THE PARSEVAL FORMULAE FOR MONOTONIC FUNCTIONS. III

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Received 10 August 1949

1. Introduction

This sequence of papers is concerned with a group of theorems on Fourier transforms, and with the corresponding results for Fourier series. The transform theorems deal with the equations

$$\left.\begin{aligned}\int_0^\infty F_c(x)\,G_c(x)\,dx &= \int_0^\infty f(t)\,g(t)\,dt, \quad (1{\cdot}1)\\ \int_0^\infty F_s(x)\,G_s(x)\,dx &= \int_0^\infty f(t)\,g(t)\,dt, \quad (1{\cdot}2)\end{aligned}\right\} \tag{1}$$

when two of the functions concerned are monotonic; and the series theorems, with the equations*

$$\tfrac{1}{2}a_0\alpha_0 + \sum_{n=1}^{\infty}(a_n\alpha_n + b_n\beta_n) = \frac{1}{\pi}\int_0^{2\pi} f(t)\,g(t)\,dt, \tag{2}$$

$$\left.\begin{aligned}\tfrac{1}{2}a_0'\alpha_0' + \sum_{n=1}^{\infty} a_n'\alpha_n' &= \frac{2}{\pi}\int_0^{\pi} f(t)\,g(t)\,dt, \quad (3{\cdot}1)\\ \sum_{n=1}^{\infty} b_n'\beta_n' &= \frac{2}{\pi}\int_0^{\pi} f(t)\,g(t)\,dt, \quad (3{\cdot}2)\end{aligned}\right\} \tag{3}$$

when two of the functions or sequences of coefficients are monotonic. A general survey of the problem was given in two earlier papers†, which we shall call M.F. (I) and M.F. (II), and the results obtained were tabulated in M.F. (II). There is thus no need to restate the position in detail here. We shall merely recall as briefly as possible what has been proved already, so as to show the scope of the present paper.

In the transform case, there were three essentially distinct sets of conditions:

[A] f and g decrease to zero, $\int_0^\infty fg$ exists,

[B] f and g decrease to zero, $\int_0^\infty F_cG_c$ or $\int_0^\infty F_sG_s$ exists,

[C] f and G_c or G_s decrease to zero, $\int_0^\infty fg$ exists.

It was shown, in a paper‡ which we denote by P.F., that equations (1) hold under conditions [A]. In M.F. (I) it was proved that (1·2) holds under conditions [B], that (1·1) does so if further restrictions are added, and that both equations hold under

* In (2), a_n, b_n are the Fourier coefficients of f in $(0, 2\pi)$, and α_n, β_n are those of g. In (3), a_n' and b_n' are the half-range cosine and sine coefficients of f in $(0, \pi)$, and α_n' and β_n' are those of g.

† 'The Parseval formulae for monotonic functions. I', *Proc. Cambridge Phil. Soc.* 43 (1947), 289–306; and 'The Parseval formulae for monotonic functions. II', *Proc. Cambridge Phil. Soc.* 46 (1950), 231–248.

‡ 'On the Parseval formulae for Fourier transforms', *Proc. Cambridge Phil. Soc.* 38 (1942), 1–19.

https://doi.org/10.1017/S0305004100025718 Published online by Cambridge University Press

conditions [C]. The corresponding series theorems of section [A] were proved in M.F. (II), where two theorems of section [B] were also included. It remains to discuss the [C] theorems for series, and the rest of the [B] theorems; this is done here. Thus the present paper is, roughly speaking, a translation of M.F. (I) into the language of series, just as M.F. (II) was a translation of P.F. It completes the proof that, with certain reservations for series, the Parseval formulae hold in all cases of this kind, except perhaps in the cosine case in [B] which remains in doubt. All the statements made in M.F. (II), pp. 234–236, are proved here or in previous papers, except those asserting that the formulae sometimes fail to hold in the 'finite or infinite' sense. These call for the construction of a certain counter-example, and will be dealt with in a further note*, M.F. (IV).

The conventions of notation which we shall use are set out in M.F. (I) and M.F. (II).

2. SINE THEOREMS, AND COSINE THEOREMS ON CONVEX FUNCTIONS

In the transform case, our results fell into two groups; theorems in which all the functions concerned were positive, and theorems in which this was not necessarily the case. The series theorems may be grouped correspondingly. The first group is easier to prove; it includes all the theorems on sine series, and some on cosine series. The methods are based on the following well-known results†.

(IV) *If $f(t)$ is a decreasing function of L in $(0, 2\pi)$, with full-range Fourier sine coefficients b_n, then $b_n \geqslant 0$.*

If $f(t)$ is a positive decreasing function of L in $(0, \pi)$, with half-range Fourier sine coefficients b_n', then $b_n' \geqslant 0$.

(V) *If b_n is any sequence which decreases to zero‡, and if*

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx, \tag{4}$$

then

$$f(x) \geqslant -\tfrac{1}{2}b_1 \tan \tfrac{1}{4}x \quad (0 \leqslant x < 2\pi). \tag{5}$$

(VI) *If $f(t)$ is a convex function of L in $(0, 2\pi)$, with full-range Fourier cosine coefficients a_n, then $a_n \geqslant 0$ when $n > 0$.*

If $f(t)$ is a decreasing convex function of L in $(0, \pi)$, with half-range Fourier cosine coefficients a_n', then $a_n' \geqslant 0$ when $n > 0$.

(VII) *If a_n is any sequence which decreases to zero and is convex, and if*

$$f(x) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx, \tag{6}$$

then $f(x) \geqslant 0$. Also $f(x)$ belongs to $L(0, 2\pi)$, and the series in (6) is the Fourier series of f.

If $f(t)$ is a negative decreasing function, we may apply (IV) to the positive decreasing function $-f(\pi - t)$; we find that $(-1)^n b_n'$ is positive. Similarly, we can show that if

* 'The Parseval formulae for monotonic functions. IV'.

† See Hardy, G. H. and Rogosinski, W. W., *Fourier series* (Cambridge, 1944), p. 25 (for (IV) and (VI)), and p. 34 (for (VII)). For (V), see Zygmund, A., *Trigonometrical series* (Warsaw, 1935), p. 111; (VII) is given on pp. 109–10. The results (V) and (VII) are proved by partial summation, and (IV) and (VI) may be proved by partial integration.

‡ Here and elsewhere, we use the phrase 'b_n decreases to zero' as an abbreviation for 'b_n decreases (for all values of n considered), and tends to zero as $n \to \infty$'.

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$f(t)$ is an increasing convex function, $(-1)^n a_n'$ is positive; we apply (VI) to the decreasing convex function $f(\pi - t)$.

The sine and cosine theorems of this group do not resemble each other as closely for series as they did for transforms, and it is therefore convenient to treat them separately here. The sine case is the more complicated; when this has been discussed in detail, it will be possible to dismiss the cosine theorems very briefly.

We begin with a theorem in which both f and g are defined by series of type (4). This is quite closely analogous to the transform theory. Indeed, the series and transform proofs would correspond almost exactly if a function defined by (4) were necessarily positive in $(0, \pi)$; as it is, we have to allow for the fact that this is not so by including in (7) a final term, $(b_1 - b_N)\sin x$, which has no counterpart in Lemma 1 of M.F. (I), and by treating the intervals $(0, \frac{1}{2}\pi)$ and $(\frac{1}{2}\pi, \pi)$ separately.

LEMMA 10. *Let $f(x$ be defined by* (4), *where the sequence b_n decreases to zero, and let*

$$f(N, x) = \sum_{n=1}^{N} (b_n - b_N)\sin nx + (b_1 - b_N)\sin x. \tag{7}$$

Then, for fixed x in $(0, \frac{1}{2}\pi)$, $f(N, x)$ is a positive increasing function of N, and tends to $f(x) + b_1 \sin x$ as $N \to \infty$.

It is easily shown that $f(N, x)$ is positive in a wider range, but we take $(0, \frac{1}{2}\pi)$ for simplicity.

Proof. We have, from (V),

$$f(N, x) \geqslant (b_1 - b_N)(\sin x - \tfrac{1}{2}\tan \tfrac{1}{4}x),$$

which is certainly positive in the range concerned. If $N' > N$,

$$f(N', x) - f(N, x) = \sum_{n=1}^{N} (b_N - b_{N'})\sin nx + \sum_{n=N+1}^{N'} (b_n - b_{N'})\sin x + (b_N - b_{N'})\sin x,$$

which is positive for the same reason; thus $f(N, x)$ is an increasing function of N. If $0 < x < 2\pi$,

$$\begin{aligned} f(x) + b_1 \sin x - f(N, x) &= \sum_{n=1}^{N} b_N \sin nx + \sum_{n=N+1}^{\infty} b_n \sin nx + b_N \sin x \\ &= O(b_N / \sin \tfrac{1}{2}x), \end{aligned}$$

and so $f(N, x) \to f(x) + b_1 \sin x$ as $N \to \infty$, the convergence being uniform in any interval strictly interior to $(0, 2\pi)$.

THEOREM 16. *If b_n, β_n are sequences which decrease to zero, and if*

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx, \quad g(x) = \sum_{n=1}^{\infty} \beta_n \sin nx,$$

then

$$\frac{1}{\pi}\int_0^{2\pi} f(x)\, g(x)\, dx = \sum_{n=1}^{\infty} b_n \beta_n,$$

finite or infinite.

Proof. We define $f(N, x)$ as in Lemma 10, and $g(N, x)$ similarly. Since they are merely trigonometrical polynomials, we have

$$\frac{2}{\pi}\int_0^{\pi} f(N, x)\, g(N, x)\, dx = 3(b_1 - b_N)(\beta_1 - \beta_N) + \sum_{n=1}^{N} (b_n - b_N)(\beta_n - \beta_N). \tag{8}$$

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Let $N \to \infty$ in this equation. By the theorem of monotonic convergence,

$$3(b_1 - b_N)(\beta_1 - \beta_N) + \sum_{n=1}^{N} (b_n - b_N)(\beta_n - \beta_N) \to 3b_1\beta_1 + \sum_{n=1}^{\infty} b_n\beta_n, \tag{9}$$

finite or infinite, and

$$\frac{2}{\pi}\int_0^{\frac{1}{2}\pi} f(N,x)\,g(N,x)\,dx \to \frac{2}{\pi}\int_0^{\frac{1}{2}\pi} (f(x) + b_1 \sin x)(g(x) + \beta_1 \sin x)\,dx,$$

finite or infinite. Also

$$\frac{2}{\pi}\int_{\frac{1}{2}\pi}^{\pi} f(N,x)\,g(N,x)\,dx \to \frac{2}{\pi}\int_{\frac{1}{2}\pi}^{\pi} (f(x) + b_1 \sin x)(g(x) + \beta_1 \sin x)\,dx$$

(necessarily finite), by the uniform convergence of $f(N,x)$ and $g(N,x)$; and so

$$\frac{2}{\pi}\int_0^{\pi} f(N,x)\,g(N,x)\,dx \to \frac{2}{\pi}\int_0^{\pi} (f(x) + b_1 \sin x)(g(x) + \beta_1 \sin x)\,dx,$$

finite or infinite.

Now $f(x)\sin x$ and $g(x)\sin x$ are certainly integrable, being in fact continuous since the series $\Sigma b_n \sin nx \sin x$ and $\Sigma \beta_n \sin nx \sin x$ are uniformly convergent in the interval $0 \leqslant x \leqslant \pi$; and, for the same reason,

$$\frac{2}{\pi}\int_0^{\pi} f(x)\sin x\,dx = b_1, \quad \frac{2}{\pi}\int_0^{\pi} g(x)\sin x\,dx = \beta_1.$$

Hence

$$\frac{2}{\pi}\int_0^{\pi} f(N,x)\,g(N,x)\,dx \to \frac{2}{\pi}\int_0^{\pi} f(x)\,g(x)\,dx + 3b_1\beta_1, \tag{10}$$

finite or infinite. The result follows from (8), (9) and (10), since $f(x)\,g(x)$ is even.

If we interchange the roles of functions and coefficients in Theorem 16, we obtain the following half-range theorem; the corresponding full-range result has been discussed already in M.F. (II) (see Theorem 14).

THEOREM 17. *If $f(t)$ and $g(t)$ are decreasing functions of L in $(0,\pi)$, with half-range sine coefficients b_n' and β_n', then*

$$\sum_{n=1}^{\infty} b_n'\beta_n' = \frac{2}{\pi}\int_0^{\pi} f(t)\,g(t)\,dt, \tag{11}$$

finite or infinite.

Proof. Suppose first that f and g are positive; and let

$$\left.\begin{aligned} f(T,t) &= f(1/T) \quad \text{if} \quad 0 \leqslant t < 1/T, \\ &= f(t) \qquad \text{if} \quad 1/T \leqslant t \leqslant \pi, \end{aligned}\right\}$$

$g(T,t)$ being defined similarly. It is easy to show that $f(T,t)$, $g(T,t)$ and their half-range sine coefficients $b_n'(T)$ and $\beta_n'(T)$ are positive, increasing functions of T, and tend to $f(t)$, $g(t)$, b_n', β_n' as $T \to \infty$; indeed, this is merely a special case of Lemma 1 (M.F. (I)). Since $f(T,t)$ and $g(T,t)$ are of bounded variation*,

$$\sum_{n=1}^{\infty} b_n'(T)\,\beta_n'(T) = \frac{2}{\pi}\int_0^{\pi} f(T,t)\,g(T,t)\,dt.$$

* We may use either the classical theorem on functions of the class L^2, or the theorem that (2) and (3) hold if f belongs to L and g is of bounded variation. Following the notation of M.F. (II), we shall refer to this latter theorem as (III).

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The result follows on letting T tend to infinity and using the theorem of monotonic convergence.

It is easy to get rid of the restriction of sign on f and g, by the method used in Theorem 12 of M.F. (II). We define $f_+(t)$ and $f_-(t)$ as the 'positive and negative parts of $f(t)$' (i.e. as $\max(f(t), 0)$ and $\min(f(t), 0)$), and define $g_+(t)$, $g_-(t)$ similarly. The equation (11) holds for $f_+(t)$ and $g_+(t)$ (by what has been proved above); it also holds for $f_-(t)$ and $g_-(t)$, as we can easily show by considering the positive decreasing functions $-f_-(\pi-t)$, $-g_-(\pi-t)$. Moreover, (11) is true in the finite sense for $f_+(t)$ and $g_-(t)$, since these functions are not both unbounded near the same point (compare M.F. (II), p. 241); similarly for $f_-(t)$ and $g_+(t)$. The full result follows on addition; this is legitimate even in the infinite cases, since 'infinity' means '$+\infty$' throughout.

Next, we consider theorems of type [C]; that is, theorems in which the condition of monotony is imposed on one of the functions f, g and one of the sequences of coefficients in (2) or (3).

THEOREM 18. *Suppose that b_n is a sequence which decreases to zero, that*

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx,$$

and that $g(x)$ is a decreasing function of L in $(0, 2\pi)$, having Fourier sine coefficients β_n. Then

$$\lim_{\delta \to +0} \int_{\delta}^{2\pi-\delta} f(x)\, g(x)\, dx = \sum_{n=1}^{\infty} b_n \beta_n, \tag{12}$$

finite or infinite.

If $\sum_{n=1}^{\infty} b_n/n$ is convergent, the left-hand side of (12) *may be replaced by*

$$\frac{1}{\pi} \int_0^{2\pi} f(x)\, g(x)\, dx \tag{13}$$

(*finite or infinite*).

Proof. Suppose first that $g(x)$ is odd, and is positive in $(0, \pi)$. Let $f(N, x)$ be defined as in Lemma 10. Since $f(N, x)$ is a trigonometrical polynomial,

$$\frac{2}{\pi}\left\{\int_0^{\frac{1}{2}\pi} + \int_{\frac{1}{2}\pi}^{\pi}\right\} f(N, x)\, g(x)\, dx = (b_1 - b_N)\beta_1 + \sum_{n=1}^{N} (b_n - b_N)\beta_n.$$

Using the uniform convergence of the integrand for the second integral, and monotonic convergence for the remaining terms, we see that the two sides of this equation tend to

$$\frac{2}{\pi} \int_0^{\pi} (f(x) + b_1 \sin x)\, g(x)\, dx \quad \text{and} \quad b_1\beta_1 + \sum_{n=1}^{\infty} b_n \beta_n,$$

finite or infinite, as $N \to \infty$. Since

$$\frac{2}{\pi} \int_0^{\pi} (b_1 \sin x)\, g(x)\, dx = b_1 \beta_1,$$

and since $f(x)\, g(x)$ is even, the result is proved in this special case, the integral being the Lebesgue integral (13).

In the general case, let

$$\gamma(x) = \tfrac{1}{2}\{g(x) - g(2\pi - x)\}.$$

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This function satisfies the above conditions, and has the same Fourier sine coefficients as g; hence

$$\frac{1}{\pi}\int_0^{2\pi} f(x)\gamma(x)\,dx = \sum_{n=1}^{\infty} b_n\beta_n,$$

finite or infinite. Also, since f is odd,

$$\lim_{\delta\to+0}\int_{\delta}^{2\pi-\delta} f(x)\,g(x)\,dx = \lim_{\delta\to+0} 2\int_{\delta}^{\pi} f(x)\gamma(x)\,dx = \int_0^{2\pi} f(x)\gamma(x)\,dx,$$

finite or infinite.

It is well known that $f(x)$ belongs to $L(0,2\pi)$ if $\Sigma b_n/n$ is convergent*. In this case the integral (13) certainly exists in the 'finite or infinite' sense; for in the neighbourhood of each end-point, g is either bounded or of the same sign as f. It is also clear that (13) exists in this sense if g is not of constant sign for $0<x<2\pi$, whether f belongs to $L(0,2\pi)$ or not. In either case, the left-hand side of (12) can obviously be replaced by this integral.

We have incidentally proved the following theorem.

THEOREM 19. *Suppose that b'_n is a sequence which decreases to zero, that*

$$f(x) = \sum_{n=1}^{\infty} b'_n \sin nx,$$

and that $g(x)$ is a positive, decreasing function of L in $(0,\pi)$, with half-range sine coefficients β'_n. Then

$$\frac{2}{\pi}\int_0^{\pi} f(x)\,g(x)\,dx = \sum_{n=1}^{\infty} b'_n\beta'_n, \tag{14}$$

finite or infinite.

It is natural to ask what happens when $g(x)$ is not positive; and this problem seems to be more troublesome†. The proof above breaks down, since the β'_n are no longer positive. However, we may note certain special cases which can be proved without difficulty. We can ignore the behaviour of g in any interval (δ,π), where $0<\delta<\pi$; for we can write $g = g_1+g_2$, where g_1 vanishes in $(0,\delta)$ and g_2 in (δ,π), and, since the series defining f converges uniformly in (δ,π), (14) holds for f and g_1. Thus in Theorem 19 it is enough for g to be positive near the origin. When g is negative everywhere, the equation (14) still holds

(*a*) if $\Sigma b'_n/n$ is convergent (since f then belongs to $L(0,\pi)$ and g is of bounded variation near the origin), or

(*b*) if the right-hand side converges, or

(*c*) if the left-hand side is $-\infty$.

We obtain (*b*) and (*c*) from Theorem 27 below. The only case which remains in doubt is that in which g is small and negative near the origin, and $f(x)$ does not belong to $L(0,\pi)$ while $f(x)g(x)$ does; even in this case, Theorem 28 gives a solution when b'_n is a convex sequence.

* Hardy and Rogosinski, op. cit. p. 33.

† In Theorem 11 of M.F. (II), we had another instance in which the relaxation of a sign condition caused difficulty. The present case is quite different. In Theorem 11, we could extend our results easily to functions g which were bounded below; the difficulty arose over functions g tending to $-\infty$ as $t\to 2\pi$. Here, the difficult case is that in which g is small and negative near the origin.

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It is perhaps of interest to show, before we leave the subject of sine series, that the inequality (5) may be improved near the origin. I do not know whether this has already been pointed out.

THEOREM 20. *If the sequence b_n decreases to zero, and*

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx,$$

then there is a corresponding interval $(0, \delta_f)$ in which $f(x)$ is positive.

Proof. By partial summation,

$$\begin{aligned} f(x) &= \sum_{n=1}^{\infty} \Delta b_n \frac{\cos \frac{1}{2}x - \cos(n+\frac{1}{2})x}{2\sin \frac{1}{2}x} \quad (0 < x < 2\pi) \\ &= \sum_{n=1}^{\infty} \Delta b_n \frac{\{1-\cos(n+\frac{1}{2})x\} - \{1-\cos \frac{1}{2}x\}}{2\sin \frac{1}{2}x} \\ &= \left\{ \sum_{n=1}^{N} + \sum_{n=N+1}^{\infty} \right\} \Delta b_n \frac{\sin^2 \frac{1}{2}(n+\frac{1}{2})x - \sin^2 \frac{1}{4}x}{\sin \frac{1}{2}x}, \end{aligned}$$

where N is chosen so that $b_{N+1} < \frac{1}{2}b_1$. If $0 < x < \pi/(N+\frac{1}{2})$, and $1 \leqslant n \leqslant N$, then

$$\sin^2 \tfrac{1}{2}(n+\tfrac{1}{2})x > \left(\frac{2}{\pi}\right)^2 \tfrac{1}{4}(n+\tfrac{1}{2})^2 x^2 > \frac{36}{\pi^2} \sin^2 \tfrac{1}{4}x > 2\sin^2 \tfrac{1}{4}x,$$

and so $$\sum_{n=1}^{N} \Delta b_n \frac{\sin^2 \frac{1}{2}(n+\frac{1}{2})x - \sin^2 \frac{1}{4}x}{\sin \frac{1}{2}x} > \frac{\sin^2 \frac{1}{4}x}{\sin \frac{1}{2}x} \sum_{n=1}^{N} \Delta b_n = \tfrac{1}{2}(b_1 - b_{N+1}) \tan \tfrac{1}{4}x.$$

As for the remaining terms, we have

$$\sum_{n=N+1}^{\infty} \Delta b_n \frac{\sin^2 \frac{1}{2}(n+\frac{1}{2})x - \sin^2 \frac{1}{4}x}{\sin \frac{1}{2}x} \geqslant -\frac{\sin^2 \frac{1}{4}x}{\sin \frac{1}{2}x} b_{N+1} = -\tfrac{1}{2}b_{N+1} \tan \tfrac{1}{4}x.$$

Thus $$f(x) > \tfrac{1}{2}(b_1 - 2b_{N+1}) \tan \tfrac{1}{4}x > 0$$

if $0 < x < \pi/(N+\frac{1}{2})$.

At first sight it appears that this result might be used to simplify the proof of Theorem 16, since some awkwardness was caused in that theorem by the fact that the right-hand side of (5) was not zero. However, in Theorem 16 we applied (V) to infinitely many functions of the form (4), and in order to use the method suggested we should need a fixed interval in which all of these were positive. Now there is certainly no interval in which all functions of the form (4) are positive; we have only to consider the function

$$\sin x + \sin 2x + \ldots + \sin nx = \frac{\cos \frac{1}{2}x - \cos(n+\frac{1}{2})x}{2\sin \frac{1}{2}x},$$

which is negative if $x = 2\pi/(n+\frac{1}{2})$. Thus we cannot make use of Theorem 20 in this way.

The remaining theorems of this section deal with cosine series. We shall find that (VII) is a more effective weapon than (V), and that the cosine proofs are more elegant than the sine ones in consequence.

THEOREM 21. *If a_n, α_n are convex sequences which decrease to zero, and if*

$$f(x) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx, \quad g(x) = \tfrac{1}{2}\alpha_0 + \sum_{n=1}^{\infty} \alpha_n \cos nx,$$

then

$$\frac{1}{\pi}\int_0^{2\pi} f(x)\,g(x)\,dx = \tfrac{1}{2}a_0\alpha_0 + \sum_{n=1}^{\infty} a_n\alpha_n,$$

finite or infinite.

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Proof. Since f and g are positive in $(0, 2\pi)$, and not merely bounded below in $(0, \pi)$ as in Theorem 16, we can follow the transform proof more closely than was possible in that theorem; there is no need for an extra term corresponding to $(b_1 - b_N)\sin x$ in (7). By defining

$$f(N,x) = \tfrac{1}{2}\{a_0 - a_N - N\Delta a_N\} + \sum_{n=1}^{N} \{a_n - a_N - (N-n)\,\Delta a_N\} \cos nx,$$

$$g(N,x) = \tfrac{1}{2}\{\alpha_0 - \alpha_N - N\Delta\alpha_N\} + \sum_{n=1}^{N} \{\alpha_n - \alpha_N - (N-n)\,\Delta\alpha_N\} \cos nx,$$

we obtain increasing sequences of functions which tend to $f(x)$ and $g(x)$ in $(0, 2\pi)$. The Parseval formula holds for $f(N,x)$ and $g(N,x)$; we deduce, by the method of monotonic convergence, that it holds for $f(x)$ and $g(x)$ in the 'finite or infinite' sense.

The result for cosine series corresponding to Theorem 17 has already been proved (M.F. (II), Theorem 15); it did not require the functions to be convex, and was dealt with by methods quite different from those of this section.

In the following theorem, the result as well as the proof is slightly simpler than in the corresponding theorem on sine series, since the integral concerned can be taken in the Lebesgue sense.

THEOREM 22. *Suppose that a_n is a convex sequence which decreases to zero, that*

$$f(x) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx,$$

and that $g(x)$ is a convex function of L in $(0, 2\pi)$, having Fourier cosine coefficients α_n. Then

$$\frac{1}{\pi}\int_0^{2\pi} f(x)\,g(x)\,dx = \tfrac{1}{2}a_0\alpha_0 + \sum_{n=1}^{\infty} a_n\alpha_n,$$

finite or infinite.

Proof. It is enough to consider positive functions g. For g is bounded below in $(0, 2\pi)$, since it is convex; let C be a lower bound. By (VII), f belongs to $L(0, 2\pi)$, and

$$\frac{1}{\pi}\int_0^{2\pi} Cf(x)\,dx = a_0 C.$$

Thus the theorem is true for f and g if it is true for f and $g + C$.

Suppose, then, that g is positive. Defining $f(N,x)$ as in Theorem 21, we have

$$\frac{1}{\pi}\int_0^{2\pi} f(N,x)\,g(x)\,dx = \tfrac{1}{2}\{a_0 - a_N - N\Delta a_N\}\alpha_0 + \sum_{n=1}^{N} \{a_n - a_N - (N-n)\,\Delta a_N\}\alpha_n,$$

and the result follows in the usual way.

Since $g(x)$ need not be decreasing in Theorem 22, we have as a corollary

THEOREM 23. *Suppose that a_n' is a convex sequence which decreases to zero, that*

$$f(x) = \tfrac{1}{2}a_0' + \sum_{n=1}^{\infty} a_n' \cos nx,$$

and that $g(x)$ is a decreasing, convex function of L in $(0, \pi)$, with half-range cosine coefficients α_n'. Then

$$\frac{2}{\pi}\int_0^{\pi} f(x)\,g(x)\,dx = \tfrac{1}{2}a_0'\alpha_0' + \sum_{n=1}^{\infty} a_n'\alpha_n',$$

finite or infinite.

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We have only to apply Theorem 22 to an even function which coincides with g in $(0, \pi)$.

If g is convex but not decreasing in $(0, \pi)$, α_n' need no longer be positive; and we might expect complications on this account, similar to those discussed on p. 254. As a matter of fact, no such difficulties arise, since f is now known to be a function of $L(0, \pi)$. As before, we may confine our attention to the behaviour of g near the origin; and here, g is either decreasing or of bounded variation. Appealing to Theorem 23 and to (III) (see footnote, p. 252), we obtain

THEOREM 24. *Theorem* 23 *remains true if the word 'decreasing' is omitted in the description of* g.

3. COSINE THEOREMS ON FUNCTIONS WHICH ARE NOT CONVEX

In view of Theorem 6 for transforms, we naturally look for cosine results of type [C] in which there is no convexity condition. We shall prove the following theorems.

THEOREM 25. *Suppose that* a_n *is a sequence which decreases to zero, that* $f(x)$ *is defined by*

$$f(x) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx,$$

and that $g(x)$ *is a decreasing function of* L *in* $(0, 2\pi)$, *with Fourier cosine coefficients* α_n. *Then*

$$\frac{1}{\pi}\int_{\to 0}^{\to 2\pi} f(x)\, g(x)\, dx = \tfrac{1}{2}a_0\alpha_0 + \sum_{n=1}^{\infty} a_n\alpha_n, \tag{15}$$

provided that

either (i) *the left-hand side exists,*

or (ii) $g(x)$ *is bounded below, and the right-hand side exists.*

THEOREM 26. *Suppose that* a_n' *is a sequence which decreases to zero, that* $f(x)$ *is defined by*

$$f(x) = \tfrac{1}{2}a_0' + \sum_{n=1}^{\infty} a_n' \cos nx,$$

and that $g(x)$ *is a decreasing function of* L *in* $(0, \pi)$, *with half-range cosine coefficients* α_n'. *Then*

$$\frac{2}{\pi}\int_{\to 0}^{\pi} f(x)\, g(x)\, dx = \tfrac{1}{2}a_0'\alpha_0' + \sum_{n=1}^{\infty} a_n'\alpha_n',$$

provided that either side exists.

A special case of these theorems was given by W. H. Young*, who showed that

(VIII) *if* f *is defined as in Theorem* 25, *then*

$$\int_{\to 0}^{X} f(x)\, dx = \tfrac{1}{2}a_0 X + \sum_{n=1}^{\infty} \frac{a_n \sin nX}{n} \quad (0 < X < 2\pi),$$

$$\int_{\to 0}^{\to 2\pi} f(x)\, dx = \pi a_0.$$

That is, the series defining f may be integrated term by term, although the integral may not be absolutely convergent. In particular, (15) holds if g is constant; and it is

* Young, W. H., 'On the Fourier series of bounded functions', *Proc. London Math. Soc.* (2), 12 (1912), 41–70 (45). The proof, which is well known, is given in Hobson, E. W., *The theory of functions of a real variable*, 2 (Cambridge, 1926), 619.

https://doi.org/10.1017/S0305004100025718 Published online by Cambridge University Press

very easy to generalize Young's argument in the following way, so as to prove (15) for functions g of bounded variation. It is enough to consider the case in which g is positive, decreasing and bounded. Since the series $\Sigma g(x)\, a_n \cos nx$ converges uniformly in $(\delta, 2\pi - \delta')$, where $0 < \delta < 2\pi - \delta' < 2\pi$, we have

$$\int_\delta^{2\pi-\delta'} f(x)\, g(x)\, dx = \tfrac{1}{2}a_0 \int_\delta^{2\pi-\delta} g(x)\, dx + \sum_{n=1}^{\infty} a_n \int_\delta^{2\pi-\delta'} g(x) \cos nx\, dx.$$

The series on the right-hand side converges uniformly for all such δ, δ'. For, by the second mean value theorem,

$$\begin{aligned}\sum_{n=N}^{N'} a_n \int_\delta^{2\pi-\delta'} g(x) \cos nx\, dx &= g(\delta) \sum_{n=N}^{N'} a_n \int_\delta^{\xi} \cos nx\, dx \quad (\delta \leqslant \xi \leqslant 2\pi - \delta') \\ &= g(\delta) \sum_{n=N}^{N'} \frac{a_n(\sin n\xi - \sin n\delta)}{n} \\ &= O(a_N)\end{aligned}$$

uniformly in δ and δ', since $\Sigma \sin nx/n$ is boundedly convergent. We may therefore take limits term by term as δ and δ' tend to zero.

This result is of course included in Theorem 25; it has been discussed separately because we shall make use of it in the general proof.

The first step is to consider the properties of series which correspond to Lemmas 3 and 4* of M.F. (I). These are as follows:

LEMMA 11. *If the sequence a_n decreases to zero, and if α_n is any sequence such that $\sum_{n=1}^{\infty} a_n \alpha_n$ converges, then*

$$\sum_{n=1}^{\infty} \Delta a_n \sum_{\nu=1}^{n} \alpha_\nu = \sum_{n=1}^{\infty} a_n \alpha_n.$$

LEMMA 12. *Suppose that b_n is a sequence which decreases to zero and satisfies*

$$\overline{\lim_{n\to\infty}} (b_{2n}/b_n) < 1. \tag{16}$$

Let

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx, \quad f_N(x) = \sum_{n=1}^{N} b_n \sin nx. \tag{17}$$

Then there is a constant C such that

$$|f_N(x)| \leqslant C f(x)$$

for all N and all sufficiently small x.

Lemma 11 is simply a special case of its counterpart for transforms; we have only to apply Lemma 3 to the functions f and g defined by

$$f(t) = a_n, \quad g(t) = \alpha_n \quad (n-1 \leqslant t < n),$$

or we could, of course, adapt the proof of Lemma 3 by using partial summation and Abel's lemma instead of partial integration and the second mean value theorem. Lemma 12 is more troublesome. Naturally, the proof is based on the same ideas as that of

* These lemmas are quoted in footnotes on p. 260.

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Lemma 4, and the main outline of the argument is quite straightforward; but there are tiresome complications of detail, which arise from the fact that the values of $\sin nx$ do not in general recur. These trivial points lengthen the discussion a good deal; and, while they plainly cannot cause any genuine difficulty, it is perhaps as well to set out the proof for the sake of completeness. It will be found on pp. 261–263. In the meantime we shall assume Lemma 12, in order that we may pass straight to the proof of Theorems 25 and 26.

It is easy to see that these two theorems are equivalent. Suppose for instance that Theorem 25 is proved. In Theorem 26, just as in the corresponding sine theorem*, the behaviour of $g(x)$ near the point π is not important, and we may therefore suppose without loss of generality that $g(x)$ is bounded below in $(0, \pi)$. Using Young's result (VIII), we find that it is then enough to consider functions g which are positive in $(0, \pi)$. In this case, we may apply Theorem 25 to the function which is equal to $g(x)$ in $(0, \pi)$ and to zero in $(\pi, 2\pi)$; this gives Theorem 26. It would also be possible to adopt the reverse procedure, proving Theorem 26 and deducing Theorem 25 from it.

Proof of Theorem 25. (i) We begin by considering the case in which $\int_{\to 0}^{\to 2\pi} fg$ is known to exist.

Suppose first that $g(x)$ is positive. Then we may assume without further loss of generality that $g(x)$ vanishes in the interval $(\delta, 2\pi)$, where δ is any given number satisfying $0 < \delta < 2\pi$. For if

$$\left.\begin{aligned} g_1(x) &= 0 \qquad (0 \leqslant x < \delta) \\ &= g(x) \quad (\delta \leqslant x \leqslant 2\pi), \end{aligned}\right\}$$

then g_1 is of bounded variation, and so (15) holds for f and g_1 by the generalized form of (VIII). We shall choose δ by means of the inequality (19) below.

Since $g(2\pi) = 0$, we have

$$\alpha_0 = -\frac{1}{\pi}\int_0^{2\pi} x\,dg(x), \quad \alpha_n = -\frac{1}{\pi}\int_0^{2\pi} \frac{\sin nx}{n}\,dg(x) \quad (n \geqslant 1),$$

and so

$$\tfrac{1}{2}a_0\alpha_0 + \sum_{n=1}^{N} a_n\alpha_n = -\frac{1}{\pi}\int_0^{2\pi}\left\{\tfrac{1}{2}a_0 x + \sum_{n=1}^{N}\frac{a_n}{n}\sin nx\right\} dg(x). \tag{18}$$

Now, by (VIII),

$$\tfrac{1}{2}a_0 x + \sum_{n=1}^{\infty}\frac{a_n}{n}\sin nx = \int_{\to 0}^{x} f(u)\,du.$$

Since a_n/n satisfies the condition (16) of Lemma 12, there are constants C and δ such that

$$\left|\tfrac{1}{2}a_0 x + \sum_{n=1}^{N}\frac{a_n}{n}\sin nx\right| \leqslant C\left\{\tfrac{1}{2}a_0 x + \sum_{n=1}^{\infty}\frac{a_n}{n}\sin nx\right\} = C\int_{\to 0}^{x} f(u)\,du \tag{19}$$

for all N, if $0 < x \leqslant \delta$. It follows, by dominated convergence, that

$$\int_0^{\delta}\left\{\tfrac{1}{2}a_0 x + \sum_{n=1}^{N}\frac{a_n}{n}\sin nx\right\} dg(x) \to \int_0^{\delta} dg(x)\int_{\to 0}^{x} f(u)\,du$$

* See the discussion immediately after Theorem 19.

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as $N \to \infty$, provided the latter integral is finite; and this is the case by Lemma 3 of M.F. (I)*, since $\int_{\to 0}^{\delta} fg$ exists. As we have seen, we can suppose that $g(x)$ vanishes throughout $(\delta, 2\pi)$, and so

$$\int_0^{2\pi} \left\{ \tfrac{1}{2}a_0 x + \sum_{n=1}^{N} \frac{a_n}{n} \sin nx \right\} dg(x) \to \int_0^{2\pi} dg(x) \int_{\to 0}^{x} f(u)\, du.$$

Thus, by (18), $\Sigma a_n \alpha_n$ converges, and

$$\tfrac{1}{2}a_0\alpha_0 + \sum_{n=1}^{\infty} a_n \alpha_n = -\frac{1}{\pi}\int_0^{2\pi} dg(x) \int_{\to 0}^{x} f(u)\, du$$
$$= \frac{1}{\pi}\int_{\to 0}^{2\pi} f(x)\, g(x)\, dx$$

by Lemma 3.

If $g(x)$ is not positive, we may apply what has already been proved to the functions $g_+(x)$ and $-g_-(2\pi - x)$ separately†, observing that the cosine coefficients of the latter function are simply those of $g_-(x)$ with the sign reversed, and that

$$\int_{\to 0}^{\to 2\pi} -g_-(2\pi - x)\, f(x)\, dx = -\int_{\to 0}^{\to 2\pi} f(x)\, g_-(x)\, dx.$$

The full result follows on subtraction.

(ii) We are now given that $\Sigma a_n \alpha_n$ converges, and that g is bounded below. It is obviously enough to consider the case in which g is positive; and, by the argument used in (i), we may also suppose that g vanishes in the interval $(\pi, 2\pi)$. Now, by partial summation,

$$f(x) = \sum_{n=0}^{\infty} \Delta a_n \frac{\sin (n+\frac{1}{2})x}{2 \sin \frac{1}{2}x},$$

the series being uniformly convergent in any interval (δ, π), where $0 < \delta < \pi$. Thus

$$\frac{1}{\pi}\int_{\delta}^{2\pi} f(x)\, g(x)\, dx = \frac{1}{\pi}\int_{\delta}^{\pi} f(x)\, g(x)\, dx = \sum_{n=0}^{\infty} \Delta a_n \cdot \frac{1}{\pi}\int_{\delta}^{\pi} g(x) \frac{\sin (n+\frac{1}{2})x}{2 \sin \frac{1}{2}x}\, dx. \qquad (20)$$

We observe next that the function $g(x)/2 \sin \frac{1}{2}x$ satisfies the 'λ condition' of Lemma 4‡, for any $\lambda > \frac{1}{2}$ and for sufficiently small x; and that it is a positive decreasing function throughout $(0, \pi)$. The argument of Lemma 4 shows that, for sufficiently large n,

$$O \leqslant \int_0^{\delta} g(x) \frac{\sin (n+\frac{1}{2})x}{2 \sin \frac{1}{2}x}\, dx \leqslant \frac{1}{1-\lambda}\int_0^{\pi} g(x) \frac{\sin (n+\frac{1}{2})x}{2 \sin \frac{1}{2}x}\, dx,$$

and so

$$\left| \int_{\delta}^{\pi} g(x) \frac{\sin (n+\frac{1}{2})x}{2 \sin \frac{1}{2}x}\, dx \right| \leqslant \left(1 + \frac{1}{1-\lambda}\right) \int_0^{\pi} g(x) \frac{\sin (n+\frac{1}{2})x}{2 \sin \frac{1}{2}x}\, dx.$$

* Lemma 3 is as follows: *If $f(t)$ decreases to zero in $(0, \infty)$, and if $g(t)$ is any measurable function such that $\int_{\to 0}^{\to \infty} f(t)\, g(t)\, dt$ exists, then*

$$\int_{\to 0}^{\to \infty} df(t) \int_{\to 0}^{t} g(u)\, du = -\int_{\to 0}^{\to \infty} f(t)\, g(t)\, dt.$$

† These are of course defined by $g_+(x) = \max(g(x), 0)$, $g_-(x) = \min(g(x), 0)$, as in Theorem 17.

‡ Lemma 4 is given in M.F. (I), and is as follows: *If $f(t)$ decreases to zero in $(0, \infty)$, and satisfies*

$$f(2t) \leqslant \lambda f(t) \quad (t > 0,\ 0 < \lambda < 1,\ \lambda \text{ constant}),$$

and if

$$F_T(x) = \sqrt{\left(\frac{2}{\pi}\right)} \int_0^T f(t) \sin xt\, dt,$$

then

$$0 \leqslant F_T(x) \leqslant F_x(x)/(1-\lambda) \quad (x \geqslant 0,\ T \geqslant 0).$$

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It follows, by Weierstrass's M-test, that the right-hand side of (20) converges uniformly in δ and tends to

$$\sum_{n=0}^{\infty} \Delta a_n \cdot \frac{1}{\pi} \int_0^{\pi} g(x) \frac{\sin (n+\frac{1}{2})x}{2 \sin \frac{1}{2}x} dx \tag{21}$$

as $\delta \to 0$, provided that the latter series is convergent. Now the series (21) is simply

$$\tfrac{1}{2}\alpha_0 \Delta a_0 + \sum_{n=1}^{\infty} \Delta a_n \left(\tfrac{1}{2}\alpha_0 + \sum_{\nu=1}^{n} \alpha_\nu \right),$$

which converges by Lemma 11, since $\Sigma a_n \alpha_n$ is convergent. Thus fg has a Cauchy integral at the origin, and

$$\begin{aligned} \frac{1}{\pi} \int_{\to 0}^{2\pi} f(x) g(x) dx &= \tfrac{1}{2}\alpha_0 \Delta a_0 + \sum_{n=1}^{\infty} \Delta a_n \left(\tfrac{1}{2}\alpha_0 + \sum_{\nu=1}^{n} \alpha_\nu \right) \\ &= \tfrac{1}{2} a_0 \alpha_0 + \sum_{n=1}^{\infty} a_n \alpha_n \end{aligned}$$

by the same lemma.

If g is not bounded below, the convergence of $\Sigma a_n \alpha_n$ does not imply the existence of $\int_{\to 0}^{\to 2\pi} fg$. For instance, if g is odd the series automatically converges, since all terms are zero; but it is certainly possible to find a g of this type, and a corresponding f, such that $\int_{\to 0}^{\pi} fg$ does not exist. It is of course true in this case that $\lim_{\delta \to +0} \int_{\delta}^{2\pi - \delta} fg$ exists and has the right value; but it is not clear whether the convergence of the series always implies the existence of this limit.

It remains to prove Lemma 12, which we assumed when dealing with Theorem 25 (i).

In order to simplify the notation, we write $\sum_{p}^{q} u_n$ for the sum over values of n satisfying $p \leqslant n \leqslant q$, whether p and q are integers or not. The definition of $f_N(x)$ given in (17) is then applicable even when N is not an integer; it gives $f_N(x) = f_{[N]}(x)$, where $[N]$ denotes the greatest integer not exceeding N.

Proof of Lemma 12. We have first to show that, if $0 < x < \pi$,

$$\left.\begin{aligned} f_N(x) &\leqslant f_{\pi/x}(x) + b_{[\pi/x]+1} \quad (\text{all } N), \\ f_N(x) &\geqslant f_{2\pi/x}(x) - b_{[2\pi/x]+1} \quad (N > \pi/x). \end{aligned}\right\} \tag{22}$$

These are the inequalities corresponding to (3) of M.F. (I). We shall in fact obtain better inequalities; but (22) will be sufficient for our purpose.

Suppose that $N > \pi/x$ (otherwise the first inequality is obvious). Writing

$$N_1 = [\pi/x] + 1,$$

we have, by Abel's lemma,

$$\begin{aligned} f_N(x) - f_{\pi/x}(x) &= \sum_{n=N_1}^{N} b_n \sin nx \\ &\leqslant b_{N_1} \max_{N_1 \leqslant N' \leqslant N} \sum_{n=N_1}^{N'} \sin nx \\ &= b_N \max_{N_1 \leqslant N' \leqslant N} \frac{\cos (N_1 - \frac{1}{2})x - \cos (N' + \frac{1}{2})x}{2 \sin \frac{1}{2}x} \\ &\leqslant b_N \frac{\cos^2 \frac{1}{2}(N_1 - \frac{1}{2})x}{\sin \frac{1}{2}x}. \end{aligned}$$

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Since $\pi/x < N_1 \leqslant \pi/x + 1$, we have

$$\tfrac{1}{2}\pi - \tfrac{1}{4}x < \tfrac{1}{2}(N_1 - \tfrac{1}{2})x \leqslant \tfrac{1}{2}\pi + \tfrac{1}{4}x,$$

which gives

$$|\cos \tfrac{1}{2}(N_1 - \tfrac{1}{2})x| \leqslant \sin \tfrac{1}{4}x.$$

Thus

$$f_N(x) \leqslant f_{\pi/x}(x) + b_{N_1}\frac{\sin^2 \tfrac{1}{4}x}{\sin \tfrac{1}{2}x} \leqslant f_{\pi/x}(x) + b_{[\pi/x]+1}. \tag{23}$$

The second part of (22) can be proved similarly.

We now come to the most important part of the argument, the proof that $f_{\pi/x}(x)$ and $f_{2\pi/x}(x)$ are of the same order of magnitude.

We write

$$f_{2\pi/x} = S_1 - S_2,$$

where

$$S_1 = \sum_1^{\pi/x} b_n \sin nx, \quad S_2 = \sum_{\pi/x}^{2\pi/x} b_n |\sin nx|.$$

If π/x is an integer*, we can follow the argument of Lemma 4 very closely, pairing off each term of S_1 with a term of S_2 having the same sine factor. This cannot be done in general; instead, we arrange a pairing of most of the terms so that the sine factor in a term of S_1 is at least as great as in the corresponding term of S_2. Fig. 1 illustrates

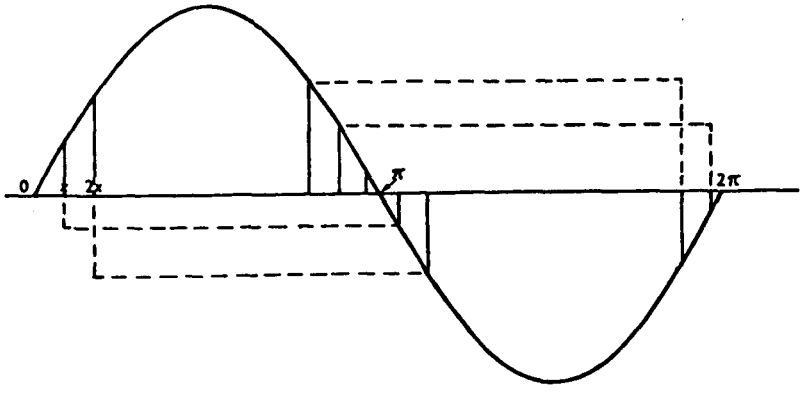


Fig. 1. Ordinates are taken at intervals of x on the graph of the function $\sin t$; thus the ordinates drawn near the point π are those at $([\pi/x]-2)x$, $([\pi/x]-1)x$, $[\pi/x]x$, $([\pi/x]+1)x$, $([\pi/x]+2)x$.

a possible method. The first term of S_1 is paired with the first of S_2, the second with the second, and so on, up to the $[\pi/2x]$th term (that is, for about half the terms of S_1). We then start at the other end of the range, and pair the last term *but one* of S_1 with the last of S_2, the last but two of S_1 with the last but one of S_2, and so on until all terms of S_1 except the last are exhausted. At most two terms of S_2 are left out, and their absolute value is less than $2b_{[\pi/x]+1}$. Taking together the paired values of n, we obtain

$$f_{2\pi/x}(x) \geqslant \sum_1^{\pi/x} (b_n - b_\nu) \sin nx - 2b_{[\pi/x]+1},$$

where ν is the integer corresponding to n (and b_ν is replaced by zero for $n = [\pi/x]$, which has no corresponding ν).

* In this case, the term given by $n = \pi/x$ appears both in S_1 and in S_2; but as it is zero, this does not matter.

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Suppose now that λ is any number satisfying

$$\overline{\lim_{n\to\infty}}\,(b_{2n}/b_n) < \lambda < 1.$$

Then there is a corresponding number n_0, such that $b_{2n}/b_n < \lambda$ for all $n > n_0$. We may suppose x so small that

$$b_{[\pi/x]+1} < \lambda b_{n_0}, \tag{24}$$

and then, *a fortiori*,

$$b_\nu < \lambda b_n \tag{25}$$

if $n \leqslant n_0$, since $\nu \geqslant [\pi/x]+1$. Moreover, (25) plainly holds for $n > n_0$, since it is obvious that $\nu > 2n$. Thus

$$\begin{aligned} f_{2\pi/x} &\geqslant (1-\lambda)\sum_1^{\pi/x} b_n \sin nx - 2b_{[\pi/x]+1} \\ &= (1-\lambda) f_{\pi/x}(x) - 2b_{[\pi/x]+1}. \end{aligned} \tag{26}$$

Now the last term is much smaller than $f_{\pi/x}(x)$ if x is small; in fact

$$f_{\pi/x}(x) \geqslant \sum_{\pi/4x}^{3\pi/4x} b_n \sin nx > \frac{1}{\sqrt{2}}\left(\frac{\pi}{2x} - 1\right) b_{[3\pi/4x]}.$$

Thus we may certainly choose δ so that

$$b_{[\pi/x]+1} < \tfrac{1}{4}(1-\lambda) f_{\pi/x}(x) \tag{27}$$

if $0 < x < \delta$ (we suppose also that (24) holds in this interval). We then have

$$f_{2\pi/x}(x) \geqslant \tfrac{1}{2}(1-\lambda) f_{\pi/x}(x) \tag{28}$$

from (26). Using (22), (27) and (28), we obtain

$$|f_N(x)| < \tfrac{5}{4} f_{\pi/x}(x),$$

$$f(x) \geqslant f_{2\pi/x}(x) - \tfrac{1}{4}(1-\lambda) f_{\pi/x}(x) \geqslant \tfrac{1}{4}(1-\lambda) f_{\pi/x}(x),$$

and so

$$|f_N(x)| < \frac{5}{1-\lambda} f(x)$$

in the interval $(0, \delta)$.

We cannot replace $(0, \delta)$ by $(0, \pi)$ in this lemma, since f need not be positive in $(0, \pi)$; nor is there any advantage to be gained in this respect by supposing that $b_{2n}/b_n < \lambda < 1$ for all n, instead of working with the weaker condition (16).

Theorems 25 and 26 are not true in the 'finite or infinite' sense; we can construct an example* in which a_n' and g decrease (g being positive), and $\sum_1^\infty a_n' \alpha_n' = \infty$, while $\int_\delta^\pi fg$ does not tend to any limit, finite or infinite, as $\delta \to 0$.

4. Further theorems on sine series

Finally, we prove two theorems of a rather different kind, which nevertheless throw some light on part of our previous work.

An essential feature of our main problem was that two of the functions concerned were monotonic. In the last section of M.F. (I), we turned from theorems of this kind to theorems on the equation (1·2) in which only one of the functions (say f) was monotonic, and a condition of a different kind was imposed upon G_s. Our conclusions were roughly as follows; we suppose in (i)–(iii) that $f(t)$ decreases to zero in $(0, \infty)$.

* See the forthcoming paper M.F. (IV).

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(i) *If $G_s(x)$ is positive and $g(t)$ is a function satisfying*

$$\int_{\to 0}^{t} g(u)\,du = \sqrt{\left(\frac{2}{\pi}\right)} \int_0^\infty G_s(x) \frac{1-\cos xt}{x}\,dx \quad (t>0),$$

then (1·2) *is true when* $\int_{\to 0}^{\to \infty} fg$ *exists.* It was pointed out that the equation connecting g and G_s is satisfied when these functions are sine transforms of each other in any of the usual senses.

(ii) *If f decreases rapidly enough for F_s to be integrable in the neighbourhood of the origin, and G_s is a bounded function of $L(0,\infty)$, then* (1·2) *is true.* Here there is no existence condition, but the other conditions make it obvious that $\int_0^\infty F_s G_s$ must exist.

(iii) *If f decreases still more rapidly, or if it decreases in a 'regular' way*, and if G_s belongs to $L(0,\infty)$, then the mere existence of $\int_0^\infty F_s G_s$ is enough to justify* (1·2)*; G_s need not be bounded.*

It was also mentioned (though not proved) that the existence condition in (i) cannot be transferred to the integral $\int_0^\infty F_s G_s$, even if f satisfies the more stringent conditions of (ii).

A theorem of this kind has only one interesting analogue for series. The difficulties are concerned with the behaviour of f at infinity and of G_s near the origin. Thus we naturally correlate f with a decreasing sequence of coefficients and G_s with an integrable function.

THEOREM 27. *Suppose that b_n' is a sequence which decreases to zero, that $f(x)$ is defined by*

$$f(x) = \sum_{n=1}^{\infty} b_n' \sin nx, \tag{29}$$

and that $g(x)$ is a positive function of $L(0,\pi)$, with half-range sine coefficients β_n'. Then

$$\frac{2}{\pi}\int_0^\pi f(x)\,g(x)\,dx \leqslant \lim_{N\to\infty} \sum_{n=1}^{N} b_n'\beta_n', \tag{30}$$

finite or infinite. If $\sum_{n=1}^{\infty} b_n'\beta_n'$ converges, then

$$\frac{2}{\pi}\int_0^\pi f(x)\,g(x)\,dx = \sum_{n=1}^{\infty} b_n'\beta_n'.$$

Proof. Let $$f_N(x) = \sum_{n=1}^{N} b_n' \sin nx.$$

Since $f_N(x) \geqslant -\frac{1}{2}b_1' \tan \frac{1}{4}x$, we have by Fatou's lemma,

$$\frac{2}{\pi}\int_0^\pi g(x)\{f(x)+\tfrac{1}{2}b_1'\tan\tfrac{1}{4}x\}\,dx \leqslant \lim_{N\to\infty}\frac{2}{\pi}\int_0^\pi g(x)\{f_N(x)+\tfrac{1}{2}b_1'\tan\tfrac{1}{4}x\}\,dx$$

$$= \lim_{N\to\infty}\left\{\sum_{n=1}^{N} b_n'\beta_n'\right\} + \frac{2}{\pi}\int_0^\pi \tfrac{1}{2}b_1' g(x)\tan\tfrac{1}{4}x\,dx$$

since $g(x)$ belongs to $L(0,\pi)$. The first part of the theorem follows at once.

* I.e. f either satisfies $\overline{\lim_{t\to\infty}} f(2t)/f(t) < 1$, or is convex.

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The proof of the second part is based on that of Theorem 8 for transforms. We start from the equation

$$f(x) = \sum_{n=1}^{\infty} \Delta b'_n \frac{1-\cos(n+\frac{1}{2})x}{2\sin\frac{1}{2}x} - \tfrac{1}{2}b'_1 \tan\tfrac{1}{4}x.$$

Since all terms of the series are positive, we may multiply by the positive function $g(x)$ and integrate term by term. We must, of course, interpret our equations in the 'finite or infinite' sense until the last step of the proof, when it becomes evident that we have been dealing with the finite case throughout. We have

$$\begin{aligned}\frac{2}{\pi}\int_0^\pi f(x)\,g(x)\,dx &= \sum_{n=1}^{\infty} \Delta b'_n \cdot \frac{2}{\pi}\int_0^\pi g(x)\,\frac{1-\cos(n+\frac{1}{2})x}{2\sin\frac{1}{2}x}\,dx - \frac{1}{\pi}b'_1\int_0^\pi g(x)\tan\tfrac{1}{4}x\,dx \\ &= \sum_{n=1}^{\infty} \Delta b'_n \cdot \frac{2}{\pi}\int_0^\pi g(x)\,\frac{\cos\frac{1}{2}x-\cos(n+\frac{1}{2})x}{2\sin\frac{1}{2}x}\,dx \\ &= \sum_{n=1}^{\infty} \Delta b'_n \sum_{\nu=1}^{n} \beta'_\nu \\ &= \sum_{n=1}^{\infty} b'_n \beta'_n\end{aligned}$$

by Lemma 11, since the last series converges.

As a matter of fact, the hypotheses of Theorem 27 can be weakened. The function g was supposed integrable in order that the coefficients β'_n might be defined; but all the requisite integrals exist if $xg(x)$ belongs to $L(0,\pi)$, and this is all we need assume.

Just as for transforms, we can modify the conditions of Theorem 27 in other ways. It is easy to show that (30) holds if g is bounded below instead of positive, provided that $\Sigma b'_n/n$ is convergent. By applying this result both to g and to $-g$, we can obtain the following theorem, which was given by W. H. Young*; his proof was apparently different from this, but was based on similar ideas.

If $g(x)$ is any bounded measurable function, and $f(x)$ is defined by (29), *where b'_n decreases to zero and $\sum_{n=1}^{\infty} b'_n/n$ is convergent, then*

$$\frac{2}{\pi}\int_0^\pi f(x)\,g(x)\,dx = \sum_{n=1}^{\infty} b'_n\beta'_n.$$

The corresponding full-range result

$$\frac{1}{\pi}\int_0^{2\pi} f(x)\,g(x)\,dx = \sum_{n=1}^{\infty} b_n\beta_n \quad (\text{where } b_n = b'_n)$$

also holds in these circumstances.

We now consider what happens if we are given the existence of the integral $\int fg$ but not the convergence of the series $\Sigma b'_n\beta'_n$. It is found that the restrictions so far imposed on the sequence b'_n will no longer serve; we can construct an example in which b'_n

* See the paper referred to on p. 257. A number of theorems of this kind were proved by Young in this and other papers; but his standpoint was slightly different from that adopted here. As far as I know, he was concerned with conditions bearing directly on the functions themselves, and not with existence conditions on the integrals or series (except when he had already proved a summability theorem). Thus he gave the result quoted here, but I have not found Theorem 27, for instance, in his work.

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decreases to zero, $\Sigma b_n'/n$ is convergent, $g(x)$ is a positive function of $L(0,\pi)$, and $f(x)\,g(x)$ belongs to $L(0,\pi)$ but $\Sigma b_n'\beta_n'$ does not converge*. However, by restricting b_n' in a different way, we do obtain a theorem of this type; the sign of $g(x)$ is now immaterial.

We have first to prove a lemma corresponding to Lemma 6 of M.F. (I). The proof is a little tiresome, for the same reasons as in the case of Lemma 12. However, it is rather interesting to note that the lemma gives conditions under which f is positive throughout $(0,\pi)$; as we have seen, this is not the case for all sequences b_n which decrease to zero. This point does not arise for transforms, since the sine transform of a decreasing function is positive.

LEMMA 13. *If b_n decreases to zero and is convex, and if*

$$f(x)=\sum_{n=1}^{\infty} b_n \sin nx, \quad f_N(x)=\sum_{n=1}^{N} b_n \sin nx,$$

then

$$-\tfrac{1}{2}b_1 \tan \tfrac{1}{4}x \leqslant f_N(x) \leqslant 2f(x)+\tfrac{3}{2}b_1 \tan \tfrac{1}{4}x$$

for all N, and all x in $(0,\pi)$. Moreover, $f(x)$ is positive for all x in $(0,\pi)$.

Proof. Suppose that x satisfies $0<x\leqslant\pi$. We have, by (5) and (23)†,

$$-\tfrac{1}{2}b_1 \tan \tfrac{1}{4}x \leqslant f_N(x) \leqslant f_{[\pi/x]}(x)+\tfrac{1}{2}b_{[\pi/x]+1} \tan \tfrac{1}{4}x. \tag{31}$$

Now
$$\begin{aligned} 2f(x)-f_{[\pi/x]}(x) &= \sum_{1}^{\infty} b_n \sin nx + \sum_{[\pi/x]+1}^{\infty} b_n \sin nx \\ &= \sum_{1}^{\infty} b_n \sin nx - \sum_{[\pi/x]+1}^{\infty} b_n \sin\left(n-\frac{\pi}{x}\right)x \\ &= \sum_{1}^{\infty} (b_n - b_{n+[\pi/x]}) \sin nx + \sum_{1}^{\infty} b_{n+[\pi/x]} \{\sin nx - \sin (n-\alpha)x\}, \end{aligned}$$

where $\alpha=(\pi/x)-[\pi/x]$. Since $b_n-b_{n+[\pi/x]}$ is a decreasing function of n for fixed x,

$$\sum_{1}^{\infty} (b_n - b_{n+[\pi/x]}) \sin nx \geqslant -\tfrac{1}{2}(b_1 - b_{[\pi/x]+1}) \tan \tfrac{1}{4}x.$$

Also
$$\begin{aligned} &\sum_{1}^{\infty} b_{n+[\pi/x]} \{\sin nx - \sin (n-\alpha)x\} \\ &\quad = 2\sin^2 \tfrac{1}{2}\alpha x \sum_{1}^{\infty} b_{n+[\pi/x]} \sin nx + \sin \alpha x \sum_{1}^{\infty} b_{n+[\pi/x]} \cos nx \\ &\quad \geqslant -b_{[\pi/x]+1} \sin^2 \tfrac{1}{2}\alpha x \tan \tfrac{1}{4}x \end{aligned}$$

by (V) and (VII) (p. 250), since $b_{n+[\pi/x]}$ decreases to zero and is convex, and since $0\leqslant\alpha<1$. It follows that

$$2f(x)-f_{[\pi/x]}(x) \geqslant -\tfrac{1}{2}(b_1+b_{[\pi/x]+1}) \tan \tfrac{1}{4}x \geqslant -b_1 \tan \tfrac{1}{4}x,$$

and so, by (31),
$$f_N(x) \leqslant 2f(x)+\tfrac{3}{2}b_1 \tan \tfrac{1}{4}x.$$

It is clear that the right-hand side could have been written in the form

$$2f(x)+\tfrac{1}{2}\{b_1+o(1)\} \tan \tfrac{1}{4}x \quad \text{as} \quad x\to 0.$$

* See M.F. (IV).

† We use square brackets to denote 'integer part of', as in Lemma 12; but we do not continue the practice of omitting these brackets in limits of summation, as this would be confusing here.

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Finally, we have to show that $f(x)$ is positive in $(0, \pi)$, and not merely bounded below. The natural method of attack is, of course, by partial summation. Applying this process twice, and observing that b_n and $n\Delta b_n$ tend to zero as $n \to \infty$, we obtain

$$\begin{aligned} f(x) &= \sum_1^\infty \Delta b_n \frac{\cos \tfrac{1}{2}x - \cos(n+\tfrac{1}{2})x}{2\sin \tfrac{1}{2}x} \qquad (0 < x \leqslant \pi) \\ &= \sum_1^\infty \Delta^2 b_n \frac{n\cos \tfrac{1}{2}x - \{(\sin(n+1)x/2\sin \tfrac{1}{2}x) - \cos \tfrac{1}{2}x\}}{2\sin \tfrac{1}{2}x} \\ &= \frac{1}{4\sin^2 \tfrac{1}{2}x} \sum_1^\infty \Delta^2 b_n . \phi_n(x), \end{aligned}$$

where $$\phi_n(x) = (n+1)\sin x - \sin(n+1)x.$$

It is elementary that $\phi_n(x)$ is positive for all positive values of n and all x in $(0, \pi)$; this may be proved, either by induction*, or by observing that the turning-points of $\phi_n(x)$ occur where $\cos x = \cos(n+1)x$, and so $|\sin x| = |\sin(n+1)x|$. Since $\Delta^2 b_n$ is also positive, we have the required result.

It is easy to see that the partial sums $f_N(x)$ are not necessarily positive in $(0, \pi)$; for instance, $\sin x + \frac{2}{3}\sin 2x = \sin x(1 + \frac{2}{3}\cos x)$, which is negative in $(\pi - \cos^{-1}\frac{2}{3}, \pi)$.

THEOREM 28. *Suppose that $g(x)$ belongs to $L(0, 2\pi)$, that b_n decreases to zero, and that either* (i) $\overline{\lim_{n\to\infty}}\,(b_{2n}/b_n) < 1$,

or (ii) *b_n is convex.*

Let $$f(x) = \sum_{n=1}^\infty b_n \sin nx,$$

and let the Fourier sine coefficients of g be β_n. Then

$$\frac{1}{\pi}\int_0^{2\pi} f(x)\,g(x)\,dx = \sum_{n=1}^\infty b_n\beta_n,$$

provided that the left-hand side exists. A similar result holds for the half-range series.

Proof. We have, as usual,

$$\frac{1}{\pi}\int_0^{2\pi} f_N(x)\,g(x)\,dx = \sum_{n=1}^N b_n\beta_n.$$

In case (i), $f_N(x)\,g(x)$ is dominated by $C\,|f(x)\,g(x)|$ in the interval $(0, \delta)$ of Lemma 12 and in $(2\pi - \delta, 2\pi)$, and is dominated by a multiple of $|g(x)|$ in $(\delta, 2\pi - \delta)$. In case (ii), $f_N(x)\,g(x)$ is dominated by $$2\,|f(x)\,g(x)| + \tfrac{3}{2}b_1\,|g(x)|\tan \tfrac{1}{2}x$$ in $(0, \pi)$, and by $$2\,|f(x)\,g(x)| + \tfrac{3}{2}b_1\,|g(x)|\tan \tfrac{1}{4}(2\pi - x)$$ in $(\pi, 2\pi)$, since f is odd. In both cases the dominating functions belong to $L(0, 2\pi)$, since $g(x)$ and $f(x)\,g(x)$ do; thus

$$\int_0^{2\pi} f_N(x)\,g(x)\,dx \to \int_0^{2\pi} f(x)\,g(x)\,dx$$

as $N \to \infty$. The half-range result is, of course, a special case of this.

* As suggested in Zygmund, op. cit. p. 12; this method requires n to be an integer, but this is of course all we need here.

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