Probability in the Engineering and Informational Sciences (2026), **40:**3 416–439
doi:10.1017/S0269964826100199

CAMBRIDGE UNIVERSITY PRESS

RESEARCH ARTICLE

Ordering of extreme order statistics among q-Weibull random variables

Ameen Mahmood K, Shyamal Ghosh and Priyanka Majumder

School of Data Science, Indian Institute of Science Education and Research Thiruvananthapuram, Thiruvananthapuram, Kerala, India
Corresponding author: Shyamal Ghosh; Email: shyamal@iisertvm.ac.in

Keywords: archimedean copula, majorization, order statistics, q-Weibull distribution, random shock, stochastic orders

The q-Weibull distribution, as a generalization of the Weibull distribution, plays an important role in the field of reliability theory, survival analysis, finance, engineering, medical science, etc. In contrast to the Weibull distribution, which is limited to describing monotonic hazard rate functions, the q-Weibull distribution offers the flexibility to model various behaviors of the hazard rate function, including unimodal, bathtub-shaped, monotonic (both increasing and decreasing), and constant. In this article, we investigated the stochastic comparison of extreme order statistics derived from independent, heterogeneous q-Weibull random variables using various stochastic orderings, including the usual stochastic order, hazard rate order, reversed hazard rate order, and likelihood ratio order. Some of these results are further extended to dependent setups by incorporating Archimedean copulas to model the dependence structure. Finally, we explored the behavior of extreme order statistics when the random variables are subjected to random shocks.

1. Introduction

The Weibull distribution holds significant importance in survival analysis, reliability theory, and modeling data of nonnegative random variables (r.v.'s) in several fields such as atmospheric science, engineering, insurance, medical sciences, hydrology, and forestry, among others [10, 18, 35, 41]. However, the inherent monotonic shape of the hazard rate function restricts the applications of the Weibull distribution. To address this limitation, various modifications and generalizations of the Weibull distribution have been proposed in the literature, aiming to accommodate non-monotonic hazard rate functions [3, 38]. These modifications and generalizations can be broadly categorized into two groups: (1) methods that introduce additional parameters to existing distributions to create classes of more flexible distributions and (2) methods that combine two or more distributions, with at least one being a Weibull distribution. In the context of the generalization of the Weibull distribution, recent advancements have emerged, such as the q-Weibull distribution introduced by Picoli *etal*. [39], which builds upon the foundational work by Tsallis [47] within the framework of non-extensive statistical mechanics.

The probability density function (p.d.f.) of the q-Weibull distribution is given by

$$f(t) = \lambda\beta(2-q)t^{\beta-1}\left[1-(1-q)\lambda t^{\beta}\right]^{\frac{1}{1-q}},\ t \in \begin{cases} \left[0, [\lambda(1-q)]^{-\frac{1}{\beta}}\right], & 0 < q < 1 \\ [0,\infty), & 1 < q < 2 \end{cases} \tag{1}$$

where $\beta > 0$ and $\lambda > 0$, and the cumulative distribution function (c.d.f.) is given by

$$F(t) = 1 - \left[1-\lambda(1-q)t^{\beta}\right]^{\frac{2-q}{1-q}}, t \geq 0.$$

© The Author(s), 2026. Published by Cambridge University Press. This is an Open Access article, distributed under the terms of the Creative Commons Attribution licence (http://creativecommons.org/licenses/by/4.0), which permits unrestricted re-use, distribution and reproduction, provided the original article is properly cited.

https://doi.org/10.1017/S0269964826100199 Published online by Cambridge University Press

In contrast to the Weibull distribution, which is limited to describing monotonic hazard rate functions, the q-Weibull distribution offers the flexibility to model various behaviors of the hazard rate function, including unimodal, bathtub-shaped, monotonic (both increasing and decreasing), and constant [4, 52]. Thus, the q-Weibull distribution bridges the gap between monotonic and non-monotonic hazard rate functions by employing a single set of parameters, providing a versatile and elegant approach for fitting failure data [6, 15, 23, 24, 43, 52]. In the context of modeling the failure phenomenon of the Brazilian Hydropower Equipment, Assis *etal*. [6] show that the q-Weibull model must be preferred over the Weibull model as the q-Weibull is capable of producing more flexible and closer to reality results. Again, in the context of modeling the first time to failure of 500 MW generators, Xu *etal*. [52] show that the q-Weibull distribution is a good alternative to the other bathtub-shaped hazard rate models, namely the modified Weibull extension and the exponetiated Nadaraja-Haghighi distribution (ENH). Apart from its exceptional performance in fitting failure time data, the q-Weibull distribution boasts a robust theoretical foundation, derived within the framework of Tsallis non-extensive entropy [48]. Additionally, the q-Weibull distribution finds application in describing complex systems with long-range interactions and long-term memory [4, 20, 52]. The q-Weibull distribution has also been employed in various models, such as the autoregressive conditional duration model [50, 51], stress-strength model [26], and max-min processes [25], positioning itself as an alternative to existing life distributions for modeling reliability data.

Apart from the extensive applications of the q-Weibull distribution in fields such as chemical science [15], engineering [5, 43], reliability [24, 26, 53], biology [45], hydrology [6], medical science [17], etc., a substantial amount of literature exists on the q-Weibull distribution and its properties. For instance, Assis *etal*. [4] delineated the parameter ranges corresponding to each type of hazard rate function, while Jose and Naik [26] explored several reliability properties and stress-strength analyses. Moreover, several authors have proposed methods for estimating the parameters of the q-Weibull distribution. Assis *etal*. [4] obtained parameter estimates by maximizing the coefficient of determination R^2 of the fitted c.d.f. Xu *etal*. [52] derived maximum likelihood estimators using a hybrid approach combining the Nelder–Mead simplex method with the artificial bee colony algorithm. Jia *etal*. [24] developed both maximum likelihood estimate (MLE)- and least-squares estimate (LSE)-based estimators and constructed corresponding confidence intervals, while Jia [23] further examined point and interval estimation under these methods. More recently, Oliveira *etal*. [37] employed a log-cumulant technique based on the Mellin transform. While numerous researchers have extensively investigated various properties of the q-Weibull distribution, the stochastic comparisons of their extreme order statistics have yet to be explored. This serves as the primary motivation for driving the current study.

Extreme order statistics hold significant importance across diverse statistical disciplines like reliability theory, survival analysis, auction theory, finance, actuarial science, etc., see [9, 49] and the references therein. Apart from their foundational role in statistical and probability theory, extreme order statistics find a wide range of applications in practical fields such as modeling extreme events like floods, droughts, and heat waves, in environmental science [46], in insurance to evaluate risk and set premiums [34], in sports to rank the athletes in a particular competition [33], and in quality control to analyze and evaluate the quality of products in manufacturing processes [7]. Consider a scenario where $U_1, U_2, \ldots, U_n$ denotes n r.v.'s representing the lifetimes of components within a system comprising n components and $U_{1:n}, U_{2:n}, \ldots, U_{n:n}$ are the corresponding order statistics. The extreme order statistics $U_{1:n}$ and $U_{n:n}$ represent the lifetime of two important configurations, such as series and parallel systems, which are particular members of m-out-of-n system that operates successfully if at least m out of the n components are operational. In particular rth order statistic $U_{r:n}$ represents the lifetime of a $(n-r+1)$-out-of-n system. Keeping in mind the significance of the extreme order statistics, stochastic comparisons between extreme order statistics is an important topic of research and have been studied rather extensively, especially for parametric univariate distributions, based on different stochastic orderings [1, 2, 14, 16, 19, 22, 27, 31, 42]. Generally, these comparisons assume that the units in the sample will surely fail. However, in reality, the units may encounter random shocks that do not necessarily ensure their failure. In the context of insurance claim amount, Barmalzan and Najafabadi [11], Barmalzan *etal*.

https://doi.org/10.1017/S0269964826100199 Published online by Cambridge University Press

[12], Balakrishnan *etal*. [8], and Kundu *etal*. [28] discuss the comparisons of the extreme order statistics. Again, in many real-life scenarios, the components have structural dependence among themselves. This structural dependence leads to a set of statistically dependent variables. Stochastic comparison of extremes in the presence of dependency is also an important research topic in the literature and has been explored by several authors. Li and Fang [30] investigated the stochastic ordering of sample extremes and adjacent order statistics arising from dependent r.v.'s characterized by Archimedean copulas. Rezapour and Alamatsaz [40] established sufficient conditions for comparing the lifetimes of two $(n-k+1)$-out-of-n systems with dependent components under the usual stochastic order. Zhang *etal*. [54] derived comparison results for the largest claim amounts from two heterogeneous and dependent insurance portfolios in terms of the usual stochastic and hazard rate orders. In addition, Zhang *etal*. [55] provided stochastic comparison results for parallel and series systems composed of dependent and independent heterogeneous resilience-scaled components.

The main focus of this work is to investigate comparisons between extreme order statistics coming from heterogeneous q-Weibull samples in terms of various notions of stochastic ordering, such as usual stochastic order, hazard rate order, reversed hazard rate order, and likelihood ratio order. Further results considering dependency among the samples and exposure to random shocks are also explored here. The organization of the rest of the paper is as follows. Section 2 describes some preliminary results pertinent to our work. In Sections 3 and 4, we explore various stochastic comparison results among the extremes when the heterogeneous q-Weibull samples are structurally independent and dependent. Further comparison results of the extremes are also derived in the presence of random shocks in Section 5.

2. Preliminaries

In this section, we will present some important definitions and concepts of stochastic orders, majorizations, and copulas together with some results that will be pertinent to our future discussion.

Let U and V be two absolute continuous nonnegative r.v.'s with c.d.f.s $F_U(t)$ and $F_V(t)$, p.d.f.s $f_U(t)$ and $f_V(t)$, survival functions $\bar{F}_U(t)(=1-F_U(t))$ and $\bar{F}_V(t)(=1-F_V(t))$, hazard rates $h_U(t)(=f_U(t)/\bar{F}_U(t))$ and $h_V(t)(=f_V(t)/\bar{G}_V(t))$, and reversed hazard rate functions $r_U(t)(=f_U(t)/F_U(t))$ and $r_V(t)(=f_V(t)/F_V(t))$, respectively.

Definition 2.1. *A r.v. U is smaller than V in:*

(i) the *usual stochastic order,* denoted by $U \leq_{st} V$, if $\bar{F}_U(t) \leq \bar{F}_V(t)$ for all t;
(ii) the *hazard rate order,* denoted by $U \leq_{hr} V$, if $\bar{F}_V(t)/\bar{F}_U(t)$ is increasing for all t, which can equivalently be written as $h_U(t) \geq h_V(t)$ for all t;
(iii) the *reversed hazard rate order,* denoted by $U \leq_{rh} V$, if $F_V(t)/F_U(t)$ is increasing in t, which can equivalently be written as $r_U(t) \leq r_V(t)$ for all t; and
(iv) the *likelihood ratio order,* denoted by $U \leq_{lr} V$, if $f_V(t)/f_U(t)$ is increasing in t.

For an extensive and comprehensive discussion on stochastic orders, one may consult the excellent texts by Shaked and Shanthikumar [44] and Belzunce *etal*. [13]. The following relationships hold among the above stochastic orders:

$$\begin{array}{ccc} U \leq_{lr} V & \Longrightarrow & U \leq_{hr} V \\ \Downarrow & & \Downarrow \\ U \leq_{rh} V & \Longrightarrow & U \leq_{st} V. \end{array}$$

One of the key concepts in stochastic inequalities is majorization. This is a pre-ordering on vectors where all the components are in nondecreasing (nonincreasing) order. Majorization serves as a useful tool for constructing different types of inequalities and bounds. Let J^n be a subset of the n-dimensional

https://doi.org/10.1017/S0269964826100199 Published online by Cambridge University Press

Euclidean space $\mathbb{R}^n$, where $J \subseteq \mathbb{R}$. Let $(u_{(1)}, u_{(2)}, \ldots, u_{(n)})$ denote the components of the vector $\boldsymbol{u} = (u_1, u_2, \ldots, u_n) \in J^n$ arranged in increasing order. The following definitions can be found in [32].

Definition 2.2. *The vector $\boldsymbol{u}$ is said to be majorized by the vector $\boldsymbol{v}$, denoted by $\boldsymbol{u} \overset{m}{\preceq} \boldsymbol{v}$, if $\sum_{i=1}^{j} u_{(i)} \geq \sum_{i=1}^{j} v_{(i)}$ for $j = 1, \ldots, n-1$ and $\sum_{i=1}^{n} u_{(i)} = \sum_{i=1}^{n} v_{(i)}$. In addition, the vector $\boldsymbol{u}$ is said to be weakly submajorized (weakly supermajorized) by the vector $\boldsymbol{v}$, denoted by $\boldsymbol{u} \preceq_w \boldsymbol{v}$ ($\boldsymbol{u} \preceq^w \boldsymbol{v}$), if $\sum_{i=j}^{n} u_{(i)} \leq \sum_{i=j}^{n} v_{(i)}$ for $j = 1, \ldots, n$ ($\sum_{i=1}^{j} u_{(i)} \geq \sum_{i=1}^{j} v_{(i)}$ for $j = 1, \ldots, n$).*

Clearly, $\boldsymbol{u} \overset{m}{\preceq} \boldsymbol{v} \Longrightarrow \boldsymbol{u} \preceq_w \boldsymbol{v}$ and $\boldsymbol{u} \overset{m}{\preceq} \boldsymbol{v} \Longrightarrow \boldsymbol{u} \preceq^w \boldsymbol{v}$, but the converse is not always true.

Definition 2.3. *A function $\psi : J^n \to \mathbb{R}$ is said to be Schur-convex (Schur-concave) on J^n if $\boldsymbol{u} \overset{m}{\preceq} \boldsymbol{v}$ implies $\psi(\boldsymbol{u}) \leq (\geq)\ \psi(\boldsymbol{v})$ for all $\boldsymbol{u}, \boldsymbol{v} \in J^n$.*

Next, we state some well-known results that will be utilized to establish our main results. Throughout the paper, we will use the notation $\mathbb{D}_+ = \{(u_1, u_2, \ldots, u_n) : u_1 \geq u_2 \geq \cdots \geq u_n > 0\}$ and $\mathbb{E}_+ = \{(u_1, u_2, \ldots, u_n) : 0 < u_1 \leq u_2 \leq \cdots \leq u_n\}$.

Lemma 2.4. *A real-valued function ψ on J^n has the property $\psi(\boldsymbol{u}) \leq (\geq)\psi(\boldsymbol{v})$ whenever $\boldsymbol{u} \preceq_w \boldsymbol{v}$ if and only if ψ is increasing (decreasing) and Schur-convex (Schur-concave) on J^n. Similarly, ψ has the property $\psi(\boldsymbol{u}) \leq (\geq)\psi(\boldsymbol{v})$ whenever $\boldsymbol{u} \preceq^w \boldsymbol{v}$ if and only if ψ is decreasing (increasing) and Schur-convex (Schur-concave) on J^n.*

Lemma 2.5 (Schur–Ostrowski criterion). *A continuously differentiable function $\psi : J^n \to \mathcal{R}$ is Schur-convex (Schur-concave) if and only if it is symmetric and*

$$(w_i - w_j)\left(\frac{\partial \psi(\boldsymbol{w})}{\partial w_i} - \frac{\partial \psi(\boldsymbol{w})}{\partial w_j}\right) \geq (\leq) 0$$

for all $i \neq j$ and $\boldsymbol{w} \in J^n$.

Now, we will briefly discuss the concept of a copula. A copula is a widely used tool for modeling and characterizing the structural dependence among r.v.'s. The mathematical description of the copula can be described as follows:

A copula associated with a multivariate d.f. G is a function $C : [0,1]^n \to [0,1]$ satisfying $G(\boldsymbol{u}) = C(G_1(u_1), G_2(u_2), \ldots, G_n(u_n))$, where G_i's, $1 \leq i \leq n$ are the univariate marginal d.f.'s of U_i's. Similarly, a survival copula associated with a multivariate survival function $\bar{G}$ is a function $\bar{C} : [0,1]^n \to [0,1]$ satisfying $\bar{G}(\boldsymbol{u}) = \bar{C}(\bar{G}_1(u_1), \bar{G}_2(u_2), \ldots, \bar{G}_n(u_n))$ where $\bar{G}_i$'s, $1 \leq i \leq n$ are the univariate marginal survival functions of U_i's. In our future discussion, we will use a particular copula known as the Archimedean copula.

Definition 2.6. *A copula C is said to be* Archimedean *if there exists a generator function $\psi : [0, \infty] \to [0,1]$ such that*

$$C(\mathbf{u}) = \psi\left(\psi^{-1}(u_1) + \psi^{-1}(u_2) + \cdots + \psi^{-1}(u_d)\right).$$

https://doi.org/10.1017/S0269964826100199 Published online by Cambridge University Press

The function ψ must satisfy the following necessary and sufficient conditions:

(i) $\psi(0) = 1$ *and* $\psi(\infty) = 0$;
(ii) ψ *is* d-monotone, *that is,* $\frac{(-1)^k d^k \psi(s)}{ds^k} \geq 0$ *for* $k \in \{0, 1, \ldots, d-2\}$, *and the* $(d-2)$*-th derivative* $\frac{(-1)^{d-2} d^{d-2} \psi(s)}{ds^{d-2}}$ *is decreasing and convex.*

For a more detailed discussion on copulas, the reader is referred to the excellent text by [36].

Lemma 2.7 (Li and Fang [30]). *For two n-dimensional Archimedean copulas $C_{\psi_1}(\boldsymbol{w})$ and $C_{\psi_2}(\boldsymbol{w})$, with $\phi_2 = \psi_2^{-1} = sup\{x \in \mathcal{R} : \psi_2(x) > u\}$ as the right continuous inverse, if $\phi_2 \circ \psi_1$ is super- additive, then $C_{\psi_1}(\boldsymbol{w}) \leq C_{\psi_2}(\boldsymbol{w})$ for all $\boldsymbol{w} \in [0,1]^n$. (Recall that a function f is said to be super-additive if $f(u+v) \geq f(u) + f(v)$ for all u and v in the domain of f.)*

3. Heterogeneous independent case

In this section, we investigate stochastic comparison results between the extremes with respect to the usual stochastic order, hazard rate order, reverse hazard rate order, and likelihood ratio order when the components are independent.

Let $\{U_i\}_{i=1}^n$ and $\{V_i\}_{i=1}^n$ be two sets of n independent r.v.'s, where U_i follows q-Weibull distribution with parameters $(q_i, \lambda_i, \beta_i)$ and V_i follows q-Weibull distribution with parameters $(q_i^\star, \lambda_i^\star, \beta_i^\star)$. Denote $U_{1:n} = \min(U_1, U_2, \ldots, U_n)$ and $V_{1:n} = \min(V_1, V_2, \ldots, V_n)$. Define $\bar{F}_{U_{1:n}}(t)$ and $\bar{F}_{V_{1:n}}(t)$ are the survival functions of the smallest order statistics $U_{1:n}$ and $V_{1:n}$, respectively, then

$$\bar{F}_{U_{1:n}}(t) = \prod_{l=1}^{n} \bar{F}_{U_l}(t) = \prod_{l=1}^{n} \left[1 - \lambda_l (1-q_l) t^{\beta_l}\right]^{\frac{2-q_l}{1-q_l}} \tag{2}$$

and

$$\bar{F}_{V_{1:n}}(t) = \prod_{l=1}^{n} \bar{F}_{V_l}(t) = \prod_{l=1}^{n} \left[1 - \lambda_l^\star (1-q_l^\star) t^{\beta_l^\star}\right]^{\frac{2-q_l^\star}{1-q_l^\star}}. \tag{3}$$

Similarly, if we denote $h_{U_{1:n}}(t)$ and $h_{V_{1:n}}(t)$ are the failure rate functions of $U_{1:n}$ and $V_{1:n}$, respectively, then

$$\begin{aligned} h_{U_{1:n}}(t) &= \sum_{l=1}^{n} h_{U_l}(t) \\ &= \sum_{l=1}^{n} \frac{\lambda_l \beta_l (2-q_l) t^{\beta_l - 1}}{1 - (1-q_l)\lambda_l t^{\beta_l}}, \quad t \in \begin{cases} \left[0, \min_l \left\{[\lambda_l (1-q_l)]^{\frac{-1}{\beta_l}}\right\}\right], & 0 < q_l < 1 \\ [0, \infty), & 1 < q_l < 2, \end{cases} \end{aligned} \tag{4}$$

and

$$\begin{aligned} h_{V_{1:n}}(t) &= \sum_{l=1}^{n} h_{V_l}(t) \\ &= \sum_{l=1}^{n} \frac{\lambda_l^\star \beta_l^\star (2-q_l^\star) t^{\beta_l^\star - 1}}{1 - (1-q_l^\star)\lambda_l^\star t^{\beta_l^\star}}, \quad t \in \begin{cases} \left[0, \min_l \left\{[\lambda_l^\star (1-q_l^\star)]^{\frac{-1}{\beta_l^\star}}\right\}\right], & 0 > q_l^\star > 1, \\ [0, \infty), & 1 > q_l^\star < 2. \end{cases} \end{aligned} \tag{5}$$

https://doi.org/10.1017/S0269964826100199 Published online by Cambridge University Press

Note that, for $0 < q_l, q_l^\star < 1$, $U_{1:n}$ and $V_{1:n}$ are defined over different domains. So, any comparison of these r.v.'s should be over a common domain. In our results, we denote the common domain of $U_{1:n}$ and $V_{1:n}$ as $(0, D)$ where $D = \min\left\{\min_l\left\{[\lambda_l(1-q_l)]^{-\frac{1}{\beta_l}}\right\}, \min_l\left\{[\lambda_l^\star(1-q_l^\star)]^{-\frac{1}{\beta_l^\star}}\right\}\right\}$.

The following theorems compare the smallest order statistics with respect to the usual stochastic order, hazard rate order, and likelihood ratio order.

Theorem 3.1. *Let $U_1, U_2, \ldots, U_n$ be a set of independent r.v.'s such that $U_i \sim q - W(q, \lambda_i, \beta)$, $i = 1, 2, \ldots, n$, and $V_1, V_2, \ldots, V_n$ be another set of independent r.v.'s such that $V_i \sim q - W(q, \lambda_i^\star, \beta)$, $i = 1, 2, \ldots, n$. Then the following results hold:*

(1) *If $0 < q < 1$ and $(\lambda_1, \lambda_2, \ldots, \lambda_n) \preceq_w (\lambda_1^\star, \lambda_2^\star, \ldots, \lambda_n^\star)$, then $V_{1:n} \leq_{hr} U_{1:n}$.*
(2) *If $1 < q < 2$ and $(\lambda_1, \lambda_2, \ldots, \lambda_n) \preceq^w (\lambda_1^\star, \lambda_2^\star, \ldots, \lambda_n^\star)$, then $V_{1:n} \geq_{hr} U_{1:n}$.*

Proof.

(1) To establish the result, it suffices to show that $h_{U_{1:n}}(t) \leq h_{V_{1:n}}(t)$ for all $t \in (0, D)$, where $h_{U_{1:n}}(t)$ and $h_{V_{1:n}}(t)$ are defined in (4) and (5), respectively. We first note that $h_{U_{1:n}}(t) = \sum_{l=1}^n \frac{\beta\lambda_l(2-q)t^{\beta-1}}{1-(1-q)\lambda_l t^\beta} := \sum_{l=1}^n \tau(\lambda_l)$. By Proposition C.1 of [32], in order to establish the Schur-convexity of $h_{U_{1:n}}(t)$, it is enough to verify that $\tau(\lambda_l)$ is a convex function of λ_l. Observe that for $t \in (0, D)$,

$$\frac{d\tau(\lambda_l)}{d\lambda_l} = \frac{\beta(2-q)t^{\beta-1}}{[1-(1-q)\lambda_l t^\beta]^2}, \tag{6}$$

and

$$\frac{d^2\tau(\lambda)}{d\lambda_l^2} = \frac{2\beta(2-q)(1-q)t^{2\beta-1}}{[1-(1-q)\lambda_l t^\beta]^3}, \tag{7}$$

both are nonnegative for all λ_l. Therefore, by application of Lemma 2.4, together with Definition 2.1, the theorem follows.

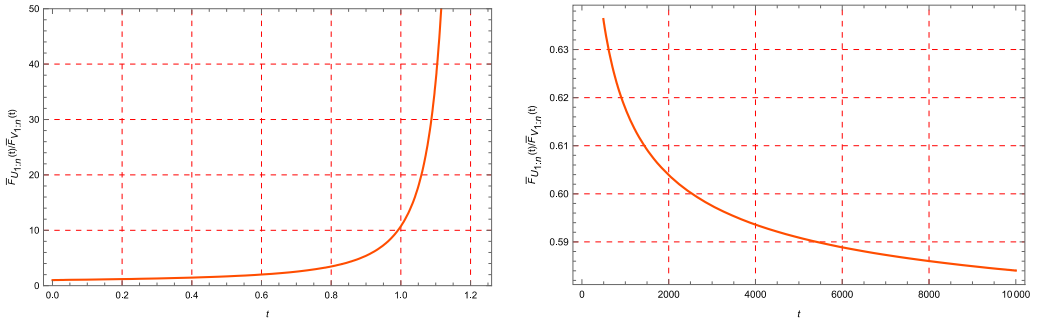
(2) From (6) and (7), observe that for $1 < q < 2$, $\frac{d\tau(\lambda_l)}{d\lambda_l} > 0$ and $\frac{d^2\tau(\lambda)}{d\lambda_l^2} < 0$ for all $t \in (0, \infty)$. Thus, $\tau(\lambda_l)$ is increasing and concave in λ_l. Consequently, by an argument analogous to that used in Part 1, the desired result follows from an application of Lemma 2.4.

□

In the following, we consider a numerical example to demonstrate the result stated in the above theorem.

Example 3.2. Let $U_i \sim q - W(q, \lambda_i, \beta)$ and $V_i \sim q - W(q, \lambda_i^\star, \beta), i = 1, 2$. First, consider he case $0 < q < 1$ with $q = 0.5, \beta = 0.5, \lambda_1 = 0.5, \lambda_2 = 1.5, \lambda_1^\star = 0.3$, and $\lambda_2^\star = 1.8$. Here $(\lambda_1, \lambda_2) \preceq_w (\lambda_1^\star, \lambda_2^\star)$. Figure 1(a) represents the plot of $\bar{F}_{U_{1:2}}(t)/\bar{F}_{V_{1:2}}(t)$ in $(0, D) = (0, 1.23)$. The plot clearly indicates that $\bar{F}_{U_{1:2}}(t)/\bar{F}_{V_{1:2}}(t)$ is increasing in t, which in turn implies $h_{U_{1:2}} \leq h_{V_{1:2}}$ and hence $U_{1:2} \geq_{hr} V_{1:2}$. Similarly, for the case of $q > 1$, consider $q = 1.5, \beta = 0.5, \lambda_1 = 0.6, \lambda_2 = 1.5, \lambda_1^\star = 0.3$, and $\lambda_2^\star = 1.7$, where $(\lambda_1, \lambda_2) \preceq^w (\lambda_1^\star, \lambda_2^\star)$. Then from Figure 1(b), it is clear that $\bar{F}_{U_{1:2}}(t)/\bar{F}_{V_{1:2}}(t)$ is decreasing in t, which leads to $U_{1:2} \leq_{hr} V_{1:2}$.

https://doi.org/10.1017/S0269964826100199 Published online by Cambridge University Press



(a) $q = 0.5, \beta = 0.5, \lambda_1 = 0.5, \lambda_2 = 1.5, \lambda_1^\star = 0.3, \lambda_2^\star = 1.8$ (b) $q = 1.5, \beta = 0.5, \lambda_1 = 0.6, \lambda_2 = 1.5, \lambda_1^\star = 0.3, \lambda_2^\star = 1.7$

Figure 1. *Plots of* $\overline{F}_{U_{1:2}}(t)/\overline{F}_{V_{1:2}}(t)$*: (a)* $q = 0.5, \beta = 0.5, \lambda_1 = 0.5, \lambda_2 = 1.5, \lambda_1^\star = 0.3, \lambda_2^\star = 1.8$ *and (b)* $q = 1.5, \beta = 0.5, \lambda_1 = 0.6, \lambda_2 = 1.5, \lambda_1^\star = 0.3, \lambda_2^\star = 1.7$.

Theorem 3.1 demonstrated that under the stated conditions, the smallest order statistics are comparable in hazard rate order. A natural question arises: Does this conclusion hold when substituting the hazard rate order by the likelihood ratio order? The following counterexample furnishes the answer.

Example 3.3. Let $U_i \sim q\text{-}W(q, \lambda_i, \beta)$ and $V_i \sim q\text{-}W(q, \lambda_i^\star, \beta), i = 1, 2$. Choose $q = 0.5, \beta = 0.5, \lambda_1 = 0.5, \lambda_2 = 1.5, \lambda_1^\star = 0.3$, and $\lambda_2^\star = 1.7$. Then, Figure 2(a) represents the plot of $f_{V_{1:2}}(t)/f_{U_{1:2}}(t)$ in a subset of $(0, D) = (0, 1.38)$, from which it is clear that $f_{V_{1:2}}(t)/f_{U_{1:2}}(t)$ is not an increasing function in t. Thus, for $0 < q < 1$, $U_{1:n}$ and $V_{1:n}$ are not comparable in likelihood ratio order when $(\lambda_1, \lambda_2) \overset{m}{\preceq} (\lambda_1^\star, \lambda_2^\star)$. A similar conclusion can also be drawn when q is larger than 1. Set, $q = 1.5, \beta = 2, \lambda_1 = 0.5, \lambda_2 = 1.5, \lambda_1^\star = 0.3$, and $\lambda_2^\star = 1.7$. Then, from Figure 2(b) it is clear that $f_{V_{1:2}}(t)/f_{U_{1:2}}(t)$ is also not an increasing function, which enables us to say that $U_{1:n}$ and $V_{1:n}$ are not comparable in likelihood ratio order.

Theorem 3.4. *Let* $U_1, U_2, \ldots, U_n$ *be a set of independent r.v.'s such that* $U_i \sim q\text{-}W(q_i, \lambda, \beta)$*,* $i = 1, 2, \ldots, n$*, and* $V_1, V_2, \ldots, V_n$ *be another set of independent r.v.'s such that* $V_i \sim q\text{-}W(q_i^\star, \lambda, \beta)$*,* $i = 1, 2, \ldots, n$*. Then the following results hold:*

(1) *If* $0 < \max_l\{q_l, q_l^\star\} < 1$ *and* $(q_1, q_2, \ldots, q_n) \preceq^w (q_1^\star, q_2^\star, \ldots, q_n^\star)$*, then* $V_{1:n} \preceq_{hr} U_{1:n}$.
(2) *If* $1 < \min_l\{q_l, q_l^\star\} < 2$ *and* $(q_1, q_2, \ldots, q_n) \preceq^w (q_1^\star, q_2^\star, \ldots, q_n^\star)$*, then* $V_{1:n} \preceq_{hr} U_{1:n}$.

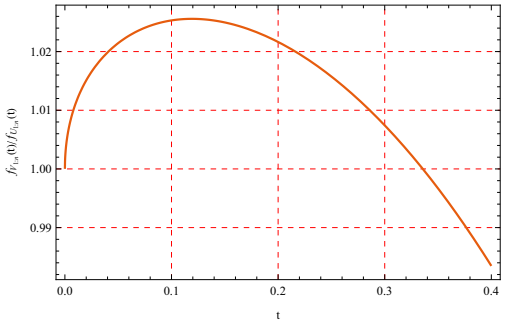
Proof. The results can be established following arguments similar to those used in Theorem 3.1. □

We now present a numerical example to illustrate the result established in Theorem 3.4.

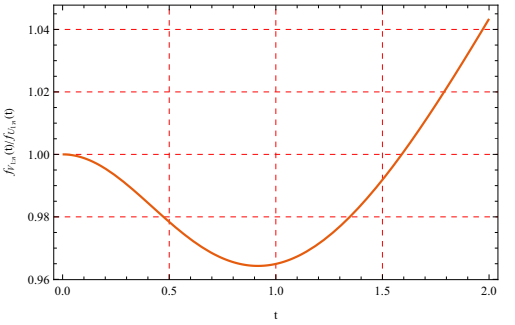
Example 3.5. Let $U_i \sim q - W(q_i, \lambda, \beta)$ and $V_i \sim q - W(q_i^\star, \lambda_i, \beta), i = 1, 2$. For the case $q < 1$, consider $q_1 = 0.3, q_2 = 0.5, q_1^\star = 0.1, q_2^\star = 0.6, \beta = 2$, and $\lambda = 0.5$, where $(q_1, q_2) \preceq^w (q_1^\star, q_2^\star)$. Figure 3(a) displays the plot of $\bar{F}_{U_{1:2}}(t)/\bar{F}_{V_{1:2}}(t)$ over the interval $(0, D) = (0, 1.49)$. The figure clearly shows that $\bar{F}_{U_{1:2}}(t)/\bar{F}_{V_{1:2}}(t)$ is increasing function in t, which implies that $U_{1:2} \geq_{hr} V_{1:2}$. Similarly, for the case of $q > 1$, let $q_1 = 1.3, q_2 = 1.5, q_1^\star = 1.1, q_2^\star = 1.6, \beta = 2$, and $\lambda = 0.5$, with $(q_1, q_2) \preceq^w (q_1^\star, q_2^\star)$. As illustrated in Figure 3(b), $\bar{F}_{U_{1:2}}(t)/\bar{F}_{V_{1:2}}(t)$ is again increasing, confirming that, $U_{1:2} \geq_{hr} V_{1:2}$.

The above theorem establishes the comparison result with respect to the hazard rate order. A natural question is whether this conclusion also holds when the hazard rate order is replaced by the likelihood ratio order. The following counterexample provides an answer.

https://doi.org/10.1017/S0269964826100199 Published online by Cambridge University Press

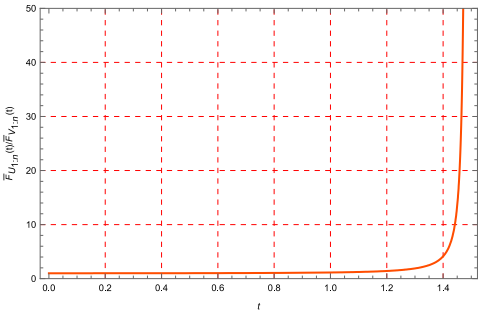


(a) $q = 0.5, \beta = 0.5, \lambda_1 = 0.5, \lambda_2 = 1.5, \lambda_1^\star = 0.3, \lambda_2^\star = 1.7$

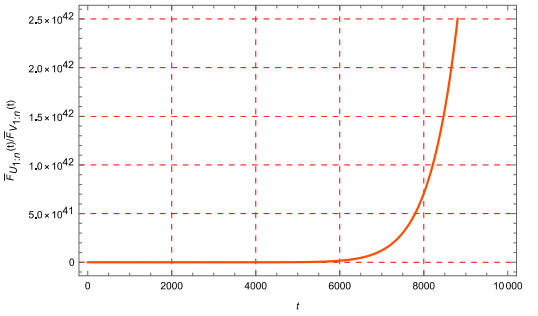


(b) $q = 1.5, \beta = 2, \lambda_1 = 0.5, \lambda_2 = 1.5, \lambda_1^\star = 0.3, \lambda_2^\star = 1.7$

Figure 2. *Plots of* $f_{V_{1:2}}(t)/f_{U_{1:2}}(t)$*: (a)* $q = 0.5, \beta = 0.5, \lambda_1 = 0.5, \lambda_2 = 1.5, \lambda_1^\star = 0.3, \lambda_2^\star = 1.7$ *and (b)* $q = 1.5, \beta = 2, \lambda_1 = 0.5, \lambda_2 = 1.5, \lambda_1^\star = 0.3, \lambda_2^\star = 1.7$.



(a) $q_1 = 0.3, q_2 = 0.5, q_1^\star = 0.1, q_2^\star = 0.6, \beta = 2, \lambda = 0.5$



(b) $q_1 = 1.3, q_2 = 1.5, q_1^\star = 1.1, q_2^\star = 1.6, \beta = 2, \lambda = 0.5$

Figure 3. *Plots of* $\bar{F}_{U_{1:2}}(t)/\bar{F}_{V_{1:2}}(t)$*: (a)* $q_1 = 0.3, q_2 = 0.5, q_1^\star = 0.1, q_2^\star = 0.6, \beta = 2, \lambda = 0.5$ *and (b)* $q_1 = 1.3, q_2 = 1.5, q_1^\star = 1.1, q_2^\star = 1.6, \beta = 2, \lambda = 0.5$.

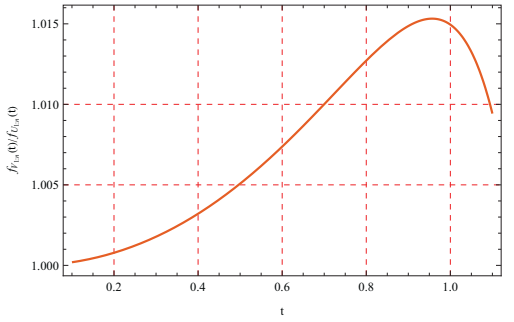
Example 3.6. Let $U_i \sim q - W(q_i, \lambda, \beta)$ and $V_i \sim q - W(q_i^\star, \lambda, \beta), i = 1, 2$. Choose $q_1 = 0.2, q_2 = 0.7, q_1^\star = 0.1, q_2^\star = 0.8, \beta = 2$, and $\lambda = 0.5$. Then, $(q_1, q_2) \overset{m}{\preceq} (q_1^\star, q_2^\star)$. Figure 4(a) represents the plot of $f_{V_{1:2}}(t)/f_{U_{1:2}}(t)$ in a subset of $(0, D) = (0, 1.49)$, from which it is clear that $f_{V_{1:2}}(t)/f_{U_{1:2}}(t)$ is not an increasing function in t. Thus, for $0 < \max_l\{q_l, q_l^\star\} < 1$, $U_{1:n}$ and $V_{1:n}$ are not comparable in likelihood ratio order. By setting, $q_1 = 1.2, q_2 = 1.7, q_1^\star = 1.1, q_2^\star = 1.8, \beta = 2$, and $\lambda = 0.5$ similar, conclusion can also be drawn from Figure 4(b), when $1 < \min_l\{q_l, q_l^\star\} < 2$.

Next, we consider the case when heterogeneity occurs in terms of the parameter β.

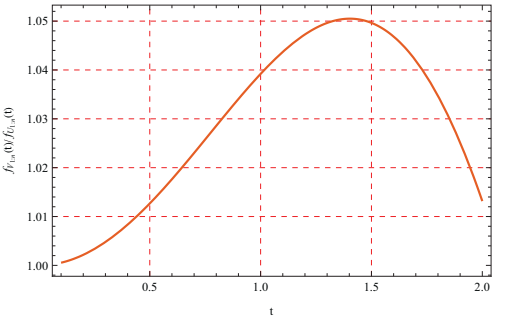
Theorem 3.7. *Let* $U_1, U_2, \ldots, U_n$ *be a set of independent r.v.'s such that* $U_i \sim q\text{-}W(q, \lambda, \beta_i)$, $i = 1, 2, \ldots, n$, *and* $V_1, V_2, \ldots, V_n$ *be another set of independent r.v.'s such that* $V_i \sim q\text{-}W(q, \lambda, \beta_i^\star)$, $i = 1, 2, \ldots, n$. *Then the following results hold:*

(1) *If* $0 < q < 1$ *and* $(\beta_1, \beta_2, \ldots, \beta_n) \overset{m}{\preceq} (\beta_1^\star, \beta_2^\star, \ldots, \beta_n^\star)$, *then* $V_{1:n} \leq_{st} U_{1:n}$.
(2) *If* $1 < q < 2$ *and* $(\beta_1, \beta_2, \ldots, \beta_n) \overset{m}{\preceq} (\beta_1^\star, \beta_2^\star, \ldots, \beta_n^\star)$, *then* $V_{1:n} \leq_{st} U_{1:n}$.

https://doi.org/10.1017/S0269964826100199 Published online by Cambridge University Press



(a) $q_1 = 0.2, q_2 = 0.7, q_1^\star = 0.1, q_2^\star = 0.8, \beta = 2, \lambda = 0.5$



(b) $q_1 = 1.2, q_2 = 1.7, q_1^\star = 1.1, q_2^\star = 1.8, \beta = 2, \lambda = 0.5$

Figure 4. *Plots of* $f_{V_{1:2}}(t)/f_{U_{1:2}}(t)$*: (a)* $q_1 = 0.2, q_2 = 0.7, q_1^\star = 0.1, q_2^\star = 0.8, \beta = 2, \lambda = 0.5$ *and (b)* $q_1 = 1.2, q_2 = 1.7, q_1^\star = 1.1, q_2^\star = 1.8, \beta = 2, \lambda = 0.5$.

Proof.

(1) To establish the theorem, it suffices to show that $\bar{F}_{U_{1:n}}(t) \le \bar{F}_{V_{1:n}}(t)$ for all $t \in (0, D)$, where $\bar{F}_{U_{1:n}}(t)$ and $\bar{F}_{V_{1:n}}(t)$ denote the survival functions of $U_{1:n}$ and $V_{1:n}$, as defined in (2) and (3), respectively. Note that

$$\bar{F}_{U_{1:n}}(t) = \prod_{l=1}^{n} \bar{F}_{U_l}(t) := \prod_{l=1}^{n} \eta(\beta_l),$$

where $\eta(x) = [1 - \lambda(1-q)t^x]^{\frac{2-q}{1-q}}$. By Proposition E.1 of [32], it is sufficient to show that $\ln(\eta(x))$ is concave in x. Observe that for all $t \in D$,

$$\frac{d^2}{dx^2} \ln(\eta(x)) = \frac{-(2-q)\lambda t^x (\ln(t))^2}{[1-(1-q)\lambda t^x]^2} \le 0,$$

as for all β_l, $0 < t < D \Rightarrow 0 < t < \min_l \left\{ \lambda(1-q)]^{-\frac{1}{\beta_l}} \right\}$. Therefore, $\bar{F}_{1:n}(t)$ is Schur-concave in $(\beta_1, \beta_2, \ldots, \beta_n)$. Hence, the result follows.

(2) By a similar argument, the proof of this part follows from the observation that for all t, $\frac{d^2}{dx^2} \ln(\eta(x)) \le 0$.

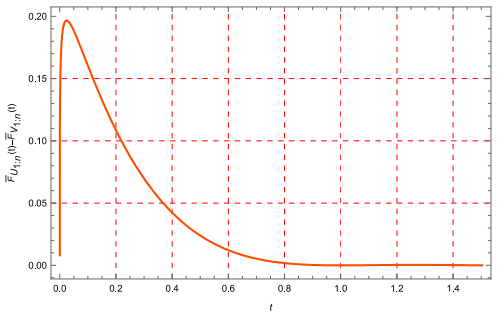
□

A numerical example is provided below to illustrate Theorem 3.7.

Example 3.8. Let $U_i \sim q - W(q, \lambda, \beta_i)$ and $V_i \sim q - W(q, \lambda_i, \beta_i^\star), i = 1, 2$. For the case $q < 1$, consider $q = 0.5, \beta_1 = 0.5, \beta_2 = 1.5, \beta_1^\star = 0.3, \beta_2^\star = 1.7$, and $\lambda = 1$, where $(\beta_1, \beta_2) \overset{m}{\preceq} (\beta_1^\star, \beta_2^\star)$. Figure 5(a) shows the plot of $\bar{F}_{U_{1:2}}(t) - \bar{F}_{V_{1:2}}(t)$ in the interval $(0, D) = (0, 1.5)$. It is evident from the figure that $\bar{F}_{U_{1:2}}(t) - \bar{F}_{V_{1:2}}(t) \ge 0$ for all t, which implies that $U_{1:2} \ge_{st} V_{1:2}$. Similarly, for the case of $q > 1$, take $q = 1.5, \beta_1 = 0.5, \beta_2 = 1.5, \beta_1^\star = 0.3, \beta_2^\star = 1.7$, and $\lambda = 0.5$, with $(\beta_1, \beta_2) \overset{m}{\preceq} (\beta_1^\star, \beta_2^\star)$. As illustrated in Figure 6(b), $\bar{F}_{U_{1:2}} - \bar{F}_{V_{1:2}} \ge 0$, confirming that $U_{1:2} \ge_{st} V_{1:2}$.

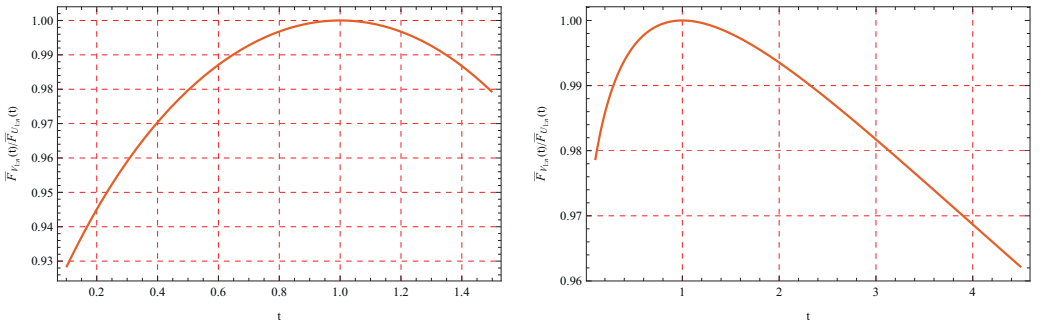
The following counterexample illustrates that the condition in Theorem 3.7 is not sufficient for the hazard rate ordering between $U_{1:n}$ and $V_{1:n}$.

https://doi.org/10.1017/S0269964826100199 Published online by Cambridge University Press



(a) $q = 0.5, \beta_1 = 0.5, \beta_2 = 1.5, \beta_1^\star = 0.3, \beta_2^\star = 1.7, \lambda = 1$ (b) $q = 1.5, \beta_1 = 0.5, \beta_2 = 1.5, \beta_1^\star = 0.3, \beta_2^\star = 1.7, \lambda = 0.5$

Figure 5. *Plots of* $\bar{F}_{U_{1:2}}(t) - \bar{F}_{V_{1:2}}(t)$*: (a)* $q = 0.5, \beta_1 = 0.5, \beta_2 = 1.5, \beta_1^\star = 0.3, \beta_2^\star = 1.7, \lambda = 1$ *and (b)* $q = 1.5, \beta_1 = 0.5, \beta_2 = 1.5, \beta_1^\star = 0.3, \beta_2^\star = 1.7, \lambda = 0.5$.



(a) $q = 0.3, \beta_1 = 0.5, \beta_2 = 0.5, \beta_1^\star = 0.3, \beta_2^\star = 0.7, \lambda = 0.5$ (b) $q = 1.3, \beta_1 = 0.5, \beta_2 = 0.5, \beta_1^\star = 0.3, \beta_2^\star = 0.7, \lambda = 0.5$

Figure 6. *Plots of* $\bar{F}_{V_{1:2}}(t)/\bar{F}_{U_{1:2}}(t)$*: (a)* $q = 0.3, \beta_1 = 0.5, \beta_2 = 0.5, \beta_1^\star = 0.3, \beta_2^\star = 0.7, \lambda = 0.5$ *and (b)* $q = 1.3, \beta_1 = 0.5, \beta_2 = 0.5, \beta_1^\star = 0.3, \beta_2^\star = 0.7, \lambda = 0.5$.

Example 3.9. Let $U_i \sim q - W(q, \lambda, \beta_i)$ and $V_i \sim q - W(q, \lambda, \beta_i^\star), i = 1, 2$. Consider $q = 0.3, \beta_1 = 0.5, \beta_2 = 0.5, \beta_1^\star = 0.3, \beta_2^\star = 0.7$, and $\lambda = 0.5$. From Figure 6(a), it is clear that $\bar{F}_{V_{1:2}}(t)/\bar{F}_{U_{1:2}}(t)$ is not monotone in a subset of $(0, D) = (0, 4.48)$. Then for $0 < q < 1$, $(\beta_1, \beta_2) \overset{m}{\preceq} (\beta_1^\star, \beta_2^\star)$ does not imply $V_{1:n} \leq_{hr} U_{1:n}$. Similarly, taking, $q = 1.3, \beta_1 = 0.5, \beta_2 = 0.5, \beta_1^\star = 0.3, \beta_2^\star = 0.7$, and $\lambda = 0.5$ we can conclude from Figure 6(b) that $U_{1:n}$ and $V_{1:n}$ are not comparable in hazard rate order when $1 < q < 2$ and $(\beta_1, \beta_2) \overset{m}{\preceq} (\beta_1^\star, \beta_2^\star)$.

Next, we carry out stochastic comparisons between the largest order statistics arising from two sets of heterogeneous q-Weibull samples U_i's and V_i's with parameters $(q_i, \lambda_i, \beta_i)$ and $(q_i^\star, \lambda_i^\star, \beta_i^\star)$, $i = 1, 2, \ldots, n$, respectively. Suppose $F_{U_{n:n}}(t)$ and $F_{V_{n:n}}(t)$ are the d.f.'s of the largest order statistics $U_{n:n}$ and $V_{n:n}$, respectively, then

$$F_{U_{n:n}}(t) = \prod_{l=1}^{n} F_{U_l}(t) = \prod_{l=1}^{n} \left[1 - \left[1 - \lambda_l (1 - q_l) t^{\beta_l}\right]^{\frac{2-q_l}{1-q_l}}\right],$$

https://doi.org/10.1017/S0269964826100199 Published online by Cambridge University Press

and

$$F_{V_{n:n}}(t) = \prod_{l=1}^{n} F_{V_l}(t) = \prod_{l=1}^{n} \left[1 - \left[1 - \lambda_l^{\star}(1-q_l^{\star})t^{\beta_l^{\star}}\right]^{\frac{2-q_l^{\star}}{1-q_l^{\star}}}\right].$$

It is important to note that, for $0 < q_l, q_l^* < 1$, $X_{n:n}$ and $Y_{n:n}$ are defined over different domains. So, for any comparison of these r.v.'s, we should take the common domain of $U_{n:n}$ and $V_{n:n}$ as $(0, D)$ where $D = \min\left\{\min_l\left\{[\lambda_l(1-q_l)]^{-\frac{1}{\beta_l}}\right\}, \min_l\left\{[\lambda_l^*(1-q_l^*)]^{-\frac{1}{\beta_l^*}}\right\}\right\}$.

Theorem 3.10. *Let $U_1, U_2, \ldots, U_n$ be a set of independent r.v.'s such that $U_i \sim q\text{-}W(q, \lambda_i, \beta)$, $i = 1, 2, \ldots, n$, and $V_1, V_2, \ldots, V_n$ be another set of independent r.v.'s such that $V_i \sim q\text{-}W(q, \lambda_i^*, \beta)$, $i = 1, 2, \ldots, n$. Then the following results hold:*

(1) *If $0 < q < 1$ and $(\lambda_1, \lambda_2, \ldots, \lambda_n) \preceq^w (\lambda_1^*, \lambda_2^*, \ldots, \lambda_n^*)$, then $U_{n:n} \leq_{st} V_{n:n}$.*
(2) *If $1 < q < 2$ and $(\lambda_1, \lambda_2, \ldots, \lambda_n) \preceq^w (\lambda_1^*, \lambda_2^*, \ldots, \lambda_n^*)$, then $U_{n:n} \leq_{st} V_{n:n}$.*

Proof. The d.f. of the largest order statistics $U_{n:n}$ can be written as

$$F_{U_{n:n}}(t) = \prod_{l=1}^{n} \left[1 - \left[1 - \lambda_l(1-q_l)t^{\beta}\right]^{\frac{2-q}{1-q}}\right] = \prod_{l=1}^{n} \xi_t(\lambda_l).$$

Now we have

$$\frac{d}{d\lambda} ln[\xi_t(\lambda)] = (2-q)t^{\beta} \frac{\left[1 - \lambda(1-q)t^{\beta}\right]^{\frac{1}{1-q}}}{1 - \left[1 - \lambda(1-q)t^{\beta}\right]^{\frac{2-q}{1-q}}}$$

and

$$\frac{d^2}{d\lambda^2} ln[\xi_t(\lambda)] \overset{sign}{=} -(2-q)\left(1 - \lambda(1-q)t^{\beta}\right)^{\frac{1}{1-q}} \times \left[\left(1 - \lambda(1-q)t^{\beta}\right)^{-1}\left\{1 - \left(1 - \lambda(1-q)t^{\beta}\right)^{\frac{2-q}{1-q}}\right\} + (2-q)\left(1 - \lambda(1-q)t^{\beta}\right)^{\frac{1}{1-q}}\right].$$

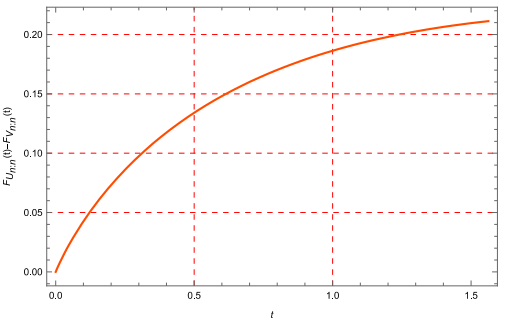
(1) To prove the result using Lemma 2.4, we need the Schur-concavity of $F_{n:n}(t)$ in λ. By Proposition E.1. of [32], this can be shown by the concavity property of $ln[\xi_t(\lambda_l)]$ in λ. Now, for $0 < q < 1$, we have $0 \leq t \leq D$, which implies $\frac{d}{d\lambda} ln[\xi_t(\lambda)] \geq 0$ and $\frac{d^2}{d\lambda^2} ln[\xi_t(\lambda)] \leq 0$. Hence, the result follows.
(2) For $1 < q < 2$, we similarly obtain $\frac{d}{d\lambda} ln[\xi_t(\lambda)] \geq 0$ and $\frac{d^2}{d\lambda^2} ln[\xi_t(\lambda)] \leq 0$, which again implies the Schur-concavity of $F_{U_{n:n}}(t)$ by Proposition E.1. of [32]. The conclusion then follows from Lemma 2.4.

□

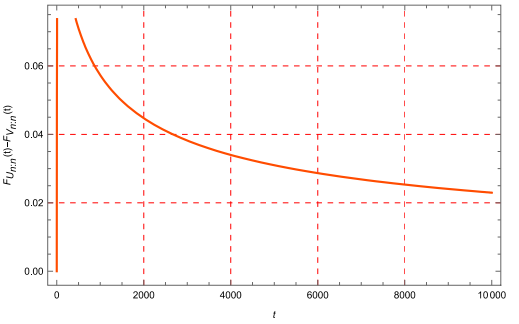
Below, we consider a numerical example to demonstrate the result stated in Theorem 3.10.

Example 3.11. Let $U_i \sim q - W(q, \lambda_i, \beta)$ and $V_i \sim q - W(q, \lambda_i^{\star}, \beta), i = 1, 2$. For the case $q < 1$, consider $q = 0.5, \beta = 0.5, \lambda_1 = 0.5, \lambda_2 = 1.5, \lambda_1^{\star} = 0.3$, and $\lambda_2^{\star} = 1.6$, where $(\lambda_1, \lambda_2) \preceq^w (\lambda_1^{\star}, \lambda_2^{\star})$.

https://doi.org/10.1017/S0269964826100199 Published online by Cambridge University Press



(a) $q = 0.5, \beta = 0.5, \lambda_1 = 0.5, \lambda_2 = 1.5, \lambda_1^\star = 0.3, \lambda_2^\star = 1.6$



(b) $q = 1.5, \beta = 0.5, \lambda_1 = 0.5, \lambda_2 = 1.5, \lambda_1^\star = 0.3, \lambda_2^\star = 1.6$

Figure 7. *Plots of* $F_{U_{2:2}}(t) - F_{V_{2:2}}(t)$*: (a)* $q = 0.5, \beta = 0.5, \lambda_1 = 0.5, \lambda_2 = 1.5, \lambda_1^\star = 0.3, \lambda_2^\star = 1.6$ *and (b)* $q = 1.5, \beta = 0.5, \lambda_1 = 0.5, \lambda_2 = 1.5, \lambda_1^\star = 0.3, \lambda_2^\star = 1.6$.

Figure 7(a) presents the plot of $F_{U_{2:2}}(t) - F_{V_{2:2}}(t)$ in the interval $(0, D) = (0, 1.56)$. The plot clearly indicates that $F_{U_{2:2}}(t) - F_{V_{2:2}}(t) \geq 0$ in t, which implies $U_{2:2} \leq_{st} V_{2:2}$. Similarly, for the case of $q > 1$, choose $q = 1.5, \beta = 0.5, \lambda_1 = 0.5, \lambda_2 = 1.5, \lambda_1^\star = 0.3$, and $\lambda_2^\star = 1.6$, with $(\lambda_1, \lambda_2) \preceq^w (\lambda_1^\star, \lambda_2^\star)$. As shown in Figure 7(b), we again find that $F_{U_{2:2}}(t) - F_{V_{2:2}}(t) \geq 0$, confirming that, $U_{2:2} \leq_{st} V_{2:2}$.

4. Heterogeneous dependent samples

In this section, we compare the extreme order statistics coming from the r.v.'s $U_1, U_2, \ldots, U_n$ and $V_1, V_2, \ldots, V_n$ having distribution functions $F_{U_i}(\cdot)$ and $F_{V_i}(\cdot)$, respectively. We consider that these random variables are dependent, and the dependence structure is modeled by an Archimedean copula.

Let us consider two sets of heterogeneous q-Weibull samples, U_i's and V_i's, with parameters $(q_i, \lambda_i, \beta_i)$ and $(q_i^\star, \lambda_i^\star, \beta_i^\star)$, $i = 1, 2, \ldots, n$, respectively. The dependence structures of these samples are modeled by Archimedean copulas with generators $\psi_1(\phi_1 = \psi_1^{-1})$ and $\psi_2(\phi_2 = \psi_2^{-1})$. As discussed earlier, for comparison results in case of $0 < q_i, q_i^\star < 1$, we should take the common domain of $U_{n:n}$ and $V_{n:n}$ as $(0, D)$, where $D = \min\left\{\min_i\left\{[\lambda_i(1-q_i)]^{-\frac{1}{\beta_i}}\right\}, \min_i\left\{[\lambda_i^\star(1-q_i^\star)]^{-\frac{1}{\beta_i^\star}}\right\}\right\}$.

Theorem 4.1. *Let* $U_i \sim q - W(q, \lambda_i, \beta)$*,* $i = 1, \ldots, n$*, be a set of dependent r.v.'s sharing an Archimedean copula with generator* ψ_1*. Similarly, let* $V_i \sim q - W(q, \lambda_i^\star, \beta)$*,* $i = 1, \ldots, n$*, be a set of dependent r.v.'s sharing an Archimedean copula with generator* ψ_2*.*

(1) *Suppose that* $\phi_2 \circ \psi_1$ *is super-additive and* ψ_1 *or* ψ_2 *is log-convex. Then, for* $0 < q < 1$ *and* $\boldsymbol{\lambda} \in \mathbb{D}_+(\mathbb{E}_+)$*,*

$$(\lambda_1^\star, \lambda_2^\star, \ldots, \lambda_n^\star) \preceq^w (\lambda_1, \lambda_2, \ldots, \lambda_n) \text{ implies } U_{n:n} \geq_{st} V_{n:n}.$$

(2) *Suppose that* $\phi_2 \circ \psi_1$ *is super-additive and* ψ_1 *or* ψ_2 *is log-convex. Then, for* $1 < q < 2$ *and* $\boldsymbol{\lambda} \in \mathbb{D}_+(\mathbb{E}_+)$*,*

$$(\lambda_1^\star, \lambda_2^\star, \ldots, \lambda_n^\star) \preceq^w (\lambda_1, \lambda_2, \ldots, \lambda_n) \text{ implies } U_{n:n} \geq_{st} V_{n:n}.$$

Proof. The d.f.s of $X_{n:n}$ and $Y_{n:n}$ can be written as follows

$$F_{U_{n:n}}(t) = \psi_1\left[\sum_{i=1}^{n} \phi_1\left[1 - (1 - \lambda_i(1-q)t^\beta)^{\frac{2-q}{1-q}}\right]\right],$$

https://doi.org/10.1017/S0269964826100199 Published online by Cambridge University Press

and

$$F_{V_{n:n}}(t) = \psi_2\left[\sum_{i=1}^{n}\phi_2\left[1-(1-\lambda_i^*(1-q)t^\beta)^{\frac{2-q}{1-q}}\right]\right].$$

Since $\phi_2 \circ \psi_1$ is assumed to be super-additive, it follows from Lemma 2.7 that

$$\psi_1\left[\sum_{i=1}^{n}\phi_1\left[1-(1-\lambda_i(1-q)t^\beta)^{\frac{2-q}{1-q}}\right]\right] \le \psi_2\left[\sum_{i=1}^{n}\phi_2\left[1-(1-\lambda_i(1-q)t^\beta)^{\frac{2-q}{1-q}}\right]\right]. \tag{8}$$

Define $\Omega(\boldsymbol{\lambda}) = \psi_2\left[\sum_{i=1}^{n}\phi_2\left[1-(1-\lambda_i(1-q)t^\beta)^{\frac{2-q}{1-q}}\right]\right]$. Differentiating $\Omega(\boldsymbol{\lambda})$ with respect to λ_k, we obtain

$$\begin{aligned}\frac{\partial\Omega(\boldsymbol{\lambda})}{\partial\lambda_k} &= \psi_2'\left[\sum_{i=1}^{n}\phi_2\left[1-(1-\lambda_i(1-q)t^\beta)^{\frac{2-q}{1-q}}\right]\right]\phi_2'\left[1-(1-\lambda_k(1-q)t^\beta)^{\frac{2-q}{1-q}}\right]\\ &\quad\times(2-q)(1-\lambda_k(1-q)t^\beta)^{\frac{1}{1-q}}t^\beta\\ &= \frac{(2-q)t^\beta(1-\lambda_k(1-q)t^\beta)^{\frac{1}{1-q}}}{1-(1-\lambda_k(1-q)t^\beta)^{\frac{2-q}{1-q}}}\psi_2'\left[\sum_{i=1}^{n}\phi_2\left[1-(1-\lambda_i(1-q)t^\beta)^{\frac{2-q}{1-q}}\right]\right]\\ &\quad\times\frac{\psi_2\left[\phi_2\left[1-(1-\lambda_k(1-q)t^\beta)^{\frac{2-q}{1-q}}\right]\right]}{\psi_2'\left[\phi_2\left[1-(1-\lambda_k(1-q)t^\beta)^{\frac{2-q}{1-q}}\right]\right]}\end{aligned} \tag{9}$$

(1) For $0 < q < 1$, we have $\frac{\partial\Omega(\boldsymbol{\lambda})}{\partial\lambda_k} \ge 0$ when $t \in [0, D)$, that is, $\Omega(\boldsymbol{\lambda})$ is increasing in $\boldsymbol{\lambda}$. Suppose $\boldsymbol{\lambda} \in \mathbb{D}_+(\mathbb{E}_+)$, that is, for $i \le j$, $\lambda_i \ge (\le)\lambda_j$ implies $(1-\lambda_i(1-q)t^\beta)^{\frac{1}{1-q}} \le (\ge)(1-\lambda_j(1-q)t^\beta)^{\frac{1}{1-q}}$, and $(1-\lambda_i(1-q)t^\beta)^{\frac{2-q}{1-q}} \le (\ge)(1-\lambda_j(1-q)t^\beta)^{\frac{2-q}{1-q}}$. This further implies

$$\phi_2\left[1-(1-\lambda_i(1-q)t^\beta)^{\frac{2-q}{1-q}}\right] \le (\ge)\phi_2\left[1-(1-\lambda_j(1-q)t^\beta)^{\frac{2-q}{1-q}}\right], \tag{10}$$

since ϕ_2 is d-monotone. Now, from our assumption that ψ_2 is log-convex, we can say that $\frac{\psi_2(t)}{\psi_2'(t)}$ is decreasing in t. Therefore,

$$\frac{\psi_2\left[\phi_2\left[1-(1-\lambda_i(1-q)t^\beta)^{\frac{2-q}{1-q}}\right]\right]}{\psi_2'\left[\phi_2\left[1-(1-\lambda_i(1-q)t^\beta)^{\frac{2-q}{1-q}}\right]\right]} \ge (\le)\frac{\psi_2\left[\phi_2\left[1-(1-\lambda_j(1-q)t^\beta)^{\frac{2-q}{1-q}}\right]\right]}{\psi_2'\left[\phi_2\left[1-(1-\lambda_j(1-q)t^\beta)^{\frac{2-q}{1-q}}\right]\right]}. \tag{11}$$

Furthermore, since ψ_2 is d-monotone, it follows that

$$\begin{aligned}&\frac{(2-q)t^\beta(1-\lambda_i(1-q)t^\beta)^{\frac{1}{1-q}}}{1-(1-\lambda_i(1-q)t^\beta)^{\frac{2-q}{1-q}}}\psi_2'\left[\sum_{i=1}^{n}\phi_2\left[1-(1-\lambda_i(1-q)t^\beta)^{\frac{2-q}{1-q}}\right]\right] \ge (\le)\\ &\frac{(2-q)t^\beta(1-\lambda_j(1-q)t^\beta)^{\frac{1}{1-q}}}{1-(1-\lambda_j(1-q)t^\beta)^{\frac{2-q}{1-q}}}\psi_2'\left[\sum_{i=1}^{n}\phi_2\left[1-(1-\lambda_i(1-q)t^\beta)^{\frac{2-q}{1-q}}\right]\right]\end{aligned} \tag{12}$$

https://doi.org/10.1017/S0269964826100199 Published online by Cambridge University Press

Multiplying (11) and (12), and using (9), we obtain

$$\frac{\partial \Omega(\boldsymbol{\lambda})}{\partial \lambda_i} \leq (\geq) \frac{\partial (\Omega \boldsymbol{\lambda})}{\partial \lambda_j}.$$

Finally, by Lemma 2 (Lemma 3) of [28], it follows that $\Omega(\boldsymbol{\lambda})$ is Schur-concave in $\boldsymbol{\lambda}$. Combining this with the fact that $\Omega(\boldsymbol{\lambda})$ is increasing and Schur-concave in $\boldsymbol{\lambda}$, we obtain the desired result from (8) and Lemma 2.4.

(2) For $1 < q < 2$, we have $\frac{\partial \Omega(\boldsymbol{\lambda})}{\partial \lambda_k} \geq 0$ for all $t \geq 0$, that is, $\Omega(\boldsymbol{\lambda})$ is increasing in $\boldsymbol{\lambda}$. Suppose $\boldsymbol{\lambda} \in \mathbb{D}_+(\mathbb{E}_+)$, that is, for $i \leq j$, $\lambda_i \geq (\leq)\lambda_j$ implies $(1-\lambda_i(1-q)t^\beta)^{\frac{1}{1-q}} \leq (\geq)(1-\lambda_j(1-q)t^\beta)^{\frac{1}{1-q}}$, and $(1-\lambda_i(1-q)t^\beta)^{\frac{2-q}{1-q}} \leq (\geq)(1-\lambda_j(1-q)t^\beta)^{\frac{2-q}{1-q}}$. This further implies

$$\phi_2\left[1-(1-\lambda_i(1-q)t^\beta)^{\frac{2-q}{1-q}}\right] \leq (\geq)\phi_2\left[1-(1-\lambda_j(1-q)t^\beta)^{\frac{2-q}{1-q}}\right]$$

since ϕ_2 is d-monotone.

From the assumption that ψ_2 is log-convex, we know that $\frac{\psi_2(t)}{\psi_2'(t)}$ is decreasing in t. Hence,

$$\frac{\psi_2\left[\phi_2\left[1-(1-\lambda_i(1-q)t^\beta)^{\frac{2-q}{1-q}}\right]\right]}{\psi_2'\left[\phi_2\left[1-(1-\lambda_i(1-q)t^\beta)^{\frac{2-q}{1-q}}\right]\right]} \geq (\leq) \frac{\psi_2\left[\phi_2\left[1-(1-\lambda_j(1-q)t^\beta)^{\frac{2-q}{1-q}}\right]\right]}{\psi_2'\left[\phi_2\left[1-(1-\lambda_j(1-q)t^\beta)^{\frac{2-q}{1-q}}\right]\right]}. \tag{13}$$

Moreover, since ψ_2 is d-monotone, we obtain

$$\begin{aligned}&\frac{(2-q)t^\beta(1-\lambda_i(1-q)t^\beta)^{\frac{1}{1-q}}}{1-(1-\lambda_i(1-q)t^\beta)^{\frac{2-q}{1-q}}}\psi_2'\left[\sum_{i=1}^n \phi_2\left(1-(1-\lambda_i(1-q)t^\beta)^{\frac{2-q}{1-q}}\right)\right] \geq (\leq)\\ &\frac{(2-q)t^\beta(1-\lambda_j(1-q)t^\beta)^{\frac{1}{1-q}}}{1-(1-\lambda_j(1-q)t^\beta)^{\frac{2-q}{1-q}}}\psi_2'\left[\sum_{i=1}^n \phi_2\left(1-(1-\lambda_i(1-q)t^\beta)^{\frac{2-q}{1-q}}\right)\right]\end{aligned} \tag{14}$$

Multiplying (13) and (14), and using (9), we conclude that

$$\frac{\partial \Omega(\boldsymbol{\lambda})}{\partial \lambda_i} \leq (\geq) \frac{\partial \Omega(\boldsymbol{\lambda})}{\partial \lambda_j}.$$

By Lemma 2 (Lemma 3) of [28], it follows that $\Omega(\boldsymbol{\lambda})$ is Schur-concave in $\boldsymbol{\lambda}$. Finally, since $\Omega(\boldsymbol{\lambda})$ is both increasing and Schur-concave in $\boldsymbol{\lambda}$, the desired result follows from (8) and Lemma 2.4.

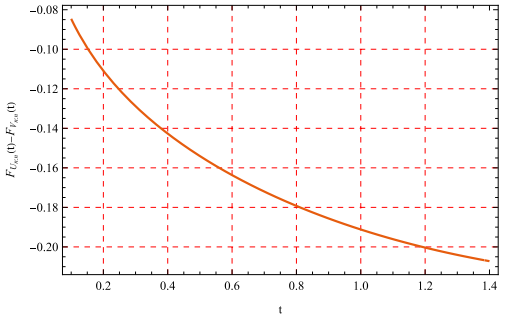
□

Remark 1. The assumption that either ψ_1 or ψ_2 is log-convex can be replaced by any of the following: (i) the underlying copula C is $TP2$; (ii) the underlying copula C is left-tail decreasing. For Archimedean copulas, these three conditions are equivalent (see Proposition 4.2 of [21]).

The next example illustrates the validity of the preceding theorem.

Example 4.2. Let $U_i \sim q-W(q,\lambda_i,\beta)$ and $V_i \sim q-W(q,\lambda_i^*,\beta)$ for $i=1,2$ be two sets of components being mutually dependent with the Clayton copula (member of the Archimedean copula family). Here, we have considered the generators of the Clayton copula as $\psi_1(u) = \frac{1}{1-u}$ and $\psi_2(u) = \frac{1}{\sqrt{2u+1}}$. Then

https://doi.org/10.1017/S0269964826100199 Published online by Cambridge University Press



(a) $q = 0.5, \beta = 0.5, \lambda_1 = 0.3, \lambda_2 = 1.7, \lambda_1^\star = 0.5, \lambda_2^\star = 1.6$

(b) $q = 1.5, \beta = 0.5, \lambda_1 = 0.3, \lambda_2 = 1.7, \lambda_1^\star = 0.5, \lambda_2^\star = 1.6$

Figure 8. *Plots of* $F_{U_{2:2}}(t) - F_{V_{2:2}}(t)$*: (a)* $q = 0.5, \beta = 0.5, \lambda_1 = 0.3, \lambda_2 = 1.7, \lambda_1^\star = 0.5, \lambda_2^\star = 1.6$ *and (b)* $q = 1.5, \beta = 0.5, \lambda_1 = 0.3, \lambda_2 = 1.7, \lambda_1^\star = 0.5, \lambda_2^\star = 1.6$.

$\phi_1(u) = \frac{1-u}{u}$ and $\phi_2(u) = \frac{1-u^2}{2u^2}$. It is easy to verify that ψ_1 and ψ_2 are log-convex and $\phi_2 \circ \psi_1$ is super-additive. Set $\lambda_1 = 0.3, \lambda_2 = 1.7, \lambda_1^\star = 0.5, \lambda_2^\star = 1.6, \beta = 0.5$ and $q = 0.5$, then Figure 8(a) demonstrates that $F_{U_{n:n}} - F_{V_{n:n}} \leq 0$ on $(0, D) = (0, 1.384)$ when $0 < q < 1$. Figure 8(a) demonstrates that the result for $1 < q < 2$ also holds. In this case, we have considered $\lambda_1 = 0.3, \lambda_2 = 1.7, \lambda_1^\star = 0.5, \lambda_2^\star = 1.6, \beta = 0.5$ and $q = 1.5$.

Theorem 4.3. *Let* $U_i \sim q - W(q, \lambda_i, \beta)$*, for* $i = 1, \ldots, n$*, be a set of dependent r.v.'s sharing an Archimedean copula with generator* ψ_1*. Similarly, let* $V_i \sim q - W(q, \lambda_i^*, \beta)$*, for* $i = 1, \ldots, n$*, be a set of dependent r.v.'s sharing an Archimedean copula with generator* ψ_2*.*

(1) *Suppose* $\phi_2 \circ \psi_1$ *is super-additive and* ψ_1 *or* ψ_2 *is log-convex. Then, for* $0 < q < 1$ *and* $\boldsymbol{\lambda} \in \mathbb{D}_+(\mathbb{E}_+)$*,*

$$(\lambda_1^*, \lambda_2^*, \ldots, \lambda_n^*) \preceq_w (\lambda_1, \lambda_2, \ldots, \lambda_n) \text{ implies } U_{1:n} \leq_{st} V_{1:n}.$$

(2) *Suppose* $\phi_2 \circ \psi_1$ *is super-additive and* ψ_1 *or* ψ_2 *is log-concave. Then, for* $1 < q < 2$ *and* $\boldsymbol{\lambda} \in \mathbb{D}_+(\mathbb{E}_+)$

$$(\lambda_1, \lambda_2, \ldots, \lambda_n) \preceq^w (\lambda_1^*, \lambda_2^*, \ldots, \lambda_n^*) \text{ implies } U_{1:n} \leq_{st} V_{1:n}.$$

Proof. The survival functions of the smallest order statistics $U_{1:n}$ and $V_{1:n}$ are given by

$$\bar{F}_{U_{1:n}}(t) = \psi_1 \left[\sum_{i=1}^{n} \phi_1 \left\{ \left(1 - (1-q)\lambda_i t^\beta\right)^{\frac{2-q}{1-q}} \right\} \right],$$

and

$$\bar{F}_{V_{1:n}}(t) = \psi_2 \left[\sum_{i=1}^{n} \phi_2 \left\{ \left(1 - (1-q)\lambda_i^* t^\beta\right)^{\frac{2-q}{1-q}} \right\} \right].$$

From the assumption $\phi_2 \circ \psi_1$ is super-additive, it follows from Lemma 2.7 that,

$$\psi_1 \left[\sum_{i=1}^{n} \phi_1 \left\{ \left(1 - (1-q)\lambda_i t^\beta\right)^{\frac{2-q}{1-q}} \right\} \right] \leq \psi_2 \left[\sum_{i=1}^{n} \phi_2 \left\{ \left(1 - (1-q)\lambda_i t^\beta\right)^{\frac{2-q}{1-q}} \right\} \right]. \tag{15}$$

https://doi.org/10.1017/S0269964826100199 Published online by Cambridge University Press

Define

$$\Psi(\boldsymbol{\lambda}) = \psi_2\left[\sum_{i=1}^{n}\phi_2\left\{\left(1-(1-q)\lambda_i t^{\beta}\right)^{\frac{2-q}{1-q}}\right\}\right].$$

Differentiating with respect to λ_i, we obtain

$$\frac{\partial\Psi(\boldsymbol{\lambda})}{\partial\lambda_i} = -\psi_2'\left[\sum_{i=1}^{n}\phi_2\left\{\left(1-(1-q)\lambda_i t^{\beta}\right)^{\frac{2-q}{1-q}}\right\}\right] \times \frac{\psi_2\left[\phi_2\left\{\left(1-(1-q)\lambda_i t^{\beta}\right)^{\frac{2-q}{1-q}}\right\}\right]}{\psi_2'\left[\phi_2\left\{\left(1-(1-q)\lambda_i t^{\beta}\right)^{\frac{2-q}{1-q}}\right\}\right]}\frac{(2-q)t^{\beta}}{1-(1-q)\lambda_i t^{\beta}}. \tag{16}$$

(1) For $0 < q < 1$, observe that $\dfrac{\partial\Psi(\boldsymbol{\lambda})}{\partial\lambda_i} < 0$, implying that $\Psi(\boldsymbol{\lambda})$ is decreasing in each λ_i. Since $\boldsymbol{\lambda} \in \mathbb{D} + (\mathbb{E}+)$, we have $\lambda_i \geq (\leq), \lambda_j$ for $i \leq j$. Therefore, for $i \leq j$,

$$\frac{(2-q)t^{\beta}}{1-(1-q)\lambda_i t^{\beta}} \geq \frac{(2-q)t^{\beta}}{1-(1-q)\lambda_j t^{\beta}}. \tag{17}$$

Moreover, $\phi_2\left[\left(1-(1-q)\lambda_i t^{\beta}\right)^{\frac{2-q}{1-q}}\right] \geq (\leq)\phi_2\left[\left(1-(1-q)\lambda_j t^{\beta}\right)^{\frac{2-q}{1-q}}\right]$. Now, under the assumption that $\psi(\cdot)$ is log-convex, it follows that for $i \leq j$

$$\frac{\psi_2\left[\phi_2\left\{\left(1-(1-q)\lambda_i t^{\beta}\right)^{\frac{2-q}{1-q}}\right\}\right]}{\psi_2'\left[\phi_2\left\{\left(1-(1-q)\lambda_i t^{\beta}\right)^{\frac{2-q}{1-q}}\right\}\right]} \leq (\geq)\frac{\psi_2\left[\phi_2\left\{\left(1-(1-q)\lambda_j t^{\beta}\right)^{\frac{2-q}{1-q}}\right\}\right]}{\psi_2'\left[\phi_2\left\{\left(1-(1-q)\lambda_j t^{\beta}\right)^{\frac{2-q}{1-q}}\right\}\right]}. \tag{18}$$

Therefore, using (17) and (18), in (16) we obtain

$$\frac{\partial\Psi(\boldsymbol{\lambda})}{\partial\lambda_i} \leq (\geq)\frac{\partial\Psi(\boldsymbol{\lambda})}{\partial\lambda_j}.$$

Hence, by Lemma 2 (Lemma 3) of [28], $\Psi(\boldsymbol{\lambda})$ is Schur-concave in $\boldsymbol{\lambda}$. Finally, the desired result follows from Lemma 2.4 together with (15).

(2) For $1 < q < 2$, we have $\frac{\partial\Psi(\boldsymbol{\lambda})}{\partial\lambda_i}$ is negative, which implies that $\Psi(\boldsymbol{\lambda})$ is a decreasing in each λ_i. Note that when $1 < q < 2$,

$$\left[\left(1-(1-q)\lambda_i t^{\beta}\right)^{\frac{2-q}{1-q}}\right] \leq (\geq)\left[\left(1-(1-q)\lambda_j t^{\beta}\right)^{\frac{2-q}{1-q}}\right]$$

for $i \leq j$ as $\boldsymbol{\lambda} \in \mathbb{D}_+(\mathbb{E}_+)$. Consequently, for $i \leq j$,

$$\frac{(2-q)t^{\beta}}{1-(1-q)\lambda_i t^{\beta}} \leq \frac{(2-q)t^{\beta}}{1-(1-q)\lambda_j t^{\beta}}. \tag{19}$$

https://doi.org/10.1017/S0269964826100199 Published online by Cambridge University Press

Moreover, for $i \leq j$,

$$\phi_2\left[\left(1-(1-q)\lambda_i t^\beta\right)^{\frac{2-q}{1-q}}\right] \geq (\leq)\phi_2\left[\left(1-(1-q)\lambda_j t^\beta\right)^{\frac{2-q}{1-q}}\right].$$

Now, under the assumption that $\psi(\cdot)$ is log-concave, it follows that for $i \leq j$,

$$\frac{\psi_2\left[\phi_2\left\{\left(1-(1-q)\lambda_i t^\beta\right)^{\frac{2-q}{1-q}}\right\}\right]}{\psi_2'\left[\phi_2\left\{\left(1-(1-q)\lambda_i t^\beta\right)^{\frac{2-q}{1-q}}\right\}\right]} \geq (\leq)\frac{\psi_2\left[\phi_2\left\{\left(1-(1-q)\lambda_j t^\beta\right)^{\frac{2-q}{1-q}}\right\}\right]}{\psi_2'\left[\phi_2\left\{\left(1-(1-q)\lambda_j t^\beta\right)^{\frac{2-q}{1-q}}\right\}\right]}. \tag{20}$$

Therefore, using (19) and (20), in (16) we obtain

$$\frac{\partial \Psi(\boldsymbol{\lambda})}{\partial \lambda_i} \geq (\leq)\frac{\partial \Psi(\boldsymbol{\lambda})}{\partial \lambda_j}.$$

Hence, by Lemma 2 (Lemma 3) of [28], $\Psi(\boldsymbol{\lambda})$ is Schur-convex in $\boldsymbol{\lambda}$. Finally, the result follows from (15).

□

Remark 2. The assumption that either ψ_1 or ψ_2 is log-concave can be replaced by the assumption that the underlying copula C is $RR2$; For Archimedean copulas, these two conditions are equivalent (see Theorem 5.1 of [29]).

Now, we present an example to demonstrate that if the smallest order statistics satisfy the conditions of Theorem 4.3, then the survival function of one will be smaller than the other.

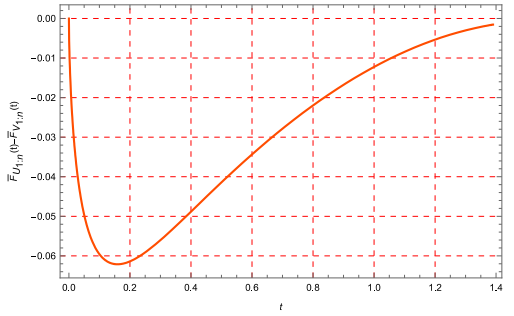
Example 4.4. Let $U_i \sim q - W(q, \lambda_i, \beta)$ and $V_i \sim q - W(q, \lambda_i^*, \beta)$ for $i = 1, 2$ be two sets of components being mutually dependent. For the case of $0 < q < 1$, we have used Clayton copula with generators $\psi_1(u) = \frac{1}{1-u}$ and $\psi_2(u) = \frac{1}{\sqrt{2u+1}}$ to model the dependence. With the choice of these generators, it is easy to check that the conditions of Theorem 4.3 hold. Figure 9(a) demonstrates that $\bar{F}_{U_{1:n}} - \bar{F}_{V_{1:n}} \leq 0$ on $(0, D) = (0, 1.384)$ where $\lambda_1 = 0.5, \lambda_2 = 1.7, \lambda_1^\star = 0.6, \lambda_2^\star = 1.5, \beta = 0.5$, and $q = 0.5$.

For the case of $1 < q < 2$, to model the dependence, we employ the Gumbel-Hougaard copula (member of Archimedean copula family) with generators $\psi_1(u) = e^{1-(1+u)^8}$ and $\psi_2(u) = e^{1-(1+u)^3}$. Then $\phi_1(u) = (1-\ln u)^{\frac{1}{8}} - 1$ and $\phi_2(u) = (1-\ln u)^{\frac{1}{3}} - 1$. Observe that both $\psi_1(u)$ and $\psi_2(u)$ are log-concave and $\phi_2 \circ \psi_1$ is super-additive. With the parameter choices $\lambda_1 = 0.5, \lambda_2 = 0.8, \lambda_1^\star = 0.3, \lambda_2^\star = 0.9, \beta = 0.5$, and $q = 1.5$, Figure 9(b) demonstrates that the result also holds for $1 < q < 2$.

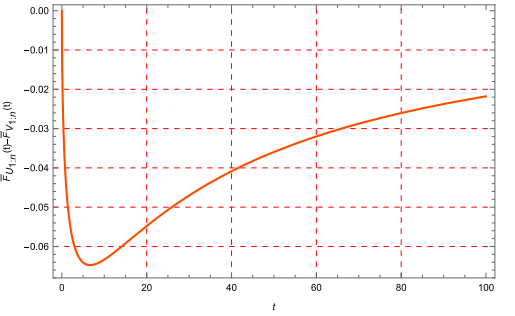
5. Independent variables random shock

In the previous section, we examined the stochastic comparisons of extreme order statistics. In this section, we extend the analysis to the stochastic comparison of two samples, where the order statistics are subject to random shocks that may or may not lead to failure. Such models find applications in various areas, including reliability theory, actuarial science, and maintenance theory. As an illustrative example, consider two groups of insurance policyholders, where the lifetimes of the insured individuals in each group follow q-Weibull distributions. Our objective is to compare the extreme order statistics of these two groups when each policyholder is exposed to random shocks that have the potential to cause failure.

https://doi.org/10.1017/S0269964826100199 Published online by Cambridge University Press



(a) $q = 0.5, \beta = 0.5, \lambda_1 = 0.5, \lambda_2 = 1.7, \lambda_1^\star = 0.6, \lambda_2^\star = 1.5$



(b) $q = 1.5, \beta = 0.5, \lambda_1 = 0.5, \lambda_2 = 0.8, \lambda_1^\star = 0.3, \lambda_2^\star = 0.9$

Figure 9. *Plots of $\overline{F}_{U_{1:2}}(t) - \overline{F}_{V_{1:2}}(t)$: (a) $q = 0.5, \beta = 0.5, \lambda_1 = 0.5, \lambda_2 = 1.7, \lambda_1^\star = 0.6, \lambda_2^\star = 1.5$ and (b) $q = 1.5, \beta = 0.5, \lambda_1 = 0.5, \lambda_2 = 0.8, \lambda_1^\star = 0.3, \lambda_2^\star = 0.9$.*

Let $U_i \sim q - W(q_i, \lambda_i, \beta_i)$ and $V_i \sim q - W(q_i^\star, \lambda_i^\star, \beta_i^\star)$, for $i = 1, 2, \ldots, n$, be two sets of independent r.v.'s and $I_i, I_i^\star$ be another two sets of independent Bernoulli r.v.'s, independent of U_i's and V_i's respectively, with $E(I_i) = p_i$ and $E(I_i^\star) = p_i^\star$. These parameters are referred to as shock parameters. Under random shocks, define $U_i^\star = U_i I_i$ and $V_i^\star = V_i I_i^\star$. Then, the survival functions of $U_i^\star$ and $V_i^\star$ are given by

$$\bar{F}_{U_i^\star}(t) = P(U_i I_i \geq t) = P(U_i I_i \geq t | I_i = 1) P(I_i = 1) = p_i (1 - \lambda_i (1 - q_i) t^{\beta_i})^{\frac{2-q_i}{1-q_i}},$$

and

$$\bar{F}_{V_i^\star}(t) = P(V_i I_i^\star \geq t) = P(V_i I_i^\star \geq t | I_i^\star = 1) P(I_i^\star = 1) = p_i^\star (1 - \lambda_i^\star (1 - q_i^\star) t^{\beta_i^\star})^{\frac{2-q_i^\star}{1-q_i^\star}}.$$

We now investigate stochastic comparisons between the largest order statistics in the presence of random shocks. The c.d.f.s of the largest order statistics $U_{n:n}^\star$ and $V_{n:n}^\star$ are given by

$$F_{U_{n:n}^\star}(t) = \prod_{i=1}^{n} \left[1 - p_i (1 - \lambda_i (1 - q_i) t^{\beta_i})^{\frac{2-q_i}{1-q_i}} \right],$$

and

$$F_{V_{n:n}^\star}(t) = \prod_{i=1}^{n} \left[1 - p_i^\star (1 - \lambda_i^\star (1 - q_i^\star) t^{\beta_i^\star})^{\frac{2-q_i^\star}{1-q_i^\star}} \right],$$

with $F_{U_{n:n}^\star}(0) = \prod_{i=1}^{n} (1 - p_i)$ and $F_{V_{n:n}^\star}(0) = \prod_{i=1}^{n} (1 - p_i^\star)$.

Theorem 5.1. *Let $U_i \sim q - W(q, \lambda_i, \beta)$ and $V_i \sim q - W(q, \lambda_i, \beta)$, $i = 1, \ldots, n$. Suppose $I_i(I_i^\star)$ be a set of independent Bernoulli r.v.'s, independent of $U_i(V_i)$ with $E(I_i) = p_i (E(I_i^\star) = p_i^\star)$. Let $\wp : [0, 1] \to \mathcal{R}_+$ be a convex, differentiable, and strictly increasing function. Assume $\boldsymbol{\lambda} \in \mathbb{D}_+(\mathbb{E}_+)$, $\wp(\boldsymbol{p}) \in \mathbb{E}_+(\mathbb{D}_+)$, where $\wp(\boldsymbol{p}) \equiv (\wp(p_1), \wp(p_2), \ldots, \wp(p_n))$. Then the following results hold:*

(1) *For $0 < q < 1$, if $\wp(\boldsymbol{p}) \preceq_w \wp(\boldsymbol{p}^\star)$, then $U_{n:n}^\star \leq_{st} V_{n:n}^\star$.*
(2) *For $1 < q < 2$, if $\wp(\boldsymbol{p}) \preceq_w \wp(\boldsymbol{p}^\star)$, then $U_{n:n}^\star \leq_{st} V_{n:n}^\star$.*

https://doi.org/10.1017/S0269964826100199 Published online by Cambridge University Press

Proof. For fixed q, β and $t \geq 0$, define

$$F_{U^{\star}_{n:n}}(t) = \prod_{k=1}^{n} \left[1 - \wp^{-1}(u_k)(1 - \lambda_k(1-q)t^{\beta})^{\frac{2-q}{1-q}}\right] = \boldsymbol{\Psi}(\boldsymbol{u}),$$

where $\wp(p_i) = u_i$. Differentiating $\boldsymbol{\Psi}(\boldsymbol{u})$ with respect to u_i, we obtain

$$\begin{aligned}\frac{\partial \boldsymbol{\Psi}(\boldsymbol{u})}{\partial u_i} = &\prod_{k=1, k\neq i}^{n} \left[1 - \wp^{-1}(u_k)(1 - \lambda_k(1-q)t^{\beta})^{\frac{2-q}{1-q}}\right] \\ &\times \left[-\frac{d\wp^{-1}(u_i)}{du_i}(1 - \lambda_i(1-q)t^{\beta})^{\frac{2-q}{1-q}}\right] \leq 0\end{aligned}$$

which shows that $\boldsymbol{\Psi}(\boldsymbol{u})$ is decreasing in each u_i. Therefore, for $t \in [0, D)$ when $0 < q < 1$, and for $t \geq 0$ when $1 < q < 2$, we have

$$\begin{aligned}&\frac{\partial \boldsymbol{\Psi}(\boldsymbol{u})}{\partial u_i} - \frac{\partial \boldsymbol{\Psi}(\boldsymbol{u})}{\partial u_j} \overset{sign}{=} \\ &\left[1 - \wp^{-1}(u_i)(1 - \lambda_i(1-q)t^{\beta})^{\frac{2-q}{1-q}}\right] \frac{d\wp^{-1}(u_j)}{du_j}(1 - \lambda_j(1-q)t^{\beta})^{\frac{2-q}{1-q}} \\ &- \left[1 - \wp^{-1}(u_j)(1 - \lambda_j(1-q)t^{\beta})^{\frac{2-q}{1-q}}\right] \frac{d\wp^{-1}(u_i)}{du_i}(1 - \lambda_i(1-q)t^{\beta})^{\frac{2-q}{1-q}}.\end{aligned}$$

Now, for all $i \leq j$, we have $\lambda_i \geq (\leq)\lambda_j, u_i \leq (\geq)u_j$ and note that $\wp^{-1}(u)$ is increasing in u. Then, for $0 < q < 1$,

$$(1 - \lambda_i(1-q)t^{\beta})^{\frac{2-q}{1-q}} \leq (\geq)(1 - \lambda_j(1-q)t^{\beta})^{\frac{2-q}{1-q}},$$

and for $1 < q < 2$,

$$(1 - \lambda_i(1-q)t^{\beta})^{\frac{2-q}{1-q}} \leq (\geq)(1 - \lambda_j(1-q)t^{\beta})^{\frac{2-q}{1-q}}.$$

From the above inequalities, it follows that

$$\left(1 - \wp^{-1}(u_i)(1 - \lambda_i(1-q)t^{\beta})^{\frac{2-q}{1-q}}\right) \geq (\leq) \left(1 - \wp^{-1}(u_j)(1 - \lambda_j(1-q)t^{\beta})^{\frac{2-q}{1-q}}\right).$$

Using the concavity of $\wp(u)$ in u along with the above inequalities, we obtain

$$\begin{aligned}&\frac{\partial \boldsymbol{\Psi}(\boldsymbol{u})}{\partial u_i} - \frac{\partial \boldsymbol{\Psi}(\boldsymbol{u})}{\partial u_j} \overset{sign}{=} \\ &\left[1 - \wp^{-1}(u_i)(1 - \lambda_i(1-q)t^{\beta})^{\frac{2-q}{1-q}}\right] \frac{d\wp^{-1}(u_j)}{du_j}(1 - \lambda_j(1-q)t^{\beta})^{\frac{2-q}{1-q}} \\ &- \left[1 - \wp^{-1}(u_j)(1 - \lambda_j(1-q)t^{\beta})^{\frac{2-q}{1-q}}\right] \frac{d\wp^{-1}(u_i)}{du_i}(1 - \lambda_i(1-q)t^{\beta})^{\frac{2-q}{1-q}} \geq (\leq)0.\end{aligned}$$

Hence, $\partial \boldsymbol{\Psi}(\boldsymbol{u})/\partial u_k$ is decreasing in $k = 1, 2, \ldots, n$. Since $u_k \in \mathbb{E}_+$, it follows that $\boldsymbol{\Psi}(\boldsymbol{u})$ is *Schur* − *concave* in $\boldsymbol{u}$ by Lemma 3 (Lemma 2) of [28]. The desired result then follows from Lemma 2.4. □

https://doi.org/10.1017/S0269964826100199 Published online by Cambridge University Press

Theorem 5.2. *Let $U_i \sim q - W(q, \lambda_i, \beta)$ and $V_i \sim q - W(q, \lambda_i^\star, \beta)$, $i = 1, \ldots, n$. Suppose $I_i(I_i^\star)$ be a set of independent Bernoulli r.v.'s, independent of $U_i(V_i)$ with $E(I_i) = p_i$ $(E(I_i^\star) = p_i^\star)$, $i = 1, 2, \ldots, n$. Let $\wp : [0,1] \to \mathcal{R}_+$ be a differentiable and strictly increasing function. Assume $\boldsymbol{\lambda}, \boldsymbol{\lambda}^\star \in \mathbb{D}_+(\mathbb{E}_+)$ and $\wp(\boldsymbol{p}) \in \mathbb{E}_+(\mathbb{D}_+)$. Then the following results hold:*

(1) *For $0 < q < 1$, if $\boldsymbol{\lambda} \preceq^w \boldsymbol{\lambda}^\star$, then $U_{n:n}^\star \leq_{st} V_{n:n}^\star$.*
(2) *For $1 < q < 2$, if $\boldsymbol{\lambda} \preceq^w \boldsymbol{\lambda}^\star$, then $U_{n:n}^\star \leq_{st} V_{n:n}^\star$.*

Proof. For fixed q, β and $t \geq 0$, define

$$F_{U_{n:n}^\star} = \prod_{k=1}^{n} \left[1 - \wp^{-1}(u_k)(1 - \lambda_k(1-q)t^\beta)^{\frac{2-q}{1-q}}\right] = \boldsymbol{\Psi}(\boldsymbol{\lambda}),$$

where $\wp(p_i) = u_i$. Differentiating $\boldsymbol{\Psi}(\boldsymbol{\lambda})$ with respect to λ_i, we obtain

$$\begin{aligned}\frac{\partial \boldsymbol{\Psi}(\boldsymbol{\lambda})}{\partial \lambda_i} = &\prod_{k=1, k\neq i}^{n} \left[1 - \wp^{-1}(u_k)(1 - \lambda_k(1-q)t^\beta)^{\frac{2-q}{1-q}}\right] \\ &\times \left[\wp^{-1}(u_i)(1 - \lambda_i(1-q)t^\beta)^{\frac{1}{1-q}}(2-q)t^\beta\right] \geq 0.\end{aligned}$$

Therefore, for $t \in [0, D)$ when $0 < q < 1$, and for $t \geq 0$ when $1 < q < 2$, we have

$$\begin{aligned}&\frac{\partial \boldsymbol{\Psi}(\boldsymbol{\lambda})}{\partial \lambda_i} - \frac{\partial \boldsymbol{\Psi}(\boldsymbol{\lambda})}{\partial \lambda_j} \overset{sign}{=} \\ &\left[1 - \wp^{-1}(u_j)(1 - \lambda_j(1-q)t^\beta)^{\frac{2-q}{1-q}}\right] \wp^{-1}(u_i)(1 - \lambda_i(1-q)t^\beta)^{\frac{1}{1-q}}(2-q)t^\beta \\ &- \left[1 - \wp^{-1}(u_i)(1 - \lambda_i(1-q)t^\beta)^{\frac{2-q}{1-q}}\right] \wp^{-1}(u_j)(1 - \lambda_j(1-q)t^\beta)^{\frac{1}{1-q}}(2-q)t^\beta.\end{aligned}$$

Now, for all $i \leq j$, we have $\lambda_i \geq (\leq)\lambda_j, u_i \leq (\geq)u_j$, and note that $\wp(u)$ is increasing in u. Then, for both $0 < q < 1$ and $1 < q < 2$,

$$(1 - \lambda_i(1-q)t^\beta)^{\frac{2-q}{1-q}} \leq (\geq)(1 - \lambda_j(1-q)t^\beta)^{\frac{2-q}{1-q}},$$

$$(1 - \lambda_i(1-q)t^\beta)^{\frac{1}{1-q}} \leq (\geq)(1 - \lambda_j(1-q)t^\beta)^{\frac{1}{1-q}},$$

and $\wp^{-1}(u_i) \leq \wp^{-1}(u_j)$. From the above relations, it follows that

$$(1 - \wp^{-1}(u_i)(1 - \lambda_i(1-q)t^\beta)^{\frac{2-q}{1-q}}) \geq (\leq)(1 - \wp^{-1}(u_j)(1 - \lambda_j(1-q)t^\beta)^{\frac{2-q}{1-q}}).$$

Then,

$$\begin{aligned}&\frac{\partial \boldsymbol{\Psi}(\boldsymbol{\lambda})}{\partial \lambda_i} - \frac{\partial \boldsymbol{\Psi}(\boldsymbol{\lambda})}{\partial \lambda_j} \overset{sign}{=} \\ &\left[1 - \wp^{-1}(u_j)(1 - \lambda_j(1-q)t^\beta)^{\frac{2-q}{1-q}}\right] \wp^{-1}(u_i)(1 - \lambda_i(1-q)t^\beta)^{\frac{1}{1-q}}(2-q)t^\beta \\ &- \left[1 - \wp^{-1}(u_i)(1 - \lambda_i(1-q)t^\beta)^{\frac{2-q}{1-q}}\right] \wp^{-1}(u_j)(1 - \lambda_j(1-q)t^\beta)^{\frac{1}{1-q}}(2-q)t^\beta \leq (\geq)0.\end{aligned}$$

Thus, $\partial \boldsymbol{\Psi}(\boldsymbol{\lambda})/\partial \lambda_k$ is increasing in $k = 1, 2, \ldots, n$. Since $\lambda_k \in \mathbb{D}_+$, it follows that $\boldsymbol{\Psi}(\boldsymbol{\lambda})$ is *Schur−concave* in $\boldsymbol{\lambda}$, by Lemma 2 (Lemma 3) of [28]. The desired results then follow from Lemma 2.4. □

https://doi.org/10.1017/S0269964826100199 Published online by Cambridge University Press

Theorem 5.3. *Let $U_i \sim q - W(q, \lambda_i, \beta)$ and $V_i \sim q - W(q, \lambda_i^\star, \beta)$, $i = 1, \ldots, n$. Suppose $I_i(I_i^\star)$ be a set of independent Bernoulli r.v.'s, independent of U_i's(V_i's) with $E(I_i) = p_i(E(I_i^\star) = p_i^\star), i = 1, 2, \ldots, n$. If $\prod_{i=1}^n p_i \le \prod_{i=1}^n p_i^\star$, then:*

(1) *For $0 < q < 1$, if $(\lambda_1, \lambda_2, \ldots, \lambda_n) \succeq_w (\lambda_1^\star, \lambda_2^\star, \ldots, \lambda_n^\star)$, then $U_{1:n}^\star \le_{hr} V_{1:n}^\star$,*
(2) *For $1 < q < 2$, if $(\lambda_1, \lambda_2, \ldots, \lambda_n) \preceq^w (\lambda_1^\star, \lambda_2^\star, \ldots, \lambda_n^\star)$, then $U_{1:n}^\star \le_{hr} V_{1:n}^\star$.*

Proof. To establish the result, it suffices to show that $\bar{F}_{V_{1:n}^\star}(t)/\bar{F}_{U_{1:n}^\star}(t)$ is increasing in $t \in [0, D)$ when $0 < q < 1$, and in $t \ge 0$ when $1 < q < 2$. We begin by expressing the ratio as follows:

$$\frac{\bar{F}_{V_{1:n}^\star}(t)}{\bar{F}_{U_{1:n}^\star}(t)} = \frac{\prod_{i=1}^n p_i^\star (1 - \lambda_i^\star (1-q)t^\beta)^{\frac{2-q}{1-q}}}{\prod_{i=1}^n p_i (1 - \lambda_i (1-q)t^\beta)^{\frac{2-q}{1-q}}}$$
$$= \frac{\prod_{i=1}^n p_i^\star \prod_{k=1}^n (1 - \lambda_k^\star (1-q)t^\beta)^{\frac{2-q}{1-q}}}{\prod_{i=1}^n p_i \prod_{k=1}^n (1 - \lambda_k (1-q)t^\beta)^{\frac{2-q}{1-q}}} = \frac{\prod_{i=1}^n p_i^\star \bar{F}_{V_{1:n}}(t)}{\prod_{i=1}^n p_i \bar{F}_{U_{1:n}}(t)}.$$

According to Theorem 3.1, if $0 < q < 1$ and $(\lambda_1, \lambda_2, \ldots, \lambda_n) \succeq_w (\lambda_1^\star, \lambda_2^\star, \ldots, \lambda_n^\star)$, then $U_{1:n} \le_{hr} V_{1:n}$. That is, the ratio $\bar{F}_{V_{1:n}}(t)/\bar{F}_{U_{1:n}}(t)$ is increasing, which implies that $\bar{F}_{V_{1:n}^\star}(t)/\bar{F}_{U_{1:n}^\star}(t)$ is increasing in $t \in (0, D)$.

Furthermore, using the fact that $\lim_{t \to 0^-} \bar{F}_{V_{1:n}^\star}(t)/\bar{F}_{U_{1:n}^\star}(t) = 1$, and noting that $1 \le \dfrac{\prod_{i=1}^n p_i^\star}{\prod_{i=1}^n p_i}$, we obtain

$$\lim_{x \to 0^-} \frac{\bar{F}_{V_{1:n}^\star}(t)}{\bar{F}_{U_{1:n}^\star}(t)} \le \frac{\prod_{i=1}^n p_i^\star \bar{F}_{V_{1:n}}(0)}{\prod_{i=1}^n p_i \bar{F}_{U_{1:n}}(0)} = \frac{\bar{F}_{V_{1:n}^\star}(0)}{\bar{F}_{U_{1:n}^\star}(0)}$$

showing that $\bar{F}_{V_{1:n}^\star}(t)/\bar{F}_{U_{1:n}^\star}(t)$ is increasing at $t = 0$. Hence, the desired result follows for $0 < q < 1$.

A similar argument applies to the case $1 < q < 2$. □

Theorem 5.4. *Let $U_i \sim q - W(q_i, \lambda, \beta)$ and $V_i \sim q - W(q_i^\star, \lambda, \beta)$, for $i = 1, 2, \ldots, n$. Suppose $I_i(I_i^\star)$ be a set of independent Bernoulli r.v.'s, independent of U_i's(V_i's) with $E(I_i) = p_i(E(I_i^\star) = p_i^\star), i = 1, 2, \ldots, n$. If $\prod_{i=1}^n p_i \le \prod_{i=1}^n p_i^\star$, then:*

(1) *For $0 < \max_l\{q_l, q_l^\star\} < 1$, if $(q_1, q_2, \ldots, q_n) \succeq^w (q_1^\star, q_2^\star, \ldots, q_n^\star)$ then $U_{1:n}^\star \le_{hr} V_{1:n}^\star$.*
(2) *For $1 < \min_l\{q_l, q_l^\star\} < 2$, if $(q_1, q_2, \ldots, q_n) \succeq^w (q_1^\star, q_2^\star, \ldots, q_n^\star)$, then $U_{1:n}^\star \le_{hr} V_{1:n}^\star$.*

Proof. The results can be proved using an approach similar to that of the previous theorem, along with the results from Theorem 3.4. □

Theorem 5.5. *Let $U_i \sim q - W(q, \lambda, \beta_i)$ and $V_i \sim q - W(q, \lambda, \beta_i^\star)$, $i = 1, \ldots, n$. Further, suppose $I_i(I_i^\star)$ be a set of Bernoulli r.v.'s, independent of U_i's(V_i's) with $E(I_i) = p_i(E(I_i^\star) = p_i^\star), i = 1, 2, \ldots, n$. If $\prod_{i=1}^n p_i \ge \prod_{i=1}^n p_i^\star$, then:*

(1) *For $0 < q < 1$, if $(\beta_1, \beta_2, \ldots, \beta_n) \overset{m}{\preceq} (\beta_1^\star, \beta_2^\star, \ldots, \beta_n^\star)$, then $V_{1:n}^\star \le_{st} U_{1:n}^\star$.*
(2) *For $1 < q < 2$, if $(\beta_1, \beta_2, \ldots, \beta_n) \overset{m}{\preceq} (\beta_1^\star, \beta_2^\star, \ldots, \beta_n^\star)$, then $V_{1:n}^\star \le_{st} U_{1:n}^\star$.*

https://doi.org/10.1017/S0269964826100199 Published online by Cambridge University Press

Proof. The survival function of $U^{\star}_{1:n}$ can be expressed as

$$\bar{F}_{U^{\star}_{1:n}}(t) = \prod_{i=1}^{n} p_i (1-\lambda(1-q)t^{\beta_i})^{\frac{2-q}{1-q}} = \prod_{i=1}^{n} p_i \prod_{k=1}^{n} (1-\lambda(1-q)t^{\beta_k})^{\frac{2-q}{1-q}} = \prod_{i=1}^{n} p_i \bar{F}_{U_{1:n}}(t).$$

Similarly, the survival function of $V^{\star}_{1:n}$ is given by

$$\bar{F}_{V^{\star}_{1:n}}(t) = \prod_{i=1}^{n} p^{\star}_i \bar{F}_{V_{1:n}}(t).$$

From Theorem 3.7, we know that $\bar{F}_{V_{1:n}}(t) \leq \bar{F}_{U_{1:n}}(t)$, whenever $(\beta_1, \beta_2, \ldots, \beta_n) \overset{m}{\preceq} (\beta^{\star}_1, \beta^{\star}_2, \ldots, \beta^{\star}_n)$. Moreover, since $\prod_{i=1}^{n} p_i \geq \prod_{i=1}^{n} p^{\star}_i$, it follows that $\bar{F}_{U^{\star}_{1:n}}(t) \geq \bar{F}_{V^{\star}_{1:n}}(t)$. Hence, the desired result follows. □

6. Conclusion

The q-Weibull distribution provides a flexible framework for modeling a wide range of hazard rate behaviors, including unimodal, bathtub-shaped, monotonic (increasing or decreasing), and constant patterns. In this paper, we study the stochastic comparison of extreme order statistics derived from heterogeneous q-Weibull distributions under both independent and dependent settings. For the dependent case, the dependence structure is modeled using Archimedean copulas. Furthermore, we examine the comparison of extreme order statistics when the r.v.'s are exposed to random shocks. Several examples and counterexamples are presented to illustrate the theoretical results.

An extension of the comparison results to general rth-order statistics with particular emphasis on the second-order statistic constitutes an important avenue for future work, and we are currently investigating this problem. Moreover, extending the results obtained under dependence to the frameworks of hazard rate order and reversed hazard rate order remains a challenging task and represents a promising direction for further research.

Acknowledgments. The authors would like to thank the Associate Editor and two reviewers for their constructive suggestions, which have led to improvements in the presentation of this article.

References

[1] Abdlahi, M., Parham, G.A., & Chinipardaz, R. (2023). Ordering results of the smallest order statistics from independent heterogeneous new modified generalized linear failure rate r.v.'s. *Communications in Statistics - Theory and Methods* 52(16): 5606–5639.

[2] Al-jabbar, A., Kelkinnama, M., & Mahmoodi, S. (2025). Stochastic comparisons of extremes with a generalization of Weibull distribution. *Statistics* 59(5): 1139–1160.

[3] Almalki, S.J., & Nadarajah, S. (2014). Modifications of the Weibull distribution: A review. *Reliability Engineering ans Syatem Safety* 124: 32–55.

[4] Assis, E.M., Borges, E.P., & Vieira de Melo, S.A. (2013). Generalized q-Weibull model and the bathtub curve. *International Journal of Quality & Reliability Management* 30(7): 720–736.

[5] Assis, E.M., Borges, E.P., Vieira de Melo, S.A. *et al*. (2015). Modeling failure rate of a robotic welding station using generalized $q-$distributions. *International Journal of Quality & Reliability Management* 32(2): 156–166.

[6] Assis, E.M., Lima, G.A.C., Prestes, A *et al*. (2019). q-Weibull applied to Brazilian hydropower equipment. *IEEE Transactions on Reliability* 68(1): 122–132.

[7] Balakrishnan, N., Triantafyllou, I.S., & Koutras, M.V. (2010). A distribution-free control chart based on order statistics. *Communications in Statistics - Theory and Methods* 39(20): 3652–3677.

[8] Balakrishnan, N., Zhang, Y., & Zhao, P. (2018). Ordering the largest claim amounts and ranges from two sets of heterogeneous portfolios. *Scandinavian Actuarial Journal* 2018(1): 23–41.

[9] Balakrishnan, N., & Zhao, P. (2013). Ordering properties of order statistics from heterogeneous populations: A review with an emphasis on some recent developments. *Probability in the Engineering and Informational Sciences* 27(4): 403–443.

https://doi.org/10.1017/S0269964826100199 Published online by Cambridge University Press

[10] Barlow, R.E., & Proschan, F. (1981). *Statistical theory of reliability and life testing: Probability models.* Silver Spring, MD: To Begin With.
[11] Barmalzan, G., & Najafabadi, A.T.P. (2015). On the convex transform and right-spread orders of smallest claim amounts. *Insurance: Mathematics and Economics* 64: 380–384.
[12] Barmalzan, G., Najafabadi, A.T.P., & Balakrishnan, N. (2017). Ordering properties of the smallest and largest claim amounts in a general scale model. *Scandinavian Actuarial Journal* 2017(2): 105–124.
[13] Belzunce, F., Mulero, J., & Riquelme, C.M. (2016). *An Introduction to Stochastic orders.* Amsterdam, The Netherlands: Elsevier/Academic Press.
[14] Chowdhury, S., & Kundu, A. (2017). Stochastic comparison of parallel systems with log-Lindley distributed components. *Operations Research Letters* 45(3): 199–205.
[15] Costa, U.M.S., Freire, V.N., Malacarne, L.C. *et al*. (2006). An improved description of the dielectric breakdown in oxides based on a generalized Weibull distribution. *Physica A* 361(1): 209–215.
[16] Das, S., Kayal, S., & Choudhuri, D. (2021). Ordering resulys on extremes of exponentiated location-scale models. *Probability in the Engineering and Informational Sciences* 35(2): 331–354.
[17] Fan L., Hu Z., & Ling Q. *et al*. (2022) Reliability analysis of computed tomography equipment using the $q-$Weibull distribution. Engineering Reports. 5: e12613.
[18] Fan, T., Wang, Y., & Ju, S. (2019). A competing risks model with multiply censored reliability data under multivariate Weibull distribution. *IEEE Transaction on Reliability* 68(2): 462–475.
[19] Fang, L., & Balakrishnan, N. (2016). Ordering results for the smallest and largest order statistics from independent heterogeneous exponential-Weibull random variables. *Statistics* 50(6): 1195–1205.
[20] Ferreira, R.S., & Silva, P.C.A. (2020). q-Weibull distributions describing commercial service routes. *Physica A* 541: 123649.
[21] Fuchs, S., & Tschimpke, M. (2023). Total positivity of copulas from a markov kernel perspective. *Journal of Mathematical Analysis and Applications* 518(1): 126629.
[22] Ghosh, S., Majumder, P., & Mitra, M. (2021). Stochastic comparisons of systems with heterogeneous log-logistic components. In A.K. Laha (ed), *Applied advanced analytics: 6th IIMA international conference on advanced data analysis, business analytics and intelligence*. Singapore: Springer, pp. 157–173.
[23] Jia, X. (2021). Reliability analysis for $q-$Weibull distribution with multiply Type-I censored data. *Quality and Reliability Engineering International* 37(6): 2790–2817.
[24] Jia, X., Nadarajah, S., & Guo, B. (2020). Inference on q-Weibull parameters. *Statistical Papers* 61: 575–593.
[25] Jose, K.K., Naik, S.R., & RistiĆ, M.M. (2010). Marshall-Olkin $q-$Weibull distribution and max-min processes. *Statistical Papers* 51: 837–851.
[26] Jose, K.K., & Naik, SR. (2009). On the $q-$Weibull distribution and its applications. *Communications in Statistics - Theory and Methods* 38(6): 912–926.
[27] Kundu, A., & Chowdhury, S. (2018). Ordering properties of sample minimum from Kumaraswamy-G random variables. *Statistics* 52(1): 133–146.
[28] Kundu, A., Chowdhury, S., & Balakrishnan, N. (2023). Ordering properties of the smallest and largest lifetimes in Gompertz–Makeham model. *Communications in Statistics - Theory and Methods* 52(3): 643–669.
[29] Li, C., & Li, X. (2021). On stochastic dependence in residual lifetime and inactivity time with some applications. *Statistics & Probability Letters* 177: 109120.
[30] Li, X., & Fang, R. (2015). Ordering properties of order statistics from random variables of Archimedean copulas with applications. *Journal of Multivariate Analysis* 133: 304–320.
[31] Majumder, P., Ghosh, S., & Mitra, M. (2020). Ordering results of extreme order statistics from heterogeneous Gompertz–Makeham random variables. *Statistics* 54(3): 595–617.
[32] Marshall, A.W., Olkin, I., & Arnold, B.C. (2011). *Inequalities: Theory of majorization and its applications.* New York: Springer series in Statistics.
[33] Martonosi, S.E., Gonzalez, M., & Oshiro, N. (2023). Predicting elite NBA lineups using individual player order statistics. *Journal of Quantitative Analysis in Sports* 19(2): 51–71.
[34] Mazzoccoli, A., & Naldi, M. (2020). The expected utility insurance premium principle with fourth-order statistics: Does it make a difference?. *Algorithms* 13(5): 116.
[35] Murthy, D.N.P., Xie, M., & Jiang, R. (2004). *Weibull models.* Hoboken, New Jersey: John Wiley & Sons.
[36] Nelsen, R.B. (2006). *Introduction to copulas.* New York: Springer.
[37] Oliveira, M., Vasconcelos, J., & Gomes-Silva, F. *et al*. (2025). Estimation and goodness-of-fit for the $q-$Weibull distribution via the Mellin transform. *Methodology and Computing in Applied Probability* 27: 38.
[38] Pham, H., & Lai, C.D. (2007). On recent generalizations of the Weibull distribution. *IEEE Transactions on Reliability* 56(3): 454–458.
[39] Picoli, S., Mendes, R.S., & Malacarne, L.C. (2003). q-exponential, Weibull and q-Weibull distributions: An emporical analysis. *Physica A* 324(3–4): 678–688.
[40] Rezapour, M., & Alamatsaz, M.H. (2014). Stochastic comparison of lifetimes of two $(n-k-1)$- out-of-n systems with heterogeneous dependent components. *Journal of Multivariate Analysis* 130: 240–251.
[41] Samanta, D., Gupta, A., & Kundu, D. (2019). Analysis of Weibull step-stress model in presence of competing risk. *IEEE Transaction on Reliability* 68(2): 420–438.

https://doi.org/10.1017/S0269964826100199 Published online by Cambridge University Press

[42] Samanta, R.J., Das, S., & Balakrishnan, N. (2024). Orderings of extremes among dependent extended Weibull random variables. *Probability in the Engineering and Informational Sciences* 38(4): 705–732.
[43] Sartori, I., Assis, E.M., & Silva, A.L. *et al*. (2009). Reliability modeling of a natural gas recovery plant using q-Weibull distribution. *Computer Aided Chemical Engineering* 27: 1797–1802.
[44] Shaked, M., & Shanthikumar, J. (2007). *Stochastic orders.* New York: Springer.
[45] Souza Vilela Podestá, T., Venzel Rosembach, T., Aparecida dos Santos, A. *et al*. (2017). Anomalous diffusion and $q-$Weibull velocity distributions in epithelial cell migration. *Plos one* 12(7): e0180777.
[46] Sura, P. (2011). A general perspective of extreme events in weather and climate. *Atmospheric Research* 101(1–2): 1–21.
[47] Tsallis, C. (1988). Possible generalization of Boltzmann-Gibbs statistics. *Journal of Statistical Physics* 52: 479–487.
[48] Tsallis, C. (2009). *Introduction to non-Extensive Statistical mechanics: Approaching a Complex world.* New York: Springer.
[49] Varghese, A.A., Mahmood, K.A., Sarkar, S. *et al*. (2025). On stochastic comparisons of α-mixture models with location-scale family of distributions. *Statistics* 57(5): 1278–1300. https://doi.org/10.1080/02331888.2025.2502946.
[50] Vuorenmaa, T. (2006). A q-Weibull autoregressive conditional duration model and threshold dependence. *Technical report, Discussion Paper 117*, 1–61, University of Helsinky.
[51] Vuorenmaa, T. (2009). A q-Weibull autoregressive conditional duration model with an application to nyse and hse data. *SSRN.* 1–41. http://dx.doi.org/10.2139/ssrn.1952550.
[52] Xu, M., Droguett, E.L., & Lins, I.D. *et al* (2017). On the $q-$Weibull distribution for reliability applications: an adaptive hybrid artificial bee colony algorithm for parameter estimation. *Reliability Engineering and System Safety* 158: 93–105.
[53] Xu, M., Herrmann, J.W., & Droguett, E.L. (2021). Modeling dependent series system with $q-$Weibull distribution and Clayton copula. *Applied Mathematical Modelling* 94: 117–138.
[54] Zhang, J., Yan, R., & Zhang, Y. (2023). Stochastic comparisons of largest claim amount from heterogeneous and dependent insurance portfolios. *Journal of Computational and Applied Mathematics* 431: 115265.
[55] Zhang, Y., Cai, X., Zhao, P. *et al*. (2019). Stochastic comparisons of parallel and series systems with heterogeneous resilience-scaled components. *Statistics* 53(2): 126–147.

Cite this article: Mahmood K A., Ghosh S. and Majumder P. (2026). Ordering of extreme order statistics among q-Weibull random variables. *Probability in the Engineering and Informational Sciences* 40(3): 416–439. https://doi.org/10.1017/S0269964826100199

https://doi.org/10.1017/S0269964826100199 Published online by Cambridge University Press