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Bend, break and count II: elliptics, cuspidals, linear genera

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In [**R2**] we showed how elementary considerations involving geometry on ruled surfaces may be used to obtain recursive enumerative formulae for rational plane curves. Here we show how similar considerations may be used to obtain further enumerative formulae, as follow. First some notation. As usual we denote by N_d^g the number of irreducible plane curves of degree d and genus g through $3d+g-1$ general points. Also, we denote by $N_{d\to}^g$ (resp. $N_{d\times}^g$) the number of such curves passing through general points $A_1,\dots,A_{3d+g-2}$ and having a given tangent direction (resp. a node) at A_1. As is well known and easy to see, we have

$$N_d^g = N_{d\to}^g + 2N_{d\times}^g. \tag{1}$$

For any d,g, these numbers are computed in [**R1**] as part of a more general recursive procedure. For N_d^0, N_d^1, relatively simple recursions have been given by Kontsevich–Manin (see [**FP**]) and Eguchi–Hori–Xiong–Getzler (see [**P1**]), respectively.

Theorem. (i) *We have*

$$N_{d\to}^0 = \frac{1}{2}\sum_{d_1+d_2=d} N_{d_1}^0 N_{d_2}^0 d_1 d_2 \binom{3d-4}{3d_1-2}. \tag{2}$$

$$N_{d\to}^1 = \frac{(d-1)(d-2)}{24}N_d^0 + \frac{1}{3d-1}\sum (3d_1-1)\, d_1 d_2 \binom{3d-1}{3d_1-1} N_{d_1}^0 N_{d_2}^1. \tag{3}$$

(ii) *The numbers K_d^0, K_d^1 of rational* (resp. *elliptic*) *plane curves of degree d with* 1 *cusp through* $3d-2$ (resp. $3d-1$) *general points satisfy*

$$K_d^0 = 3N_d^0 + \frac{1}{2}\sum_{d_1+d_2=d} N_{d_1}^0 N_{d_2}^0 d_1 d_2 \left[(3d-6)\binom{3d-4}{3d_1-2} - \binom{3d-2}{3d_1-1}\right], \tag{4}$$

$$K_d^1 = 3N_d^1 + \frac{(d-1)(d-2)(d-4)}{8}N_d^0 + \sum (3d_1-2)\, d_1 d_2 \binom{3d-1}{3d_1-1} N_{d_1}^0 N_{d_2}^1. \tag{5}$$

(iii) *The respective linear genera* (= *geometric genera of generic curve-sections*) g_d^0, g_d^1 *of the Severi varieties V_d^0, V_d^1, satisfy*

$$2g_d^0 - 2 = K_d^0 - (3d/2)\sum N_{d_1}^0 N_{d_2}^0 d_1 d_2 \binom{3d-4}{3d_1-2} + (3d-2)N_d^0, \tag{6}$$

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$$2g^1_d-2=(3d-\tfrac{5}{2})N^1_d+\frac{(d-1)(d-2)((3d/2)-13)}{24}N^0_d$$
$$+\sum(3d_1-2)d_1d_2N^0_{d_1}N^1_{d_2}\binom{3d-1}{3d_1-1}. \quad (7)$$

For genus 0, closely related results were also obtained by Pandharipande [**P2**].

1. *Elliptics*

Let N^g_d denote the Severi degree, i.e. the number of irreducible plane curves of degree d and genus g through $3d+g-1$ general points. In [**R1**] we gave a recursive procedure for computing N^g_d. However this procedure really is for a much larger set of numbers, including N^g_d, and does not, as it stands, permit a recursion involving N^g_d alone. Moreover the procedure is of exponential size in the number of nodes hence seems to be at its most complicated in low-genus cases. On the other hand for N^0_d Kontsevich *et al.* have more recently given a simple recursive formula, for which a short elementary proof is given in [**R2**]. Quite recently an equally simple recursive formula for N^1_d was discovered by the physicists Eguchi, Hori and Xiong and proven by Getzler (see [**P1**]). Surprisingly, the formula (3) for the infinitely-near analogue $N^1_{d\to}$ is both simple and elementary to prove and the proof will also yield other numbers of interest.

We begin by recalling some facts about the local geometry in codimension 1 of the Severi variety V^g_d, which may be found in [**DH**, **P2**]. Let $\bar{B}$ denote the (closure of) the locus of curves in V^g_d passing through $3d+g-2$ general points $A_1,\dots,A_{3d+g-2}\in\mathbb{P}^2$, $B\to\bar{B}$ the normalization, $\bar{X}\to\bar{B}$ the tautological family of plane curves, $X\to\bar{X}$ the normalization $\pi: X\to B$ the natural map and $s_i=s_{A_i}\subset X$ the tautological section mapping to A_i for $i=1,\dots,3d+g-2$. Then B is a smooth curve, X is a smooth surface and any fibre of π is either a smooth irreducible curve of genus g, or a 1-nodal curve of arithmetic genus g. The map $B\to\bar{B}$ is simply ramified over the points corresponding to cuspidal curves, which themselves are cusps of $\bar{B}$ and unramified elsewhere.

Now as Vakil kindly pointed out to me, and is explained in [**P2**], the curve $\bar{B}$ is more singular than the generic curve-section of V^g_d, the extra singularities being nodes corresponding to the curves $\bar{C}$ in $\bar{B}$ having a node at some A_i: at such curves V^g_d itself is smooth and the hyperplane in the ambient projective space, corresponding to the condition of passing through A_i, is (simply) tangent to V^g_d. Alternatively, we may blow up $A_1,\dots,A_{3d+g-2}$ and realize $\bar{B}$ as a suitable 'Severi' variety associated to the blown-up surface, so that a curve $\bar{C}$ as above, nodal at A_i, will correspond to $C'+E$, C' being the proper transform of $\bar{C}$ and E the exceptional curve at A_i, which meets C' transversely twice. The two branches of $\bar{B}$ at C' correspond as usual to preserving one or the other of the nodes of $C'+E$ on E as nodes. In any event, we see that each A_i thus gives rise to $N^g_{d\times}$ nodes of $\bar{B}$ which disappear in the generic curve-section, while all other singularities of $\bar{B}$ come from singularities of V^g_d itelf and persist in the generic curve-section. Consequently

$$g^g_d=g(B)+(3d+g-2)N^g_{d\times}. \quad (8)$$

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We have an exact sequence

$$0 \to T_{\mathrm{v}} \to T_X \to \pi^* T_B \to \mathcal{O}_{\mathrm{sing}\,(\pi)} \to 0, \tag{9}$$

where sing (π) is the set of singular points of fibres and T_{v} is the vertical tangent sheaf of X, for which (9) may be taken as a definition. Note that T_{v} is invertible and

$$c_1(T_{\mathrm{v}}) \sim -K_X + (2g(B)-2)\,F,$$

where $\mathrm{F} = \pi^{-1}(pt)$.

Now consider a 2-dimensional 'thickening' $\bar{B}^+$ of $\bar{B}$ in a small neighbourhood of $\bar{B}$ in the Severi variety, corresponding to omitting the base condition A_1, and let $\pi^+ : X^+ \to B^+$ be the normalization of the tautological family, $f^+ : X^+ \to \mathbb{P}^2$ the natural map and $B_1 \subset B^+$ the image of B, which has nodes precisely at the curves having a node at A_1 and is smooth elsewhere. As A_1 moves in the plane, B_1 moves on B^+ so that

$$(B_1)^2_{B^+} = N^g_d,$$
$$\deg\,(N_{B \to B^+}) = N^g_{d\to},$$

and as these numbers clearly differ by exactly twice the number of nodes of $\bar{B}$, i.e. by $2N^g_{d\times}$, this proves (1). Moreover, the natural map $s_1 \to X^+$ is an embedding whose image is a generic fibre of f^+, so that the normal bundle $N_{s_1 \to X^+}$ is trivial. In view of the exact sequence

$$0 \to T_{\mathrm{v}} \,|\, s_1 \to N_{s_1 \to X^+} \to \pi^* N_{B \to B^+} \to 0,$$

we conclude

$$(s_1^2)_X = -N^g_{d\to}. \tag{10}$$

Now consider the special case $g = 1$. Then the singular fibres of π are of one of the following two types.

(i) Irreducible rational curves, lying over the intersection points of $\bar{B}$ with V^0_d, which are N^0_d in number; at these points the Severi variety V^1_d, hence $\bar{B}$, has $\frac{1}{2}[(d-1)\,(d-2)]$ smooth branches corresponding to the choice of a node of the rational plane curve which remains a node on the corresponding fibre of π; hence the total number of singular fibres of this type is $\frac{1}{2}[(d-1)\,(d-2)]N^0_d$;

(ii) Fibres of the form $R_i + E_i$, where R_i maps to an irreducible rational curve $\bar{R}_i$ of degree d_1 through $3d_1 - 1$ of $A_1, \ldots, A_{3d-1}$ and E_i maps to an elliptic curve $\bar{E}_i$ of degree d_2 through the remaining As and exactly one of the intersection points of $\bar{E}_i$, $\bar{R}_i$ comes from $E_i \cap R_i$.

Our surface X is the blow-up of a smooth relatively minimal elliptic surface X' with exceptional divisor $R = \sum R_i$. As X has no multiple fibres we have, by the well-known canonical bundle formula [**BPV**],

$$K_X \sim (\omega + 2g - 2)\,F + R,$$

where $g = g(B)$ and 12ω is the degree of the j-function $j : B \to M_1 = \mathbb{P}^1$ or equivalently $12\chi(\mathcal{O}_{X'})$. Considering the fibre of j over $\infty \in \mathbb{P}^1$, we have

$$\omega = \frac{1}{12}\,\frac{(d-1)\,(d-2)}{2}\,N^0_d.$$

(Alternatively, this also follows from Noether's formula $\chi = (c_1^2 + c_2)/12$ and the multiplicativity formula for Euler numbers $e(X') = e(B)\,e(F) + \#(\text{sing. fibres})$.) Note

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also that (e.g. by adjunction) $-\omega$ may be identified as the self-intersection of any section of X'. In particular

$$s_{A_i}^2 = -\omega - s_{A_i}.R. \tag{11}$$

Now for any function u of d_1 we define a new function $T[u]$ of d by the formula

$$T[u](d) = \sum_{d_1+d_2=d} u(d_1)\, d_1\, d_2 \binom{3d-1}{3d_1-1} N_{d_1}^0 N_{d_2}^1.$$

Then as each R_i of degree d_1 contains precisely $3d_1-1$ of the As, we have

$$s_{A_i}^2 = -\omega - \frac{1}{3d-1} T[3d_1-1](d).$$

Together with (10), this proves (3). Now consider the ramification divisor ρ of the natural map $f: X \to \mathbb{P}^2$; this contains a vertical part $\sum_1^\kappa F_i$, $\kappa = K_d^1$, consisting of the fibres mapping to cuspidal curves. For a smooth fibre $F = \pi^{-1}(b)$, $\rho \cap F$ coincides with the divisor of zeros of the natural map $N_{F\to X} \to N_{f(F)\to\mathbb{P}^2}$, i.e. the residual intersection of $f(F)$ with the unique (mod $f(F)$) plane curve C of degree d through the nodes of $f(F)$, and is given by $A_1, \dots, A_{3d-1}$ together with one further point $\sigma(b)$. For a singular fibre F the situation is similar except C is only required to pass through the nodes of $f(F)$ not coming from the node of F; moreover, it is easy to see in case $F = E_i + R_i$ that the 'last' point $\sigma(b)$ lies on $f(E_i)$, the elliptic part. Note that the curves $f(F)$ having a node at some A_i do not require special consideration: as noted above, what happens there is that $\bar{B}$ itself will have a node at the point $\{f(F)\}$, with branches corresponding to the branches of $f(F)$ through A_i and $s_{A_i}(b)$ will be the point of F above A_i lying on the branch corresponding to the branch of $\bar{B}$ containing b. In any event, the points $\sigma(b)$ glue together to a section σ with

$$\sigma \sim f^*(-K_{\mathbb{P}^2}) + K_X - \sum_1^{3d-1} s_{A_i} - \sum_1^\kappa F_i \sim 3L + (\omega + 2g - 2 - \kappa) F - \sum s_{A_i} + R$$

where $L = f^*\mathcal{O}_{\mathbb{P}^2}(1)$, so that $L^2 = N_d^1$. As $\sigma \cap R = \emptyset$, we have

$$\begin{aligned} \sigma^2 &= -\omega \\ &= (3L + (\omega + 2g - 2 - \kappa) F - \sum s_{A_i} + R)^2. \end{aligned}$$

We simplify this, using

$$L^2 = N_d^1, \quad L.F = d, \quad L.s_{A_i} = 0, \quad L.R = T[d_1](d), \quad F.s_{A_i} = 1,$$
$$F.R = 0, \quad s_{A_i}.R = 0 = s_{A_i}.s_{A_j}, \quad i \neq j, \quad R^2 = -T[1](d)$$

plus the above formula for $s_{A_i}^2$, and we get

$$2g - 2 - \kappa = \tfrac{1}{2}((3d-4)\omega - 9N_d^1 + T(3d_1-2)). \tag{12}$$

To obtain K_d^1, consider the natural map

$$T_V \to f^*(T_{\mathbb{P}^2}).$$

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Comparing with (6), we see that this map vanishes (simply) precisely at the singular points of fibres of π plus the preimages of cusps. Consequently

$$\begin{aligned} K_d^1 + 12\omega + T(1) &= c_2(f^*T_{\mathbb{P}^2} \otimes T_{\mathrm{v}}^*) \\ &= 3N_d^1 + 3d\omega + 3T(d_1) - T(1), \end{aligned}$$

which yields (5). Finally (12) and (5) yield (7).

2. *Rationals*

We now consider analogous questions for rational curves. As above, and as in [**R2**], the idea is to study the fibred surface $\pi: X \to B$ corresponding to the 1-parameter family of rational curves of degree d through a set of general points $A_1, \dots, A_{3d-1} \in \mathbb{P}^2$. X may be realized in many ways as a blowup of a geometrically ruled surface. One such way, $b: X \to X' = \mathbb{P}(E)$ is the blowing down of the sum R of all fibre components not meeting $s_1 = s_{A_1}$. Now we have

$$s_1^2 = -m < 0,$$

where by [**R2**] we have

$$2m = \sum_{d_1+d_2=d} N_{d_1}^0 N_{d_2}^0 d_1 d_2 \binom{3d-4}{3d_1-2}.$$

In view of (10), this proves (2). Now we may assume E has a nonvanishing section $\mathcal{O}$ with $\mathbb{P}(\mathcal{O}) = s_1 \subset X'$ and $M^* = E/\mathcal{O}$ of degree $-m$. In fact as X' possesses another section s_{A_2} disjoint from s_{A_1}, clearly $E = \mathcal{O} \oplus M^*$, but we will not need this here. Now let $\pi^*E^* \to \mathcal{O}_{X'}(1)$ be the canonical invertible quotient (in the anti-Grothendieck sense). Then the composite $\pi^*M \to \mathcal{O}_{X'}(1)$ vanishes precisely on s_1, hence

$$c_1(\mathcal{O}_{X'}(1)) \sim s_1 + mf', \quad f' = \text{fibre of } \pi',$$

hence

$$c_1(\mathcal{O}_X(1)) \coloneqq c_1(b^*\mathcal{O}_X(1)) \sim s_1 + mf, \quad f = \text{fibre of } \pi.$$

Consequently, with T_{v} as in (9), we have $c_1(T_{\mathrm{v}}) = 2s_1 + mf - R$. As above, the natural map $T_{\mathrm{v}} \to f^*T_{\mathbb{P}^2}$ vanishes precisely on singular points of fibres and cusp preimages, hence

$$\begin{aligned} \frac{1}{2}\sum N_{d_1}^0 N_{d_2}^0 d_1 d_2 \binom{3d-2}{3d_1-1} + K_d^0 &= c_2(f^*T_{\mathbb{P}^2} \otimes T_{\mathrm{v}}^*) \\ &= 3N_d^0 - 3dm + 3f^*\mathcal{O}_{\mathbb{P}^2}(1).R + R^2. \end{aligned}$$

Simplifying, we obtain (2). For instance, $K_3^0 = 24$, as is well known [**KS**].

Finally, note that the ramification divisor of $f: X \to \mathbb{P}^2$, i.e. the degeneracy locus of the differential $T_X \to f^*T_{\mathbb{P}^2}$, consists now precisely of the cuspidal fibres plus the sections $s_{A_1}, \dots, s_{A_{3d-2}}$. Consequently

$$-2s_1 - mf + R + (2g-2)f + 3f^*\mathcal{O}(1) \sim \sum s_{A_i} + K_d^0 f, g = g_d^0.$$

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Hence taking dot product with s_1 we get

$$m+2g-2=-m+K_d^0,$$

which yields (4).

Remark. For what it is worth, the *arithmetic* linear genus of V_d^g is given by

$$p_d^g = g_d^g + K_d^g + \left(\binom{d-1}{2} - g + 1\right) N_d^{g-1} + \sum_{\substack{g_1+g_2=g \\ d_1+d_2=d}} \epsilon_{g_1 g_2} \binom{3d+g-1}{3d_1+g_1-1} d_1 d_2 N_{d_1}^{g_1} N_{d_2}^{g_2},$$

where $\epsilon_{g_1 g_2} = 1/(1+\delta_{g_1 g_2})$ is a factor inserted to ensure that each reducible configuration counts just once. For $g=0$ this number was computed by Pandharipande [**P2**]; the above results compute it for $g=1$ as well.

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