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KEISLER'S ORDER IS NOT LINEAR, ASSUMING A SUPERCOMPACT

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Abstract. We show that if there is a supercompact cardinal, then Keisler's order is not linear. More specifically, let $T_{n,k}$ be the theory of the generic n-clique free k-ary graph for any $n > k \geq 3$, and let T_{Cas} be the simple nonlow theory described by Casanovas in [2]. Then we show that $T_{Cas} \not\trianglelefteq T_{n,k}$ always, and if there is a supercompact cardinal then $T_{n,k} \not\trianglelefteq T_{Cas}$.

Keisler's order is a pre-order $\trianglelefteq$ defined on complete countable theories, introduced by Keisler in [3]. It is defined by setting $T_1 \trianglelefteq T_2$ if and only if for all λ and for all regular ultrafilters $\mathcal{U}$ on λ and for all $M_i \models T_i$, if $M_2^\lambda/\mathcal{U}$ is λ^+-saturated then so is $M_1^\lambda/\mathcal{U}$. The regularity assumption shows that this does not depend on the choice of the models M_i, so we can say that $\mathcal{U}$ λ^+-saturates T if $M^\lambda/\mathcal{U}$ is λ^+-saturated for any $M \models T$; then we can rephrase Keisler's order as saying $T_1 \trianglelefteq T_2$ if and only if for all λ and for all regular ultrafilters $\mathcal{U}$ on λ, if $\mathcal{U}$ λ^+-saturates T_2 then $\mathcal{U}$ λ^+-saturates T_1.

In his original article [3], Keisler proved that there is a maximal class and a minimal class in Keisler's order; then in [10] Shelah obtained several results, including determining the least two classes of Keisler's order (namely the theories without the finite cover property, and the stable theories with the finite cover property). Progress was slow until Malliaris entered the scene with [5], proving that Keisler's order is local; this was the first of a series of articles by Malliaris that analyzed the connection between Keisler's order and the complexity of φ-types. Most recently, Malliaris and Shelah have obtained several results in joint work; in particular, in [9] they prove that Keisler's order has infinitely many classes, and in [8] they prove that if there is a supercompact cardinal then simplicity is a dividing line in Keisler's order, that is if T is simple and $T' \trianglelefteq T$ then T' is simple.

Using the technology developed in these articles, we observe that under the existence of a supercompact cardinal, Keisler's order is not linear.[1] More specifically, let $T_{n,k}$ be the theory of the generic n-clique free k-ary hypergraph for any $n > k \geq 3$; up to an indexing shift, these are the theories used in [9] to get infinitely many classes. Also, let T_{Cas} be the simple nonlow theory described by Casanovas in [2]. Then we show that $T_{Cas} \not\trianglelefteq T_{n,k}$ always, and if there is a supercompact cardinal then $T_{n,k} \not\trianglelefteq T_{Cas}$.

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[1] In private communications, I have learned that Malliaris and Shelah had independently obtained this result, but it was not disseminated.

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We first recall the general setup introduced in [6], following the notation of [6,8,9]. We assume familiarity with distributions on Boolean algebras, as defined in these articles. Let $\mathcal{B}$ be a complete Boolean algebra, let T be a complete countable theory, let λ be a cardinal and let $\mathcal{U}$ be an ultrafilter on $\mathcal{B}$. Then $\mathcal{U}$ is $(\lambda, \mathcal{B}, T)$-*moral* if whenever $\langle \mathbf{a}_s : s \in [\lambda]^{<\aleph_0} \rangle$ is a certain kind of distribution from $\mathcal{U}$ (namely, a $(\mathcal{B}, T, \overline{\varphi})$-possibility for some sequence of formulas $\overline{\varphi}$) then $\langle \mathbf{a}_s \rangle$ has a multiplicative refinement in $\mathcal{U}$. This definition is actually a generalization of λ^+-saturation, that is: suppose $\mathcal{B} = \mathcal{P}(\lambda)$ and $\mathcal{U}$ is λ-regular. Then $\mathcal{U}$ λ^+-saturates T if and only if $\mathcal{U}$ is $(\lambda, \mathcal{B}, T)$-moral.

The following are the Boolean algebras with which we will be dealing.

Definition 1. Given $\mu \geq \theta$ with θ regular and $\mu = \mu^{<\theta}$, and given a set X, let $P_{X,\mu,\theta}$ be the set of all partial functions from X to μ of cardinality less than θ, ordered by reverse inclusion; let $\mathcal{B}_{X,\mu,\theta}$ be the Boolean algebra completion of $P_{X,\mu,\theta}$. So $\mathcal{B}_{X,\mu,\theta}$ has the μ^+-c.c. and is $< \theta$-distributive. For each $f \in P_{X,\mu,\theta}$ let $\mathbf{x}_f$ be the corresponding element of $\mathcal{B}_{X,\mu,\theta}$.

The key fact about this setup is Theorem 6.13 from [6]:

Theorem 2. *Let $\mathcal{B} = \mathcal{B}_{2^\lambda,\mu,\theta}$, for some $\lambda \geq \mu \geq \theta$ with θ regular and $\mu = \mu^{<\theta}$, and let T_0, T_1 be complete countable theories. Suppose there is an ultrafilter $\mathcal{U}$ on $\mathcal{B}$ such that $\mathcal{U}$ is $(\lambda, \mathcal{B}, T_1)$-moral but not $(\lambda, \mathcal{B}, T_0)$-moral. Then $T_0 \not\trianglelefteq T_1$.*

Actually, with a little cardinal arithmetic one can show that this holds if $\mathcal{B}_{2^\lambda,\mu,\theta}$ is replaced by any complete Boolean algebra $\mathcal{B}$ with $|\mathcal{B}| \leq 2^\lambda$, but we won't need this.

We now indicate the various facts from [6, 8, 9] we will need to get our theorem. First, we say that the complete countable theory T is low if it is simple and for every formula $\varphi(\overline{x}, \overline{y})$, there is some k such that for all $\overline{b}$, if $\varphi(\overline{x}, \overline{b})$ does not k-divide over $\emptyset$ then it does not divide over $\emptyset$. This is the standard definition of low, for instance it is equivalent to the definition in [1], where the concept of lowness is introduced. Malliaris defined low slightly differently starting in [4], namely not requiring T to be simple (this definition is then also used in later articles, for instance in [6]). To clarify, let us say that T has the *finite dividing property* if there is some formula $\varphi(\overline{x}, \overline{y})$ such that for every k there is some indiscernible sequence $(\overline{b}_n : n < \omega)$ over the emptyset such that $\{\varphi(\overline{x}, \overline{b}_n) : n < \omega\}$ is k-consistent but not consistent. What Malliaris calls low is what we call not having the finite dividing property; and we say that a theory T is low if it is simple and does not have the finite dividing property.

One reason to prefer the original definition is the following: it is easy to check that $(\mathbb{Q}, <)$ does not have the finite dividing property, and so the finite dividing property is not a dividing line in Keisler's order. On the other hand, we show in [11] that lowness is a dividing line in Keisler's order.

The following theorem is a special case of Conclusion 9.10 from [6]:

Theorem 3. *Let T be any nonlow theory and let $\lambda > \aleph_0$. Then no ultrafilter $\mathcal{U}$ on $\mathcal{B} := \mathcal{B}_{2^\lambda,\aleph_0,\aleph_0}$ is $(\lambda, \mathcal{B}, T)$-moral.*

Fix $n > k \geq 3$ for the rest of this article. Recall $T_{n,k}$ is the theory of the random n-clique free k-ary hypergraph. (In [9], the indexing is shifted by one, so for instance, their $T_{3,2}$ is our $T_{4,3}$. In particular this affects the indicing for all the theorems we quote.) We will need the following two theorems from [9]. The following is (a special case of) Theorem 4.1.

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THEOREM 4. *Write $\lambda = \aleph_{k-2}$. There is an ultrafilter $\mathcal{U}$ on $\mathcal{B} := \mathcal{B}_{2^\lambda,\aleph_0,\aleph_0}$ which is $(\lambda, \mathcal{B}, T_{n,k})$-moral.*

The following is not literally a special case of Claim 5.1 from [9], but it has exactly the same proof. Namely, that claim addresses the case of $\mathcal{B}_{2^\lambda,\mu,\aleph_0}$ for some cardinal μ; but the only place in the proof where the third coordinate matters is the final paragraph, where we need $|S_\beta| \leq \mu$. But this clearly works also for $\mathcal{B}_{2^\lambda,\mu,\mu}$, provided that $\mu = \mu^{<\mu}$ (so that $\mathcal{B}_{2^\lambda,\mu,\mu}$ has the μ^+-c.c.).

THEOREM 5. *Suppose $\mu = \mu^{<\mu}$. Write $\lambda = \mu^{+(n-1)}$. Then no ultrafilter $\mathcal{U}$ on $\mathcal{B} := \mathcal{B}_{2^\lambda,\mu,\mu}$ is $(\lambda, \mathcal{B}, T_{n,k})$-moral.*

Thus, by Theorems 3 and 4, we have (in just ZFC) that if T is any nonlow theory then $T \not\trianglelefteq T_{n,k}$. (This is also a special case of the main theorem from [11], as mentioned previously: low theories form a dividing line in Keisler's order.)

Going forward, we take the most natural example of a simple nonlow theory T_{Cas}, and show that if there is a supercompact cardinal σ, then $T_{n,k} \not\trianglelefteq T_{Cas}$. By Theorem 5 note it will suffice to show that if σ is supercompact and $\lambda \geq \sigma$, then there is an ultrafilter on $\mathcal{B}_{2^\lambda,\sigma\sigma}$ which is $(\lambda, \mathcal{B}, T_{Cas})$-moral.

T_{Cas} was introduced by Casanovas [2] and was in fact the first example of a simple nonlow theory. The language $\mathcal{L}_{Cas}$ is $(R, P, Q, Q_n : 1 \leq n < \omega)$, where P, Q, Q_n are each unary relation symbols and R is binary. We adopt the convention that $a, a', \ldots$ are elements of P, $b, b', \ldots$ are elements of Q.

1. The universe is the disjoint union of P and Q, both infinite;
2. Each $Q_n \subseteq Q$, and the Q_n's are infinite and disjoint;
3. $R \subseteq P \times Q$;
4. For each $a \in P$ and for each $n < \omega$, there are exactly n elements $b \in Q_n$ such that $R(a, b)$;
5. Whenever B_0, B_1 are finite disjoint subsets of Q such that each $|B_1 \cap Q_n| \leq n$, there is $a \in P$ such that $R(a, b)$ for all $b \in B_1$ and $\neg R(a, b)$ for all $b \in B_0$;
6. For all A_0, A_1 finite disjoint subsets of P, there is $b \in Q$ such that $R(a, b)$ for all $a \in A_1$ and $\neg R(a, b)$ for all $a \in A_0$.

Actually, if in the definition of $\mathcal{L}_{Cas}$ we allow $n = 0$ then Q_0 will be completely harmless, so for notational convenience we let $\mathcal{L}_{Cas}$ be $(R, P, Q, Q_n : n < \omega)$.

In [2] it is shown that T_{Cas} is complete, and is the model companion of the theory axiomatized by the first four items above. In particular, it is shown that T_{Cas} has quantifier elimination in an expanded language, where we add predicates $S_{\ldots}$ that express, given $A_0, A_1 \subset P$ finite disjoint with $A_1 \neq \emptyset$, how many $b \in Q_n$ are there such that $R(a, b)$ for all $a \in A_1$ and $\neg R(a, b)$ for all $a \in A_0$. It follows that the algebraic closure of a set X is $X \cup \bigcup\{b \in \bigcup_n Q_n : \text{ there is } a \in X \cap P \text{ such that } R(a, b)\}$, and every formula over a set X is equivalent to a quantifier-free formula over $\mathrm{acl}(X)$.

Casanovas also shows that T_{Cas} is simple with the following forking relation: $X \mathop{\smile\!\!\!\!\!\!|}_Z Y$ iff $\mathrm{acl}(X) \cap \mathrm{acl}(Y) \subseteq \mathrm{acl}(Z)$. Also the formula $R(x, y)$ witnesses that T_{Cas} is not low. In [11] we show that T_{Cas} is a minimal nonlow theory in Keisler's order.

The following lemma is immediate from the quantifier elimination in the expanded language discussed above:

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LEMMA 6. *Let $M \models T_{Cas}$. As notation let Q_ω denote $Q \setminus \bigcup_n Q_n$.*

- *For each $n < \omega$, there is a unique nonalgebraic type $p(x)$ over M with $Q_n(x) \in p(x)$. It is isolated by the formulas $Q_n(x)$ together with $\neg R(a, x)$ for each $a \in P^M$.*
- *For each $A \subseteq P^M$ let $p_A(x)$ be the type over M that says $Q_\omega(x)$ holds, $x \neq b$ for each $b \in Q^M$, and finally for each $a \in P^M$, $R(a, x)$ holds iff $a \in A$. Then $p_A(x)$ generates a complete type over M that does not fork over $\emptyset$. Moreover, all nonalgebraic complete types over M extending $\{Q(x)\} \cup \bigcup_n \{\neg Q_n(x)\}$ are of this form.*
- *Suppose $B \subseteq Q^M$ is such that each $|B \cap Q_n^M| \leq n$. Let $p_B(x)$ be the type over M that says $P(x)$ holds, and $x \neq a$ for each $a \in P^M$, and for each $b \in Q^M$, $R(x, b)$ holds iff $b \in B$. Then $p_B(x)$ generates a complete type over M, and moreover every complete nonalgebraic type over M extending $P(x)$ is of this form. Further, given $M_0 \subseteq M$, we have that $p(x)$ does not fork over M_0 iff for each $n < \omega$, $B \cap Q_n^{M_0} = B \cap Q_n^M$.*

From this lemma we get the following characterization of the saturated models of T_{Cas}.

LEMMA 7. *$M \models T_{Cas}$ is λ^+-saturated iff*:

1. *$|Q_n| \geq \lambda^+$ for each $n < \omega$, and also $|Q_\omega| \geq \lambda^+$;*
2. *For all $B_0, B_1 \subseteq Q^M$ disjoint with each $|B_i| \leq \lambda$, and with $|B_1 \cap Q_n| \leq n$ for each $n < \omega$, there is $a \in P$ such that $R(a, b)$ for each $b \in B_1$, and $\neg R(a, b)$ for each $b \in B_0$;*
3. *For all $A_0, A_1 \subseteq P^M$ disjoint with each $|A_i| \leq \lambda$, there is $b \in Q_\omega$ such that $R(a, b)$ for each $a \in A_1$ and $\neg R(a, b)$ for each $a \in A_0$.*

Fix a supercompact cardinal σ and a cardinal $\lambda = \lambda^{<\sigma}$ for the rest of this article. Write $\mathcal{B} = \mathcal{B}_{2^\lambda, \sigma, \sigma}$ and for each $\alpha < 2^\lambda$ let $\mathcal{B}_\alpha = \mathcal{B}_{\alpha, \sigma, \sigma}$.

We say that the sequence $(\mathbf{a}_s : s \in [\lambda]^{<\sigma})$ from $\mathcal{B}$ is *continuous* if $s \subseteq t$ implies $\mathbf{a}_s \geq \mathbf{a}_t > 0$ and also each $\mathbf{a}_s = \bigcap_{t \in [s]^{<\aleph_0}} \mathbf{a}_t$. We say that $(\mathbf{a}_s : s \in [\lambda]^{<\sigma})$ *is in the filter $\mathcal{D}$* if each $\mathbf{a}_s \in \mathcal{D}$.

The following is a special case of $(\lambda, \mu, \theta, \sigma)$-optimality in [8] (where we have set $\mu = \theta = \sigma$).

DEFINITION 8. Suppose $\mathcal{U}$ is an ultrafilter on $\mathcal{B}$. Then $\mathcal{U}$ is $(\lambda, \sigma, \sigma, \sigma)$-*optimal* if $\mathcal{U}$ is σ-complete and, for every continuous sequence $(\mathbf{b}_s : s \in [\lambda]^{<\sigma})$ in $\mathcal{U}$, (A) implies (B):

(A) There is some closed unbounded $\Omega \subset [\lambda]^{<\sigma}$ such that for each $\delta < 2^\lambda$ with $\langle \mathbf{b}_s : s \in [\lambda]^{<\sigma} \rangle \subseteq \mathcal{B}_\delta$, and there is some multiplicative refinement $\langle \mathbf{b}'_s : s \in [\lambda]^{<\sigma} \rangle$ of $\langle \mathbf{b}_s : s \in [\lambda]^{<\sigma} \rangle$ from $\mathcal{B}$ so that for each $s \in \Omega$, and for each $\mathbf{a} \in \mathcal{B}_\delta$, if $\mathbf{b}_s \cap \mathbf{a}$ is nonzero then $\mathbf{b}'_s \cap \mathbf{a}$ is nonzero.

(B) $\langle \mathbf{b}_s : s \in [\lambda]^{<\aleph_0} \rangle$ has a multiplicative refinement $\langle \mathbf{b}'_s : s \in [\lambda]^{<\aleph_0} \rangle$ in $\mathcal{U}$.

Note, in (A) it is equivalent to replace "for each $\delta < 2^\lambda, \ldots$" by "for arbitrarily large $\delta < 2^\lambda, \ldots$."

The following is a special case of Theorem 5.9 from [8] (note that since σ is supercompact and $\lambda = \lambda^{<\sigma}$, we get that $(\lambda, \sigma, \sigma, \sigma)$ is suitable in the sense of Definition 1.1 of [8]):

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THEOREM 9. *There is a $(\lambda, \sigma, \sigma, \sigma)$-optimal ultrafilter on $\mathcal{B}$.*

So to prove our goal, it suffices to establish the following:

LEMMA 10. *Suppose $\mathcal{U}$ is a $(\lambda, \sigma, \sigma, \sigma)$-optimal ultrafilter on $\mathcal{B}$. Then $\mathcal{U}$ is $(\lambda, \mathcal{B}, T_{Cas})$-moral.*

PROOF. Actually, it is easy to check that T_{Cas} is $(\lambda, \sigma, \sigma, \sigma)$-explicitly simple, and so we could apply Theorem 7.3 from [8] and be done. For the reader's convenience we give a direct proof, using the niceness of T_{Cas}. Our proof follows a template that Malliaris and Shelah employ several times; it first appears in Lemma 9.9 of [6], then in more complicated forms in Theorem 7.3 of [8] and Theorem 4.1 of [9].

Choose a regular good filter $\mathcal{D}_0$ on $\mathcal{P}(\lambda)$ and an isomorphism $\mathbf{j} : \mathcal{P}(\lambda)/\mathcal{D}_0 \cong \mathcal{B}$ (this is possibly by Theorem 7.1 of [6]). Write $\mathcal{U}_* = \mathbf{j}^{-1}(\mathcal{U})$. We want to show that $\mathcal{U}_*$ λ^+-saturates T_{Cas}.

Let $M \models T_{Cas}$, and let $\mathbf{M} = M^\lambda/\mathcal{D}$; we want to show that $\mathbf{M}$ is λ^+-saturated. Since $\mathcal{U}_*$ is λ-regular, we know that $|Q_n^{\mathbf{M}}| \geq \lambda^+$ for each $n < \omega$, and also $|Q_\omega^{\mathbf{M}}| \geq \lambda^+$. So it suffices to realize types $p(x)$ as in items two or three from Lemma 7. We just consider case two; case three is easier. So choose $B_0, B_1 \subseteq Q^{\mathbf{M}}$ disjoint with each $|B_i| \leq \lambda$ and each $|B_1 \cap Q_n^{\mathbf{M}}| \leq n$. We show there is $f \in M^\lambda$ such that $[[f/\mathcal{U}_*]] \in P^{\mathbf{M}}$ and $\mathbf{M} \models R([[f/\mathcal{U}_*]], b)$ for each $b \in B_1$, and $\mathbf{M} \models \neg R([[f/\mathcal{U}_*]], b)$ for each $b \in B_0$. Note that by extending B_1, we can suppose each $|B_1 \cap Q_n^{\mathbf{M}}| = n$. So actually we can also suppose that $B_0 \subseteq Q_\omega^{\mathbf{M}}$, as the other elements are redundant. Further we can suppose $|B_1| = \lambda$.

Enumerate $B_0 \cup B_1 = \overline{b} = (b_\alpha : \alpha < \lambda)$ without repetitions. For each $\alpha < \lambda$ choose $g_\alpha \in (Q^M)^\lambda$ with $[[g_\alpha/\mathcal{U}_*]] = b_\alpha$. For each $\alpha < \lambda$, let $i_\alpha < 2$ be such that $b_\alpha \in B_{i_\alpha}$, and let $\gamma_\alpha \leq \omega$ be such that $b_\alpha \in Q_{\gamma_\alpha}^{\mathbf{M}}$. Also, let Γ be the set of all $\alpha < \lambda$ such that $b_\alpha \in B_1 \cap \bigcup_{n<\omega} Q_n^{\mathbf{M}}$, i.e., such that $\gamma_\alpha < \omega$; so Γ is countable.

Note that there are two roles λ is playing: as set of the ultrafilter $\mathcal{U}$, and as the index set of $\overline{b}$. We use the variables $i, j, \ldots$ for elements of λ when λ is in the first role, and typical ordinal variables $\alpha, \beta, \ldots$ for elements of λ when λ is in the second role.

Now, by the Separation of Variables technology from [7] (see Section 5), $p(x)$ is realized in $\mathbf{M}$ if and only if the distribution $\langle \mathbf{a}_s : s \in [\lambda]^{<\aleph_0} \rangle$ has a multiplicative refinement in $\mathcal{U}_*$, where $\mathbf{a}_s = \{i < \lambda : M \models \exists x \bigwedge_{\alpha \in s} R(x, b_\alpha)^{i_\alpha}\}$.

We will want to view this process more systematically: for each $\alpha_0 < \cdots < \alpha_{n-1} < \lambda$ and for each formula $\varphi(y_{\alpha_0}, \ldots, y_{\alpha_{n-1}})$, let

$$||\varphi(y_{\alpha_0}, \ldots, y_{\alpha_{n-1}})|| := \mathbf{j}(\{i < \lambda : M \models \varphi(g_{\alpha_0}(i), \ldots, g_{\alpha_{n-1}}(i))\}).$$

For instance, note that each $||Q(y_\alpha)|| = 1$. For each $s \in [\lambda]^{<\aleph_0}$ we can write $\mathbf{a}_s = ||\exists x \bigwedge_{\alpha \in s} R(x, y_\alpha)^{i_\alpha}||$. Then each $\mathbf{a}_s \in \mathcal{U}$, and it suffices to show that $\langle \mathbf{a}_s : s \in [\lambda]^{<\aleph_0} \rangle$ has a multiplicative refinement in $\mathcal{U}$.

For the rest of the proof, we aim to apply $(\lambda, \sigma, \sigma, \sigma)$-optimality. First, write

$$\mathbf{b}_* = \bigcap_{\alpha \neq \beta \in \Gamma} ||y_\alpha \neq y_\beta|| \cap \bigcap_{\alpha \in \Gamma} ||Q_{\gamma_\alpha}(y_\alpha)||.$$

So $\mathbf{b}_* \in \mathcal{U}$ since $\mathcal{U}$ is σ-complete. For each $s \in [\lambda]^{<\aleph_0}$ let $\mathbf{b}_s = \mathbf{b}_* \cap \mathbf{a}_s$. We will show that $\langle \mathbf{b}_s : s \in [\lambda]^{<\aleph_0} \rangle$ has a multiplicative refinement in $\mathcal{U}$.

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For each $s \in [\lambda]^{<\sigma}$ let $\mathbf{b}_s := \bigcap_{t\in[s]^{<\aleph_0}} \mathbf{b}_t$; then each $\mathbf{b}_s \in \mathcal{U}$ and $\langle \mathbf{b}_s : s \in [\lambda]^{<\sigma}\rangle$ is continuous. Let Ω be the club set of all $s \in [\lambda]^{<\sigma}$ with $\Gamma \subseteq s$. It suffices to show that condition (A) holds in the definition of optimality with respect to Ω. So let δ be given; we can suppose by increasing δ that whenever φ is a formula with parameters from $\overline{y} = (y_\alpha : \alpha < \lambda)$, then $||\varphi|| \in \mathcal{B}_\delta$.

We are thus trying to find a multiplicative refinement $\langle \mathbf{b}'_s : s \in [\lambda]^{<\sigma}\rangle$ of $\langle \mathbf{b}_s : s \in [\lambda]^{<\sigma}\rangle$ so that for each $s \in \Omega$, and for each $\mathbf{a} \in \mathcal{B}_\delta$, if $\mathbf{b}_s \cap \mathbf{a}$ is nonzero then so is $\mathbf{b}'_s \cap \mathbf{a}$. We give a preview of the end of the argument. We will eventually have defined a special family $\mathcal{F}$ of pairs $(\mathbf{a}, s)$ where $s \in [\lambda]^{<\sigma}$ and $\mathbf{a} \leq \mathbf{b}_s$ is a nonzero element of $\mathcal{B}_\delta$, and we will have defined, for each $(\mathbf{a}, s) \in \mathcal{F}$, an element $h_{\mathbf{a},s} \in P_{\lambda,\sigma,\sigma}$. (see Definition 1). We need to be guaranteeing that given any finite sequence $(\mathbf{a}_i, s_i) : i < n$, if $\bigcup_{i<n} h_{\mathbf{a}_i,s_i}$ is a function then $\bigcap_{i<n} \mathbf{a}_i \leq \mathbf{b}_{\bigcup_i s_i}$.

Now, roughly speaking, each pair $(\mathbf{a}_i, s_i)$ will induce a type $p_i(x) \in S(A_i)$, where A_i is a subset of the monster model of T_{Cas} of size less than σ; we will need to check that $\bigcup_i p_i(x)$ is consistent. We will have that each $p_i(x)$ is nonalgebraic and extends $P(x)$. Further, by choice of $\mathbf{b}_*$, for each $m < \omega$ there are m-many $b \in Q_m$ such that for each $i < n$, $p_i(x) \models R(x, b)$. Note then that if $(p_i(x) : i < n)$ are pairwise consistent, then $\bigcup_{i<n} p_i(x)$ is consistent; this will be easy to arrange with the functions $h_{\mathbf{a}_i,s_i}$.

However, we will have an additional difficulty, arising from the fact that we may have $||y_\alpha = y_\beta|| > 0$ for distinct α, β. One way to think about this is that we should be using an artificial equality relation $\equiv$, so we additionally have to worry that there are $i, j < n$ and $b_i \in A_i \backslash A_j, b_j \in A_j \backslash A_i$ such that $p_i(x)$ asserts that $R(x, b_i)$, and $p_j(x)$ asserts that $\neg R(x, b_j)$, and $b_i \equiv b_j$. Malliaris and Shelah refer to this as a "collision;" we now describe their technique for "collision detection." Given $s \in [\lambda]^{<\sigma}$ and given $\mathbf{a} \in \mathcal{B}_\delta$ nonzero, say that $\mathbf{a}$ is *strong for* s if for all $\alpha \in s$, if $\beta \leq \alpha$ is least such that $\mathbf{a} \cap ||y_\alpha = y_\beta|| \neq 0$, then $\beta \in s$ and $\mathbf{a} \leq ||y_\alpha = y_\beta||$. The idea here is that any equalities that could happen have already happened, and so we have anticipated any possible collisions.

Note that for all $s \in [\lambda]^{<\sigma}$ and for all nonzero $\mathbf{a} \in \mathcal{B}_\delta$, there is $t \supseteq s$ in $[\lambda]^{<\sigma}$ and $\mathbf{b} \leq \mathbf{a}$ such that $\mathbf{b}$ is a nonzero element of $\mathcal{B}_\delta$ and $\mathbf{b}$ is strong for t. This is because $\mathcal{B}_\delta$ has a σ-closed dense subset, namely $P_{\delta,\sigma,\sigma}$, and so we can construct $\mathbf{b}$ iteratively.

Suppose $\mathbf{a}$ is strong for s. Define $\pi_{\mathbf{a},s} : s \to s$ by $\pi_{\mathbf{a}}(\alpha) =$ the least $\beta \leq \alpha$ with $||y_\alpha = y_\beta|| \cap \mathbf{a}$ nonzero. This β is an element of s by definition of strongness, and further we always have $\mathbf{a} \leq ||y_\alpha = y_\beta||$. Note also that if $\mathbf{a}$ is strong for s and $\mathbf{b}$ is strong for t and $\mathbf{a} \cap \mathbf{b} \neq 0$, then $\pi_{\mathbf{a},s}$ and $\pi_{\mathbf{b},t}$ agree on $s \cap t$ and $\mathbf{a} \cap \mathbf{b}$ is strong for $s \cup t$. Further, if $\mathbf{a}$ is strong for s, then for each $\alpha, \alpha' \in s$, $\mathbf{a} \leq ||y_\alpha = y_{\alpha'}||$ iff $\mathbf{a} \cap ||y_\alpha = y_{\alpha'}|| \neq 0$ iff $\pi_{\mathbf{a},s}(\alpha) = \pi_{\mathbf{a},s}(\alpha')$.

We now describe how to build $\langle \mathbf{b}'_s\rangle$.

For each $s \in \Omega$ let $\{\mathbf{a}_{s,\xi} : \xi < \xi(s)\}$ and $\{w_{s,\xi} : \xi < \xi(s)\}$ satisfy

- $\{\mathbf{a}_{s,\xi} : \xi < \xi(s)\}$ is a maximal antichain of $\mathcal{B}_\delta$ (and hence of $\mathcal{B}$) below $\mathbf{b}_s$;
- Each $\mathbf{a}_{s,\xi}$ is strong for $w_{s,\xi}$;
- Set $\pi_{s,\xi} = \pi_{\mathbf{a}_{s,\xi},w_{s,\xi}}$. Then $w_{s,\xi} = s \cup \{\pi_{s,\xi}(\alpha) : \alpha \in s\}$.

For each s, ξ define $g_{s,\xi} : w_{s,\xi} \to 2$ by: $g_{s,\xi}(\alpha) = i$ iff there is $\beta \in s$ with $\pi_{s,\xi}(\beta) = \pi_{s,\xi}(\alpha)$ and $i_\beta = i$. This is well-defined since $\mathbf{a}_{s,\xi} \leq \mathbf{b}_s$.

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For each s, ξ define $h_{s,\xi}$ by: $h_{s,\xi} = \{\langle \delta + \alpha, i\rangle : \langle \alpha, i\rangle \in g_{s,\xi}\}$. Then, for each $\alpha < \lambda$ let $\mathbf{b}'_{\{\alpha\}} = \mathbf{b}_* \cap \bigcup\{\mathbf{a}_{s,\xi} \cap \mathbf{x}_{h_{s,\xi}} : \alpha \in s \in \Omega, \xi < \xi(s)\}$, and for each $s \in [\lambda]^{<\sigma}$ let $\mathbf{b}'_s = \bigcap_{\alpha\in s} \mathbf{b}'_{\{\alpha\}}$. Then it suffices to show $\langle \mathbf{b}'_s\rangle$ is as in (A) from the definition of optimal ultrafilters. Multiplicativity is clear. Also, suppose $s \in \Omega$ and $\mathbf{a} \in \mathcal{B}_\delta$ is such that $\mathbf{a} \cap \mathbf{b}_s$ is nonzero. Then there is some $\xi < \xi(s)$ such that $\mathbf{a} \cap \mathbf{a}_{s,\xi}$ is nonzero. Then $\mathbf{a} \cap \mathbf{a}_{s,\xi} \cap \mathbf{x}_{h_{s,\xi}}$ is nonzero, but $\mathbf{a}_{s,\xi} \cap \mathbf{x}_{h_{s,\xi}} \leq \mathbf{b}'_s$.

So it remains to show that for $s \in [\lambda]^{<\sigma}$, $\mathbf{b}'_s \leq \mathbf{b}_s$. It suffices to show this for finite s, since both $\langle \mathbf{b}'_s\rangle$ and $\langle \mathbf{b}_s\rangle$ are continuous. Choose $s \in [\lambda]^{<\aleph_0}$ and suppose towards a contradiction that $\mathbf{b}'_s \not\leq \mathbf{b}_s$. Since $\mathbf{b}'_s \leq \mathbf{b}_*$, it follows that $\mathbf{b}'_s \not\leq \mathbf{a}_s$. Write $\mathbf{c}_0 = \mathbf{b}'_s \cap (1 - \mathbf{a}_s)$ and write $s = \{\alpha_0, \ldots, \alpha_{n-1}\}$. We can inductively choose $\mathbf{c}_0 \geq \cdots \geq \mathbf{c}_n > 0$ such that for each $i < n$, writing $\alpha = \alpha_i$, there is s_α and ξ_α such that $\alpha \in s_\alpha$ and $\mathbf{c}_{i+1} \leq \mathbf{a}_{s_\alpha,\xi_\alpha} \cap \mathbf{x}_{h_{s_\alpha,\xi_\alpha}}$. For each $\alpha \in s$, let $w_\alpha = w_{s_\alpha,\xi_\alpha}$; and let $w = \bigcup_{\alpha\in s} w_\alpha$. Also choose $\mathbf{c} < \mathbf{c}_n$ nonzero such that for each $\beta \in w$ and for each $m < \omega$, $\mathbf{c}$ decides $||Q_m(y_\beta)||$. This is possible because $\mathcal{B}_\delta$ has a σ-closed dense subset (and $\sigma > \aleph_0$).

Since the h_{s_α,ξ_α}'s are compatible, so must the g_{s_α,ξ_α}'s be. Put $g := \bigcup_{\alpha\in s} g_{s_\alpha,\xi_\alpha}$; clearly $g : w \to 2$. Similarly, since $\mathbf{c}$ is strong for each $w_{s_\alpha,\xi_\alpha} = w_\alpha$, we have that $\mathbf{c}$ is strong for w; let $\pi := \bigcup_{\alpha\in s} \pi_{s_\alpha,\xi_\alpha}$.

We will now translate this to the situation described previously, where we were trying to show consistency of a union of types. Indeed, first note that for every finite tuple $\beta_0, \ldots, \beta_{m-1}$ from w, and for every formula $\varphi(y_{\beta_0}, \ldots, y_{\beta_{m-1}})$, $\mathbf{c}$ decides $||\varphi(y_{\beta_0}, y_{\beta_{m-1}})||$; that is, either $\mathbf{c} \leq ||\varphi(y_{\beta_0}, \ldots, y_{\beta_{m-1}})||$ or $\mathbf{c} \leq -||\varphi(y_{\beta_0}, \ldots, y_{\beta_{m-1}})|| = ||\neg\varphi(y_{\beta_0}, \ldots, y_{\beta_{m-1}})||$. This is because each $||Q(y_\beta)|| = 1$, and so the only additional things we can express are whether $y_\beta = y_{\beta'}$, which is decided by strongness, and whether $Q_k(y_\beta)$ holds for $k < \omega$, which we have decided by choice of $\mathbf{c}$.

Thus we get a complete type $q(y_\beta : \beta \in w)$. Choose a realization $\overline{b}' := (b'_\beta : \beta \in w)$ in the monster model. For each $\alpha \in s$, let $A_\alpha = \{b'_\beta : \beta \in w_\alpha\}$. Let $p_\alpha(x)$ be the complete type over A_α induced by g_α; that is, $p_\alpha(x)$ asserts that $P(x)$ holds, and for every $\beta \in s_\alpha$, $R(x, b'_\beta)$ holds iff $g(b'_\beta) = 1$. Note that for each $\alpha \in s$, $\mathbf{c} \leq \mathbf{a}_{s_\alpha}$, so in particular each $p_\alpha(x)$ is consistent.

Now, since $\mathbf{c} \leq 1 - \mathbf{a}_s = ||\neg\exists x \bigwedge_{\alpha\in s} R(x, y_\alpha)^{i_\alpha}||$, it must be the case that $\bigcup_{\alpha\in s} p_\alpha(x)$ is inconsistent (since $g(\alpha) = i_\alpha$ for each $\alpha \in s$). We show this is impossible.

First, note that there cannot be any collisions. Indeed, suppose we had $\beta \in w_\alpha$ and $\gamma \in w_{\alpha'}$ with $b'_\beta = b'_\gamma$. This means $\pi(\beta) = \pi(\gamma)$; we want to show that $g(\beta) = g(\gamma)$. But $g(\beta) = g(\pi(\beta))$ by definition of g_{s_α,ξ_α}, and similarly $g(\gamma) = g(\pi(\gamma))$ by definition of $g_{s_{\alpha'},\xi_{\alpha'}}$.

The only other possibility for an inconsistency is that for some $m < \omega$, $\bigcup_{\alpha\in s} p_\alpha(x)$ asserts that there are $m + 1$-many $b \in Q_m$ with $R(x, b)$; in other words, there is some $t \in [w]^{m+1}$ such that for each $\beta \in t$, $Q_m(y_\beta)$ holds and $g(\beta) = 1$, and for each $\beta, \gamma \in t$, $b'_\beta \neq b'_\gamma$. Let t' be the set of all $\beta < \lambda$ such that $\gamma_\beta = m$. Then $t' \in [\Gamma]^m$, so $t' \subseteq s_\alpha$ for each $\alpha \in s$ (since each $s_\alpha \in \Omega$). Then we can choose $\beta_* \in t$ such that $\pi(\beta_*) \neq \pi(\beta)$ for each $\beta \in t'$, and choose $\alpha_* \in s$ such that $\beta_* \in s_{\alpha_*}$. Then clearly $p_{\alpha_*}(x)$ is inconsistent; but this is a contradiction to $\mathbf{c} \leq \mathbf{a}_{s_{\alpha_*}}$. ⊣

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THEOREM 11. *Suppose there is a supercompact cardinal. Then Keisler's order is not linear.*

PROOF. By Theorems 3 and 4 we know that $T_{Cas} \not\trianglelefteq T_{n,k}$, and by Theorems 5 and 9 and Lemma 10 we conclude that $T_{n,k} \not\trianglelefteq T_{Cas}$. ⊣

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