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# Hausdorff dimension of sets of generic points for non-statistical dynamical systems

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*Abstract.* We consider one-dimensional maps with several neutral fixed points that do not admit any physical measures. We show that there is a simplex of measures so that every measure in this simplex has a basin that has full Hausdorff dimension.

Key words: dimension theory, low dimensional dynamics, symbolic dynamics
2020 Mathematics Subject Classification: 37C45 (Primary); 37E05, 37A25, 37D25, 37A50 (Secondary)

## 1. *Introduction*

We study generic points for non-ergodic measures for certain non-statistical maps of the interval. Our results apply to a large class of interval maps admitting $d \geq 2$ equally sticky neutral fixed points. We first present our results in the context of a simple example. The full description will be given in §2.

Following [SY19], consider the transformation $f : [0, 1] \to [0, 1]$ given by

$$f(x) := \begin{cases} x + c_1 x^3, & x \in [0, 1/3], \\ x + c_2(x - \frac{1}{2})^3, & x \in (1/3, 2/3), \\ x + c_3(x - 1)^3, & x \in [2/3, 1], \end{cases} \tag{1.1}$$

where $c_1 = c_3 = 18$ and $c_2 = 72$, so that $f$ is full-branched (see Figure 1). The map (1.1) has derivative $f'(\cdot) > 1$, except at the neutral fixed points $\xi_1 = 0$, $\xi_2 = 1/2$, and $\xi_3 = 1$. This map admits a unique $\sigma$-finite absolutely continuous ergodic invariant measure $\mu$ which gives infinite mass to the unit interval. (Unique up-to scaling by a positive constant.)

For $x \in X$, let $\delta_x$ denote the Dirac delta measure at $x$. By [CMT24], the map $f$ given by (1.1) is *non-statistical* in the sense that for Lebesgue almost every $x$, the *empirical measures*

$$e_n(x) := \frac{1}{n} \sum_{k=0}^{n-1} \delta_{f^k x}$$

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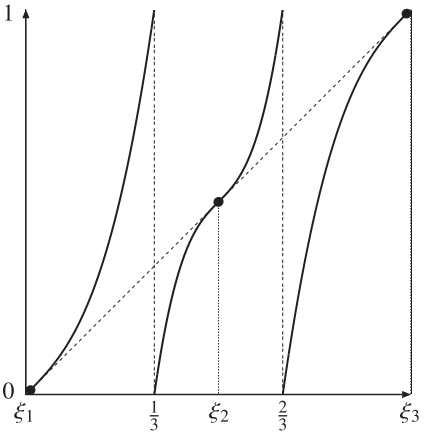



FIGURE 1. The map $f$ given in (1.1).

do not converge in the space of Borel probability measures in the weak$*$ topology, as $n \to \infty$. Even more, for Lebesgue-almost every point $x$, these measures distribute along the simplex $\pi\mathcal{S}$ of invariant measures supported at the fixed points, where

$$\mathcal{S} := \{\overline{p} \in [0,1]^3 \colon p_1 + p_2 + p_3 = 1\} \quad \text{and} \quad \pi : \mathcal{S} \to \mathcal{M}(f), \quad \pi(\overline{p}) = \nu_{\overline{p}} := \sum_{j=1}^{3} p_i \delta_{\xi_i},$$

and where $\mathcal{M}(f)$ denotes the space of $f$-invariant Borel probability measures. Indeed, letting

$$\mathcal{V}(x) := \{\text{limit points of } (e_n(x))_{n\in\mathbb{N}}\}, \tag{1.2}$$

by [**CMT24**, Theorem A], one has

$$\mathcal{V}(x) = \pi\mathcal{S} \quad \text{for Lebesgue almost every } x. \tag{1.3}$$

Given an $f$-invariant probability measure $\nu$, consider its set of *generic points* or *basin of attraction*,

$$\mathcal{G}(\nu) := \{x \colon e_n(x) \to \nu \text{ in the weak} * \text{ topology as } n \to \infty\}. \tag{1.4}$$

By [**CMT24**], the basin of attraction of any measure in $\pi\mathcal{S}$ is negligible in terms of the Lebesgue measure

$$\text{Leb}\,(\mathcal{G}(\nu)) = 0 \quad \text{for all } \nu \in \pi\mathcal{S}. \tag{1.5}$$

Moreover, by [**PS07**, Theorem 4.1] (see also (1.9)), it is also negligible in terms of the topological entropy (here, we use the concept of topological entropy $h_{\text{top}}(f, \cdot)$ on general (not necessarily compact) subsets following [**Bow74**]) of $f$

$$h_{\text{top}}(f, \mathcal{G}(\nu)) = 0 \quad \text{for all } \nu \in \pi\mathcal{S}. \tag{1.6}$$

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Our main result, applied to the map (1.1), tells us that despite (1.5) and (1.6), these basins of attraction are in fact maximal in terms of Hausdorff dimension (we recall its definition in §6.5).

THEOREM 1.1. *For every* $\overline{p} \in \mathcal{S}$, $\dim_{\mathrm{H}} \mathcal{G}(\nu_{\overline{p}}) = 1$.

Note that the set $\mathcal{V}(x)$ defined in (1.2) is a non-empty closed connected subset of $\mathcal{M}(f)$. Our second result tells us that for *any* closed connected subset of $\pi\mathcal{S}$, the conclusion of the above theorem remains true.

THEOREM 1.2. *For every non-empty closed connected subset* $\mathcal{C} \subset \mathcal{S}$, $\dim_{\mathrm{H}}\{x\colon \mathcal{V}(x) = \pi\mathcal{C}\} = 1$.

Theorems 1.1 and 1.2 fall within the scope of multifractal analysis, which investigates the 'size' of certain level sets of asymptotic quantifiers such as Birkhoff averages and local dimensions. In this type of analysis, two sorts of behavior are frequently studied: when a limit, say the average of a function $\phi\colon X \to \mathbb{R}$, exists,

$$\mathcal{L}_{\phi}(\alpha) := \left\{x\colon \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \phi(f^k(x)) = \alpha\right\}, \tag{1.7}$$

or when a limit does not exist, and the full range of irregular behavior with lower and upper limits $\alpha < \beta$ is investigated,

$$\mathcal{L}_{\phi,\,\mathrm{irr}}(\alpha,\beta) := \left\{x\colon \liminf_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \phi(f^k(x)) = \alpha, \limsup_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \phi(f^k(x)) = \beta\right\}. \tag{1.8}$$

For a map $f$ being a full shift and a function $\phi$ being Hölder continuous, the sets (1.7) and (1.8) can be characterized in terms of their Hausdorff dimension and topological entropy, and studied in terms of certain Gibbs measures which indeed govern the decay of deviations from the expected asymptotic average. See, for example, [**BPS97**] for references to some classical results.

The investigation of entropy of (1.7) and (1.8) is done in terms of certain rate functions that govern the decay of the distribution of Birkhoff averages over space. This type of analysis is also referred to as level-1 large deviation results (see [**Ell85**, **You90**] for details on rate functions and [**TV03**] for applications to multifractal analysis).

Here, we study level-2 deviations, focusing on empirical *measures* and their distributions. The study of divergence points, such as in (1.8), from the point of view of distributional measures (1.4) and quantification in terms of entropy, is given, for example, in [**EKL05**]. In a similar spirit, for dynamical systems satisfying the specification property, [**TV03**] studies the entropy of Birkhoff averages-level sets, with a particular focus also on the Manneville–Pomeau map. As a precursor to [**TV03**, **EKL05**], [**Bow74**] showed that for a continuous map $f$ on a compact metric space $X$, the entropy of $f$ on the set of generic points (1.4) of an *ergodic* probability measure $\nu$ is

$$h_{\mathrm{top}}(f, \mathcal{G}(\nu)) = h_{\nu}(f).$$

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Non-generic points can also be described. By [PS07, Theorem 4.1] for any non-empty connected compact set $\mathcal{C} \subset \mathcal{M}(f)$ in the space of invariant Borel probability measures,

$$h_{\text{top}}(f, \{x : \mathcal{V}(x) = \mathcal{C}\}) \leq \inf\{h_\mu(f) : \mu \in \mathcal{C}\}. \tag{1.9}$$

The equality in (1.9) also holds, but under stronger hypotheses, see [PS07]. In the setting of Theorem 1.1, one can show that (1.9) implies (1.6).

The study of Hausdorff dimension of level sets (1.7) and (1.8) usually requires some type of hyperbolicity as the results collected in [BPS97] illustrate. In [GR09, JJÖP10], the authors study non-uniformly hyperbolic interval maps with indifferent fixed points. These maps can be considered as on the boundary to hyperbolicity as they have the specification property and have a finite number of ergodic non-hyperbolic measures supported on parabolic fixed points. The studies in [GR09, JJÖP10] follow an approach of *bridging measures* based on 'exhausting' the non-uniformly hyperbolic part of dynamics by 'uniformly hyperbolic sub-dynamics', see also [BS00]. Theorems 1.1 and 1.2 focus on empirical measures and the *non-hyperbolic* part. The approach in [GR09] will also be a key step to prove the assertion about the dimension of (1.4) in Theorem 1.1.

The phenomenon (1.3) for a map with just two parabolic fixed points was shown in [ATZ05, CL24, BC24] and in [CMT24] for maps with several parabolic fixed points. More generally, non-statistical behavior was observed in other contexts. For example, in the quadratic family, Hofbauer and Keller [HK95] prove that there are uncountably many parameters which correspond to a map $f$ which has 'maximal oscillation' in the sense that for Lebesgue almost-every initial condition, the empirical measures accumulate on the full simplex of $f$-invariant Borel probability measures. In [Tal22], the maximal oscillation-property was shown to hold generically for a certain class of rational maps. Finally, [AG22] establishes an analogous version of (1.3) for certain irrational linear flows of the two-torus (see also the references therein for further results in this direction).

The paper is organized as follows. In §2, we introduce the more general setting which we study (for which (1.1) is a particular example) and state our main result, Theorem A. Theorems 1.1 and 1.2 are special cases of Theorem A. Section 4 provides details on repellers and return times. In §§3, 5, and 6, we prove Theorem A. In §8, we provide further examples to which Theorem A applies.

## 2. *Assumptions and statement of results*

2.1. *Assumptions.* We recall some classical definitions and make precise the assumptions we use throughout the paper. Throughout, we let $X$ denote either the circle $\mathbb{S}^1$ or the interval $[0, 1]$. We say that a non-singular transformation $f : X \to X$ is *Markov* if there exists a countable partition (modulo Leb) $\mathcal{P}$ of $X$ into closed sub-intervals such that the restriction of $f$ to any partition element is a bijection onto a union of partition elements. A Markov map $f$ with partition $\mathcal{P}$ is *topologically mixing* if for every $a, b \in \mathcal{P}$, there exists an $N \in \mathbb{N}$ such that $f^n(a) \cap b \neq \emptyset$ for every $n \geq N$.

*Assumption 2.1.* Assume $f : X \to X$ is a Markov map with respect to a partition $\mathcal{P}$.

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*Assumption 2.2.* There exists a union of partition elements $Y$ so that the first return map $F\colon Y \to Y$ given by

$$F\colon Y \to Y, \quad F(x) \coloneqq f^{\tau(x)}(x),$$

where

$$\tau(x)\colon X \to \mathbb{N}, \quad \tau(x) \coloneqq \min\{n \in \mathbb{N}\colon f^n(x) \in Y\},$$

is a topologically mixing $C^2$ uniformly expanding Gibbs–Markov map with respect to a refinement $\mathcal{Q}$ of $\mathcal{P}$. That is, $F$ is Markov with respect to $\mathcal{Q}$, topologically mixing, and the following additional properties hold.

- *(Finite images)* $\mathrm{Card}\{F(a)\colon a \in \mathcal{Q}\} < \infty$, that is, there exist $Y_1, \ldots, Y_L$ so that each $Y_i$ is a union of elements of $\mathcal{Q}$ and $\{F(a)\colon a \in \mathcal{Q}\} = \{Y_1, \ldots, Y_L\}$.
- *(Bounded distortion)* There exist $\theta \in (0, 1)$ and $C > 0$ so that the function $\log(dm/dm \circ F)$ is $d_\theta$-Lipschitz on elements of $\mathcal{Q}$, where $d_\theta$ is the metric $d_\theta(x, y) \coloneqq \theta^{s(x,y)}$ and

$$s(x, y) \coloneqq \inf\{n \in \mathbb{N}_0\colon F^n(x), F^n(y) \text{ lie in different elements of } \mathcal{Q}\}.$$

- *($C^2$ branches)* For every $a \in \mathcal{Q}$, the map $F|_{\mathrm{int}(a)}\colon \mathrm{int}(a) \to F(\mathrm{int}(a))$ is a $C^2$ diffeomorphism. (Notice that we do not necessarily assume that $F$ has $C^2$ extension to the closure of $a$.)
- *(Uniform expansion)* There exists $\lambda > 1$ such that $\inf_{x \in Y} |F'(x)| > \lambda$.

LEMMA 2.3. *Up to taking a possible refinement of the partition $\mathcal{Q}$, we may assume that $F(\mathcal{Q}) = \{Y_1, \ldots, Y_n\}$ is a collection of disjoint subintervals which partition $Y$.*

*Proof.* Let $\{y_1 < y_2 \cdots < y_{N-1} < y_N\}$ be the collection of end points of the intervals $Y_i$ ordered so that $y_i < y_{i+1}$ and let $\mathcal{Y}$ denote the partition of $Y$ given by all intervals of the form $[y_i, y_{i+1}]$ that intersect $Y$. As each $Y_i$ is a union of elements of $\mathcal{P}$, it follows that each element in $\mathcal{Y}$ is also a union of elements of $\mathcal{P}$.

As each $Y_i$ is a finite union of elements in $\mathcal{Y}$, $f$ is a bijection on elements of $\mathcal{P}$ and we can refine $\mathcal{P}$ to form a new partition $\widetilde{\mathcal{Q}}$ so that $f(a) \in \mathcal{Y}$ for all $a \in \widetilde{\mathcal{Q}}$. □

*Remark 2.4.* Up to taking a refinement of $\mathcal{Q}$, we can assume that $\tau$ takes a finite and constant value on the interior of every $a \in \mathcal{Q}$.

*Remark 2.5.* As the first return map $F$ is Gibbs–Markov, it preserves an invariant probability measure $\mu_Y$ which is equivalent to Lebesgue (see, for example, [**Aar97**, Ch. 4]). This implies that $f$ preserves the $\sigma$-finite $f$-invariant ergodic measure $\mu$ equivalent to Lebesgue given by

$$\mu(A) \coloneqq \sum_{n=0}^{\infty} \mu_Y(\{\tau > n\} \cap f^{-n}A). \tag{2.1}$$

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Defining $\mu$ in this way, one notes that $\mu|_Y = \mu_Y$ (if $A \subset Y$, then one has that $\{\tau > n\} \cap f^{-n})A = \emptyset$ for every $n \geq 1$ as $Y \cap f^{-n}A \subset \{\tau \leq n\}$) and that

$$\mu(X) < \infty \quad \text{if and only if} \int \tau \, d\mu_Y < \infty.$$

Notice that in our definition of Gibbs–Markov in Assumption 2.2, we assume the finite image property instead of the slightly weaker 'big image property' which also commonly appears in the literature (see, for example, [**Zwe98**, **Zwe00**, **Gou10**]). This stronger assumption guarantees that $\mu_Y$, and hence $\mu$, is equivalent to Lebesgue as opposed to only being absolutely continuous with respect to Lebesgue.

*Assumption 2.6.* There exist constants $d \in \mathbb{N}$, $\alpha \in (0, 1)$, and $\gamma_1, \ldots, \gamma_d > 0$, and a partition (mod Leb) of $X \setminus Y$ into open sets $X_1, \ldots, X_d$ such that for every $i, j \in \{1, \ldots, d\}$:

(1) there is a single fixed point $\xi_j \in X_j$;
(2) for $i \neq j$, orbits cannot pass from $X_i$ to $X_j$ without first passing through $Y$, that is, if $f^n(x) \in X_i$ and $f^{n+m}(x) \in X_j$ for some $i \neq j$ and $m \in \mathbb{N}$, then there is $k \in \{1, \ldots, m-1\}$ such that $f^{n+k}(x) \in Y$,
(3) $\mu_Y(\tau^{(j)} > n) \sim \gamma_j n^{-\alpha}$, where

$$\tau^{(j)}(y) \coloneqq \mathrm{Card}\{n \leq \tau(y) \colon f^n(y) \in X_j\}.$$

*Remark 2.7.* Notice that each connected component of $J_n \coloneqq \{x \in X \setminus Y \colon \tau(x) = n\}$ is the preimage of an interval (a connected component of $Y$) by one of the branches of $f$ (a $C^2$ diffeomorphism). Thus, the sets $J_n$ necessarily accumulate at fixed points of $f$, in the sense that

$$\bigcap_{n \in \mathbb{N}} \overline{\bigcup_{k > n} J_k} = \{\xi_1, \ldots, \xi_d\}.$$

*Remark 2.8.* Item (2) in Assumption 2.6 tells us that for any $x \in Y$, only one of the values $\tau^{(1)}(x), \ldots, \tau^{(d)}(x)$ can be non-zero. In particular,

$$\tau = \tau^{(1)} + \cdots + \tau^{(d)} + 1, \tag{2.2}$$

and so

$$\mu_Y(\tau > n) \sim \gamma n^{-\alpha} \quad \text{where } \gamma \coloneqq \gamma_1 + \cdots + \gamma_d.$$

In particular, $\int \tau \, d\mu_Y = \infty$ and so $\mu(X) = \infty$.

2.2. *Statement of results.* Throughout the remainder of the article, we assume that $f$ is a map satisfying Assumptions 2.1–2.6. Compare also Figure 2. We consider

$$\mathcal{S} \coloneqq \left\{ \overline{p} \in [0, 1]^d \colon \sum_{j=1}^{d} p_j = 1 \right\} \quad \text{and} \quad \pi \colon \mathcal{S} \to \mathcal{M}(f), \quad \pi(\overline{p}) = \nu_{\overline{p}} \coloneqq \sum_{j=1}^{d} p_j \delta_{\xi_j}.$$

Denoting by $e_n(x) \coloneqq \sum_{k=0}^{n-1} \delta_{f^k(x)}$ the $n$th empirical measure of $x \in X$, we consider

$$\mathcal{V}(x) \coloneqq \{\text{limit points of } (e_n(x))_{n \in \mathbb{N}}\}$$

of weak$*$ limit points of the empirical measures. The following is our main result.

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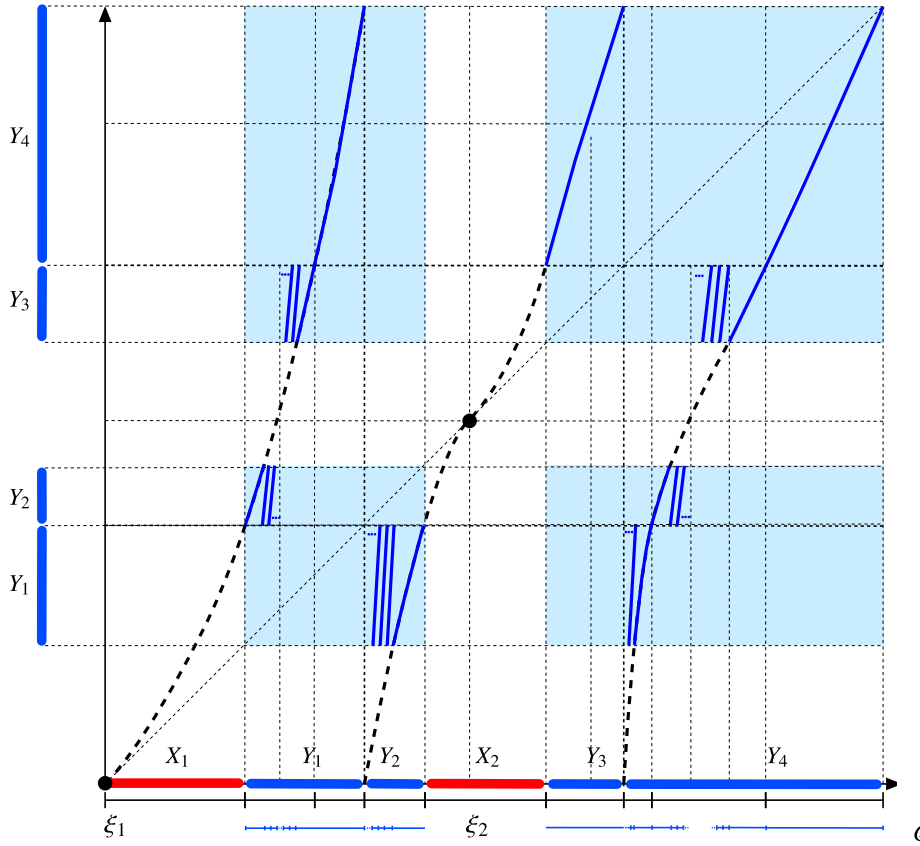



FIGURE 2. An example of a Markov map $f : [0, 1] \to [0, 1]$ (dashed curve) with two parabolic fixed points $\xi_1, \xi_2$ and its Gibbs–Markov first return map $F : Y \to Y$ (solid curve) on the set $Y$ (union of intervals $Y_1, \ldots, Y_4$). See Example 8.3. The collection $\{Y_1, \ldots, Y_4\}$ of sets $\{F(a) : a \in \mathcal{Q}\}$ cover $Y$, see Assumption 2.2. (Color online.).

THEOREM A. *If Assumptions 2.1–2.6 hold, then for every closed connected set $\mathcal{C} \subset \mathcal{S}$,*

$$\dim_{\mathrm{H}}\{x : \mathcal{V}(x) = \pi\mathcal{C}\} = 1.$$

Considering the special case that $\mathcal{C} = \{\overline{p}\}$, one obtains the following corollary.

COROLLARY 2.9. *If Assumptions 2.1–2.6 hold, then* $\dim_{\mathrm{H}} \mathcal{G}(\nu_{\overline{p}}) = 1$ *for every* $\overline{p} \in \mathcal{S}$.

*Remark 2.10.* We only treat the case that $\alpha \in (0, 1)$ even though the case $\alpha = 1$ is also known to lead to non-statistical behavior [**CMT24**]. Our method of proof will rely heavily on knowing that occupation times of neighborhoods of the fixed points converge in distribution to a non-degenerate limit. For $\alpha \in (0, 1]$, it is known that in our present setting, these occupation times will converge in distribution to a generalized arcsine law (cf. [**SY19**, **Ser20**] and Lemma 5.3 for a precise statement). However, when $\alpha = 1$, this arcsine law is degenerate and our methods break down.

One can also consider the case $\alpha = 0$, which corresponds to the map making flat tangencies at the neutral fixed points (that is, the derivative at the fixed points converge

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to 1 exponentially fast). In this case, the tails of the return time $\mu_Y(\tau > n)$ decay logarithmically [**Des23**] and our methods (which rely heavily on $\mu_Y(\tau > n)$ decaying polynomially) break down.

Sections 3–7 are devoted to proving Theorem A. It will be convenient to first prove Corollary 2.9, which is a special case.

The first key step is to establish a coding of generic points in terms of the asymptotic behavior of their return times. For this, we will consider the joint behavior of the return times $\tau^{(1)}, \ldots, \tau^{(d)}$ and their Birkhoff sums under $F$. To this end, we introduce

$$\overline{\tau} := (\tau^{(1)}, \ldots, \tau^{(d)}) \colon Y \to \mathbb{N}^d,$$

and let

$$\overline{\tau}_k := \sum_{j=0}^{k-1} \overline{\tau} \circ F^j \quad \text{and} \quad \tau_k := \sum_{j=0}^{k-1} \tau \circ F^j,$$

and define the $\tau_k^{(i)}$ analogously for $i = 1, \ldots, d$. In §3, we will show

$$e_n(x) \to \nu_{\overline{p}} \iff \frac{\overline{\tau}_k}{\tau_k}(x) \to \overline{p}. \tag{2.3}$$

Equation (2.3) reduces the problem of understanding the observables $e_n \colon X \to \mathcal{M}(f)$ under the dynamics of $f$ to understanding the Birkhoff sums of the function $\overline{\tau} \colon Y \to \mathbb{N}^d$ under the dynamics of a uniformly expanding Gibbs–Markov map $F$.

By Remark 2.4, $\overline{\tau}$ is piecewise constant. So, together with (2.3), we obtain a symbolic description of $\mathcal{G}(\nu_{\overline{p}})$ (albeit one that requires an infinite alphabet) and can start building approximations of $\mathcal{G}(\nu_{\overline{p}})$. Namely, we will construct collections $\mathcal{A}$ of cylinder sets where the return times have a certain prescribed behavior up to some finite time. This will be done in such a way that: the limit points of $(\overline{\tau}_k/\tau_k)_k$ will all be close to $\overline{p}$ for points in the maximal invariant set of $\mathcal{A}$; and the dimension of this maximal invariant set is close to 1. See §§4 and 5.

In §6, we then consider a sequence $(\mathcal{A}_i)_i$ of increasingly better approximations to $\mathcal{G}(\nu_{\overline{p}})$ to build a set $\Gamma \subset \mathcal{G}(\nu_{\overline{p}})$. We conclude by arguing that $\dim_{\mathrm{H}} \Gamma = 1$.

Finally, in §7, we explain how the proof of Corollary 2.9 can be strengthened to prove Theorem A.

## 3. *Coding of $\nu_{\bar{p}}$-generic points*

In this section, we explore equivalent characterizations of the convergence of the empirical measures in terms of the return times $\overline{\tau}$ by relating $\mathcal{V}(x)$ and the set

$$\mathcal{T}(x) := \left\{ \text{limit points of the sequence } \left( \frac{\overline{\tau}_k}{\tau_k}(x) \right)_{k \in \mathbb{N}} \right\} \subset [0,1]^d.$$

THEOREM 3.1. *Let be $\mathcal{T}(x)$ a closed connected subset of $\mathcal{S}$ and suppose that*

$$\lim_{k \to \infty} \left| \frac{\overline{\tau}_k}{\tau_k}(x) - \frac{\overline{\tau}_{k+1}}{\tau_{k+1}}(x) \right| = 0. \tag{3.1}$$

*Then, $\mathcal{V}(x) = \pi(\mathcal{T}(x))$.*

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The above theorem will be proven at the end of this section. When $\mathcal{C}$ contains a single point $\overline{p}$, Theorem 3.1 reduces to the following result.

COROLLARY 3.2. *For all $x \in X$ and $\overline{p} \in \mathcal{S}$,*

$$e_n(x) \to \nu_{\overline{p}} \quad \Longleftrightarrow \quad \frac{\overline{\tau}_k}{\tau_k}(x) \to \overline{p}.$$

We begin with the following intermediate result.

PROPOSITION 3.3. *If $\mathcal{T}(x) \subset \mathcal{S}$, then $\mathcal{V}(x) \subset \pi\mathcal{S}$.*

To prove Proposition 3.3, we first collect some preliminary results. From the ergodic theorem for infinite measures, we know that

$$\lim_{n\to\infty} e_n(x)(K) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} 1_K \circ f^i(x) = 0 \tag{3.2}$$
$$\text{for every compact set } K \text{ with } K \cap \{\xi_1, \ldots, \xi_d\} = \emptyset,$$

holds for Lebesgue-almost every $x$. (The ergodic theorem (see [**Aar97**, Exercise 2.2.1]) here tells us that $e_n(x)(A) \to 0$ for $\mu$-almost every $x$ and for all $A$ with $\mu(A) < +\infty$. The fact that any compact set which does not contain any of the neutral fixed points has finite measure is an easy consequence of Remark 2.7 and (2.1).) Recall that $e_n(x)(K)$ is just the proportion of time up to time $n$ that the orbit of $x$ spends in $K$. So, if (3.2) holds for some $x$, then its orbit must spend proportion 1 of its time near the fixed points. This, in turn, suggests $\mathcal{V}(x) \subset \pi\mathcal{S}$. Indeed, we get the following result.

LEMMA 3.4. *For any $x \in X$, $\mathcal{V}(x) \subset \pi\mathcal{S}$ and (3.2) are equivalent.*

*Proof.* The Portmanteau theorem tells us that $\mathcal{V}(x) \subset \pi\mathcal{S}$ if and only if the only limit point of $(e_n(x)(A))_n$ is 0 whenever $A \subset X$ is such that $\partial A \cap \{\xi_1, \ldots, \xi_d\} = \emptyset$. The result quickly follows. □

From our discussion above, it is now clear that $\mathcal{V}(x) \subset \pi\mathcal{S}$ for Lebesgue-almost every $x$. (Indeed, note that by [**CMT24**], the reverse inclusion $\pi\mathcal{S} \subset \mathcal{V}(x)$ also holds for Lebesgue-typical $x$). We are interested, though, in the behavior of non-typical points. Moreover, we want to encode condition (3.2) using the symbolic dynamics of the induced map. For that, we introduce the condition

$$\lim_{k\to\infty} \frac{\tau_k(x)}{k} = +\infty. \tag{3.3}$$

LEMMA 3.5. *For any $x \in X$, conditions (3.2) and (3.3) are equivalent.*

*Proof.* Assume (3.2). Notice that

$$\frac{k}{\tau_k(x)} = e_{\tau_k(x)}(x)(Y).$$

Hence, (3.3) follows immediately from (3.2) by taking $K = Y$.

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Now, assume (3.3). As $K \subset X$ is a compact set which does not contain any of the neutral fixed points, we know from Remark 2.7 that $\sup_{K\setminus Y} \tau < \infty$. Setting

$$N := \sup_{x\in K\setminus Y} \tau(x),$$

the number of entrances to $K$ between successive returns to $Y$ is at most $N$, that is,

$$\sup_{k\in\mathbb{N}} \sum_{i=\tau_{k-1}(x)+1}^{\tau_k(x)-1} \mathbb{1}_K \circ f^i(x) \le \sup_{x\in K\setminus Y} \tau(x) = N < \infty.$$

For each $n \in \mathbb{N}$, choose $k_n \in \mathbb{N}$ such that

$$\tau_{k_n}(x) \le n < \tau_{k_n+1}(x).$$

By the definition of $k_n$ and $N$, we have

$$e_n(x)(K) = \frac{1}{n}\sum_{i=0}^{n-1} \mathbb{1}_K \circ f^i(x) \le \frac{k_n N + N}{\tau_{k_n}(x)} \to 0$$

by our assumption on $x$. This proves (3.2). □

*Proof of Proposition 3.3.* By Lemmas 3.4 and 3.5, it is enough to show that $\mathcal{T}(x) \subset \mathcal{S}$ implies (3.3). We first claim that

$$\lim_{k\to\infty} \frac{\sum_{j=1}^d \tau_k^{(j)}(x)}{\tau_k(x)} = 1. \tag{3.4}$$

From (2.2), we know that $\sum_{j=1}^d \tau_k^{(j)}(x)/\tau_k(x) \le 1$. So, the sequence in the limit in (3.4) is bounded and, hence, we can choose a subsequence $(k_\ell)_\ell$ so that

$$\lim_{\ell\to\infty} \frac{\sum_{j=1}^d \tau_{k_\ell}^{(j)}(x)}{\tau_{k_\ell}(x)} = c$$

for some $c \in [0, d]$. By the compactness of $[0, 1]^d$, up to passing to a further subsequence, we can assume that $\lim_{\ell\to\infty}(\overline{\tau}_{k_\ell}/\tau_{k_\ell})(x) = (p_1, \dots, p_d)$. However, as all the limit points $\mathcal{T}(x)$ of $((\overline{\tau}_k/\tau_k)(x))_k$ are contained in $\mathcal{S}$, we find that

$$c = p_1 + \cdots + p_d = 1,$$

which proves (3.4).

Now, using (2.2), we see that

$$\frac{\sum_{j=1}^d \tau_k^{(j)}(x)}{\tau_k(x)} = \frac{\tau_k(x) - k}{\tau_k(x)} = 1 - \frac{k}{\tau_k(x)}.$$

Thus, (3.4) yields (3.3). This proves the proposition. □

Define

$$\widetilde{\mathcal{T}}(x) := \{\text{limit points of } ((e_n(x)(X_1), \dots, e_n(x)(X_d)))_n\} \subset [0, 1]^d.$$

LEMMA 3.6. *If $\mathcal{V}(x) \subset \pi\mathcal{S}$, then $\mathcal{V}(x) = \pi(\widetilde{\mathcal{T}}(x))$.*

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*Proof.* Suppose that $\mathcal{V}(x) \subset \pi\mathcal{S}$.

As the observable $(\mathbb{1}_{X_1}, \dots, \mathbb{1}_{X_d})$ is continuous at the continuity points of any measure in $\pi\mathcal{S}$, the Portmanteau theorem tells us that if $e_{n_k}(x) \to \nu_{\overline{p}}$ for a subsequence $(n_k)_k$ and some $\overline{p} \in \mathcal{S}$, then

$$\begin{aligned}\overline{p} = \int (\mathbb{1}_{X_1}, \dots, \mathbb{1}_{X_d})\, d\nu_{\overline{p}} &= \lim_{k\to\infty} \int (\mathbb{1}_{X_1}, \dots, \mathbb{1}_{X_d})\, de_{n_k}(x) \\ &= (e_{n_k}(x)(X_1), \dots, e_{n_k}(x)(X_d)). \end{aligned} \tag{3.5}$$

Thus, $\mathcal{V}(x) \subset \pi\widetilde{\mathcal{T}}(x)$.

Now, suppose that $\lim_{k\to\infty}(e_{n_k}(x)(X_1), \dots, e_{n_k}(x)(X_d)) = \overline{p}$ for a subsequence $(n_k)_k$. Reasoning as in (3.5), we see that the only possible limit point of $(e_{n_k}(x))_k$ is $\nu_{\overline{p}}$. By compactness, $\lim_{k\to\infty} e_{n_k}(x) = \nu_{\overline{p}}$, which gives $\widetilde{\mathcal{T}}(x) \subset \pi^{-1}\mathcal{V}(x)$. □

*Proof of Theorem 3.1.* Suppose that $\mathcal{T}(x)$ is a closed connected subset of $\mathcal{S}$. By Proposition 3.3, $\mathcal{V}(x) \subset \pi\mathcal{S}$. Thus, from Lemma 3.6, it is enough to show that $\widetilde{\mathcal{T}}(x) = \mathcal{T}(x)$. As

$$\frac{\overline{\tau}_k}{\tau_k}(x) = (e_{\tau_k}(x)(X_1), \dots, e_{\tau_k}(x)(X_d)),$$

it is clear that $\mathcal{T}(x) \subset \widetilde{\mathcal{T}}(x)$.

By Assumption 2.6(2), the orbit of $x$ spends all of its time in at most one of the sets $X_1, \dots, X_d$ between any two successive entrances to $Y$. Thus, for any given $j = 1, \dots, d$, $k \in \mathbb{N}$, and $0 \le \ell < \tau_{k+1}(x) - \tau_k(x)$, one has for $n = \tau_k(x) + \ell$ that either

$$e_n(x)(X_j) = \frac{\tau_k^{(j)}(x) + \ell}{\tau_k(x) + \ell} \quad \text{or} \quad e_n(x)(X_j) = \frac{\tau_k^{(j)}(x)}{\tau_k(x) + \ell},$$

and the former is increasing in $\ell$ and the latter is decreasing in $\ell$. (Notice that the derivative of $t \mapsto (a + t/b + t)$ is $(b - a/(b + t)^2)$, which is non-negative whenever $b \ge a$.) Thus, the sequence $(e_n(X_j))_n$ is strictly monotonic on each time interval $[\tau_k(x), \tau_{k+1}(x)] \cap \mathbb{N}$. This tells us that

$$\min\left\{\frac{\tau_k^{(j)}(x)}{\tau_k(x)}, \frac{\tau_{k+1}^{(j)}(x)}{\tau_{k+1}(x)}\right\} \le e_n(x)(X_j) \le \max\left\{\frac{\tau_k^{(j)}(x)}{\tau_k(x)}, \frac{\tau_{k+1}^{(j)}(x)}{\tau_{k+1}(x)}\right\}. \tag{3.6}$$

Suppose that (3.1) is satisfied and let

$$\varepsilon_k := \left|\frac{\tau_k^{(j)}(x)}{\tau_k(x)} - \frac{\tau_{k+1}^{(j)}(x)}{\tau_{k+1}(x)}\right|.$$

Hence, as $\mathcal{T}(x)$ are the limit points of $(\tau_k^{(j)}(x)/\tau_k(x))_k$, for all $k$ sufficiently large, we get

$$\frac{\tau_k^{(j)}(x)}{\tau_k(x)} \in B_{2\varepsilon_k}(\mathcal{T}(x)).$$

Thus, we get that

$$(e_n(x)(X_1), \dots, e_n(x)(X_d)) \in B_{3\varepsilon_k}(\mathcal{T}(x)),$$

which implies that $\widetilde{\mathcal{T}}(x) \subset \mathcal{T}(x)$. This concludes the proof. □

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## 4. *Auxiliary results on repellers and return times*

In this section, we collect some auxiliary results on 'sub-repellers' for some (high enough) iterate $F^n$ of our initial map $F$.

### 4.1. *Cylinders and associated maximal invariant sets.*

Given $n \in \mathbb{N}$, we let

$$\mathcal{Q}^n := \bigvee_{i=0}^{n-1} F^{-i}\mathcal{Q}$$

denote the refinement of $\mathcal{Q}$ under $F$ into $n$-cylinders. Write

$$\mathcal{Q}^* := \{a \in \mathcal{Q}^n : n \in \mathbb{N}_0\}$$

for the collection of all cylinders. Given $a \in \mathcal{Q}^n$ and $b \in \mathcal{Q}^*$, we write $ab := a \cap F^{-n}b$.

We denote by $|a|$ the length of the interval $a \in \mathcal{Q}^*$.

Unless there is a risk of misunderstanding, given a collection of $n$-cylinders $\mathcal{A} \subset \mathcal{Q}^n$, we use the symbol $\mathcal{A}$ also to denote the union of all partition elements that $\mathcal{A}$ contains. For $\mathcal{A} \subset \mathcal{Q}^n$, consider the associated maximal invariant set

$$\Lambda(\mathcal{A}) := \{x : F^{kn}(x) \in \mathcal{A} \text{ for all } k \in \mathbb{N}_0\}. \tag{4.1}$$

Given any $A \subset X$, we denote by $\mathcal{Q}^\ell_A$ the set of $\ell$-cylinders whose interiors intersect $A$,

$$\mathcal{Q}^\ell_A := \{a \in \mathcal{Q}^\ell : \operatorname{int}(a) \cap A \neq \emptyset\}.$$

*Remark 4.1.* Notice that if $\mathcal{A} \subset \mathcal{Q}^n$, then for every $k \in \mathbb{N}_0$ and $s \in \{0, \dots, n-1\}$,

$$\mathcal{Q}^{kn+s}_{\Lambda(\mathcal{A})} = \{a_1 \cdots a_k b : a_1, \dots, a_k \in \mathcal{A} \text{ and } b \in \mathcal{Q}^s_{\mathcal{A}}\}.$$

### 4.2. *Geometric properties of repellers formed from maximal invariant sets.*

Given a finite collection $\mathcal{A} \subset \mathcal{Q}^n$, we define the *virtual dimension* $\operatorname{vdim}\Lambda(\mathcal{A})$ of its maximal invariant set $\Lambda(\mathcal{A})$ defined in (4.1) by setting

$$\operatorname{vdim}\Lambda(\mathcal{A}) \text{ is the unique } s > 0 \text{ satisfying } \sum_{a\in\mathcal{A}} |a|^s = 1. \tag{4.2}$$

By Assumption 2.2, the map $F$ is a $C^2$ uniformly expanding Markov map with bounded distortion. Hence, there exist a $\lambda > 1$ and a constant $D > 0$ such that for every $n \in \mathbb{N}$,

$$\inf_{x\in Y} |(F^n)'(x)| \geq \lambda^n \quad \text{and} \quad \sup_{a\in\mathcal{Q}^n} \sup_{x,y\in a} \log \frac{|(F^n)'(x)|}{|(F^n)'(y)|} \leq D. \tag{4.3}$$

In particular, map $g = F^n|_{\Lambda(\mathcal{A})}$ is an *expanding repeller* in the sense of [**PU10**]. That is, equipping $Y$ with the topology induced by the separation time distance, $\Lambda(\mathcal{A})$ is a *repeller* for $g$:

- $g(\Lambda(\mathcal{A})) = \Lambda(\mathcal{A})$;
- $g|_{\mathcal{A}}$ is continuous;
- $\mathcal{A} \subset Y$ is open and $\Lambda(\mathcal{A}) = \bigcap_{k=0}^{\infty} g^{-k}\mathcal{A}$; and
- $g$ is *expanding*: $|(g^k)'(x)| \geq \lambda^{nk}$ for all $x \in \Lambda(\mathcal{A})$.

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*Remark 4.2.* Estimates (4.3) imply that

$$D^{-1}\lambda^{-n} \leq |a| \leq \lambda^{-n} \quad \text{for all } n \in \mathbb{N} \text{ and } a \in \mathcal{Q}^n, \tag{4.4}$$

and

$$D^{-1}|a||b| < |ab| < D|a||b| \quad \text{whenever } a, b, ab \in \mathcal{Q}^*. \tag{4.5}$$

From [**PT93**, proof of Theorem 3, §4.2], we then obtain the following lemma.

LEMMA 4.3. *Let $\mathcal{A} \subset \mathcal{Q}^n$ be a finite subcollection. If $F^n\colon \Lambda(\mathcal{A}) \to \Lambda(\mathcal{A})$ is topologically mixing, then*

$$|\dim_{\mathrm{H}} \Lambda(\mathcal{A}) - \mathrm{vdim}_{\mathrm{H}}\Lambda(\mathcal{A})| \leq \frac{D}{n \log \lambda - D},$$

*where $D > 0$ and $\lambda > 1$ are as in (4.3).*

A Borel probability measure $m$ is *geometric* for $F^n|_{\Lambda(\mathcal{A})}$ if it is ergodic and invariant for $F^n$, $m(\Lambda(\mathcal{A})) = 1$, and if there exists a constant $E > 0$ so that

$$\dim_{\mathrm{H}} \Lambda(\mathcal{A}) - \frac{E}{\log 2r} \leq \frac{\log m(B_r(x))}{\log 2r} \leq \dim_{\mathrm{H}} \Lambda(\mathcal{A}) + \frac{E}{\log 2r} \tag{4.6}$$

for all $x \in \Lambda(\mathcal{A})$ and all $r > 0$. The following is provided, for example, by [**PU10**, Theorem 9.1.6] and [**PU10**, Theorem 8.1.6], by taking $m$ to be the Gibbs state corresponding to the function $-d \log |(F^n)'|$ (with respect to the mixing expanding repeller $F^n|_{\Lambda(\mathcal{A})}$), with $d = \dim_{\mathrm{H}} \Lambda(\mathcal{A})$.

LEMMA 4.4. *For any $\mathcal{A} \subset \mathcal{Q}^n$ so that $F^n|_{\Lambda(\mathcal{A})}$ is topologically mixing, there exists a corresponding geometric measure $m$.*

4.3. *Return times on cylinders.* By Remark 2.4, the function $\tau$ (and, hence, the vector-valued function $\overline{\tau}$) is constant on the interior of each element of $\mathcal{Q}$. Thus, for any $n \in \mathbb{N}$, the value of $\overline{\tau}_n(x)$ is completely determined by the $n$-cylinder which contains $x$. For $a \in \mathcal{Q}^n$, we will thus write $\overline{\tau}_n(a)$ for the value of $\overline{\tau}_n|_{\mathrm{int}(a)}$.

*Remark 4.5.* For $x, y, v, w > 0$,

$$\text{if} \quad \frac{x}{y} < \frac{v}{w} \quad \text{then} \; \frac{x}{y} \leq \frac{x+v}{y+w} \leq \frac{v}{w}. \tag{4.7}$$

For $\overline{x}, \overline{y} \in \mathbb{R}^d$, and $v, w \neq 0$, we have that

$$\frac{\overline{x}+\overline{y}}{v+w} - \frac{\overline{x}}{v} = \left(\frac{\overline{y}}{w} - \frac{\overline{x}}{v}\right)\frac{w}{v+w}. \tag{4.8}$$

From Remark 4.5, we obtain the following simple inequalities. Here, all equalities involving vectors in $\mathbb{R}^d$ are to be understood coordinate-wise. For $\overline{x} \in \mathbb{R}^d$, let $|\overline{x}| := \max_{i=1,\ldots,d} |x_i|$.

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LEMMA 4.6. *For every $a \in \mathcal{Q}^n$ and $b \in \mathcal{Q}^k$ such that $ab \in \mathcal{Q}^{n+k}$:*

(1) $|(\overline{\tau}_{n+k}/\tau_{n+k})(ab) - (\overline{\tau}_n/\tau_n)(a)| \le (2\tau_k(b))/(\tau_n(a) + \tau_k(b))$;

(2) $|(\overline{\tau}_{n+k}/\tau_{n+k})(ab) - (\overline{\tau}_k/\tau_k)(b)| \le (2\tau_n(a))/(\tau_n(a) + \tau_k(b))$;

(3) $\min\{(\overline{\tau}_n/\tau_n)(a), (\overline{\tau}_k/\tau_k)(b)\} \le (\overline{\tau}_{n+k}/\tau_{n+k})(ab) \le \max\{(\overline{\tau}_n/\tau_n)(a), (\overline{\tau}_k/\tau_k)(b)\}$.

*Proof.* First, notice that

$$\frac{\overline{\tau}_{n+k}}{\tau_{n+k}}(ab) = \frac{\overline{\tau}_n(a) + \overline{\tau}_k(b)}{\tau_n(a) + \tau_k(b)}.$$

The inequality in item (3) then follows from (4.7). Also note that $|\overline{\tau}_j/\tau_j| \le 1$. Hence, (4.8) implies

$$\left|\frac{\overline{\tau}_{n+k}}{\tau_{n+k}}(ab) - \frac{\overline{\tau}_n}{\tau_n}(a)\right| = \left|\left(\frac{\overline{\tau}_k(b)}{\tau_k(b)} - \frac{\overline{\tau}_n(a)}{\tau_n(a)}\right)\frac{\tau_k(b)}{\tau_n(a) + \tau_k(b)}\right| \le 2\frac{\tau_k(b)}{\tau_n(a) + \tau_k(b)}$$

together with the analogous lower bound. This proves item (1). The proof of item (2) is analogous. □

## 5. *Approximation by repellers*

Recall the big image property for $F$ from Assumption 2.2 stating that $\{F(a) : a \in \mathcal{Q}\} = \mathcal{Y} := \{Y_1, \dots, Y_L\}$ for some $L \in \mathbb{N}$. It hence follows that

$$\{F^n(a) : a \in \mathcal{Q}^n\} = \mathcal{Y} \quad \text{for every } n \ge 1. \tag{5.1}$$

In this section, we will prove the following proposition.

PROPOSITION 5.1. *For every $\overline{p} \in \mathcal{S}$ and every $\varepsilon > 0$, there exist $n, N \in \mathbb{N}$, a finite subset $\mathcal{A} \subset \mathcal{Q}^n$, and a Borel probability measure $m$ on $Y$ such that: the image of $\mathcal{A}$ under $F^n$ covers $Y$ and, in particular, $F^n(\mathcal{A}) = \mathcal{Y}$; $m(\Lambda(\mathcal{A})) = 1$; and $m(\Lambda(\mathcal{A}) \cap Y_i) > 0$ for each $i = 1, \dots, L$. Moreover,*

$$\frac{\overline{\tau}_\ell}{\tau_\ell}(a) \in B_\varepsilon(\overline{p}) \quad \text{and} \quad \frac{\log m(a)}{\log |a|} \in B_\varepsilon(1) \quad \text{for all } \ell > N \text{ and all } a \in \mathcal{Q}^\ell_{\Lambda(\mathcal{A})}. \tag{5.2}$$

Throughout this section, we fix $\overline{p} \in \mathcal{S}$ and $\varepsilon \in (0, 1)$. In §5.1, we construct, for each $n \in \mathbb{N}$ sufficiently large, a collection of $n$-cylinders $\mathcal{A}(n)$. In §§5.2 and 5.3, we show, for all $\ell \in \mathbb{N}$ sufficiently large, that cylinders $a \in \mathcal{Q}^\ell_{\Lambda(\mathcal{A}(n))}$ satisfy the assertion (5.2) of Proposition 5.1 with respect to a certain measure $m(n)$ on $\Lambda(\mathcal{A}(n))$.

### 5.1. *Construction of $\mathcal{A}(n)$.*

PROPOSITION 5.2. *There exists $C \in (0, 1)$ so that for every $n$ sufficiently large, there is a finite collection $\mathcal{A}(n) \subset \mathcal{Q}^n$ such that:*

(1) $\mathcal{A}(n) \subset \{a \in \mathcal{Q}^n : \overline{\tau}_n(a)/\tau_n(a) \in B_{3\varepsilon/4}(\overline{p})\}$;

(2) $\mathrm{Leb}(\mathcal{A}(n) \cap Y_i) > C$ *for all* $i = 1, \dots, L$;

(3) $F^n\mathcal{A}(n) = \mathcal{Y}$.

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We divide the proof of Proposition 5.2 into several intermediate lemmas. First, we give the following uniform lower bound for the Lebesgue measure of the set of points $x$ where the ratio $(\overline{\tau}_n/\tau_n)(x)$ is close to $\overline{p}$.

LEMMA 5.3. *There exists $C' > 0$ such that*

$$\mathrm{Leb}\{x \in Y : \overline{\tau}_n(x)/\tau_n(x) \in B_{\varepsilon/2}(\overline{p})\} > C'$$

*for all $n$ sufficiently large.*

*Proof.* Let us first prove that there exists a random variable $\boldsymbol{Z}$ whose distribution $\boldsymbol{Z}_*\mathbb{P}$ on $\mathcal{S}$ is equivalent to Leb and such that the random variables

$$x \mapsto (e_n(x)(X_1), \ldots, e_n(x)(X_d))$$

converge in distribution to $\boldsymbol{Z}$ as $n \to \infty$, that is,

$$\begin{aligned}&\mu\{x : (e_n(x)(X_1), \ldots, e_n(x)(X_d)) \in A\}\\ &\quad \to \mathbb{P}(\boldsymbol{Z} \in A) \quad \text{for every } A \text{ so that } \mathbb{P}(\boldsymbol{Z} \in \partial A) = 0.\end{aligned}$$

This follows from [**Ser20**, Corollary 4.2 and Theorem 3.3], provided one can check [**Ser20**, Assumptions 2.1, 2.2, and 2.6]. Assumption 2.6 yields [**Ser20**, Assumption 2.1], and then Assumption 2.6(3) together with [**Ser20**, Lemma 2.4] gives [**Ser20**, Assumption 2.2]. Finally, as $F$ is topologically mixing, it is exponentially continued fraction mixing (see, for example, [**Aar97**, §4]), which yields [**Ser20**, Assumption 2.6].

Recall that, as in the proof of Theorem 3.1, we have that for all $x \in X$,

$$\frac{\overline{\tau}_k}{\tau_k}(x) = (e_{\tau_k}(x)(X_1), \ldots, e_{\tau_k}(x)(X_d)).$$

Thus, as the distribution of $\boldsymbol{Z}_*\mathbb{P}$ of $\boldsymbol{Z}$ is equivalent to Lebesgue on $\mathcal{S}$, we can take $A = B_{\varepsilon/2}(\overline{p})$ in the above to obtain

$$\lim_{n\to\infty} \mu\{x : \overline{\tau}_n(x)/\tau_n(x) \in B_{\varepsilon/2}(\overline{p})\} = \mathbb{P}(\boldsymbol{Z} \in B_{\varepsilon/2}(\overline{p})) > 0.$$

This implies the conclusion of the lemma for the measure $\mu$. Finally, as (see Remark 2.5) $\mu$ is equivalent to Leb, this implies the assertion. □

LEMMA 5.4. *There is $k_0 \in \mathbb{N}$ such that for every $i, j \in \{1, \ldots, L\}$, there exists $a_{ij} \subset Y_i$ such that $a_{ij} \in \mathcal{Q}^{k_0}$ and $F^{k_0}(a_{ij}) = Y_j$.*

*Proof.* Fix $i, j \in \{1, \ldots, L\}$ and fix $a_i \in \mathcal{Q}$ with $a_i \subset Y_i$. As $F$ is topologically mixing, we can choose $k_{ij} \in \mathbb{N}$ to be such that, for all $n \geq k_{ij}$,

$$F^n(a_i) \cap Y_j \neq \emptyset.$$

Let $k_0 := \max_{1\leq i,j\leq L} k_{ij}$. Hence, there is $a_{ij} \in \mathcal{Q}^{k_0} \cap a_i$ such that $F^{k_0}(a_{ij}) \cap Y_j \neq \emptyset$. However, we know that $F^{k_0}(a_{ij}) \in \{Y_1, \ldots, Y_L\}$, Hence, we get $F^{k_0}(a_{ij}) = Y_j$. □

*Proof of Proposition 5.2.* Given $n \in \mathbb{N}$, let

$$\mathcal{B}(n) := \{a \in \mathcal{Q}^n : \overline{\tau}_n(a)/\tau_n(a) \in B_{\varepsilon/2}(\overline{p})\}$$

and let $\mathcal{B}(n)' \subset \mathcal{B}(n)$ be a finite sub-collection so that $\mathrm{Leb}(\mathcal{B}(n)') \geq \frac{1}{2}\mathrm{Leb}(\mathcal{B}(n))$. Thus, by Lemma 5.3, there exists some constant $C' > 0$ so that

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$$\text{Leb } \mathcal{B}(n)' \geq C' \quad \text{for all } n \text{ sufficiently large.} \tag{5.3}$$

Let $k_0$ and $\{a_{ij}\}_{i,j=1}^{L}$ be as given by Lemma 5.4. As $F$ is assumed to be a topologically mixing Gibbs–Markov map, each element of $\mathcal{Y}$ is a union of elements of $\mathcal{Q}$, and the union of all elements in $\mathcal{Y}$ equals $Y$. So, for any cylinder $b \in \mathcal{Q}^n$, we know from (5.1) that there exist $p(b), s(b) \in \{1, \ldots, L\}$ so that $b \subset Y_{p(b)}$ and $F^n(b) = Y_{s(b)}$. Thus, for all $i, j \in \{1, \ldots, L\}$, the cylinder $a_{ip(b)}ba_{s(b)j} \in \mathcal{Q}^{n+2k_0}$ is always non-empty and satisfies

$$a_{ip(b)}ba_{s(b)j} \subset Y_i \quad \text{and} \quad F^{n+2k_0}(a_{ip(b)}ba_{s(b)j}) = Y_j.$$

To choose now our collection, given $n > 2k_0$, we set

$$\mathcal{A}(n) := \{a_{ip(b)}ba_{s(b)j} : b \in \mathcal{B}(n-2k_0)',\ i, j = 1, \ldots, L\}.$$

By construction, $\mathcal{A}(n)$ satisfies item (3) of the proposition.

Letting $i = 1, \ldots, L$,

$$\begin{aligned}
\text{Leb}(\mathcal{A}(n) \cap Y_i) &= \sum_{j=1}^{L} \sum_{b \in \mathcal{B}(n-2k_0)'} |a_{ip(b)}ba_{s(b)j}| \\
\text{(by (4.5))} \quad &\geq D^{-2} \sum_{j=1}^{L} \sum_{b \in \mathcal{B}(n-2k_0)'} |a_{ip(b)}||b||a_{s(b)j}| \\
&\geq D^{-2} \left( \min_{\ell,j=1,\ldots,L} |a_{\ell,j}| \right)^2 \text{Leb}(\mathcal{B}(n-2k_0)').
\end{aligned}$$

So, for all $n$ large enough that (5.3) holds, there is a $C > 0$ such that $\text{Leb}(\mathcal{A}(n) \cap Y_i) > C$. This concludes item (2).

Now, consider an arbitrary element $a_{ip}ba_{sj} \in \mathcal{A}(n)$ for some $n \geq 1$. From Lemma 4.6(2) and the fact that $\tau_n \geq n$, we know that

$$\left| \frac{\overline{\tau}_n}{\tau_n}(a_{ip}ba_{sj}) - \frac{\overline{\tau}_{n-k_0}}{\tau_{n-k_0}}(ba_{sj}) \right| < 2 \frac{\tau_{k_0}(a_{ip})}{\tau_{n-k_0}(ba_{sj}) + \tau_{k_0}(a_{ip})} \leq \frac{2M}{n}, \tag{5.4}$$

where

$$M := \max_{\ell,m=1,\ldots,L} \tau_{k_0}(a_{\ell m}).$$

Similarly, by Lemma 4.6(1),

$$\left| \frac{\overline{\tau}_{n-k_0}}{\tau_{n-k_0}}(ba_{sj}) - \frac{\overline{\tau}_{n-2k_0}}{\tau_{n-2k_0}}(b) \right| < 2 \frac{\tau_{k_0}(a_{sj})}{\tau_{n-2k_0}(b) + \tau_{k_0}(a_{sj})} \leq \frac{2M}{n-k_0}. \tag{5.5}$$

So, as $b \in \mathcal{B}(n-2k_0)' \subset \mathcal{B}(n-2k_0)$, we have

$$\frac{\overline{\tau}_n}{\tau_n}(b) \in B_{\varepsilon/2}(\overline{p}). \tag{5.6}$$

Thus, combining (5.4), (5.5), and (5.6), we find that

$$\mathcal{A}(n) \subset \{a \in \mathcal{Q}^n : (\overline{\tau}_n/\tau_n)(a) \in B_{3\varepsilon/4}(\overline{p})\}$$

provided that $n$ is sufficiently large. This yields item (1) and completes the proof. □

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5.2. *Return time on $\mathcal{A}(n)$.* Let $n$ be large enough so that the conclusions of Proposition 5.2 hold. Let $\mathcal{A}(n)$ be as provided by Proposition 5.2 and let $\Lambda(n) := \Lambda(\mathcal{A}(n))$.

LEMMA 5.5. *There exists $N_0 = N_0(\mathcal{A}(n)) \in \mathbb{N}$ such that*

$$|(\overline{\tau}_\ell/\tau_\ell)(a) - \overline{p}| < \varepsilon \quad \textit{for every } \ell > N_0 \textit{ and every } a \in \mathcal{Q}^\ell_{\Lambda(n)}. \tag{5.7}$$

*Proof.* We first claim that

$$\left|\frac{\overline{\tau}_{kn}}{\tau_{kn}}(a) - \overline{p}\right| < \frac{3\varepsilon}{4} \quad \text{for every } a \in \mathcal{Q}^{kn}_{\Lambda(\mathcal{A}(n))} \text{ and every } k \geq 1. \tag{5.8}$$

Notice that item (1) of Proposition 5.2 implies (5.8) for $k = 1$. Proceeding inductively on $k$, let $a \in \mathcal{Q}^{(k+1)n}_{\Lambda(n)}$. By Remark 4.1, we have $a = a_1 \cdots a_k a_{k+1}$, for $a_i \in \mathcal{A}(n)$. It follows from Lemma 4.6(3) that

$$\frac{\overline{\tau}_{(k+1)n}}{\tau_{(k+1)n}}(a) = \frac{\overline{\tau}_{kn+n}}{\tau_{kn+n}}(a) \leq \max\left\{\frac{\overline{\tau}_{kn}(a_1 \cdots a_k)}{\tau_{kn}(a_1 \cdots a_k)}, \frac{\overline{\tau}_n(a_{k+1})}{\tau_n(a_{k+1})}\right\}$$

together with the analogous lower bound. Hence, from (5.8) and our inductive hypothesis,

$$\frac{\overline{\tau}_{(k+1)n}}{\tau_{(k+1)n}}(a) \in B_{3\varepsilon/4}(\overline{p}\,),$$

yielding (5.8).

We finally conclude (5.7). As $\mathcal{A}(n)$ only contains finitely many cylinders, the value

$$M := \max_{b \in \mathcal{A}(n)} \tau_n(b)$$

is finite. Let $b \in \mathcal{Q}^s \cap \mathcal{A}(n)$ with $s \in \{1, \ldots, n\}$. It follows from Lemma 4.6(1) that

$$\left|\frac{\overline{\tau}_{kn+s}}{\tau_{kn+s}}(bc) - \frac{\overline{\tau}_{kn}}{\tau_{kn}}(b)\right| \leq \frac{2\overline{\tau}_s(c)}{\overline{\tau}_{kn}(b) + \overline{\tau}_s(c)} \leq 2\frac{M}{kn+s}. \tag{5.9}$$

Choose $N_0 := N_0(\mathcal{A}(n)) > n$ so that $2M/N_0 < \varepsilon/4$. For every $\ell > N_0$ and $a \in \mathcal{Q}^\ell_{\Lambda(n)}$, relation (5.9) together with (5.8) implies that $(\overline{\tau}_\ell/\tau_\ell)(a) \in B_\varepsilon(\overline{p})$. This proves the lemma. □

5.3. *Geometric properties of $\mathcal{A}(n)$.* For each $n$ large enough so that Proposition 5.2 holds, let $\mathcal{A}(n)$ be as provided by Proposition 5.2 and let $\Lambda(n) = \Lambda(\mathcal{A}(n))$.

LEMMA 5.6. *There exists a $C' > 0$ such that $\dim_{\mathrm{H}} \Lambda(n) > 1 - C'/n$ for all $n$ sufficiently large.*

*Proof.* Consider the function $h\colon [0, 1] \to \mathbb{R}$ given by

$$h(t) := \sum_{a \in \mathcal{A}(n)} |a|^{1-t}.$$

Let $s(n) := 1 - \operatorname{vdim} \Lambda(n) \in [0, 1]$ so that, by (4.2), $h(s(n)) = 1$. It follows from item (2) in Proposition 5.2 that

$$h(0) = \sum_{a \in \mathcal{A}(n)} |a| = \operatorname{Leb}(\mathcal{A}(n)) > C, \tag{5.10}$$

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and this property holds for all $n$ sufficiently large. We get

$$\begin{aligned} h'(t) &= -\sum_{a\in\mathcal{A}(n)} \log|a|\cdot|a|^{1-t} \\ &\geq n\log\lambda\cdot\sum_{a\in\mathcal{A}(n)}|a|^{1-t} \geq n\log\lambda\cdot\sum_{a\in\mathcal{A}(n)}|a| \quad \text{by (4.4)} \\ &\geq Cn\log\lambda \quad \text{by (5.10)} \end{aligned}$$

giving that $h'(t)\geq Cn\log\lambda$. It follows that

$$s(n)\leq\frac{1-h(0)}{\|h'\|_\infty}\leq\frac{1-C}{Cn\log\lambda}.$$

Using Lemma 4.3, with the above, we get

$$\begin{aligned} \dim_{\mathrm{H}}\ \Lambda(n) &\geq \mathrm{vdim}\Lambda(n)-\frac{D}{n\log\lambda-D} = 1-s(n)-\frac{D}{n\log\lambda-D} \\ &\geq 1-\frac{1-C}{Cn\log\lambda}-\frac{D}{n\log\lambda-D}. \end{aligned}$$

Letting $C' := (1-C)/(C\log\lambda)-D/(\log\lambda-D)$, the assertion follows. □

Fix $m := m(n)$ to be the geometric measure for $F^n|_{\Lambda(n)}$ whose existence is guaranteed by Lemma 4.4.

LEMMA 5.7. *For all $n$ sufficiently large, there exists an $N_1 = N_1(\mathcal{A}(n))$ so that*

$$\left|\frac{\log m(a)}{\log|a|}-1\right|<\varepsilon \quad \textit{for all } \ell>N_1 \textit{ and all } a\in\mathcal{Q}^\ell_{\Lambda(n)}.$$

*Proof.* As $m$ is geometric for $F^n|_{\Lambda(n)}$, by (4.6), there exists an $E(n)>0$ so that

$$\dim_{\mathrm{H}}\ \Lambda(n)+\frac{E(n)}{\log|a|}\leq\frac{\log m(a)}{\log|a|}\leq\dim_{\mathrm{H}}\ \Lambda(n)-\frac{E(n)}{\log|a|}. \tag{5.11}$$

Using (4.4), we can choose $N_1 = N_1(\mathcal{A}(n))\in\mathbb{N}$ so that for $\ell\geq N_1$,

$$\sup_{a\in\mathcal{Q}^\ell}\frac{E(n)}{-\log|a|}\leq\frac{E(n)}{\ell\log\lambda}<\varepsilon/2. \tag{5.12}$$

If $n>2C'/\varepsilon$, then Lemma 5.6 ensures $\dim_{\mathrm{H}}\ \Lambda(n)\geq 1-\varepsilon/2$. Combining this with (5.11) and (5.12) concludes the proof. □

5.4. *Proof of Proposition 5.1.* Fix $n$ large enough so that Lemma 5.7 and Proposition 5.2 hold. Let $\mathcal{A}=\mathcal{A}(n)\subset\mathcal{Q}^n$ be as in Proposition 5.2 and let $N_0$, $N_1$ be as in Lemmas 5.5 and 5.7. Fix $N := \max\{N_0, N_1\}$ and let $m$ be the geometric measure for $F^n|_{\Lambda(\mathcal{A})}$ whose existence is guaranteed by Lemma 4.4. Then, Lemmas 5.5 and 5.7 yield assertion (5.2). Moreover, $m(\Lambda(\mathcal{A}))=1$ and $F^n(\mathcal{A})=\mathcal{Y}$ by item (3) of Proposition 5.2.

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Fix $i \in \{1, \dots, L\}$. As $Y_i \cap \mathcal{A}$ is a finite collection of intervals, we know from (4.6) that

$$m(\Lambda(\mathcal{A}) \cap Y_i) = m(\mathcal{A} \cap Y_i) = \sum_{a \in \mathcal{A} \cap Y_i} m(a) \geq E' \sum_{a \in \mathcal{A} \cap Y_i} |a|^d \geq E' r^{d-1} \sum_{a \in \mathcal{A} \cap Y_i} \mathrm{Leb}(a) \geq E' r^{d-1} \mathrm{Leb}(\mathcal{A} \cap Y_i),$$

where $r = \min_{a \in \mathcal{A} \cap Y_i} |a|$ for some $E' > 0$. The latter is strictly positive by item (2) of Proposition 5.2. This concludes the proof.

## 6. *Proof of Corollary 2.9*

Throughout this section, we fix an arbitrary $\overline{p} \in \mathcal{S}$. We are going to show that

$$\dim_{\mathrm{H}}(\mathscr{G}(\nu_{\overline{p}})) = 1. \tag{6.1}$$

To do so, we first construct a fractal set $\Gamma$, see §6.1. In §6.2, we construct a Borel probability measure supported on $\Gamma$. Section 6.3 states some auxiliary results. In §6.4, we show $\Gamma \subset \mathscr{G}(\nu_{\overline{p}})$. In §6.5, we prove $\dim_{\mathrm{H}}(\Gamma) = 1$, which implies (6.1).

### 6.1. *Construction of a fractal set* $\Gamma$.

Fix a sequence $(\varepsilon_i)_{i \geq 0} \subset (0, 1)$ such that $\lim_{i \to \infty} \varepsilon_i = 0$. For each $i \geq 0$, fix

$$n_i = n(\varepsilon_i, \overline{p}), \quad \mathcal{A}_i = \mathcal{A}(\varepsilon_i, \overline{p}) \subset \mathcal{Q}^{n_i}, \quad \Lambda_i = \Lambda(\mathcal{A}_i), \quad N_i = N(\varepsilon_i, \overline{p}), \quad m_i = m(\varepsilon_i, \overline{p}) \tag{6.2}$$

as given by Proposition 5.1.

Let $(k_i)_{i \geq 0} \subset \mathbb{N}$ be an arbitrary sequence of integers. In the remainder of this subsection, we construct a fractal set $\Gamma = \Gamma((k_i)_i)$ which will depend on this sequence. The precise choice of $(k_i)_i$, and hence the precise choice of $\Gamma$, will be fixed in §6.3. Set

$$t_0 \coloneqq 0 \quad \text{and} \quad t_i \coloneqq k_0 n_0 + \cdots + k_{i-1} n_{i-1},$$

and define

$$\mathcal{B}_i \coloneqq \bigvee_{j=0}^{i-1} F^{-t_j} \mathcal{Q}_{\Lambda_j}^{k_j n_j} \quad \text{and} \quad \Gamma = \Gamma((k_i)_i) \coloneqq \bigcap_{i \in \mathbb{N}} \bigcup_{b \in \mathcal{B}_i} b. \tag{6.3}$$

Heuristically, the set $\Gamma$ consists of points that enter a neighborhood of $\Lambda_i$ at time $t_i$ and spend time $k_i n_i$ close to $\Lambda_i$ before moving close to $\Lambda_{i+1}$ at time $t_{i+1}$. Note that for each $i \geq 0$, $\mathcal{B}_i \subset \mathcal{Q}^{t_i}$ and that the collection $\mathcal{B}_i$ covers $\Gamma$. By definition, the collections $(\mathcal{B}_i)_i$ are 'nested' in the sense that every element in $\mathcal{B}_{i+1}$ is a subset of an element of $\mathcal{B}_i$ for every $i$.

LEMMA 6.1. *$\Gamma$ is non-empty and compact.*

*Proof.* It is enough to check that for every index $i \geq 0$ for every $a \in \mathcal{A}_{i+1}$, there exists $a' \in \mathcal{A}_i$ such that $F^{n_i}(a') \supset a$. This follows from the fact that Proposition 5.1 ensures $F^{n_i}(\mathcal{A}_i) = \mathcal{Y}$. So, every $\mathcal{B}_i$ is non-empty and the intersection in (6.3) is the intersection of a nested sequence of compact sets. This implies the assertion. □

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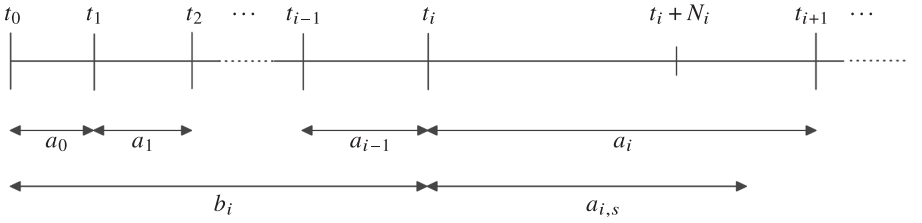



FIGURE 3. A sketch showing cylinders $b_i$, $a_i$, and $a_{i,s}$ as defined in (6.8) and (6.9).

6.2. *Construction of m.* We now construct a Borel measure $m$ on $Y$ satisfying $m(\Gamma) = 1$. We first define $m$ on cylinders $ba \in \mathcal{Q}^{t_i+s}$, where $b \in \mathcal{Q}^{t_i}$ and $a \in \mathcal{Q}^s$ for $s = 1, \ldots, k_i n_i$, inductively on $i$. For $i = 0$, $s = 1, \ldots, k_1 n_1$, and $a \in \mathcal{Q}^s$, we set

$$m(a) = m_0(a).$$

Proceeding inductively, for $i \geq 1$, $s = 1, \ldots, k_i n_i$, and $ba \in \mathcal{Q}^{t_i+s}$ with $b \in \mathcal{Q}^{t_i}$ and $a \in \mathcal{Q}^s$, we define

$$m(ba) = C_i(b) m(b) m_i(a), \tag{6.4}$$

where

$$C_i(b) \coloneqq (m_i(\Lambda_i \cap F^{t_i}(b)))^{-1} = (m_i(F^{t_i}(b)))^{-1}. \tag{6.5}$$

*Remark 6.2.* As $F^{t_i}(b) \in \mathcal{Y}$, Proposition 5.1 ensures that $C_i(b) > 0$ for all $b \in \mathcal{B}_i$.

This defines $m$ on all of $\mathcal{Q}^*$. One readily checks that this extends to a Borel probability measure on $Y$. By construction, $m(\mathcal{B}_i) = 1$ for every $i \geq 1$, and, in particular, $m(\Gamma) = 1$.

6.3. *Choice of the sequence* $(k_i)_i$. In the section, we fix the choice of the sequence $(k_i)_i$. First, we introduce

$$M_i \coloneqq \max_{a \in \mathcal{A}_i} \max_{0 \leq s \leq n_i - 1} \tau(F^s(a)) \tag{6.6}$$

and

$$\widetilde{C}_i \coloneqq \sup_{b \in \mathcal{Q}_\Gamma^{t_i}} |\log C_i(b)| = \max_{j=1,\ldots,L} |\log m_i(Y_j)|. \tag{6.7}$$

For every $x \in \Gamma$ and each $i \geq 0$, and $s = 1, \ldots, k_i n_i$, define the cylinders

$$a_{i,s}(x) \in \mathcal{Q}^s_{\Lambda_i}, \quad a_i(x) \in \mathcal{Q}^{k_i n_i}_{\Lambda_i} \quad \text{and} \quad b_i(x) \in \mathcal{B}_i \tag{6.8}$$

such that

$$x \in b_i(x) a_{i,s}(x), \quad a_i(x) = a_{i,k_i n_i}(x) \quad \text{and} \quad b_i(x) = a_0(x) \cdots a_{i-1}(x). \tag{6.9}$$

In what follows, we will regularly suppress the dependence of the associated $b_i$ and $a_i$ on $x$ to ease notation. See Figure 3.

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We can now state our precise conditions on $(k_i)_i$.

$$k_i > \frac{N_i}{n_i}, \tag{6.10}$$

$$\frac{\sum_{j=0}^{i-1} M_j k_j n_j}{k_i n_i} < \varepsilon_i, \tag{6.11}$$

$$\frac{N_{i+1} M_{i+1}}{t_{i+1}} < \varepsilon_i, \tag{6.12}$$

$$\sup_{x\in\Gamma} \frac{|\log m(b_i(x))| + \widetilde{C}_i}{|\log m_i(a_i(x))|} \le \varepsilon_i, \tag{6.13}$$

$$\sup_{x\in\Gamma} \frac{|\log |b_i(x)|| + |\log D|}{|\log |a_i(x)||} \le \varepsilon_i, \tag{6.14}$$

$$\sup_{x\in\Gamma} \sup_{1\le s\le N_i} \frac{|\log m_i(a_{i,s}(x))| + \widetilde{C}_i}{|\log m(b_i)|} < \varepsilon_{i-1}, \tag{6.15}$$

$$\sup_{x\in\Gamma} \sup_{1\le s\le N_i} \frac{|\log m_i(a_{i,s}(x))| + \widetilde{C}_i}{|\log m(b_i)|} < \varepsilon_{i-1}, \tag{6.16}$$

*Remark 6.3.* Condition (6.10) enables the application of Proposition 5.1, compare (6.2).

By our recursive decomposition into symbolic blocs $b_{i+1} = b_i a_i$, the asymptotic growth of $\overline{\tau}/\tau$ is essentially determined by the 'immediate past' $a_i$: see the proof of Lemma 6.8 where conditions (6.11) and (6.12) enable estimates (6.20) and (6.22), respectively. Analogous arguments are used to prove Lemma 7.1.

Conditions (6.13)–(6.14) and (6.15)–(6.16) control that arising distortion terms are small (see the proof of Lemma 6.6 and estimates (6.24) and (6.25), respectively).

In the remainder of the section, we prove the following proposition.

PROPOSITION 6.4. *There exists a sequence $(k_i)_i$ such that (6.10)–(6.16) hold.*

We divide the proof into the following lemmas.

LEMMA 6.5. *If the sequence $(k_i)_i$ grows sufficiently quickly, then (6.13) and (6.14) hold.*

*Proof.* Note that the collection $\{b_i(x) : x \in \Gamma\}$ is finite. We also know that the quantities on the tops of the fractions in (6.13) and (6.14) depend on $k_0, \ldots, k_{i-1}$, but not $k_i$. So, there exists some constant $L_i < +\infty$ which is independent of $k_i$ (but may depend on $k_0, \ldots, k_{i-1}$) such that

$$\sup_{x\in\Gamma} |\log m(b_i(x))| + \widetilde{C}_i < L_i \quad \text{and} \quad \sup_{x\in\Gamma} |\log |b_i(x)|| + |\log D| < L_i.$$

We know from Remark 4.2 that $|\log |a_i(x)|| \ge k_i n_i \log \lambda$. We also know from (6.17) that $|\log m_i(a_i)|/\log |a_i| \in B_{\varepsilon_i}(1)$. In particular, we must have that

$$|\log m_i(a_i)| \ge (1 - \varepsilon_i) k_i n_i \log \lambda.$$

So, we can choose $k_i$ recursively to grow fast enough so that (6.13) and (6.14) hold. □

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For what follows, recall from Proposition 5.1 that

$$\frac{\log m_i(a_{i,s}(x))}{\log |a_{i,s}(x)|} \in B_{\varepsilon_i}(1) \quad \text{for all } N_i \le s \le k_i n_i. \tag{6.17}$$

LEMMA 6.6. *If (6.13) and (6.14) hold, then for every $x \in \Gamma$,*

$$\lim_{i\to\infty} \frac{\log m(b_i(x))}{\log |b_i(x)|} = 1.$$

*Proof.* Let $x \in \Gamma$ and consider the associated sequence $(a_i)_i$ and $(b_i)_i$ in (6.8) and (6.9). Using the definition of $m$ in (6.4), the bounded distortion of $F$ in (4.4), we get

$$\begin{aligned}\frac{\log m(b_{i+1})}{\log |b_{i+1}|} = \frac{\log m(b_i a_i)}{\log |b_i a_i|} &\ge \frac{\log m(b_i) + \log m_i(a_i) + \log C_i(b_i)}{\log |b_i| + \log |a_i| - \log D} \\ &\ge \frac{\log m_i(a_i)}{\log |a_i|} \cdot \left( \frac{1 + (\log m(b_i) + \widetilde{C}_i)/(\log m_i(a_i))}{1 + (\log |b_i| - \log D)/(\log |a_i|)} \right) \quad \text{(by (6.7))}.\end{aligned}$$

Using (6.17), together with (6.13) and (6.14), we find the lower bound

$$\frac{\log m(b_{i+1})}{\log |b_{i+1}|} \ge (1 - \varepsilon_i) \cdot \frac{1 - \varepsilon_i}{1 + \varepsilon_i}.$$

Analogously, we obtain the upper bound

$$\frac{\log m(b_{i+1})}{\log |b_{i+1}|} \le (1 + \varepsilon_i) \cdot \frac{1 + \varepsilon_i}{1 - \varepsilon_i}.$$

Taking the limit as $i \to \infty$ concludes the proof. □

LEMMA 6.7. *If the sequence $(k_i)_i$ grows sufficiently quickly, then (6.15) and (6.16) hold.*

*Proof.* Suppose that $(k_i)_i$ is chosen to grow fast enough so that (6.13) and (6.14) hold by Lemma 6.5. The top of the fractions in (6.15) and (6.16) do not depend on the choice of sequence $(k_i)_i$. However, we know that

$$|\log |b_i(x)|| \ge k_{i-1} n_{i-1} \log \lambda.$$

Moreover, Lemma 6.6 tells us that then,

$$|\log |b_i(x)|| \ge (1 - \gamma \varepsilon_{i-1}) k_{i-1} n_{i-1} \log \lambda.$$

Choosing $(k_i)_i$ to grow sufficiently quickly ensures that (6.15) and (6.16) hold. □

*Proof of Proposition 6.4.* Conditions (6.10) and (6.12) are satisfied whenever $k_i$ is chosen to be large enough. Moreover, the top of the fraction in (6.11) depends only on $k_0, \ldots, k_{i-1}$ and not on $k_i$. These observations together with Lemmas 6.5 and 6.7 ensure that $(k_i)_i$ can be chosen such that (6.10)–(6.16) hold. This completes the proof. □

For the remainder of this section, we fix $(k_i)_i$ satisfying (6.10)–(6.16) and fix the corresponding choices of $\mathcal{B}_i$, $\Gamma$, and $m$.

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6.4. *Limit behavior of points in* $\Gamma$.

LEMMA 6.8. *For all* $x \in \Gamma$, $\mathcal{V}(x) = \nu_{\overline{p}}$.

*Proof.* Let $x \in \Gamma$. By Corollary 3.2, it is enough to show that $\mathcal{T}(x) = \overline{p}$.

Recall the definition of the associated cylinders $a_{i,s} \in \mathcal{Q}^s_{\Lambda_i}$, $a_i \in \mathcal{Q}^{k_i n_i}_{\Lambda_i}$, and $b_i \in \mathcal{B}_i$ in (6.8) and (6.9). Recall from Proposition 5.1 that

$$|(\overline{\tau}_s/\tau_s)(a_{i,s}) - \overline{p}| < \varepsilon_i \quad \text{for all } N_i \le s \le k_i n_i. \tag{6.18}$$

We also make use of the following inequalities which come from the definition (6.6) and the fact that $\tau \ge 1$:

$$t_i \le \tau_{t_i}(b_i) \le \sum_{j=0}^{i-1} M_j k_j n_j \quad \text{and} \quad s \le \tau_s(a_{i,s}) \le s M_i \tag{6.19}$$

for all $s = 1, \dots, k_i n_i$.

Recalling that $b_i = a_0 \cdots a_{i-1}$, we can write $b_{i+1} = b_i a_i$. Then, from Lemma 4.6(2) together with (6.19) and (6.11),

$$\left| \frac{\overline{\tau}_{t_{i+1}}}{\tau_{t_{i+1}}}(b_{i+1}) - \frac{\overline{\tau}_{k_i n_i}}{\tau_{k_i n_i}}(a_i) \right| \le \frac{2\tau_{t_i}(b_i)}{\tau_{t_i}(b_i) + \tau_{n_i k_i}(a_i)} < \frac{2\sum_{j=0}^{i-1} M_j k_j n_j}{k_i n_i} < 2\varepsilon_i. \tag{6.20}$$

So, using (6.18) with $s = k_i n_i$, we obtain

$$(\overline{\tau}_{t_{i+1}}/\tau_{t_{i+1}})(b_{i+1}) \in B_{3\varepsilon_i}(\overline{p}) \quad \text{for all } i \ge 0. \tag{6.21}$$

We now fix $i \ge 1$ and consider the intermediate times $t_i + s$ where $s = 1, \dots, k_i n_i - 1$. Let us first consider the range $s = 1, \dots, N_i - 1$. By Lemma 4.6(1) together with (6.19) and (6.12), we have that

$$\left| \frac{\overline{\tau}_{t_i+s}}{\tau_{t_i+s}}(b_i a_{i,s}) - \frac{\overline{\tau}_{t_i}}{\tau_{t_i}}(b_i) \right| \le 2\frac{\tau_s(a_{i,s})}{\tau_{t_i}(b_i) + \tau_s(a_{i,s})} \le \frac{2N_i M_i}{t_i} < 2\varepsilon_{i-1}. \tag{6.22}$$

So, using (6.21), we know that

$$(\overline{\tau}_{t_i+s}/\tau_{t_i+s})(b_i a_{i,s}) \in B_{5\varepsilon_{i-1}}(\overline{p}).$$

Now, let us consider the range $s = N_i, \dots, k_i n_i - 1$. By Lemma 4.6(3), we know that

$$\min\left\{ \frac{\overline{\tau}_{t_i}}{\tau_{t_i}}(b_i), \frac{\overline{\tau}_s}{\tau_s}(a_{i,s}) \right\} \le \frac{\overline{\tau}_{t_i+s}}{\tau_{t_i+s}}(b_i a_{i,s}) \le \max\left\{ \frac{\overline{\tau}_{t_i}}{\tau_{t_i}}(b_i), \frac{\overline{\tau}_s}{\tau_s}(a_{i,s}) \right\}.$$

Thus, by (5.7) and (6.21), we find that

$$\frac{\overline{\tau}_{t_i+s}}{\tau_{t_i+s}}(b_i a_{i,s}) \in B_{\delta_i}(\overline{p}) \quad \text{where } \delta_i := \max\{3\varepsilon_{i-1}, \varepsilon_i\}.$$

As we assume that $\varepsilon_i \to 0$, this proves $\mathcal{T}(x) = \overline{p}$. □

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6.5. *Hausdorff dimension of* $\Gamma$. Let us recall the definition of Hausdorff dimension (see [Fal97] for further details). Given $S \subset [0,1]$ and $d \geq 0$, let

$$H(S,d) := \lim_{\varepsilon \to 0} \inf \left\{ \sum_{k \geq 1} (\mathrm{diam} U_k)^d : \bigcup_{k \geq 1} U_k \supset S, \mathrm{diam} U_k < \varepsilon \right\},$$

where the infimum is taken over all countable covers $\{U_k\}$ of $S$. The *Hausdorff dimension* of $S$ is defined by $\dim_{\mathrm{H}} S := \inf\{d \geq 0 : H(S,d) = 0\}$.

LEMMA 6.9. $\dim_{\mathrm{H}} \Gamma = 1$.

*Proof.* Let $x \in \Gamma$. Recall the definition of the associated cylinders $a_{i,s} \in \mathcal{Q}^s_{\Lambda_i}$, $a_i \in \mathcal{Q}^{k_i n_i}_{\Lambda_i}$, and $b_i \in \mathcal{B}_i$ in (6.8) and (6.9). We will show that

$$\lim_{i \to \infty} \inf_{s=0,\ldots,k_i n_i - 1} \frac{\log m(b_i a_{i,s})}{\log |b_i a_{i,s}|} = \lim_{i \to \infty} \sup_{s=0,\ldots,k_i n_i - 1} \frac{\log m(b_i a_{i,s})}{\log |b_i a_{i,s}|} = 1. \tag{6.23}$$

Set

$$\ell_i := \frac{\log m(b_i)}{\log |b_i|}$$

and recall from Lemma 6.6 that $\lim_{i \to \infty} \ell_i = 1$. Consider first the range $s = 0, \ldots, N_i$. We proceed as in the proof of Lemma 6.6. Using the definition (6.4), the bounded distortion (see Remark 4.2)

$$\begin{aligned} \frac{\log m(b_i a_{i,s})}{\log |b_i a_{i,s}|} &\geq \frac{\log m(b_i) + \log m_i(a_{i,s}) + \log C_i(b_i)}{\log |b_i| + \log |a_{i,s}| - \log D} \\ &\geq \frac{\log m(b_i)}{\log |b_i|} \left( \frac{1 + (\log m(a_{i,s}) + \widetilde{C}_i)/(\log m(b_i))}{1 + (\log |a_{i,s}| - \log D)/(\log |b_i|)} \right) \\ &\geq \ell_i \frac{1 - \varepsilon_{i-1}}{1 + \varepsilon_{i-1}}, \end{aligned} \tag{6.24}$$

where in the final inequality, we have used (6.15) and (6.16).

Now, let us consider the range $s = N_i, \ldots, k_i n_i$. We begin as before, but this time, we make use of (4.7) to obtain

$$\begin{aligned} \frac{\log m(b_i a_{i,s})}{\log |b_i a_{i,s}|} &\geq \frac{\log m(b_i) + \log m_i(a_{i,s}) + \widetilde{C}_i}{\log |b_i| + \log |a_{i,s}| - \log D} \\ &\geq \min \left\{ \frac{\log m(b_i) + \widetilde{C}_i}{\log |b_i| - \log D}, \frac{\log m_i(a_{i,s})}{\log |a_{i,s}|} \right\}. \end{aligned} \tag{6.25}$$

Then, we notice, using (6.15) and (6.16), that

$$\frac{\log m(b_i) + \widetilde{C}_i}{\log |b_i| - \log D} = \frac{\log m(b_i)}{\log |b_i|} \left( \frac{1 + (\widetilde{C}_i)/(\log m(b_i))}{1 + (\log D)/(\log |b_i|)} \right) \geq \ell_i \frac{1 - \varepsilon_{i-1}}{1 + \varepsilon_{i-1}}. \tag{6.26}$$

Recalling (6.17), we have $\log m_i(a_{i,s}) / \log |a_{i,s}| \geq 1 - \varepsilon_i$. Combining this with (6.24), (6.25), and (6.26), we obtain

$$\inf_{s=0,\ldots,k_i n_i - 1} \frac{\log m(b_i a_{i,s})}{\log |b_i a_{i,s}|} \geq \underline{\delta}_i \quad \text{where } \underline{\delta}_i := \min \left\{ \ell_i \frac{1 - \varepsilon_{i-1}}{1 + \varepsilon_{i-1}}, 1 - \varepsilon_i \right\}.$$

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Proceeding in the same way as above, one can find analogous upper bounds for (6.24), (6.25), and (6.26), which yield

$$\sup_{s=0,\dots,k_i n_i-1} \frac{\log m(b_i a_{i,s})}{\log |b_i a_{i,s}|} \le \overline{\delta}_i \quad \text{where } \overline{\delta}_i := \max\left\{ \ell_i \frac{1+\varepsilon_{i-1}}{1-\varepsilon_{i-1}}, 1+\varepsilon_i \right\}.$$

As $\lim_{i\to\infty} \underline{\delta}_i = \lim_{i\to\infty} \overline{\delta}_i = 1$, we obtain (6.23).

Denote by $\mathcal{Q}^\ell(x)$ the element of the partition $\mathcal{Q}^\ell$ which contains $x$. Recall that, by the above, for every $x \in \Gamma$, for $\ell = t_i = k_0 n_0 + k_1 n_1 + \cdots + k_{i-1} n_{i-1}$ associated with $x$, there is a sequence $b_i(x) = a_0(x) a_1(x) \cdots a_{i-1}(x)$, where $a_j(x) \in \mathcal{Q}_{\Lambda_j}^{k_j n_j}$ and

$$\begin{aligned} \mathcal{Q}^\ell(x) &= \mathcal{Q}^{t_i}(x) = \mathcal{Q}^{k_0 n_0 + k_1 n_1 + \cdots + k_{i-1} n_{i-1}}(x) \\ &= a_0(x) \cap F^{-k_0 n_0}(a_1(x)) \cap \cdots \cap F^{-(k_0 n_0 + k_1 n_1 + \cdots + k_{i-1} n_{i-1})}(a_{i-1}(x)). \end{aligned}$$

Likewise, for $\ell = t_i + s$, with $s = 1, \dots, k_i n_i$,

$$\mathcal{Q}^\ell(x) = \mathcal{Q}^{t_i}(x) \cap F^{-(t_i+s)}(a_{i,s}(x)).$$

From the above, we get

$$\lim_{\ell\to\infty} \frac{\log m(\mathcal{Q}^\ell(x))}{\log |\mathcal{Q}^\ell(x)|} = 1.$$

As the refinements of the partition $\mathcal{Q}$ separate points, the assertion of the lemma then follows from the mass distribution principle (compare [**Fal97**, Ch. 2]). □

## 7. *Proof of Theorem A*

In this section, we prove Theorem A. We will follow the same structure as for the proof of the special case in §6. The only significant difference is how the sets $\mathcal{A}_i$ are chosen in the construction of the fractal set $\Gamma$. Throughout this section, we fix a closed connected subset $\mathcal{C} \subset \mathcal{S}$. In §7.1, we construct a set $\Gamma$. In §7.2, we show that $\Gamma \subset \{x : \mathcal{V}(x) = \pi\mathcal{C}\}$. In §7.3, we conclude $\dim_{\mathrm{H}} \Gamma = 1$, which proves Theorem A.

7.1. *Construction of a fractal set* $\Gamma$. Let $(\overline{p}_i)_{i\ge 0} \subset \mathcal{C}$ be a sequence whose limit points are equal to $\mathcal{C}$ and so that $\lim_{i\to\infty} |\overline{p}_i - \overline{p}_{i+1}| = 0$. Let $(\varepsilon_i)_{i\ge 0} \subset (0,1)$ be such that $\lim \varepsilon_i = 0$. For each $i \ge 0$, let $n_i := n_i(\varepsilon_i, \overline{p}_i)$, $\mathcal{A}_i = \mathcal{A}(\varepsilon_i, \overline{p}_i) \subset \mathcal{Q}^{n_i}$, $\Lambda_i = \Lambda(\mathcal{A}_i)$, $N_i = N(\varepsilon_i, \overline{p}_i)$, and $m_i = m(\varepsilon_i, \overline{p}_i)$ be as given in Proposition 5.1. With these definitions in place, we define $M_i, E_i$ as in (6.6). Then, we may repeat verbatim the proof of Proposition 6.4 to give the existence of a sequence $(k_i)_{i\ge 0}$ satisfying the conditions (6.10)–(6.16). Fixing such a sequence $(k_i)_i$, we set

$$t_0 := 0 \quad \text{and} \quad t_i := k_0 n_0 + \cdots + k_{i-1} n_{i-1},$$

and define

$$\mathcal{B}_i := \bigvee_{j=0}^{i-1} F^{-t_j} \mathcal{Q}_{\Lambda_j}^{k_j n_j} \quad \text{and} \quad \Gamma := \bigcap_{i\in\mathbb{N}} \bigcup_{b\in\mathcal{B}_i} b.$$

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Repeating the proof of Lemma 6.1, we see that $\Gamma$ is non-empty and compact. Following §6.2, we construct a probability measure $m$ on $\Gamma$ exactly as before.

### 7.2. *Limit behavior of points in* $\Gamma$.

LEMMA 7.1. *$\mathcal{V}(x) = \pi\mathcal{C}$ for every $x \in \Gamma$.*

*Proof.* Let $x \in \Gamma$. From Theorem 3.1, it is enough to show that $\mathcal{T}(x) = \mathcal{C}$ and (3.1) holds.

Recall the definition of the cylinders $a_{i,s}(x) \in \mathcal{Q}^s_{\Lambda_i}$, $a_i(x) \in \mathcal{Q}^{k_i n_i}_{\Lambda_i}(x)$, and $b_i(x) \in \mathcal{B}_i$ in (6.8) and (6.9) (see also Figure 3). We suppress the dependence of $a_{i,s}$, $a_i$, and $b_i$ on $x$ to ease notation. Recall from item (1) of Proposition 5.1 that

$$|(\overline{\tau}_s/\tau_s)(a_{i,s}) - \overline{p}_i| < \varepsilon_i \quad \text{for all } N_i \le s \le k_i n_i. \tag{7.1}$$

Writing $b_{i+1} = b_i a_i$, it follows from Lemma 4.6(2) together with (6.19) and (6.11) that

$$\left| \frac{\overline{\tau}_{t_{i+1}}}{\tau_{t_{i+1}}}(b_{i+1}) - \frac{\overline{\tau}_{k_i n_i}}{\tau_{k_i n_i}}(a_i) \right| \le \frac{2\tau_{t_i}(b_i)}{\tau_{t_i}(b_i) + \tau_{k_i n_i}(a_i)} < \frac{2\sum_{j=0}^{i-1} k_j n_j M_j}{k_i n_i} < 2\varepsilon_i. \tag{7.2}$$

So, using (7.1) with $s = k_i n_i$, we obtain

$$(\overline{\tau}_{t_{i+1}}/\tau_{t_{i+1}})(b_{i+1}) \in B_{3\varepsilon_i}(\overline{p}_i) \quad \text{for all } i \ge 0. \tag{7.3}$$

This gives the conclusion of the lemma along the subsequence $t_i$, in the sense that

$$\{\text{limit points of } (\overline{\tau}_{t_i}(x)/\tau_{t_i}(x))_{i\in\mathbb{N}}\} = \{\text{limit points of } (\overline{p}_i)_{i\in\mathbb{N}}\} = \mathcal{C}.$$

We now fix $i \ge 1$ and consider the intermediate times $t_i + s$, where $s = 1, \ldots, k_i n_i - 1$. Let us first consider the range $s = 1, \ldots, N_i - 1$. By Lemma 4.6(1) together with (6.19) and (6.12), we have that

$$\left| \frac{\overline{\tau}_{t_i+s}}{\tau_{t_i+s}}(b_i a_{i,s}) - \frac{\overline{\tau}_{t_i}}{\tau_{t_i}}(b_i) \right| \le 2\frac{\tau_s(a_{i,s})}{\tau_{t_i}(b_i) + \tau_s(a_{i,s})} \le \frac{2N_i M_i}{t_i} < 2\varepsilon_{i-1}. \tag{7.4}$$

So, using (6.21), we know that

$$\frac{\overline{\tau}_{t_i+s}}{\tau_{t_i+s}}(b_i a_{i,s}) \in B_{5\varepsilon_{i-1}}(\overline{p}_{i-1}).$$

Now, let us consider the range $s = N_i, \ldots, k_i n_i - 1, k_i n_i$. By Lemma 4.6(3), we know that

$$\min\left\{ \frac{\overline{\tau}_{t_i}}{\tau_{t_i}}(b_i), \frac{\overline{\tau}_s}{\tau_s}(a_{i,s}) \right\} \le \frac{\overline{\tau}_{t_i+s}}{\tau_{t_i+s}}(b_i a_{i,s}) \le \max\left\{ \frac{\overline{\tau}_{t_i}}{\tau_{t_i}}(b_i), \frac{\overline{\tau}_s}{\tau_s}(a_{i,s}) \right\},$$

where the inequality holds coordinate-wise. Let $\delta_i = \max\{3\varepsilon_{i-1}, 3\varepsilon_i\}$, and recall from (7.3) and (7.1) that $(\overline{\tau}_{t_i}/\tau_{t_i})(b_i) \in B_{\delta_i}(\overline{p}_{i-1})$ and $(\overline{\tau}_s/\tau_s)(a_{i,s}) \in B_{\delta_i}(\overline{p}_i)$. Thus,

$$\frac{\overline{\tau}_{t_i+s}}{\tau_{t_i+s}}(b_i a_{i,s}) \in R_i^{\delta_i}, \tag{7.5}$$

where $R_i^{\delta_i}$ is the $\delta_i$-neighborhood of the $d$-dimensional rectangle $R_i$ that has $\overline{p}_i$ and $\overline{p}_{i+1}$ as vertices and is given by

$$R_i^{\delta_i} := \{(x_1, \ldots, x_d) \in \mathbb{R}^d : \ \min\{p_{i-1}^{(j)}, p_i^{(j)}\} - \delta_i < x_j < \max\{p_{i-1}^{(j)}, p_i^{(j)}\} + \delta_i\}.$$

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As $\lim_{i\to\infty} |\overline{p}_i - \overline{p}_{i+1}| = 0$, and as $\lim_{i\to\infty} \delta_i = 0$, we know that the rectangles $R_i^{\delta_i}$ accumulate precisely at the limit points $\mathcal{C}$ of $(\overline{p}_i)_i$. This concludes that $\mathcal{T}(x) = \mathcal{C}$.

To check that (3.1) holds, let $k \in \mathbb{N}$ and choose $i$ such that $t_i \le k < t_{i+1}$. Hence, it follows from (7.5) that

$$\left| \frac{\overline{\tau}_k}{\tau_k}(x) - \frac{\overline{\tau}_{k+1}}{\tau_{k+1}}(x) \right| \le 2 \operatorname{diam}(R_i^{\delta_i}).$$

As $k \to \infty$, we get $i \to \infty$ and hence, the latter tends to 0. This implies that (3.1) holds and hence proves the lemma. □

7.3. *Proof of Theorem A.* To conclude the proof of Theorem A, it remains only to show that $\dim_{\mathrm{H}}(\Gamma) = 1$. This follows verbatim from the proof of Lemma 6.9.

## 8. *Examples*

We consider an interval map $f : X \to X$ that admits a Markov partition $\mathcal{P}$ which has $d \ge 2$ neutral fixed points $\{\xi_1, \dots, \xi_d\}$. At these neutral fixed points, we will always assume that we have the following expansion:

$$\text{there exist } \alpha \in (0,1),\ b_1, \dots, b_d > 0 \quad \text{such that } f(x) - \xi_i \sim b_i (x - \xi_i)^{1+1/\alpha}. \tag{8.1}$$

We assume that there is a partition $I_1, I_2, \dots$ of $X$ (modulo Leb) into $d + d'$ open sub-intervals, where $0 \le d' \le +\infty$ such that each $I_i$ is a union of elements in $\mathcal{P}$ and each restriction $f_i := f|_{I_i}$ is a $C^2$ diffeomorphism onto its image. (Notice that we do not impose that $f_i$ extends to a $C^2$ diffeomorphism on the closure of $I_i$.)

*Example 8.1.* Suppose that $d' = 0$ and that each branch of the map is full in the sense that $f_i(I_i) = (0,1)$ and monotone increasing. Moreover, we assume that

$$|f'(x)| > 1 \quad \text{for all } x \notin \{\xi_1, \dots, \xi_d\}, \tag{8.2}$$

and that Adler's distortion condition holds:

$$\sup_{1\le i\le d} \sup_{x\in I_i} \frac{|f''(x)|}{(f'(x))^2} < \infty. \tag{8.3}$$

In this case, we obtain a map with $d$ full orientation-preserving branches, each of which contains a neutral fixed point satisfying (8.1). In this case, we define

$$Y := \begin{cases} (\gamma, f\gamma) & \text{if } d = 2, \\ \bigcup_{i=1}^{d} I_i \setminus f_i^{-1} I_i & \text{if } d \ge 3, \end{cases} \tag{8.4}$$

where in the case $d = 2$, $\{\gamma, f\gamma\}$ is the unique orbit of period 2. The fact that this map satisfies Assumptions 2.1–2.6 for this choice of $Y$ can be found in [**CMT24**, §4.1].

*Example 8.2.* We can generalize the previous example by allowing an additional $1 \le d' \le +\infty$ uniformly expanding branches. As in Example 8.1, we assume that both

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(8.2) and (8.3) hold, and we set

$$Y := \bigcup_{i=1}^{d} I_i \setminus f_i^{-1} I_i,$$

regardless of the value of $d$. The fact that such maps satisfy Assumptions 2.1–2.6 can be found in [**CM25**, Example 5.14].

*Example 8.3.* Another way which can generalize Example 8.1 is by allowing for critical and/or singular points at the preimages of the neutral fixed points. For simplicity, we will stick to the case that $d = 2$, which was introduced in [**CLM23**]. The fact that these maps satisfy Assumptions 2.1–2.6 can be found in [**CMT24**, §4.2].

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