y) \in E_X^A$ implies $(\varphi(x), \varphi(y)) \in E_T^A$. On the other hand, suppose that $(\varphi(x), \varphi(y)) \in E_T^A$ and let ψ be the arc connecting $\varphi(x)$ and $\varphi(y)$ in T. If ψ were not contained in $\varphi(X)$, then the space $\varphi(X) \cup \psi$ would be a subcontinuum of T that is not unicoherent, which is not the case since every tree-like continuum is hereditarily unicoherent. The second part is obvious. □

It can be easily seen that the equivalence relation from Proposition 6.5 is unique up to Borel bireducibility. Thus, a natural question is what the exact complexity of this equivalence relation is.

7. Questions

In this section, we list some open questions that arise from our investigation. Since we were able to show some of the results only for locally compact planar Polish spaces, a natural question is whether they hold in the greatest possible generality.

Question 1. Is E_X^C Borel for any G_δ subspace $X \subseteq \mathbb{R}^2$?

In light of Theorem 4.3, this question is equivalent to asking whether every equivalence class of E_X^C is Borel for any G_δ subspace $X \subseteq \mathbb{R}^2$. If the answer to the question is positive, we can then further ask if the complexity of the continuum-connection relation is bounded for planar spaces.

Question 2. Is E_X^C hypersmooth for any G_δ subspace $X \subseteq \mathbb{R}^2$?

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Analogous questions can be asked about the chain-connection relation.

Question 3. Is E_X^{CC} Borel for any G_δ subspace $X \subseteq \mathbb{R}^2$?

Although a positive answer to this question is not enough to have any reasonable upper bound for the complexity of the chain-connection relation, we suspect that a positive answer to the question will also yield a positive answer to the following question (see the discussion after Theorem 5.6).

Question 4. Is it true that $E_X^{CC} \leq_B E_0$ for any G_δ subspace $X \subseteq \mathbb{R}^2$?

Acknowledgement

We are grateful to the anonymous referee for proposing a more general version of Theorem 3.1 and for many suggestions and comments that significantly improved this paper.

Funding

The first author was supported by the grant SVV-2025-260827 and by Czech Science Foundation grant 24-10705S. The second author was supported by TÜBİTAK-BİDEB through 2210-E Program during his Master's degree education. The third author was supported by Czech Science Foundation grant 24-10705S

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Department of Mathematical Analysis
Faculty of Mathematics and Physics, Charles University, Prague, Czechia
OORCID: 0009-0003-4192-2407
Email address: hevessy@karlin.mff.cuni.cz

Department of Mathematics
Faculty of Arts and Science, Middle East Technical University, Ankara, Turkey
ORCID: 0009-0005-8082-2972
Email address: yuyar@metu.edu.tr

Department of Mathematical Analysis
Faculty of Mathematics and Physics, Charles University, Prague, Czechia
ORCID: 0000-0002-2833-5385
Email address: vejnar@karlin.mff.cuni.cz

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