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Asymptotic behaviour of positive solutions of semilinear elliptic problems with increasing powers

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We prove existence results of two solutions of the problem

$$\begin{cases} L(u) + u^{m-1} = \lambda u^{p-1} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $L(v) = -\mathrm{div}(M(x)\nabla v)$ is a linear operator, $p \in (2, 2^*]$ and λ and m sufficiently large. Then their asymptotical limit as $m \to +\infty$ is investigated showing different behaviours.

Keywords: Increasing powers in the lower order term; Asymptotical analysis; Semilinear elliptic equations

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1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$. We study the asymptotical behaviour for m tending to infinity of some positive solutions of the following semi-linear problem

$$\begin{cases} L(u) + u^{m-1} = \lambda u^{p-1} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $L(v) = -\mathrm{div}(M(x)\nabla v) : H_0^1(\Omega) \to H^{-1}(\Omega)$ is a linear operator in divergence form. The matrix $M(x) = (m_{ij}(x))$ is symmetric, bounded and positive definite,

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i.e., there exist positive constants $0 < \alpha < \beta$ such that

$$\alpha|\xi|^2 \leqslant M(x)\xi \cdot \xi \leqslant \beta|\xi|^2. \tag{1.2}$$

The exponents p, m such that

$$2 < p \leqslant 2^* < m, \quad \text{where } 2^* = \begin{cases} \dfrac{2N}{N-2} & \text{for } N > 2, \\ 2^* = +\infty & \text{for } N = 1, 2. \end{cases} \tag{1.3}$$

and λ is a positive parameter.

In order to perform our asymptotical analysis, we first study (1.1) for m large but fixed and we prove the following result.

THEOREM 1.1. *Assume conditions* (1.2) *and* (1.3). *Then, there exists* $\underline{\lambda} > 0$ *such that for each* $\lambda > \underline{\lambda}$ *there is* $m_0 > 2^*$ *such that for every* $m \geqslant m_0$, *problem* (1.1) *has, at least, two positive solutions* $u_m \not\equiv z_m \in H_0^1(\Omega) \cap L^m(\Omega)$.

The solutions u_m and z_m are found as nontrivial critical points of the functional

$$J_m(v) := \frac{1}{2}\int_\Omega M(x)\nabla v \cdot \nabla v + \frac{1}{m}\|v\|_m^m - \frac{\lambda}{p}\|v^+\|_p^p. \tag{1.4}$$

As m does not satisfy any bound from above [see (1.3)] we are naturally lead to study J_m defined on $H_0^1(\Omega) \cap L^m(\Omega)$, and we show that J_m has a global minimum point u_m at a negative level, and another critical point, found applying the Ambrosetti-Rabinowitz Theorem at a positive minimax level, that is z_m. Even at this stage we can see some crucial difference between these two solutions: the existence of a nontrivial global minimum point u_m follows assuming the weaker hypothesis on p

$$2 < p < m; \tag{1.5}$$

while, in order to have the existence of z_m at a positive action level we need (1.3).

Then we perform the asymptotical analysis of the solutions found for $m \to +\infty$. In this study a crucial role will be played by an $L^\infty(\Omega)$ a priori bound, which will prevent any blow up of the sequences of u_m and z_m even when $p = 2^*$. This marks a strong difference with the well-known explosion phenomenon for the Lane-Emden problem in bounded domain when p approaches 2^* for $N \geqslant 3$, as shown in [**4**, **6**, **11**, **13**], and even with the finite asymptotical behaviour of the $L^\infty(\Omega)$ norm of least energy solution in dimension 2 as obtained in [**1**]. Here, we see a completely different scenario which is more related with the results obtained in [**7**] and in [**5**] in the case of a datum $f(x)$ in the place of the nonlinearity u^{p-1}.

The absence of blowing up family of solution is due to the fact the diverging exponent is the one that involves the 'absorbing term' u^{m-1} on the left-hand side instead of the 'reaction one' u^{p-1} on the right-hand side.

As a consequence of the a priori bounds we are able to show the following result

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THEOREM 1.2. *There exist* $u, z \in \mathscr{K} := \{v \in H_0^1(\Omega) : 0 \leqslant v(x) \leqslant 1\}$, *such that* $u, z \not\equiv 0$, *and* $u_m \rightharpoonup u$ *weakly in* $H_0^1(\Omega)$, $z_m \to z$ *strongly in* $H_0^1(\Omega)$ *and the convergence is for both* u_m *and* z_m *strongly in every Lebesgue space. Moreover,* u *and* z *satisfy the following variational inequalities*

$$\int_\Omega M(x)\nabla w \cdot \nabla(v-w) \geqslant \lambda \int_\Omega w^{p-1}(v-w), \quad \forall\, v \in \mathscr{K}, \tag{1.6}$$

In addition, there exist $g_u, g_z \in L^\infty(\Omega)$, *such that*

$$0 \leqslant g_u,\, g_z \leqslant \lambda \quad g_u(x)[1-u(x)] = 0,\ g_z(x)[1-z(x)] = 0 \ \textit{a. e. in } \Omega,$$

and it results

$$L(u) + g_u = \lambda u^{p-1}, \quad L(z) + g_z = \lambda z^{p-1}, \quad \textit{in } \Omega. \tag{1.7}$$

Theorem 1.2 will be proved as a consequence of theorems 4.1 and 4.4 and it shows that problem (1.1) works as a nonlinear penalization procedure to obtain solutions of the variational inequality (1.6) (see [**5**, **7**]). The role of the functions g_u and g_z appearing in the limit equation (1.7) is well described by the equalities $g_u(1-u) = 0$ and $g_z(1-z) = 0$ almost everywhere. Indeed, these conditions tell us that g_u and g_z weight the set in which $u \equiv 1$ and $z \equiv 1$, respectively.

In the convergence procedure we can see other difference between the behaviour of u_m and z_m: First of all, we do not need to prove that u_m strongly converges in $H_0^1(\Omega)$ to a non-trivial limit u and even more we get this limit for every $p > 2$; on the other hand, we need to show that z_m strongly converges to z in $H_0^1(\Omega)$ in order to show that z is not trivial. Moreover, this is obtained by a comparison with the minimax value of the limit functional

$$J_\infty := \frac{1}{2}\int_\Omega M(x)\nabla v \cdot \nabla v - \frac{\lambda}{p}\int_\Omega |v|^p,$$

and the assumption $p \leqslant 2^*$ is crucial at this point (see for more details remarks 4.5 and 4.6).

However, the main difference between the two asymptotical behaviours concerns the function g_u and g_z. Indeed, while it is quite simple to show that g_u is not trivial thanks to an a priori uniform bound on the level of the global minimum point u_m of J_m, the analogous analysis on g_z seems to be really delicate and strongly depending on the domain and on the exponent p. For example, when Ω is star-shaped and $p = 2^*$ then g_z must be not trivial and the same occurs when Ω is a ball and $p \to 2^*$ from below or $p \to 2$ from above (see proposition 4.8); but whether or not g_z is trivial for a general $p \in (2, 2^*)$ remains an open question: we can only prove an abstract result (see proposition 4.10) when the domain Ω is a ball that actually shows that g_z may be trivial or not (see for more details remark 4.11).

The paper is organized as follows: in § 2 we show the existence of u_m and z_m for fixed m (sufficiently large), in § 3 the crucial a-priori bounds and in § 4 we perform the asymptotical analysis.

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2. Existence results for fixed m

We denote with $\|\cdot\|_r$ the norm in the Lebesgue space $L^r(\Omega)$ for $1 \leqslant r \leqslant \infty$ and with $\|\cdot\|$ the usual norm in the Sobolev space $H_0^1(\Omega)$. We will study problem (1.1) by variational methods, so that we consider the functional space $X_m := H_0^1(\Omega) \cap L^m(\Omega)$ endowed with the norm $\|\cdot\|_{X_m} = \|\cdot\| + \|\cdot\|_m$ and the functional $J_m : X_m \to \mathbb{R}$ defined in (1.4). A solution of problem (1.1) is a nonzero critical point of J_m, i.e., a function $w \in X_m$, $w(x) > 0$, a.e. $x \in \Omega$, such that the following equation is satisfied for every $v \in X_m$

$$\int_\Omega M(x)\nabla w \cdot \nabla v + \int_\Omega |w|^{m-2} w\, v - \lambda \int_\Omega |w|^{p-2} w\, v = 0 \tag{2.1}$$

We will prove theorem 1.1 as a consequence of two existence results, the first one is concerned with the existence of the minimal solution u_m

THEOREM 2.1. *Assume conditions* (1.2) *and* (1.5). *Then, there exists* $\underline{\lambda}_1 > 0$ *such that for each* $\lambda > \underline{\lambda}_1$ *there is* $m_0 > 2^*$ *such that for every* $m \geqslant m_0$, *problem* (1.1) *has a nonnegative, minimal solutions* $u_m \in H_0^1(\Omega) \cap L^m(\Omega)$, $u_m \not\equiv 0$ *with*

$$J_m(u_m) \leqslant -\delta < 0. \tag{2.2}$$

REMARK 2.2. Let us observe that $\underline{\lambda}_1 > 0$ is explicitly given in (2.11).

Proof. From Hölder and Young inequalities we obtain

$$\begin{aligned} J_m(v) &\geqslant \frac{\alpha}{2}\|v\|^2 + \frac{1}{m}\|v\|_m^m - \frac{\lambda}{p}\|v\|_m^p |\Omega|^{1-p/m} \\ &\geqslant \frac{\alpha}{2}\|v\|^2 + \frac{1}{2m}\|v\|_m^m - \frac{m-p}{p^2 m}|\Omega|\lambda^{\frac{m}{m-p}} 2^{\frac{p}{m-p}}, \end{aligned}$$

which implies that J_m is coercive in X_m. Moreover, if $v_m \rightharpoonup v$ in X_m, then $v_m \to v$ almost everywhere (up to a subsequence), so that $v_m \to v$ strongly in $L^p(\Omega)$ thanks to Egorov Theorem; this fact, and the weak lower semicontinuity of the norm in X_m imply that J_m is weakly lower semicontinuous. Then, there exists a global minimum point u_m of J_m. In order to prove (2.2), for any function $\psi \in H_0^1(\Omega) \cap L^\infty(\Omega)$, we consider the one variable, positive, real function $g_m(t)$ defined by

$$g_m(t) = \frac{a}{t^{p-2}} + b_m t^{m-p}, \quad a = \frac{\beta}{2}\frac{\|\psi\|^2}{\|\psi\|_p^p}, \quad b_m = \frac{1}{m}\frac{\|\psi\|_m^m}{\|\psi\|_p^p}. \tag{2.3}$$

Note that $m > p > 2$ implies that $\lim_{t\to 0} g(t) = \lim_{t\to+\infty} g(t) = +\infty$ so that, g attains its global minimum at the point T_m given by

$$T_m = \left[\frac{p-2}{m-p}\frac{a}{b_m}\right]^{\frac{1}{m-2}} = \left[\frac{\beta}{2}\frac{m(p-2)}{m-p}\|\psi\|^2\right]^{\frac{1}{m-2}} \|\psi\|_m^{-\frac{m}{m-2}}. \tag{2.4}$$

Notice that

$$J_m(T_m\psi) < 0 \;\Leftrightarrow\; \lambda > \lambda_m(\psi) := p \min_{(0,+\infty)} g_m = p g_m(T_m) \tag{2.5}$$

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where $g_m(T_m)$ is given by

$$\begin{aligned} g_m(T_m) &= \left(\frac{p-2}{m-p}\right)^{\frac{2-p}{m-2}} \frac{a^{\frac{m-p}{m-2}}}{b_m^{\frac{2-p}{m-2}}} + \left(\frac{p-2}{m-p}\right)^{\frac{m-p}{m-2}} \frac{a^{\frac{m-p}{m-2}}}{b_m^{\frac{2-p}{m-2}}} \\ &= a^{\frac{m-p}{m-2}} b_m^{\frac{p-2}{m-2}} \left[\left(\frac{p-2}{m-p}\right)^{\frac{2-p}{m-2}} + \left(\frac{p-2}{m-p}\right)^{\frac{m-p}{m-2}}\right] \\ &= a^{\frac{m-p}{m-2}} b_m^{\frac{p-2}{m-2}} \left(\frac{p-2}{m-p}\right)^{\frac{2-p}{m-2}} \frac{m-2}{m-p}. \end{aligned}$$

Notice that it results

$$\lim_{m\to+\infty} \lambda_m(\psi) = p\frac{\beta}{2}\frac{\|\psi\|^2\|\psi\|_\infty^{p-2}}{\|\psi\|_p^p} =: \Lambda(\psi), \tag{2.6}$$

and, taking into account (2.3) and (2.4), one has

$$\lim_{m\to+\infty} T_m = \lim_{m\to+\infty}\left[\frac{1}{b_m}\right]^{\frac{1}{m-2}} = \lim_{m\to+\infty}\left[\frac{1}{m}\frac{\|\psi\|_m^m}{\|\psi\|_p^p}\right]^{-\frac{1}{m-2}} = \frac{1}{\|\psi\|_\infty}. \tag{2.7}$$

Moreover, again recalling (2.3)

$$\begin{aligned} \lim_{m\to+\infty} [T_m\|\psi\|_m]^m &= \lim_{m\to+\infty}\left[(p-2)\frac{\beta}{2}\frac{m}{m-p}\right]^{\frac{m}{m-2}} \|\psi\|^{\frac{2m}{m-2}}\|\psi\|_m^{-\frac{2m}{m-2}} \\ &= \frac{\beta}{2}(p-2)\|\psi\|^2\|\psi\|_\infty^{-2}. \end{aligned} \tag{2.8}$$

This, joint with (1.4) and (2.7) yields

$$\lim_{m\to+\infty} J_m(T_m\psi) = J_\infty(T_\infty\psi) = \frac{\|\psi\|_\infty^{-2}}{2}\int_\Omega M(x)\nabla\psi\cdot\nabla\psi - \lambda\frac{\|\psi\|_\infty^{-p}}{p}\|\psi\|_p^p,$$

where J_∞ is defined by

$$J_\infty(v) = \frac{1}{2}\int_\Omega M(x)\nabla v\cdot\nabla v - \frac{\lambda}{p}\|v\|_p^p. \tag{2.9}$$

Notice that, due to (1.5) J_∞ may not be finite, but we are computing J_∞ only on $\psi \in H_0^1(\Omega)\cap L^\infty(\Omega)$. Moreover, it is easy to see that

$$J_\infty(T_\infty\psi) < 0 \;\Leftrightarrow\; \lambda > \Lambda(\psi). \tag{2.10}$$

Then, $\underline{\lambda}_1$ given by

$$\underline{\lambda}_1 := \inf_{\psi\in H_0^1(\Omega)\cap L^\infty(\Omega)} \Lambda(\psi) = \inf_{\psi\in H_0^1(\Omega)\cap L^\infty(\Omega)} p\frac{\beta}{2}\frac{\|\psi\|^2\|\psi\|_\infty^{p-2}}{\|\psi\|_p^p} \tag{2.11}$$

is well defined as $\Lambda(\varphi_1)$ is finite where φ_1 is the positive eigenfunction associated with the first positive eigenvalue of the laplacian operator with homogeneous

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Dirichlet boundary conditions, λ_1^{Dir}. Moreover, $\underline{\lambda}_1$ is positive as

$$\int_\Omega |\psi^p| = \int_\Omega |\psi|^{p-2}|\psi|^2 \leqslant \|\psi\|_\infty^{p-2}\int_\Omega |\psi|^2 \leqslant \lambda_1^{\mathrm{Dir}}\|\psi\|_\infty^{p-2}\|\psi\|^2$$

so that $\underline{\lambda_1} \geqslant p\frac{\beta}{2}\lambda_1^{\mathrm{Dir}}$. Thus, for every $\lambda > \underline{\lambda}_1$, we can fix

$$\psi_0 \in H_0^1(\Omega)\cap L^\infty(\Omega) \quad \text{such that} \quad \Lambda(\psi_0) < \lambda \tag{2.12}$$

yielding (2.10). Taking into account (2.8), we can find $\sigma > 0$ and sufficiently small and $m_1 > 2^*$ such that for $m \geqslant m_1 > 2^*$

$$J_m(T_m\psi_0) \leqslant J_\infty(T_\infty\psi_0) + \sigma = J_\infty\left(\frac{\psi_0}{\|\psi_0\|_\infty}\right) + \sigma < 0 \tag{2.13}$$

and the conclusion follows choosing $-\delta = J_\infty(\psi_0/\|\psi_0\|_\infty)$.

Finally, it is possible to obtain $u_m \geqslant 0$ by considering the modified functional

$$J(v) = \frac{1}{2}\int_\Omega M(x)\cdot\nabla v\cdot\nabla v + \frac{1}{m}\int_\Omega |v|^m - \frac{\lambda}{p}\int_\Omega (v^+)^p.$$

The same argument of the proof of theorem 2.1 yields a nonnegative minimum point u_m. □

Remark 2.3. Let us point out that the existence of u_m does not require m sufficiently large. On the other hand, the assumption on m sufficiently large is crucial to obtain the a priori upper bound on the level of the minimum point u_m.

Moreover, concerning the exponent p, the existence of the minimum point u_m also holds for p super-critical (with respect to the Sobolev embeddings).

The existence of the second nontrivial solution is given by the following existence result.

Theorem 2.4. *Assume conditions* (1.2) *and* (1.3). *Then, there exists* $\underline{\lambda} \geqslant \underline{\lambda}_1 > 0$ *[see* (2.11)*] such that for each* $\lambda > \underline{\lambda}$ *there is* $m_0 > \max\{2^*, m_1\}$ *(*m_1 *introduced in theorem* 2.1*), such that for every* $m \geqslant m_0$, *problem* (1.1) *has a nonnegative, critical point* $z_m \in H_0^1(\Omega)\cap L^m(\Omega)$ *with* $z_m \not\equiv 0$ *and* $z_m \not\equiv u_m$.

Proof. We will obtain the existence of z_m by applying the Ambrosetti-Rabinowitz theorem [**3**]. First of all, note that

$$\begin{aligned} J_m(v) &\geqslant \frac{1}{2}\|\nabla v\|_2^2 + \frac{1}{m}\|v\|_m^m - \lambda\frac{\mathscr{S}^{p/2}}{p}|\Omega|^{1-p/2^*}\|\nabla v\|_2^p \\ &= \frac{1}{m}\|v\|_m^m + \|\nabla v\|_2^2\left(\frac{1}{2} - \lambda\frac{\mathscr{S}^{p/2}}{p}|\Omega|^{1-p/2^*}\|\nabla v\|_2^{p-2}\right). \end{aligned}$$

Now, fix r_λ such that

$$r_\lambda := \min\left\{1, \left(\frac{p}{4\lambda\mathscr{S}^{p/2}|\Omega|^{1-p/2^*}}\right)^{1/p-2}\right\}$$

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and consider $v \in X_m$ such that $\|v\|_{X_m} = r_\lambda$. In case $r_\lambda \leqslant 1$ one obtains (for $m \geqslant 4$)

$$J_m(v) \geqslant \frac{1}{m}\|v\|_m^m + \frac{1}{4}\|\nabla v\|_2^2 \geqslant \frac{1}{m}\left(\|v\|_m^m + \|\nabla v\|_2^m\right)$$
$$\geqslant \frac{1}{m2^{m-1}}\left(\|v\|_m + \|\nabla v\|_2\right)^m = \frac{1}{m2^{m-1}} r_\lambda^m =: \rho_{m,\lambda}. \tag{2.14}$$

In this way we have proved that there exists r_λ and $\rho_{m,\lambda}$ such that

$$J_m(v) \geqslant \rho_{m,\lambda} \quad \forall v \in X_m : \|v\|_{X_m} = r_\lambda.$$

Then, we can consider the family of paths

$$\Gamma_m = \{\gamma : [0,1] \to X_m, : \gamma \text{is continuous and } \gamma(0) = 0,\ \gamma(1) = T_m\psi_0\}.$$

where T_m and ψ_0 are defined in (2.4) and (2.12) respectivley and

$$c_m := \inf_{\Gamma_m} \max_{[0,1]} J_m(\gamma(t)).$$

Notice that in order to have $T_m\|\psi_0\| > r_\lambda$ it is sufficient to have that

$$\lambda > \frac{\alpha}{4\sigma}\frac{1}{\|\psi\|^{p-2}}\left[\frac{2(m-p)\|\psi\|_m^m}{m(p-2)\beta\|\psi\|^2}\right]^{\frac{p-2}{m-2}}$$

where $\sigma = \frac{\mathscr{S}^{p/2}|\Omega|^{1-p/2^*}}{p}$. Since the right-hand side in the last inequality tends as m goes to $+\infty$ to

$$R \equiv \frac{\alpha}{4\sigma}\left[\frac{\|\psi\|_\infty}{\|\psi\|}\right]^{p-2},$$

it follows that for every $\lambda > R$, there exists $m_2(\lambda) > 2^*$ such that if $m \geqslant m_2(\lambda)$, then $T_m\|\psi\|_{X_m} > r_\lambda$, consequently, every path $\gamma \in \Gamma_m$ crosses the set $\|v\|_{X_m} = r_\lambda$, and we obtain

$$c_m \geqslant \rho_{m,\lambda},$$

[where $\rho_{m,\lambda}$ is defined in (2.14)]. Therefore, the geometrical hypotheses of the Mountain Pass Theorem are fulfilled provided that

$$m \geqslant m_0 \equiv \max\{m_1(\lambda), m_2(\lambda)\}, \quad \lambda \geqslant \underline{\lambda}_2 \equiv \max\{\underline{\lambda}_1, R\}. \tag{2.15}$$

It is only left to show that J_m satisfies the Palais-Smale condition, this is a straightforward argument; indeed, take a sequence w_n satisfying

$$J_m(w_n) \overset{n\to+\infty}{\longrightarrow} c_m, \quad J_m'(w_n) \overset{n\to+\infty}{\longrightarrow} 0 \quad \text{in } X_m^*, \tag{2.16}$$

where X_m^* denotes the dual space of X_m. As J_m is coercive in X_m, the first convergence in (2.16) implies that w_n is bounded in X_m, then, up to a subsequence, there exists $w \in X_m$ such that

$$w_n \rightharpoonup w \quad \text{in } X_m,$$

which means that $w_n \rightharpoonup w$ in $L^m(\Omega)$ and in $H_0^1(\Omega)$. Thus, by the Rellich theorem, w_n is strongly convergent to w in $L^2(\Omega)$ and then, by interpolation, we deduce also

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the strong convergence of w_n in $L^r(\Omega)$ for all $r \in [2, m)$. Now, taking $w_n - w \in H_0^1(\Omega) \cap L^m(\Omega) \subset H_0^1(\Omega) \cap L^{m'}(\Omega) \subset X_m^*$ (since $m > 2$) as test function in (2.16), we obtain

$$\int_\Omega M(x)\nabla w_n \cdot \nabla(w_n - w) + \int_\Omega |w_n|^{m-2} w_n (w_n - w) - \lambda \int_\Omega |w_n|^{p-2} w_n (w_n - w) \to 0.$$

Observing that the strong convergence of w_n in $L^p(\Omega)$ implies that the third adding term is tending to zero and, by subtracting the term

$$\int_\Omega M(x)\nabla w \cdot \nabla(w_n - w) + \int_\Omega |w|^{m-2} w (w_n - w),$$

which is converging to zero thanks to the weak convergence of w_n in $H_0^1(\Omega)$, we get

$$\lim_{n \to +\infty} \int_\Omega |\nabla(w_n - w)|^2 + \int_\Omega \left[|w_n|^{m-2} w_n - |w|^{m-2} w \right] (w_n - w) = 0,$$

from which the strong convergence of w_n to w in X_m is deduced. Then, the Ambrosetti-Rabinowitz theorem implies the existence of a mountain pass critical point z_m with critical level $J_m(z_m) = c_m \geqslant 0 > J_m(u_m)$. Then, z_m and u_m are distinct solutions of problem (1.1). Finally, arguing as at the end of theorem 2.1 it is possibile to obtain that$z_m \geqslant 0$. □

Remark 2.5. Let us observe that the result holds for every $2 < p \leqslant 2^*$ and for m sufficiently large. This assumption on p is crucial to show that zero is a strict local minimum so that $c_m > 0$ for every m fixed.

Remark 2.6. Notice that for m sufficiently large $\rho_{m,\lambda}$ defined in (2.14) converges to zero as $m \to +\infty$, so that we do not have an immediate a priori bound from below on the action level of the critical point z_m.

3. A priori bounds

Let's first show the following a priori L^∞-estimate that will be fundamental in the asymptotical analysis.

Proposition 3.1. *Assume conditions* (1.2) *and* (1.5). *Then every positive solution* w *of problem* (1.1) *satisfies the following estimate*

$$\|w\|_\infty \leqslant \lambda^{\frac{1}{m-p}}. \tag{3.1}$$

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Proof. For every $t, \varepsilon > 0$, let ψ_ε the function given by

$$\psi_\varepsilon(s) = \begin{cases} 0, & s < t, \\ \dfrac{s-t}{\varepsilon}, & t < s < t+\varepsilon, \\ 1, & t+\varepsilon \leqslant s. \end{cases}$$

Let us introduce the notation $E_t = \{x \in \Omega : w(x) > t\}$ and take $v = \psi_\varepsilon(w)$ as test function in (2.1) yielding

$$\int_{E_t} w^{m-1}\psi_\varepsilon(w) \leqslant \int_{\{x\in\Omega : t<w(x)<t+\varepsilon\}} M(x)\nabla w \cdot \nabla\psi_\varepsilon(w) + \int_{E_t} w^{m-1}\psi_\varepsilon(w) = \lambda \int_{E_t} w^{p-1}\psi_\varepsilon(w) \leqslant \lambda \int_{E_t} w^{p-1}.$$

Passing to the liminf as ε goes to zero and applying the Fatou Lemma and Hölder inequality,

$$\int_{E_t} w^{m-1} \leqslant \lambda \int_{E_t} w^{p-1} \leqslant \lambda \left[\int_{E_t} w^{m-1}\right]^{\frac{p-1}{m-1}} |E_t|^{\frac{m-p}{m-1}}.$$

That implies

$$t^{m-1}|E_t| \leqslant \int_{E_t} w^{m-1} \leqslant \lambda^{\frac{m-1}{m-p}}|E_t|. \tag{3.2}$$

Consequently, taking into account that $|E_t| \neq 0$ for every $t \in (0, \|w\|_\infty)$ we obtain

$$t \leqslant \lambda^{\frac{1}{m-p}}, \quad \forall t \in (0, \|w\|_\infty)$$

and hence (3.1) is proved. □

Remark 3.2. The same conclusion of proposition 3.1 would hold for every solution of problem (1.1); the hypothesis on positiveness is actually not needed. In addition, notice that the nonlinearity u^{m-1} satisfies all the hypotheses of theorem 1 in [**15**], then it actually holds that $u_m > 0$ and $z_m > 0$.

Remark 3.3. The a priori bound (3.1) prevents any blow-up phenomenon on a sequence of positive solutions of problem (1.1), as we will see in the following.

In addition to proposition 3.1 we can also show the following.

Proposition 3.4. *Assume conditions* (1.2) *and* (1.5). *Then every solution* w *of problem* (1.1) *satisfies the following estimates*

$$\|w\|_m^m \leqslant \lambda^{\frac{m}{m-p}}|\Omega|, \tag{3.3}$$

$$\alpha\|w\|^2 + \|w\|_m^m \leqslant |\Omega|\lambda^{\frac{m}{m-p}}. \tag{3.4}$$

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Proof. Taking w as test function in (2.1) and using (1.2) we obtain

$$\int_\Omega |w|^m \leqslant \lambda \int_\Omega |w|^p \leqslant \lambda \left[\int_\Omega |w|^m\right]^{p/m} |\Omega|^{1-\frac{p}{m}}, \tag{3.5}$$

which implies that

$$\left[\int_\Omega |w|^m\right]^{1-p/m} \leqslant \lambda |\Omega|^{1-\frac{p}{m}},$$

and (3.3) is clearly deduced. Now, we choose again $v = w$ in (2.1), to get from Hölder inequality and (3.3) that

$$\alpha \int_\Omega |\nabla w|^2 + \int_\Omega |w|^m \leqslant \lambda \left[\lambda^{\frac{m}{m-p}}\right]^{\frac{p}{m}} |\Omega|^{\frac{p}{m}} |\Omega|^{1-\frac{p}{m}} = \lambda^{\frac{m}{m-p}} |\Omega|$$

□

4. Asymptotical analysis

Let us start this section studying the convergence of the sequence of solutions of minimum points $\{u_m\}$

THEOREM 4.1. *Assume (1.2) and (1.5). There exists $\underline{\lambda}_1$ such that for every $\lambda > \underline{\lambda}_1$, there exists $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ such that $u \not\equiv 0$, $u_m \rightharpoonup u$ weakly in $H_0^1(\Omega)$, strongly in every Lebesgue space and u satisfies*

$$u \in \mathscr{K} := \{v \in H_0^1(\Omega) : 0 \leqslant v(x) \leqslant 1\}, \tag{4.1}$$

$$\int_\Omega M(x)\nabla u \cdot \nabla(v-u) \geqslant \lambda \int_\Omega u^{p-1}(v-u), \quad \forall\, v \in \mathscr{K}. \tag{4.2}$$

In addition, there exists $g_u \in L^\infty(\Omega)$, such that

$$0 \leqslant g_u \leqslant \lambda, \quad g_u \not\equiv 0, \quad g_u(x)[1-u(x)] = 0, \ \text{a. e. in } \Omega, \tag{4.3}$$

and it results

$$\int_\Omega M(x)\nabla u \cdot \nabla \varphi + \int_\Omega g_u \varphi = \lambda \int_\Omega u^{p-1}\varphi, \quad \forall \varphi \in H_0^1(\Omega). \tag{4.4}$$

REMARK 4.2. The properties of the function g_u expressed in (4.3) show that g_u weights the set where $u \equiv 1$, as $g_u(x) = 0$ for almost every $x \in \{x \in \Omega : u(x) < 1\}$ and g_u is not trivial.

Proof. We first apply theorem 2.1 to obtain a sequence of minimum points u_m of J_m satisfying (2.2). From (2.2) and (3.1) we deduce that there exists $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ with $0 \leqslant u(x) \leqslant 1$ such that $u_m \rightharpoonup u$ weakly in $H_0^1(\Omega)$, strongly in $L^q(\Omega)$

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for $q \in [1, +\infty)$. Moreover, from (2.2) we also deduce that

$$J_\infty(u_m) = \frac{1}{2}\int_\Omega M(x)\nabla u_m \cdot \nabla u_m - \frac{\lambda}{p}\int_\Omega u_m^p \leqslant -\delta < 0.$$

Then, as $u_m \rightharpoonup u$ weakly in $H_0^1(\Omega)$, and strongly in $L^q(\Omega)$ for every $q \in [1, +\infty)$.

$$0 > -\delta \geqslant \liminf_{m\to+\infty} J_m(u_m) \geqslant \frac{1}{2}\int_\Omega M(x)\nabla u \cdot \nabla u - \frac{\lambda}{p}\int_\Omega u^p$$

which implies that $u \not\equiv 0$. Let now v be any element in $\mathscr{K}$, and let θ be any real number such that $0 < \theta < 1$. Using $\theta v - u_m$ as test function in (2.1) we obtain

$$\begin{aligned}\int_\Omega M(x)\nabla u_m \cdot (\theta\nabla v - \nabla u_m) + \int_\Omega {u_m}^{m-1}(\theta v - u_m)\\ = \lambda \int_\Omega {u_m}^{p-1}(\theta v - u_m).\end{aligned} \tag{4.5}$$

We write the second term as

$$\begin{aligned}\int_\Omega {u_m}^{m-1}(\theta v - u_m) = \int_{\{x:0\leqslant u_m(x)<\theta v\}} {u_m}^{m-1}(\theta v - u_m)\\ + \int_{\{x:u_m(x)\geqslant\theta v\}} {u_m}^{m-1}(\theta v - u_m),\end{aligned}$$

where the first term of the right-hand side is bounded by

$${u_m}^{m-1}(\theta v - u_m) \leqslant [\theta v]^{m-1}(\theta v + \theta v) \leqslant 2\theta^m,$$

which tends to zero when n tends to infinity because $\theta < 1$; while

$${u_m}^{m-1}(\theta v - u_m) \leqslant 0 \quad \text{on} \quad \{x : u_m(x) \geqslant \theta v\};$$

and this implies that

$$\limsup_{m\to\infty} \int_\Omega {u_m}^{m-1}(\theta v - u_m) \leqslant 0.$$

Now we write (4.5) as

$$\begin{aligned}\theta \int_\Omega M(x)\nabla u_m \cdot \nabla v + \int_{\{x:0\leqslant u_m(x)<\theta v\}} {u_m}^{m-1}(\theta v - u_m)\\ \geqslant \lambda \int_\Omega {u_m}^{p-1}(\theta v - u_m) + \int_\Omega M(x)\nabla u_m \cdot \nabla u_m,\end{aligned}$$

we pass to the limit as $n \to \infty$ and we observe that thanks to (1.2) we can exploit the weak lower semicontinuity of the norm to obtain

$$\theta \int_\Omega M(x)\nabla u \cdot \nabla v \geqslant \lambda \int_\Omega u^{p-1}(\theta v - u) + \int_\Omega M(x)\nabla u \cdot \nabla u,$$

for any v in $\mathscr{K}$ and any θ with $0 < \theta < 1$. Letting θ tend to 1 we get (4.2).

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In order to prove the second part of the result, we take into account (3.1) and we deduce that there exists $g_u \in L^\infty(\Omega)$ such that (up to a subsequence)$\{(u_m)^{m-1}\} \overset{*}{\rightharpoonup} g_u$ weakly-star in $L^\infty(\Omega)$. As a consequence, we obtain that $g_u \geqslant 0$; in addition, considering χ_E the characteristic function of the set $E := \{x \in \Omega : g_u > \lambda\}$ and exploiting again (3.1) we get

$$\lambda^{\frac{m-1}{m-p}}|E| \geqslant \int_\Omega u_m^{m-1}\chi_E \Longrightarrow \lambda|E| \geqslant \int_E g_u > \lambda|E|$$

showing that

$$g_u \leqslant \lambda.$$

Taking $\varphi \in H_0^1(\Omega)$ as test function in (2.1) and passing to the limit we get that u satisfies (4.4). Finally, let us take as test function in (4.4) $v - u$ with $v \in \mathscr{K}$. We obtain, that the equation

$$\int_\Omega M(x)\nabla u \cdot \nabla(v-u) + \int_\Omega g_u(v-u) - \lambda\int_\Omega u^{p-1}(v-u) = 0$$

is satisfied for every $v \in \mathscr{K}$. Then, using (4.2) we deduce that

$$\int_\Omega g_u(u-v) \geqslant 0 \quad \forall\, v \in \mathscr{K}.$$

Then, we can take a sequence $v_j \in \mathscr{K}$ such that $v_j \to 1$ in $L^1(\Omega)$ obtaining

$$\int_\Omega g_u(u-1) \geqslant 0$$

and this immediately implies that $g_u(1-u) \equiv 0$, as $g_u \geqslant 0$ and $u \leqslant 1$. In order to conclude, it is only left to show that $g_u \not\equiv 0$. To this aim, we take into account that $J_\infty(u) \leqslant -\delta < 0$, to get

$$\lambda\int_\Omega u^p \geqslant p\delta + \frac{p}{2}\int_\Omega M(x)\nabla u \cdot \nabla u.$$

On the other hand, choosing $\varphi = u$ in (4.4) we get

$$\lambda\int_\Omega u^p = \int_\Omega g_u u + \int_\Omega M(x)\nabla u \cdot \nabla u$$

so that

$$\int_\Omega g_u u \geqslant p\delta + \Big(\frac{p}{2} - 1\Big)\int_\Omega M(x)\nabla u \cdot \nabla u$$

and since g_u, $u \geqslant 0$, $u \not\equiv 0$, this shows that $g_u \not\equiv 0$, or equivalently $|\{x \in \Omega : u(x) = 1\}| > 0$. □

Remark 4.3. Let us point out that the previous result holds for every $p > 2$, so that the nontrivial limit solution u exists even for $p > 2^*$.

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The previous results show that, a similar phenomenon to the one observed in [**5**, **7**] also occurs for this nonlinear problem. Now, let us move to the study of the asymptotic behaviour of the sequence of the critical points z_m, showing the following result.

THEOREM 4.4. *Assume* (1.2) *and* (1.3). *There exists* $\underline{\lambda} \geqslant \underline{\lambda}_1$ *such that for every* $\lambda > \underline{\lambda}$, *there exists* $z \in H_0^1(\Omega) \cap L^\infty(\Omega)$ *such that* $z \not\equiv 0$, $z_m \to z$ *strongly in* $H_0^1(\Omega)$ *and in every Lebesgue space. The function* z *satisfies*

$$z \in \mathscr{K} := \{v \in H_0^1(\Omega) : 0 \leqslant v(x) \leqslant 1\}, \tag{4.6}$$

$$\int_\Omega M(x)\nabla z \cdot \nabla(v - z) \geqslant \lambda \int_\Omega z^{p-1}(v - z), \quad \forall\, v \in \mathscr{K}. \tag{4.7}$$

In addition, there exists $g_z \in L^\infty(\Omega)$, *such that*

$$0 \leqslant g_z \leqslant \lambda, \quad g_z(x)[1 - z(x)] = 0, \ \textit{a. e. in } \Omega, \tag{4.8}$$

and it results

$$\int_\Omega M(x)\nabla z \cdot \nabla\varphi + \int_\Omega g_z\varphi = \lambda \int_\Omega z^{p-1}\varphi, \quad \forall\, \varphi \in H_0^1(\Omega). \tag{4.9}$$

Proof. We first apply theorem 2.4 to obtain the existence of a sequence of critical points z_m of J_m for m sufficiently large; then we follow the same argument as in the proof of theorem 4.1 getting the existence of a function $z \in \mathscr{K}$ such that $z_m \rightharpoonup z$ weakly in $H_0^1(\Omega)$ and strongly in every Lebesgue space; in addition z satisfies (4.7). Now, taking $z_m - z$ as test function (2.1) yields

$$\int_\Omega M(x)\nabla z_m \cdot (\nabla z_m - \nabla z) + \int_\Omega |z_m|^{m-1}(z_m - z) - \lambda \int_\Omega |z_m|^{p-1}(z_m - z) = 0. \tag{4.10}$$

Let us observe that

$$\left| \int_\Omega |z_m|^{m-1}(z_m - z) \right| \leqslant \int_\Omega |z_m|^{m-1}|z_m - z| \leqslant \lambda^{\frac{m-1}{m-p}} \int_\Omega |z_m - z| \to 0$$

and the same argument shows that the last term in (4.10) goes to zero. Using this information in (4.10) we get

$$\int_\Omega M(x) \cdot \nabla z_m(\nabla z_m - \nabla z) \to 0$$

yielding

$$\alpha \int_\Omega |\nabla z_m - \nabla z|^2 \leqslant \int_\Omega M(x) \cdot \nabla z_m(\nabla z_m - \nabla z) - \int_\Omega M(x) \cdot \nabla z(\nabla z_m - \nabla z) = o(1) + \int_\Omega M(x) \cdot \nabla z_m(\nabla z_m - \nabla z)$$

so that $z_m \to z$ strongly in $H_0^1(\Omega)$. In order to show that $z \not\equiv 0$, let us consider again the functional J_∞ introduced in (2.9). Notice that there exists $\overline{T}$ and $\psi \in$

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$H_0^1(\Omega) \cap L^\infty(\Omega)$ such that

$$J_m(\overline{T}\psi) < 0 \quad \text{and} \quad J_\infty(\overline{T}\psi) < 0$$

and the corresponding set of paths

$$\Gamma_\infty = \left\{\gamma : [0,1] \to H_0^1(\Omega), : \gamma \text{ is continuous and } \gamma(0) = 0,\ J_\infty(\gamma(1)) < 0\right\}.$$

Moreover, we define the mountain pass value

$$c_\infty := \inf_{\Gamma_\infty} \max_{[0,1]} J_\infty(\gamma(t)).$$

We claim that $\Gamma_m \subset \Gamma_\infty$. Indeed, take $\gamma \in \Gamma_m$ then γ is evidently continuous in $H_0^1(\Omega)$ as it is continuous in $H_0^1(\Omega) \cap L^m(\Omega)$, and $\gamma(0) = 0$. Moreover, as $\gamma(1) = T_m\psi$ with $J_m(T_m\psi) < 0$, it is sufficient to notice that $J_\infty(T_m\psi) < J_m(T_m\psi) < 0$ to obtain that $\gamma \in \Gamma_\infty$. As a consequence, we get

$$\max_{[0,1]} J_m(\gamma(t)) \geqslant \max_{[0,1]} J_\infty(\gamma(t)) \quad \text{(because } J_\infty(\gamma) \leqslant J_m(\gamma) \text{ for every } \gamma).$$

Then

$$c_m = \inf_{\Gamma_m} \max_{[0,1]} J_m(\gamma(t)) \geqslant c_\infty = \inf_{\Gamma_\infty} \max_{[0,1]} J_\infty(\gamma(t)) \geqslant \rho_\infty \tag{4.11}$$

because $\Gamma_m \subset \Gamma_\infty$ and where $\rho_\infty > 0$ can be obtained as

$$J_\infty(u) \geqslant \frac{1}{2}\|u\|^2 - C_0\|u\|^p \geqslant \rho_\infty > 0$$

for $\|u\|$ sufficiently small. Therefore,

$$J_m(z_m) = c_m \geqslant \rho_\infty > 0.$$

Then, exploiting (3.1) and recalling that $z_m \to z$ strongly in $H_0^1(\Omega)$ and in every Lebesgue space, we can pass to the limit to obtain that

$$J_\infty(z) \geqslant \rho_\infty > 0$$

yielding that $z \not\equiv 0$. Finally, the existence of g_z satisfying (4.8) and (4.9) can be proved as in the proof of theorem 4.1. □

Remark 4.5. Notice that the strong $H_0^1(\Omega)$ convergence can be also proved in an analogous way for the sequence of minimum points u_m. However, while in the proof of theorem 4.4 this strong convergence is crucial to show that z is not trivial, in theorem 4.1 the weak convergence in $H_0^1(\Omega)$ is sufficient to obtain that $u \not\equiv 0$.

Remark 4.6. Notice that theorem 4.4 holds for $p \leqslant 2^*$. This marks another difference between the asymptotic behaviour of u_m and z_m. Indeed, $p \leqslant 2^*$ is not needed to pass to the limit, but it is crucial to obtain that $z \not\equiv 0$.

Remark 4.7. Let us point out that $g_u \not\equiv 0$ for every $p > 2$. While, we cannot show that $g_z \not\equiv 0$ for any $2 < p \leqslant 2^*$, we can only produce some partial result which aims to show that g_z could be trivial or not depending on p and on the domain.

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PROPOSITION 4.8. *Assume $M(x) = Id$. The following conclusions hold*

(1) *Let Ω be star-shaped, $p = 2^*$, then $g_z \not\equiv 0$.*

(2) *Suppose that $\Omega = B_1(0)$, $N \geqslant 3$ and $p = 2^* - \epsilon$. Then, for $\epsilon > 0$ sufficiently small $g_z \not\equiv 0$.*

(3) *Suppose that Ω is a smooth bounded convex domain in $\mathbb{R}^2$. Then, for $p > 2$ sufficiently large $g_z \not\equiv 0$.*

Proof. **Conclusion (1).** Suppose by contradiction that $g_z \equiv 0$, then z is a positive solution to the problem

$$\begin{cases} -\Delta z = \lambda z^{2^*-1} & \text{in } \Omega \\ z > 0 & \text{in } \Omega \\ z = 0 & \text{on } \partial\Omega \end{cases}$$

which cannot be by the Pohozaev identity (see [12, 14]).

Conclusion (2). It is known that (see [2, 10]) there exists a unique positive radially symmetric solution $U_\lambda = \lambda^{-1/p-2}U$, where $U : B_1(0) \mapsto \mathbb{R}$ is the unique positive solution of the problem

$$\begin{cases} -\Delta U = U^{p-1} & \text{in } B_1(0) \\ U = 0 & \text{on } \partial B_1(0). \end{cases} \tag{4.12}$$

Moreover, as shown in [4] $\|U_\lambda\|_\infty \to +\infty$ as $\epsilon \to 0^+$. Then, for $\epsilon > 0$ sufficiently small $U_\lambda > 1$ is a set of positive measure, but $z \leqslant 1$ in the whole Ω, so that $g_z \not\equiv 0$.

Conclusion (3). We argue as in the proof of case (2): let u_p be a positive solution of

$$\begin{cases} -\Delta u_p = \lambda u_p^{p-1} & \text{in } \Omega \\ u_p > 0 & \text{in } \Omega \\ u_p = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.13}$$

with $\lambda = 1$; then exploiting the result contained in [9] u_p is unique for p sufficiently large. So that $u_{\lambda,p} = \lambda^{-\frac{1}{p-2}} u_p$ is the unique, positive solution of problem (4.13). But, as shown in [1]

$$\|u_{\lambda,p}\|_\infty = \lambda^{-\frac{1}{p-2}} \|u_p\|_\infty \to \sqrt{e} \text{ as } p \to +\infty$$

then, again $u_{\lambda,p} > 1$ in a set of positive measure, for p sufficiently large, so that $g_z \not\equiv 0$ in this case too. □

Moreover, an analogous result holds for p approaching 2 from above.

PROPOSITION 4.9. *Let Ω be any smooth bounded convex domain in $\mathbb{R}^N$ with $N \geqslant 2$. Assume that $M(x) = Id$. Then there exists $p_0 > 2$ such that for every $p \in (2, p_0) g_z \not\equiv 0$.*

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Proof. The result can be obtained following the argument in lemma 3.9 in [**8**]. Indeed, consider u_p a positive solution of problem (4.13) with $\lambda = 1$, and $u_{\lambda,p} = \lambda^{-\frac{1}{p-2}} u_p$ solution of problem (4.13) for a given $\lambda > \underline{\lambda}$. Then, it is possible to show that, taking p_n a sequence converging to 2 from above, it results that $M_n := \|u_{p_n}\|_\infty \to +\infty$ as $n \to +\infty$, and $M_n^{p_n-2} \to \lambda_1$; then

$$\lim_{n\to+\infty} \|u_{\lambda,p_n}\|_\infty^{p_n-2} = \lim_{n\to+\infty} \lambda^{-1} \|u_{p_n}\|_\infty^{p_n-2} = \frac{\lambda_1}{\lambda}$$

so that $\|u_{\lambda,p_n}\|_\infty \to +\infty$ too. Moreover, for $p \in (2, p_0)$ this solution is unique, yielding again that $g_z \not\equiv 0$, as $\|z\|_\infty \leqslant 1$. □

Unfortunately, we cannot show that $g_z \not\equiv 0$ for a generic $p \in (2, 2^*)$, and we conjecture that this may depend on the domain and on the exponent p, let us give some partial observations in this direction.

PROPOSITION 4.10. *Assume that $M(x) = Id$. Moreover, suppose that there exists $\lambda > 0$ and $R > 0$ such that taking $\Omega = B_R(0)$, the open ball of radius R centred at zero, the following conditions are satisfied*

(a) $\lambda > \underline{\lambda}$

(b) $\left(\frac{1}{\lambda R^2}\right)^{\frac{1}{p-2}} U(0) > 1$.

Then $g_z \not\equiv 0$.

Proof. Assume by contradiction that $g_z \equiv 0$, then z is a solution to the problem

$$\begin{cases} -\Delta z = \lambda z^{p-1} & \text{in } \Omega \\ z = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.14}$$

with $\Omega = B_R(0)$. Then, taking into account theorem (see [**2**, **10**])

$$z(x) = U_{\lambda,R} := \left(\frac{1}{\lambda R^2}\right)^{\frac{1}{p-2}} U\left(\frac{x}{R}\right)$$

where U is defined in (4.12); indeed $U_{\lambda,R}$ is defined in $B_R(0)$, vanishes at the boundary and solves

$$-\Delta U_{\lambda,R} = \lambda^{-\frac{1}{p-2}} R^{-\frac{2}{p-2}-2} (-\Delta U)\left(\frac{x}{R}\right) = \lambda^{-\frac{1}{p-2}} R^{-\frac{2}{p-2}-2} \left[U\left(\frac{x}{R}\right)\right]^{p-1} = \lambda U_{\lambda,R}^{p-1}$$

In addition, recalling (4.6) and hypothesis (b)

$$1 \geqslant z(0) = \max_{B_R(0)} z = \left(\frac{1}{\lambda R^2}\right)^{\frac{1}{p-2}} U(0) > 1$$

yielding the conclusion. □

REMARK 4.11. Unfortunately, we are not able to give an example in which proposition 4.10 applies showing that $g_z \not\equiv 0$ in the subcritical regime as well. With this

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respect, let us observe that hypotheses (a) and (b) go in opposite directions, as (b) requires λ sufficiently small, while in theorem 2.4 we have seen if (a) is satisfied then $\lambda > \underline{\lambda}_1$, that is

$$\lambda > \underline{\lambda}_1 := \inf_{H_0^1(\Omega)\cap L^\infty(\Omega)} \Lambda(\varphi), \quad \Lambda(\varphi) := \frac{p}{2}\frac{\|\varphi\|^2\|\varphi\|_\infty^{p-2}}{\|\varphi\|_p^p}.$$

For example, supposing that there exists φ defined in $B_1(0)$ such that $\Lambda(\varphi) < \lambda$, then $\varphi_R(x) := \varphi(x/R)$ defined in $B_R(0)$ satisfies

$$\Lambda(\varphi_R) = \frac{1}{R^2}\frac{p}{2}\frac{\|\varphi\|^2\|\varphi\|_\infty^{p-2}}{\|\varphi\|_p^p} = \frac{1}{R^2}\Lambda(\varphi)$$

then, in order to have $g_z \not\equiv 0$ in $B_R(0)$, we would need

$$\lambda > \frac{1}{R^2}\Lambda(\varphi), \quad \text{and} \quad \left(\frac{1}{\lambda R^2}\right)^{\frac{1}{p-2}} U(0) > 1$$

which are equivalent to

$$\Lambda(\varphi) < \lambda R^2 < [U(0)]^{p-2}.$$

On the other hand, let us observe that $\Lambda(U) = \frac{p}{2}[U(0)]^{p-2}$, so that choosing $\varphi = U$ would imply that the assumptions of proposition 4.10 cannot be satisfied. So that finding an optimal φ such that the interval $(\Lambda(\varphi), [U(0)]^{p-2})$ is not empty seems to be a delicate question.

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