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Bratteli–Vershik models for Cantor minimal systems: applications to Toeplitz flows

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Abstract. We construct Bratteli–Vershik models for Toeplitz flows and characterize a class of properly ordered Bratteli diagrams corresponding to these flows. We use this result to extend by a novel approach—using basic theory of dimension groups—an interesting and non-trivial result about Toeplitz flows, first shown by Downarowicz. (Williams had previously obtained preliminary results in this direction.) The result states that to any Choquet simplex K, there exists a 0–1 Toeplitz flow (Y, ψ), so that the set of invariant probability measures of (Y, ψ) is affinely homeomorphic to K. Not only do we give a conceptually new proof of this result, we also show that we may choose (Y, ψ) to have zero entropy and to have full rational spectrum.

Furthermore, our Bratteli–Vershik model for a given Toeplitz flow explicitly exhibits the factor map onto the maximal equicontinuous (odometer) factor. We utilize this to give a simple proof of the existence of a uniquely ergodic 0–1 Toeplitz flow of zero entropy having a given odometer as its maximal equicontinuous factor and being strongly orbit equivalent to this factor. By the same token, we show the existence of 0–1 Toeplitz flows having the 2-odometer as their maximal equicontinuous factor, being strong orbit equivalent to the same, and assuming any entropy value in $[0, \ln 2)$.

Finally, we show by an explicit example, using Bratteli diagrams, that Toeplitz flows are not preserved under Kakutani equivalence (in fact, under inducing)—contrasting what is the case for substitution minimal systems. In fact, the example we exhibit is an induced system of a 0–1 Toeplitz flow which is conjugate to the Chacon substitution system, thus it is prime, i.e. it has no non-trivial factors.

The thrust of our paper is to demonstrate the relevance and usefulness of Bratteli–Vershik models and dimension group theory for the study of minimal symbolic systems. This is also exemplified in recent papers by Forrest and by Durand, Host and Skau, treating substitution minimal systems, and by papers by Boyle, Handelman and by Ormes.

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1. *Bratteli diagrams, dimension groups and Bratteli–Vershik models for Cantor minimal systems*

We give a brief survey of Bratteli diagrams and how they are employed to give models for Cantor minimal systems. We also state some basic facts about dimension groups and their relation to Bratteli diagrams and to Cantor minimal systems. For further details, the reader should consult [**8, 9, 11, 14**].

We let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ denote the natural numbers $\{1, 2, 3, \dots\}$, integers, rational numbers and real numbers, respectively. If M is a set, we let $|M|$ denote its cardinality, and if M is a subset of a topological space, we let $\overline{M}$ denote its closure.

1.1. *Topological dynamical systems and Cantor minimal systems.*

A *topological dynamical system* is a pair (X, ϕ), where X is a compact metric space and ϕ is a homeomorphism on X. We say that ϕ is *minimal*, if for any point $x \in X$ the *orbit* of x, $\mathrm{O}_\phi(x) = \{\phi^n(x) : n \in \mathbb{Z}\}$, is dense in X. If (X, ϕ) and (Y, ψ) are topological dynamical systems, we say that (Y, ψ) is an *extension* of (X, ϕ), or that (X, ϕ) is a *factor* of (Y, ψ), if there exists a continuous surjective map $F : Y \to X$ such that $F \circ \psi = \phi \circ F$. In particular, if F is injective, then (Y, ψ) and (X, ϕ) are *conjugate*. If (Y, ψ) is conjugate to either (X, ϕ) or (X, ϕ^{-1}), we say that (Y, ψ) and (X, ϕ) are *flip conjugate*.

We say that (X, ϕ) is a *Cantor minimal system* if ϕ is minimal and X is a Cantor set, i.e. X is totally disconnected without isolated points. All such X are homeomorphic, and they can also be characterized by the property of having a countable basis of clopen sets and having no isolated points.

Examples of Cantor minimal systems that we shall meet later will be minimal subshifts over a finite alphabet Σ. Specifically, let $S : \prod_{-\infty}^{\infty} \Sigma \to \prod_{-\infty}^{\infty} \Sigma$ denote the *shift* $(Sx)(n) = x(n + 1)$ for $x \in \prod_{-\infty}^{\infty} \Sigma$, where $\prod_{-\infty}^{\infty} \Sigma$ is given the product topology. Then (X, ϕ) is a minimal *subshift* over the alphabet Σ if $X \subseteq \prod_{-\infty}^{\infty} \Sigma$ is shift invariant and the restriction ϕ of S to X is minimal.

1.2. *Bratteli diagrams.*

Definition 1. A *Bratteli diagram* is an infinite directed graph (V, E), where V is the *vertex set* and E is the *edge set*. These sets are partitioned into non-empty disjoint finite sets $V = V_0 \cup V_1 \cup \cdots$ and $E = E_1 \cup E_2 \cup \cdots$, where $V_0 = \{v_0\}$ is a one-point set. There are two maps $r, s : E \to V$ such that $r(E_n) \subseteq V_n$ and $s(E_n) \subseteq V_{n-1}$ for $n \in \mathbb{N}$. Furthermore, $s^{-1}(v) \neq \varnothing$ for all $v \in V$ and $r^{-1}(v) \neq \varnothing$ for all $v \in V \setminus V_0$. We call r and s the *range map* and the *source map* of (V, E), respectively. We say that $u \in V_n$ is *connected* to $v \in V_{n+1}$ if there is an edge $e \in E_{n+1}$ such that $s(e) = u$ and $r(e) = v$.

If $V_n = \{u_1, \dots, u_k\}$ and $V_{n+1} = \{v_1, \dots, v_m\}$, we define an $(m \times k)$ matrix $A_n = (a_{ij})$, where a_{ij} is the number of edges connecting v_i to u_j. We call A_n the nth *incidence matrix* of (V, E).

It is convenient to give a diagrammatic presentation of the Bratteli diagram with V_n the vertices at (horizontal) level n and E_{n+1} the edges (downward directed) connecting the vertices at level n with those at level $n + 1$, as illustrated in Figure 1. The vertices of $V_n = \{u_1, \dots, u_k\}$ are placed in the given order from left to right.

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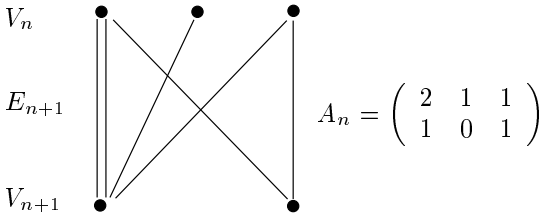


FIGURE 1. Two levels of a Bratteli diagram and the corresponding incidence matrix.

Given $e_k \in E_k$, $e_{k+1} \in E_{k+1}, \ldots, e_{k+m} \in E_{k+m}$ such that $r(e_i) = s(e_{i+1})$ for $i = k, k+1, \ldots, k+m-1$, we call the sequence $(e_k, \ldots, e_{k+m})$ a *path* (of length $m+1$) in (V, E) starting at $s(e_k) \in V_{k-1}$ and terminating at $r(e_{k+m}) \in V_{k+m}$. In the same manner, we define infinite paths $(e_1, e_2, e_3, \ldots)$, starting from the top vertex $v_0 \in V_0$. We say that two infinite paths $(e_1, e_2, \ldots)$ and $(f_1, f_2, \ldots)$ are *cofinal* if they agree from a certain level on, say k, i.e. $e_i = f_i$ for all $i > k$.

Let $\{n_k\}_{k=0}^{\infty}$ be a subsequence of $\{0, 1, 2, \ldots\}$, where we assume that $n_0 = 0$. We *telescope* (V, E) into a new Bratteli diagram (V', E') by letting $V'_k = V_{n_k}$ and letting $A'_k = A_{n_{k+1}-1} A_{n_{k+1}-2} \cdots A_{n_k}$ be the new incidence matrices. Notice that E', r' and s' are obtained in a natural way from the incidence matrices A'_k. Furthermore, we see that the edges E'_k from V'_{k-1} to V'_k correspond to the paths from $V_{n_{k-1}}$ to V_{n_k} in (V, E). *Microscoping* is the converse operation of telescoping. In other words, (V', E') is a microscoping of (V, E) if (V', E') can be telescoped into (V, E).

A Bratteli diagram is *simple* if it can be telescoped so that all the incidence matrices have only (strictly) positive entries. Equivalently, a Bratteli diagram (V, E) is simple if and only if for each $n \in \mathbb{N} \cup \{0\}$ there exists an $l_n \in \mathbb{N}$ such that there is at least one path from each vertex in V_n to each vertex in V_{n+l_n}.

1.3. *Ordered Bratteli diagrams.* For each $v \in V \setminus V_0$ we may give $r^{-1}(v)$ a linear ordering. This induces a partial order $\geq$ in E, and we obtain an *ordered Bratteli diagram*, which we denote by $(V, E, \geq)$. Let $E_{\max}$ and $E_{\min}$ denote the set of maximal and minimal edges, respectively. An infinite path is *maximal* (*minimal*) if all the edges making up the path are elements of $E_{\max}$ ($E_{\min}$). An ordered Bratteli diagram is *properly ordered* if it is simple and if it has a unique maximal and a unique minimal path, denoted respectively by $x_{\max}$ and $x_{\min}$. The partial order induces a (lexicographic) order on infinite cofinal paths. Specifically, if $x = (e_1, e_2, \ldots) \neq x_{\max}$, let k be the smallest integer such that $e_k \notin E_{\max}$. Let f_k be the successor of e_k (and so $r(e_k) = r(f_k)$). Then the *successor* of x is the path $(f_1, f_2, \ldots, f_k, e_{k+1}, e_{k+2}, \ldots)$, where $f_1, \ldots, f_{k-1} \in E_{\min}$. There is an obvious way to induce an order $\geq'$ on any telescoping (V', E') of the ordered Bratteli diagram $(V, E, \geq)$, namely by the (bottom-up) lexicographic order on paths from $V_{n_{k-1}} (= V'_{k-1})$ to $V_{n_k} (= V'_k)$ terminating at the same vertex. It is easily verified that $(V', E', \geq')$ will be properly ordered if $(V, E, \geq)$ is so.

It is fairly straightforward to show that if $(V, E, \geq)$ is properly ordered, then there exists a telescoped diagram with the following property [**14**, Proposition 2.8]: if e and e' are both

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in $E_{\max}$ ($E_{\min}$) with $r(e) = s(e')$, then e is in $x_{\max}$ ($x_{\min}$). In particular, the edges in $E_n \cap E_{\max}$ ($E_n \cap E_{\min}$) have the same source $v \in V_{n-1}$ ($u \in V_{n-1}$) for all $n \in \mathbb{N}$.

An ordered Bratteli diagram is said to be *left/right ordered* (or *vertex ordered*) if whenever $e_1 \geq e_2$ (hence necessarily $r(e_1) = r(e_2)$) and $s(e_1) = v_i$, $s(e_2) = v_j$, we have $i \geq j$. In other words, if two edges emanate from different points, the lesser one is the one further to the left. A left/right ordering is not preserved by telescoping, as is easily seen by examples.

1.4. *Bratteli–Vershik models for Cantor minimal systems.* Let X be the set of all (infinite) paths associated to a properly ordered Bratteli diagram $(V, E, \geq)$. A *cylinder set* consists of those (infinite) paths that agree from the top vertex v_0 to some $v \in V_n$, $n \in \mathbb{N}$. It is easily seen that if we topologize X by choosing as a basis of open sets the cylinder sets, then X becomes a Cantor set, and the cylinder sets will be clopen sets. Note that the topology is also obtained by the metric $d(x, y) = 1/(m+1)$, where the paths x, $y \in X$ agree to level m but not to level $m+1$. (We will tacitly assume that the Bratteli diagrams we consider are non-trivial, avoiding the cases where X is a finite set.)

Let $\phi : X \to X$ be the map that takes an infinite path to its successor in the lexicographic ordering. This is well defined for all paths except for the unique maximal path $x_{\max}$, and we define $\phi(x_{\max}) = x_{\min}$, the unique minimal path. It is straightforward to verify that ϕ, called the *Vershik map* associated to $(V, E, \geq)$, is a minimal homeomorphism. Hence (X, ϕ) is a Cantor minimal system which we call the *Bratteli–Vershik system* associated to $(V, E, \geq)$.

There is an obvious conjugacy between the Bratteli–Vershik systems associated respectively to $(V, E, \geq)$ and $(V', E', \geq')$, where $(V', E', \geq')$ is obtained from $(V, E, \geq)$ by telescoping.

It is a theorem [**14**, Theorem 4.7] that any Cantor minimal system (Y, ψ) is conjugate to a Bratteli–Vershik system, which we will call a *Bratteli–Vershik model* for (Y, ψ). We briefly recapitulate the construction. We start by choosing a decreasing sequence $\{C_n\}_{n=0}^{\infty}$ of clopen sets, where $C_0 = Y$, shrinking down to a base point y, and constructing towers over these determined by first returns. The floors of the towers over C_n will be a finite partition $\mathcal{P}_n$ of Y consisting of clopen sets, which we will call the *Kakutani–Rohlin partition* associated to C_n. The clopen set C_n will correspond to level n in the properly ordered Bratteli diagram $(V, E, \geq)$ we construct. Each tower over C_n corresponds to a vertex in V_n, and the way towers over C_{n+1} traverse towers over C_n determines the edges E_{n+1} and the orders on these. If a tower over C_n corresponds to the vertex $v \in V_n$, then the height of the tower equals the number of paths from the top vertex $v_0 \in V_0$ to v. In the general case, we may have to subdivide vertically the various towers to make sure that the Kakutani–Rohlin partition $\mathcal{P}_{n+1}$ is a refinement of $\mathcal{P}_n$ and that the floors of the towers generate the topology of the space. In our situation, such a subdivision will automatically be achieved by the way we construct the towers. By the above construction, the base point y will correspond to the unique minimal path $x_{\min}$ of the resulting Bratteli–Vershik system. (Note: in [**11, 14**], the roles of the base and top floors are reversed, so that there y will correspond to $x_{\max}$.) It is important that the Bratteli–Vershik system we obtain is the same up to conjugacy regardless of the choice of base point and the choice of

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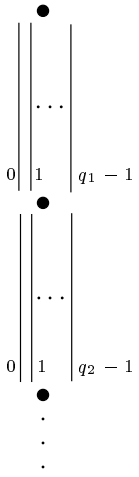


FIGURE 2. The Bratteli–Vershik model for an odometer system.

the sets shrinking down to the base point. This fact yields a great deal of freedom when we are going to perform the construction for concrete examples. A judicious choice of the shrinking sequence may hugely simplify the construction and yield a 'nice' Bratteli–Vershik model.

For the odometer system associated to the $\mathfrak{a}$-adic group $G_{\mathfrak{a}}$, $\mathfrak{a} = (q_i)_{i=1}^{\infty}$, which we shall encounter in §§2–4 we obtain a Bratteli–Vershik model with a single vertex at each level as shown in Figure 2. The ordering of the edges is indicated, starting with zero.

1.5. *Kakutani equivalence.* Let (X, ϕ) be a Cantor minimal system, and let $A \subseteq X$ be a clopen subset. Let $\phi_A : A \to A$ be the 'first return map,' i.e. if $x \in A$ then

$$\phi_A(x) = \phi^{r_A(x)}(x), \quad \text{where } r_A(x) = \inf\{m \in \mathbb{N} : \phi^m(x) \in A\}.$$

We call ϕ_A the *induced transformation* on A and say that (A, ϕ_A) is the *induced system* (or *derivative*) of (X, ϕ) with respect to A. We observe that (A, ϕ_A) is again a Cantor minimal system.

Definition 2. Let (X_1, ϕ_1) and (X_2, ϕ_2) be Cantor minimal systems. We say that the systems are *Kakutani equivalent* if they have a common derivative (up to conjugacy).

Kakutani equivalent systems have a succinct and elegant description in terms of Bratteli–Vershik systems. In fact, any system Kakutani equivalent to (X, ϕ) is obtained by making a finite change (i.e. adding and/or removing any finite number of edges and then making arbitrary choices of linear orderings of edges meeting at the same vertex for a finite number of vertices) to a Bratteli–Vershik model for (X, ϕ) [**11**, Theorem 3.8].

1.6. *From Bratteli–Vershik systems to minimal subshifts over a finite alphabet.* Recall that (X, ϕ) is *expansive* if there exists an $a > 0$ such that for every $x \neq y$

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$\sup_n d(\phi^n(x), \phi^n(y)) > a$. Here d is a metric yielding the topology of X. Expansiveness is independent of the metric d [**23**, Corollary 5.22.1]. (However, the *expansive constant* a depends on d.) It is a fact that the family of expansive Cantor minimal systems is the same, up to conjugacy, as the family of minimal subshifts over a finite alphabet (i.e. minimal *symbolic systems*). This follows easily from a well-known result for expansive minimal systems [**23**, Theorem 5.24].

Let (X, ϕ) be the Bratteli–Vershik system associated to the properly ordered Bratteli diagram $(V, E, \geq)$. For $k \geq 1$ let Σ_k be the set of paths from V_0 to V_k, i.e. the set of paths starting at $v_0 \in V_0$ and terminating at some $v \in V_k$. There is an obvious *truncation map* $\pi_k : X \to \Sigma_k$, obtained by restriction of paths to their initial segments of length k. To each $x \in X$ one associates a bisequence $\tilde{\pi}_k(x) = (\pi_k(\phi^n(x)))_n$ in $\Sigma_k^{\mathbb{Z}}$. The map $\tilde{\pi}_k$ is clearly continuous. Clearly, $\tilde{\pi}_k(\phi(x)) = S_k(\tilde{\pi}_k(x))$, where S_k denotes the shift. Hence we obtain that the subshift (Y_k, S_k) over Σ_k, where Y_k is the compact space $\tilde{\pi}_k(X)$, is a factor of (X, ϕ), the factor map being $\tilde{\pi}_k$. One verifies that (X, ϕ) is the inverse limit of $\{(Y_k, S_k)\}_{k \in \mathbb{N}}$.

If (X, ϕ) is expansive, it is easy to see that there exists a k such that the map $\tilde{\pi}_k : X \to Y_k$ is one-to-one. (The converse is also true.) Hence (X, ϕ) and (Y_k, S_k) are conjugate. By telescoping the diagram between level 0 and k, we may assume that $k = 1$.

1.7. *Basic facts about dimension groups.* To a Bratteli diagram (V, E) is associated a *dimension group* $G = K_0(V, E)$. In fact, to (V, E) is associated the system of (simplicialy) ordered groups (cf. Definition 1)

$$\mathbb{Z}^{|V_0|} \xrightarrow{A_0} \mathbb{Z}^{|V_1|} \xrightarrow{A_1} \mathbb{Z}^{|V_2|} \xrightarrow{A_2} \cdots,$$

where the order-preserving homomorphisms are given by matrix multiplication with the incidence matrices $\{A_i\}_{i=0}^{\infty}$, and $\mathbb{Z}^{|V_i|}$ denotes the set of $(|V_i| \times 1)$-column vectors over $\mathbb{Z}$. By definition, $K_0(V, E)$ is the direct limit of the system endowed with the induced order. The ordered group $K_0(V, E)$ has a distinguished *order unit*, namely the element of the positive cone $G^+ = K_0(V, E)^+$ corresponding to the element $1 \in \mathbb{Z}^{|V_0|} = \mathbb{Z}$. One can show that two dimension groups $K_0(V, E)$ and $K_0(V', E')$ are order isomorphic, by a map preserving distinguished order units, if and only if (V', E') is obtained from (V, E) by a finite succession of telescopings and microscopings. This gives us a great degree of leverage in choosing a suitable Bratteli diagram presentation of a given dimension group G. We will take advantage of this in the following.

It is a theorem by Effros, Handelman and Chen [**9**] that dimension groups may be abstractly characterized as countable, ordered abelian groups (G, G^+), where $G^+ = \{a \in G : a \geq 0\}$ is the positive cone, with the properties:

(i) (G, G^+) is *unperforated*, i.e. if $a \in G$ and $na \in G^+$ for some $n \in \mathbb{N}$, then $a \in G^+$. (Note that this implies that G is torsion free.)

(ii) (G, G^+) satisfies the *Riesz interpolation property*, i.e. given $a_1, a_2, b_1, b_2 \in G$ such that $a_i \leq b_j$ $(i, j = 1, 2)$, then there exists a $c \in G$ such that $a_i \leq c \leq b_j$ for all i, j.

A dimension group (G, G^+) is *simple* if G contains no non-trivial *order ideal*. (A subgroup J of G is an order ideal if $J = J^+ - J^+$ (where $J^+ = J \cap G^+$) and

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$0 \leq a \leq b \in J$ implies that $a \in J$.) It is a fact that the dimension group $K_0(V, E)$ associated to the Bratteli diagram (V, E) is simple if and only if (V, E) is a simple Bratteli diagram. Since we want to avoid trivial cases, we will tacitly assume that all the simple dimension groups we shall encounter are acyclic, i.e. not isomorphic to $\mathbb{Z}$.

If (G, G^+) is a simple dimension group, then any $u \in G^+ \setminus \{0\}$ is an *order unit*, i.e. $G^+ = \{a \in G : 0 \leq a \leq nu, \text{for some } n \in \mathbb{N}\}$. Fixing an order unit u, we say that a homomorphism $p : G \to \mathbb{R}$ is a *state* if p is positive (i.e. $p(g) \geq 0$ whenever $g \in G^+$) and $p(u) = 1$. Denote the collection of all states by $S_u(G)$, which is a compact, convex set. It is a fact that $S_u(G)$ determines the order on G. We have, namely,

$$G^+ = \{a \in G : p(a) > 0 \text{ for all } p \in S_u(G)\} \cup \{0\}.$$

If $\epsilon = p/q$, where $p, q \in \mathbb{N}$, then $a \leq \epsilon u$ means that $qa \leq pu$. The *infinitesimal subgroup* $\operatorname{Inf} G$ of G consists of those $a \in G$ such that $-\epsilon u \leq a \leq \epsilon u$ for all $0 < \epsilon \in \mathbb{Q}^+$. One verifies that

$$\operatorname{Inf} G = \{a \in G : p(a) = 0 \text{ for all } p \in S_u(G)\}.$$

It is easy to see that $\operatorname{Inf} G$ does not depend on which order unit u we choose. The quotient $G/\operatorname{Inf} G$ with the induced ordering is again a simple dimension group.

1.8. *Dimension groups associated to Cantor minimal systems and orbit equivalence.* The family of simple dimension groups coincides with the family

$$\{(K^0(Y, \psi), K^0(Y, \psi)^+) : (Y, \psi) \text{ a Cantor minimal system}\},$$

where $K^0(Y, \psi) = C(Y, \mathbb{Z})/\partial_\psi C(Y, \mathbb{Z})$, with the induced order from $C(Y, \mathbb{Z})$, and where $C(Y, \mathbb{Z})$ denotes the continuous functions on Y with values in $\mathbb{Z}$. Here

$$\partial_\psi : C(Y, \mathbb{Z}) \to C(Y, \mathbb{Z})$$

is the *coboundary map* defined by $\partial_\psi(f) = f - f \circ \psi^{-1}$. (There is a distinguished order unit $\mathbf{1}$ for $K^0(Y, \psi)$, corresponding to the constant function 1.) The proof of this follows immediately from the result about Bratteli–Vershik models for Cantor minimal systems, cf. §1.4. In fact, $K^0(Y, \psi)$ is order isomorphic to the dimension group $K_0(V, E)$, with $\mathbf{1}$ mapping to the distinguished order unit of $K_0(V, E)$, where $(V, E, \geq)$ is the properly ordered Bratteli diagram associated to (Y, ψ). For example, the K^0-group of the odometer (cf. §1.4) associated to the $\mathfrak{a}$-adic group $G_\mathfrak{a}$, $\mathfrak{a} = (q_i)_{i=1}^\infty$, is

$$\mathbb{Z}[1/\mathfrak{a}] \stackrel{\text{def}}{=} \{m/q_1 q_2 \cdots q_k : k \in \mathbb{N}, m \in \mathbb{Z}\} \subseteq \mathbb{Q},$$

with the obvious ordering and with distinguished order unit equal to one.

If (A, ψ_A) is an induced system of (Y, ψ), then $K^0(A, \psi_A) \cong K^0(Y, \psi)$ as ordered groups, but the distinguished order units are different. In fact, the distinguished order unit of $K^0(A, \psi_A)$ is $[\chi_A]$ in $K^0(Y, \psi)$. (Here χ_A denotes the characteristic function of A, and $[f]$ is the element of $K^0(Y, \psi)$ corresponding to $f \in C(Y, \mathbb{Z})$.) This fact is an immediate consequence of the Bratteli–Vershik description of Kakutani equivalence, cf. §1.5.

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Definition 3. The Cantor minimal systems (Y, ψ) and (X, ϕ) are *orbit equivalent* if there exists a homeomorphism $F : Y \to X$ such that $F(\mathrm{O}_\psi(y)) = \mathrm{O}_\phi(F(y))$ for all $y \in Y$. We call F an *orbit map*.

Let (Y, ψ), (X, ϕ), and F be as in Definition 3. For each point y in Y, there exists a unique integer $n(y)$ such that $F(\psi(y)) = \phi^{n(y)}(F(y))$. Likewise, there exists a unique integer $m(y)$ such that $F(\psi^{m(y)}(y)) = \phi(F(y))$. We call the maps $m, n : Y \to \mathbb{Z}$ the *orbit cocycles* associated to the orbit map F.

Definition 4. Let (Y, ψ) and (X, ϕ) be orbit equivalent Cantor minimal systems. We say that (Y, ψ) and (X, ϕ) are *strong orbit equivalent* if there exists an orbit map $F : Y \to X$ such that the associated orbit cocycles $m, n : Y \to \mathbb{Z}$ each have at most one point of discontinuity.

Strong orbit equivalence is a notion lying between orbit equivalence and flip conjugacy and is the mildest possible weakening of the latter [**11**, Theorem 1.4]. It is a theorem [**11**, Theorem 2.1] that (Y, ψ) and (X, ϕ) are strong orbit equivalent if and only if $K^0(Y, \psi) \cong K^0(X, \phi)$, as ordered groups with distinguished order units.

There is a bijective (affine) correspondence between the convex set $M(Y, \psi)$ of ψ-invariant probability measures on Y and the set of states of $(K^0(Y, \psi), K^0(Y, \psi)^+, \mathbf{1})$, given by the map

$$f \longrightarrow \int_Y f \, d\mu, \quad f \in C(Y, \mathbb{Z}),$$

where $\mu \in M(Y, \psi)$. So the K^0-group of (Y, ψ) gives complete information about the set of ψ-invariant probability measures on Y.

In particular, we note that if (Y, ψ) is uniquely ergodic with $M(Y, \psi) = \{\mu\}$, then $K^0(Y, \psi)/\mathrm{Inf}\, K^0(Y, \psi)$ is order isomorphic to the subgroup

$$\left\{ \int_Y f \, d\mu : f \in C(Y, \mathbb{Z}) \right\}$$

of $\mathbb{R}$, with the order inherited from $\mathbb{R}$. This linearly ordered group, with order unit 1, is a complete invariant for orbit equivalence for uniquely ergodic Cantor minimal systems [**11**, Theorem 2.2].

1.9. *Simple dimension groups and Choquet simplices.* Recall that a compact convex set, K, in a locally convex space is called a *Choquet simplex* if each $p \in K$ is the barycenter of a unique probability measure on the set of extreme points of K. The relation between simple dimension groups and Choquet simplices is rather intimate, and we refer to Effros [**9**] for a detailed discussion. We will just state the one result that we shall need in the following. First of all, if (G, G^+, u) is a simple dimension group with order unit u, then the state space $S_u(G)$ is a Choquet simplex, when $S_u(G) \subseteq \mathbb{R}^G$ is given the relative topology from $\mathbb{R}^G$ with the usual product topology.

THEOREM 5. *Let K be a Choquet simplex and let G be a countable, dense (in the uniform topology) subgroup of* $\mathrm{Aff}(K)$*, the additive group of real, affine, and continuous*

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functions on K. Then (G, G^+) is a simple dimension group without infinitesimal elements, where $G^+ = \{a \in G : a(k) > 0 \text{ for } k \in K\} \cup \{0\}$. Furthermore, if G contains the constant function $u = 1$, then K is affinely homeomorphic to $S_u(G)$ by the evaluation map $\hat{k}(a) = a(k)$, $a \in G$, $k \in K$.

2. *Bratteli–Vershik models for Toeplitz minimal flows*

Toeplitz minimal flows were first introduced by Jacobs and Keane [**16**]. Subsequently, Williams [**22**] contributed an important work on Toeplitz flows. We will make use of some of her notations and results. A *Toeplitz sequence* is a non-periodic element $x = (x(n))_{n=0}^{\infty}$ of $\prod_{-\infty}^{\infty} \Sigma$, where Σ is a finite alphabet (Σ could also be taken to be a compact metric space [**22**]) so that for each $m \in \mathbb{Z}$ there exists an $n \in \mathbb{N}$ such that $x(m) = x(m + kn)$ for all $k \in \mathbb{Z}$. We let $S : \prod_{-\infty}^{\infty} \Sigma \to \prod_{-\infty}^{\infty} \Sigma$ be the shift operator, $Sx(n) = x(n + 1)$, and we endow $Y = \prod_{-\infty}^{\infty} \Sigma$ with the product topology. Then a *Toeplitz flow* is the topological dynamical system $(Y, S) = (\overline{O_S(\eta)}, S)$, where η is a Toeplitz sequence and $O_S(\eta)$ is the orbit of η. It is known that a Toeplitz flow is minimal. If $x \in Y$, $p \in \mathbb{N}$ and $\sigma \in \Sigma$, we let

$$\mathrm{Per}_p(x, \sigma) = \{n \in \mathbb{Z} : x(m) = \sigma \text{ for all } m \equiv n \pmod p\},$$
$$\mathrm{Per}_p(x) = \bigcup_{\sigma \in \Sigma} \mathrm{Per}_p(x, \sigma).$$

The *p-skeleton* of x is the part of x which is periodic with period p. We say that p is an *essential* period of x if the p-skeleton of x is not periodic with any smaller period. The least common multiple of two essential periods is again an essential period, a fact which is easily verified. A *periodic structure* for a Toeplitz sequence η is a strictly increasing sequence $(p_i)_{i \in \mathbb{N}}$, such that p_i is an essential period of η for all i, $p_i | p_{i+1}$, and $\bigcup_{i=1}^{\infty} \mathrm{Per}_{p_i}(\eta) = \mathbb{Z}$. A periodic structure always exists for a Toeplitz sequence. In fact, order the essential periods, $r_1 < r_2 < r_3 < \cdots$, and let p_i be the least common multiple of $\{r_1, r_2, \ldots, r_i\}$. A periodic structure is obtained by deleting repeated terms from the sequence (p_i).

Now let $(p_i)_{i \in \mathbb{N}}$ be a periodic structure for the Toeplitz sequence η. Then the *odometer* or the *adding machine* ('add $(1, 0, 0, \ldots)$') associated to the $\mathfrak{a}$-adic group

$$G_{\mathfrak{a}} = \prod_{i=1}^{\infty} \{0, 1, \ldots, p_i/p_{i-1} - 1\},$$

where $\mathfrak{a} = (p_i/p_{i-1})_{i \in \mathbb{N}}$ (set $p_0 = 1$), is the maximal equicontinuous factor of the Toeplitz flow (Y, S) associated to η. (Note that $G_{\mathfrak{a}}$ is naturally isomorphic to the inverse limit group $\mathbb{Z}/p_1\mathbb{Z} \xleftarrow{\phi_1} \mathbb{Z}/p_2\mathbb{Z} \xleftarrow{\phi_2} \mathbb{Z}/p_3\mathbb{Z} \xleftarrow{\phi_3} \cdots$, where $\phi_i(n)$ is the residue of n modulo p_i.) If $\mathfrak{a} = (q_i)_{i \in \mathbb{N}}$, where $q_i = p$ for all i, we call $G_{\mathfrak{a}}$ the *p-adic group*, and the associated odometer is called the *p-odometer*. For detailed information on $\mathfrak{a}$-adic groups ('odometer groups'), we refer to Hewitt and Ross [**15**].

Furthermore, (Y, S) is an almost one-to-one extension of its maximal equicontinuous factor (in the literature sometimes called 'an automorphic flow'). Moreover, Toeplitz minimal systems can be characterized as expansive systems which are minimal almost one-to-one extensions of odometer systems [**5**].

For each $i \in \mathbb{N}$, $n \in \mathbb{Z}/p_i\mathbb{Z}$, let $A_n^i = \{S^m\eta : m \equiv n \pmod{p_i}\}$. Williams [**22**] showed that:

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(i) $\overline{A^i_n}$ is exactly the set of all $\omega \in \overline{O_S(\eta)}$ with the same p_i-skeleton as $S^n\eta$;

(ii) $\{\overline{A^i_n} : n \in \mathbb{Z}/p_i\mathbb{Z}\}$ is a partition of $\overline{O_S(\eta)}$ into clopen sets;

(iii) $\overline{A^i_m} \supset \overline{A^j_n}$ for $i < j$ and $m \equiv n \pmod{p_i}$;

(iv) $S\overline{A^i_n} = \overline{A^i_{n+1}}$;

(v) if $(p_n)_{n\in\mathbb{N}}$ is a periodic structure for η and $\bigcap_{i=1}^{\infty} \overline{A^i_{n_i}}$ (where $n_i \equiv n_j \pmod{p_i}$ for $j \geq i$) contains a Toeplitz sequence ω, then $\bigcap_{i=1}^{\infty} \overline{A^i_{n_i}} = \{\omega\}$. In particular, $\bigcap_{i=1}^{\infty} \overline{A^i_0} = \{\eta\}$. This implies that the factor map onto $G_{\mathfrak{a}}$ is one-to-one on the set of Toeplitz sequences in $Y = \overline{O_S(\eta)}$. (In general, we have that $x, y \in \bigcap_{i=1}^{\infty} \overline{A^i_{n_i}}$ if and only if x and y have the same p_i-skeleton for all $i \in \mathbb{N}$.)

For $x \in \prod_{-\infty}^{\infty} \Sigma$, we let $x[m, n]$ denote the subword $x(m)x(m+1)\cdots x(n)$ of x of length $n-m+1$. We let $L(\eta)$ denote the *language* associated to η, i.e. the set of all (finite) subwords of η.

Before we proceed with the proofs, we exhibit an example of a Toeplitz sequence whose periodic structure is particularly simple. The Toeplitz flow associated to this sequence will reappear later in this paper, and the Bratteli–Vershik model is shown in Figure 6 below. Consider the constant length substitution σ over the alphabet $\Sigma = \{a, b\}$ defined by $\sigma(a) = aab, \sigma(b) = abb$. The bisequence

$$\eta = \lim_n \overleftarrow{\sigma^n(b)} \,.\, \lim_n \overrightarrow{\sigma^n(a)} = \ldots aababbabb.aabaababb\ldots,$$

where the dot . is placed between the -1 and 0 coordinate, is easily seen to be a Toeplitz sequence with periodic structure $\{3, 3^2, 3^3, \ldots, 3^n, \ldots\}$. In fact, the 3^{n+1}-skeleton is generated by $\eta[-3^n, 3^n - 1] = \sigma^n(b).\sigma^n(a)$.

LEMMA 6. *Let η be a Toeplitz sequence. For k, $n \in \mathbb{N} \cup \{0\}$, there exists an essential period $p_{n,k}$ for η such that $\eta[n, n+k] = \eta[n + mp_{n,k}, n + k + mp_{n,k}]$ for all $m \in \mathbb{Z}$. Moreover, $p_{n,i} | p_{n,j}$, when $i \leq j$.*

Proof. Let $p_{n,0}$ be an essential period such that $\eta(n) = \eta(n + mp_{n,0})$ for all $m \in \mathbb{Z}$, which exists since η is a Toeplitz sequence. By induction, assume that we have constructed $p_{n,k}$. Let $p_{n,k+1}$ be the least common multiple of $p_{n,k}$ and $p_{n+k+1,0}$. The result follows. □

Definition 7. Let (V, E) be a simple Bratteli diagram. The map $P : V \setminus V_0 \to \mathbb{N}$ is defined by letting $P(v)$ be the number of paths from $v_0 \in V_0$ to v. If (V, E) is such that $P(u) = P(v)$ if $u, v \in V_n$, $n \geq 1$, we say that (V, E) has the *equal path number property*. An equivalent definition would be to say that for each $n \geq 1$, $|r^{-1}(u)| = |r^{-1}(v)|$ for every $u, v \in V_n$. (In other words, if A is any incidence matrix of (V, E) and A_i denote row i of A, then the sum of the elements of A_i equals the sum of the elements of A_j for every i, j.)

Observe that the equal path number property is preserved under telescoping.

THEOREM 8. *The family of expansive Bratteli–Vershik systems associated to Bratteli diagrams with the equal path number property coincides with the family of Toeplitz flows up to conjugacy.*

Proof. Suppose that (X, ϕ) is an expansive Bratteli–Vershik system with the equal path number property. By expansiveness, we may telescope the diagram such that the orbits

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of two distinct infinite paths x and y are separated between level 0 and 1, i.e. there exists an $n \in \mathbb{Z}$ such that $\phi^n(x) = (e_1, e_2, \dots)$ and $\phi^n(y) = (f_1, f_2, \dots)$, where $e_1 \neq f_1$. Furthermore, by telescoping, we may assume that all elements of $E_n \cap E_{\max}$ ($E_n \cap E_{\min}$) have the same source $u^{(n-1)} \in V_{n-1}$ ($v^{(n-1)} \in V_{n-1}$) for $n \in \mathbb{N}$ (cf. §1.3). It is easily verified that $\tilde{\pi}_1(x_{\min})$ is a Toeplitz sequence, and so $\tilde{\pi}_1$ is a conjugacy map between (X, ϕ) and a Toeplitz minimal flow, where $\tilde{\pi}_1$ is the map associated to the truncation map π_1 described in §1.6.

Now we assume that we have a Toeplitz flow $(\overline{O_S(\eta)}, S)$ over the alphabet Σ, and we want to construct a Bratteli–Vershik system with the equal path number property, which is conjugate to $(\overline{O_S(\eta)}, S)$. We will make use of the results about Toeplitz sequences summarized above, retaining the notation already introduced. We assume that $(q_i)_{i\in\mathbb{N}}$ is a periodic structure for η. Let $C_0 = Y (= \overline{O_S(\eta)})$. Let p_1 be the least common multiple of $p_{0,0}$ (recall the notation from Lemma 6) and q_1. We shall make p_1 the first element in a new periodic structure for η. Let W_1 be the collection of words in the language $L(\eta)$ associated to η of length p_1, beginning with $\eta(0)$. For each $w \in W_1$ we define

$$B_w^1 = \{x \in \overline{A_0^1} : x[0, p_1 - 1] = w\}.$$

Each $w \in W_1$ such that $B_w^1 \neq \varnothing$ gives rise to a tower over $C_1 = \overline{A_0^1}$ of height p_1, the floors being the clopen sets

$$\{S^j B_w^1 : j = 0, 1, \dots, p_1 - 1\}.$$

We obtain a Kakutani–Rohlin partition of Y,

$$\mathcal{P}_1 = \{S^j B_w^1 : w \in W_1;\ j = 0, 1, \dots, p_1 - 1\},$$

over the base $\overline{A_0^1}$. (Note that $(S^j B_w^1) \cap \overline{A_0^1} = \varnothing$ for $1 \leq j < p_1 - 1$ by (ii) and (iv) above.) By induction, let p_{n+1} be the least common multiple of p_{0,p_n-1} and q_{n+1}. (Notice that $\eta[0, p_n - 1]$ is part of the p_{n+1}-skeleton of η.) Set $C_{n+1} = \overline{A_0^{n+1}}$ and let W_{n+1} be the collection of words in $L(\eta)$ of length p_{n+1} beginning with $\eta[0, p_n - 1]$. Then as above we obtain a Kakutani–Rohlin partition:

$$\mathcal{P}_{n+1} = \{S^j B_w^{n+1} : w \in W_{n+1};\ j = 0, 1, \dots, p_{n+1} - 1\},$$

where $B_w^{n+1} = \{x \in \overline{A_0^{n+1}} : x[0, p_{n+1}-1] = w\}$. Also, $\mathcal{P}_{n+1}$ is a refinement of $\mathcal{P}_n$, and the heights of the various towers over $C_{n+1} = \overline{A_0^{n+1}}$ are all equal to p_{n+1}. Carrying on in this manner, we see that each p_i is an essential period of η and that $p_i | p_{i+1}$ for all i. (The latter follows because $p_{0,i} | p_{0,j}$ when $j > i$ (by Lemma 6) and $q_i | q_{i+1}$.) Furthermore, since (q_i) is a periodic structure for η and $q_i | p_i$ for all i, it follows that $\bigcup_{i=1}^{\infty} \mathrm{Per}_{p_i}(\eta) = \mathbb{Z}$, so (p_i) is a periodic structure for η. According to the construction outlined in §1.4 we obtain a properly ordered Bratteli diagram $(V, E, \geq)$ with the constant path number property, the number of paths from $v_0 \in V_0$ to $v \in V_n$ being p_n.

To verify that this diagram with its Vershik map models the Toeplitz system, we must show that $\bigcup_{n=1}^{\infty} \mathcal{P}_n$ generates the topology of $Y = \overline{O_S(\eta)}$. This amounts to showing that $\bigcup_{n=1}^{\infty} \mathcal{P}_n$ separates the points of Y. In other words, if $x \in \bigcap_{i=1}^{\infty} S^{j_i} B_{w_i}^i$, we must show

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that x is uniquely determined. Clearly, $x \in S^{j_i} B^i_{w_i}$, $0 \le j_i \le p_i - 1$, implies that $x(-j_i)$, $x(-j_i+1), \ldots, x(p_i - 1 - j_i)$ is uniquely determined.

There are three possibilities.

First, $\sup_i(j_i) = \infty$ and $\sup_i(p_i - j_i) = \infty$. In this case, $x(n)$ is determined for all $n \in \mathbb{Z}$ by the $S^{j_i} B^i_{w_i}$'s, and x is unique.

Second, $\sup_i(j_i) < \infty$. Then there exists an $N \in \mathbb{N}$ such that $j_i = N$ for all but finitely many i. (In fact, since $S^N B^i_{w_i} \subseteq \overline{A^i_N}$ by (iv), this is a consequence of (ii) and (iii).) It follows that $x(m) = \eta(m+N)$ for all $m \ge 0$. In other words, $(S^N\eta)(m) = x(m)$ for $m \ge 0$. This implies that $S^N\eta \in \bigcap_{i=1}^{\infty} S^{j_i} B_{w_i} \subseteq \bigcap_{i=1}^{\infty} \overline{A^i_{j_i}}$. As $S^N\eta$ is a Toeplitz sequence, we conclude from (v) that $x = S^N\eta$, and hence x is uniquely determined.

Third, $\sup_i(p_i - j_i) < \infty$. This case is treated similarly as the second case to conclude that x is uniquely determined, and the proof is complete. □

Recall that a substitution θ on a finite alphabet A is *proper* if there exist a $p \in \mathbb{N}$ and two symbols $r, l \in A$ such that for every $a \in A$, r is the last and l is the first symbol of $\sigma^p(a)$.

COROLLARY 9. *Substitution minimal systems associated to (primitive, aperiodic) proper substitutions of constant length are Toeplitz flows (up to conjugacy).*

Proof. It may be shown that the Bratteli diagram associated to a proper substitution of constant length is stationary and has the equal path number property [**8**, Proposition 16]. Combining this with Theorem 8, the result is obvious. □

Remarks. It is not true that substitution minimal systems associated to (primitive, aperiodic) substitutions of constant length are Toeplitz flows; the Morse–Thue substitution is an example. In fact, Martin [**19**, 6.03 Theorem] has given a general criterion to decide when a substitution of constant length gives rise to a Toeplitz flow.

Downarowicz and Lacroix [**5**] proved that (X, S) is a Toeplitz flow if and only if it is a minimal and expansive almost one-to-one extension of some odometer, which then is the maximal equicontinuous factor of (X, S). (This result was originally stated by Markley and Paul [**18**] without proof.) Our Bratteli–Vershik model gives us the odometer and the factor map right away by 'collapsing' the vertices at each level to one vertex. Figure 3 illustrates how this works. (An edge $e \in E_n$ with ordinal number k is mapped to the edge $f \in E'_n$ with ordinal number k (as illustrated with E_3 and E'_3 in the figure). This induces an obvious map on infinite paths, which, by an easy verification, is the factor map. Observe that the pre-image of the unique minimal path of the odometer is the unique minimal path of the 'Toeplitz diagram', which by construction corresponds to the Toeplitz sequence η.)

It is natural to ask whether it is necessary to assume that the Bratteli–Vershik system in Theorem 8 is expansive. In fact, if one excludes the special cases of odometer systems (which are not expansive), it could be that the equal path-number property implies that the system is expansive. However, this is not the case as may be seen from the example illustrated in Figure 4. The figure exhibits a properly ordered Bratteli diagram $(V, E, \ge)$, where $|V_n| = |V_{n-1}| + 2$ for all $n \ge 2$. Each vertex at level n is connected by exactly one edge to each vertex at level $n-1$. The edges are left/right ordered (cf. §1.3) with the exceptions noted in the figure.

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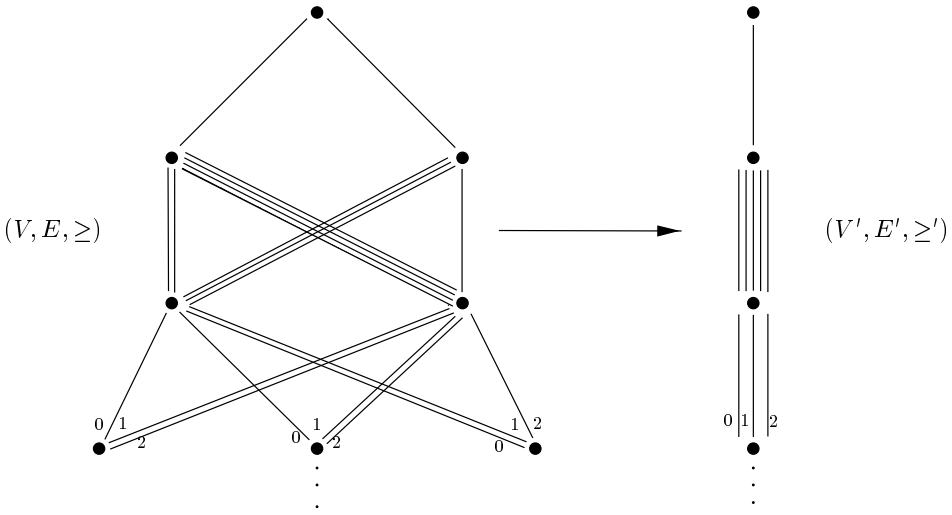


FIGURE 3. 'Collapsing' the Toeplitz system to obtain its maximal equicontinuous factor.

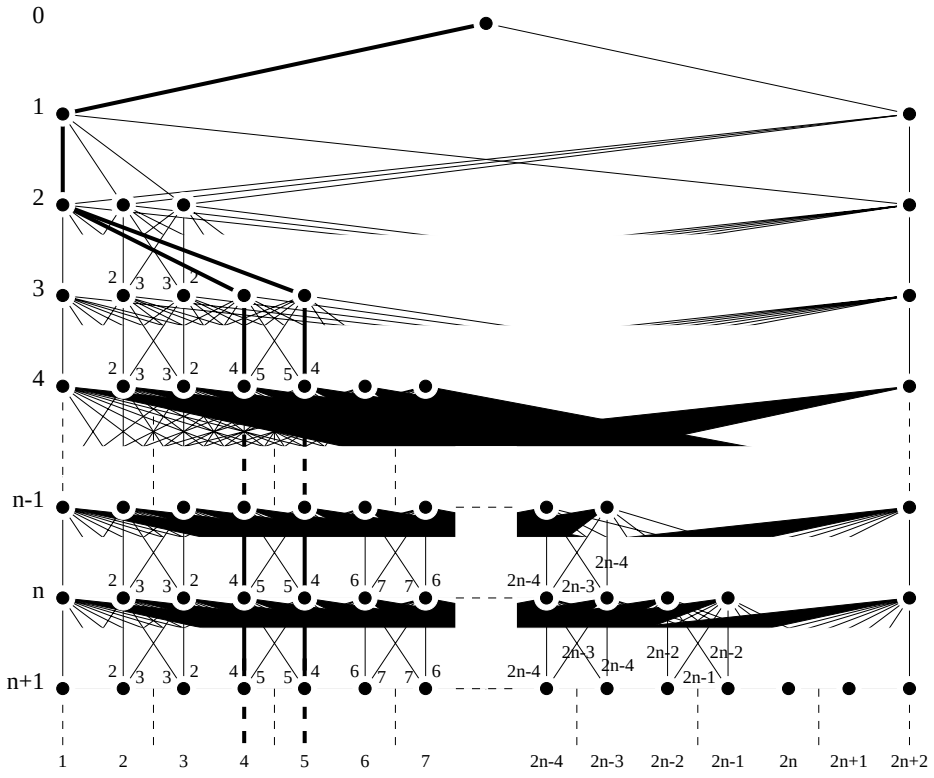


FIGURE 4. A 'Toeplitz-like' diagram yielding a non-expansive system which is not an odometer.

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The left-most vertex and the three right-most vertices have all left/right ordering of edges. Now the two paths drawn with thick lines in the figure cannot be separated above level two by (the same iteration of) the Vershik map. More generally, a path which follows the unique minimal path down to level n, and then passes through vertex number $2n$ for all $k > n$, cannot be separated above level n from another path which follows the unique minimal path down to level n, and then passes through vertex $2n+1$ for all $k > n$. It follows that this 'Toeplitz-like' diagram yields a Cantor minimal system which is not expansive, and therefore it cannot be a Toeplitz system. However, the example does exhibit an almost one-to-one extension of an odometer (the 2-odometer) with the factor map as explained previously. Furthermore, the system has a Toeplitz flow as an 'intermediate factor', which is obtained via one of the truncation maps of §1.6. In fact, not all the 'truncation factors' (Y_k, S_k) of §1.6 can be finite since the inverse limit of these is not an odometer. Hence one of them has to be Toeplitz. (Cf. the first part of the proof of Theorem 8.) It is easy to verify that the K^0-groups, as ordered groups with distinguished order units, of the 'Toeplitz-like' system and the odometer factor are isomorphic, both being isomorphic to the dyadic rationals $\mathbb{Z}[1/2]$, and so they are strong orbit equivalent. In particular, the 'Toeplitz-like' system is uniquely ergodic.

We show that the 'Toeplitz-like' system is not (conjugate to) an odometer system. To see this, assume that it is conjugate to an odometer system with a Bratteli–Vershik model $(V', E', \geq')$, where $|V'_i| = 1$ for all i. Then there exists an 'aggregate diagram' $(V'', E'', \geq'')$, i.e. $(V'', E'', \geq'')$ is a properly ordered diagram which when telescoped to odd levels yields a telescoping of $(V, E, \geq)$ and when telescoped to even levels yields a telescoping of $(V', E', \geq')$ [**11**, Theorem 3.6]. This implies that there exists a telescoping of $(V, E, \geq)$ having the property that edges ranging at the same level and with the same ordinal number have the same source. This is easily seen to be impossible.

Open problem. Let (V, E) be a simple Bratteli diagram with the equal path number property. Does there exist a Toeplitz flow (Y, ψ) such that $K^0(Y, \psi) \cong K_0(V, E)$ as ordered groups with distinguished order units?

If this were the case, we would have a Bratteli diagram 'presentation' of all K^0-groups that arise from Toeplitz flows. (We may assume without loss of generality that $|V_n| \geq 2$ for $n \geq 1$. The difficulty lies in giving (V, E)—or a telescoping of (V, E)—a proper ordering such that the associated Bratteli–Vershik system is expansive, and so is a Toeplitz flow according to Theorem 8. If the number of vertices $|V_n|$ at level n does not increase 'too fast' compared to the number of edges ranging at each $v \in V_n$, it can be done, cf. the first part of the proof of Lemma 10.)

It is noteworthy that there does exist an expansive Cantor minimal system (Z, θ) of zero entropy such that $K^0(Z, \theta) \cong K_0(V, E)$ [**2**]. The Bratteli diagram used to construct (Z, θ) does not necessarily satisfy the equal path number property however.

3. *Realizing Choquet simplices as sets of invariant probability measures of Toeplitz flows*

We will prove that any Choquet simplex may be obtained as the set of invariant probability measures of a Toeplitz flow, in fact, a 0–1 Toeplitz flow, i.e. with $\Sigma = \{0, 1\}$. This result has already been proved by Downarowicz, by a long and intricate argument [**3**]. We will use

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the Bratteli–Vershik model we have constructed for a Toeplitz flow to extend this result and show that it is a consequence of basic dimension group theory, specifically as highlighted in Theorem 2. The manipulations with Bratteli diagrams that are performed in the proof are fairly standard procedures.

We want to express our gratitude to David Handelman for providing us with a key idea for the proof of the theorem.

We shall need the following technical lemma. (Recall that π_1 is the truncation map to level 1 (cf. §1.6).)

LEMMA 10. *Let (V, E) be a simple Bratteli diagram such that all pairs of vertices at consecutive levels are connected by edges, i.e. all incidence matrices have strictly positive entries. Assume that $|V_k| \geq 2$ for $k \geq 1$. Let n_k be the minimum number of edges between a pair of vertices in V_k and V_{k+1}. If $n_k > 6|V_{k+1}|$ for all $k \geq 1$, then it is possible to give the diagram a proper ordering such that the corresponding Bratteli–Vershik system (X, ϕ) is expansive. Furthermore, for any two distinct paths $x, y \in X$ there is an $m \in \mathbb{Z}$ such that $\pi_1(\phi^m(x)) \neq \pi_1(\phi^m(y))$, in other words, x and y are separated between level 0 and level 1 by ϕ.*

If, moreover,

$$\lim_{k\to\infty} \frac{\ln(n_k|V_k|)}{n_k} = 0,$$

then (X, ϕ) has zero entropy.

Proof. Let $V_k = \{u_1, \ldots, u_s\}$, $V_{k+1} = \{v_1, \ldots, v_t\}$, where $k \in \mathbb{N}$. For $v_i \in V_{k+1}$ order the edges $r^{-1}(v_i)$ in the following manner. The first $\lfloor n_k/3 \rfloor + i$ edges shall have source at u_1. (Here $\lfloor l \rfloor$ denotes the greatest integer less than or equal to l.) The next $\lfloor n_k/2 \rfloor$ edges in the ordering shall have source at u_2. The remaining edges of $r^{-1}(v_i)$ are ordered left/right.

Endowing (V, E) with this ordering, we first observe that we obtain a properly ordered diagram. In fact, the unique minimal (maximal) path will pass through the left-most (right-most) vertex at each level. Furthermore, it is easily verified that if x and y are two paths that agree down to level k, but not down to level $k+1$, then we may find an $m \in \mathbb{Z}$ such that $\phi^m(x)$ and $\phi^m(y)$ do not agree down to level k for $k \geq 1$. In fact, they will pass through distinct vertices at level k, i.e. they will be separated at level k. We see this in the following manner. Assume x (y) passes through vertex v_i (v_j) at level $k+1$, where i and j may be equal. By applying (the same (negative) iteration of) the Vershik map ϕ to x and y, we may reduce the problem to the case that, say, $r(x) = v_i$, $s(x) = u_1$, and the edge e_{k+1} in E_{k+1}, which is part of $x = (e_1, e_2, \ldots)$, lies in $E_{\min}$, and that $r(y) = v_j$, $s(y) = u_1$. (Otherwise, x and y would already have been separated at level k by some (other) iteration of ϕ, and we would have been done.) Now a simple argument, treating a few cases, will show that by applying a suitable iteration of ϕ to both x and y they will separate at level k. By repeating this procedure, we eventually obtain that x and y will be separated at level 1 by ϕ. In particular, (X, ϕ) is expansive.

To see that (X, ϕ) has entropy 0, we refer the reader to a result by Boyle and Handelman [**1**, Proposition 2.2]. It is easy to see that their proof, which assumes left/right ordering, may be adapted to our setting. In fact, our ordering has been chosen so that any $e \in E_{k+1}$

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lies in a subset (of E_{k+1}) consisting of consecutively ordered edges with the same source and with cardinality at least $\lfloor n_k/3 \rfloor$. □

Remark 11. For a Bratteli diagram (V, E) where $|V_k|$ is bounded, the last condition of Lemma 10 is redundant because we may telescope the diagram such that $(n_k)_{k\geq 1}$ becomes an increasing sequence.

THEOREM 12. *Let K be a Choquet simplex. There exists a Toeplitz flow (Y, ψ) of zero entropy over a two symbol alphabet (i.e. a 0–1 Toeplitz flow) such that K is affinely homeomorphic to the set $M(Y, \psi)$ of ψ-invariant probability measures. Furthermore, we may choose (Y, ψ) so that the (unique) maximal equicontinuous factor is the odometer (X, ϕ) associated to the $\mathfrak{a}$-group $G_{\mathfrak{a}}$, where $\mathfrak{a} = (2, 3, 4, 5, \dots)$. Hence $K^0(X, \phi) \cong \mathbb{Q}$ as ordered groups with distinguished order units, and the set of (topological) eigenvalues of (Y, ψ) is $\{e^{2\pi i r} : r \in \mathbb{Q}\}$.*

Proof. Choose a countable, dense subgroup G of $\mathrm{Aff}(K)$, the additive group of affine continuous functions on K, such that G is a vector space over $\mathbb{Q}$ containing the constant function $u = 1$. So G is a divisible group. By Theorem 5, K is affinely homeomorphic to $S_u(G)$.

We will first show that G has a Bratteli diagram presentation which has the equal path number property. In fact, let (V', E') be a simple Bratteli diagram such that $G \cong K_0(V', E')$ (cf. §1.7), where $\cong$ denotes order isomorphism preserving the distinguished order units. Hence G is the ordered direct limit of the system

$$\mathbb{Z} \xrightarrow{A_0} \mathbb{Z}^{|V_1'|} \xrightarrow{A_1} \mathbb{Z}^{|V_2'|} \xrightarrow{A_2} \cdots,$$

where we assume that all the incidence matrices $(A_n)_{n=0}^{\infty}$ have strictly positive entries. We may also assume that $|V_n'| \geq 2$ for $n \geq 1$ and that $|V_1'| = 2$ with a single edge connecting each of the vertices in V_1' with $v_0 \in V_0'$. (This is easily obtained by the procedure described in §1.7.) Since G is divisible, it is clear that G is also the ordered inductive limit of the system

$$\mathbb{Q} \xrightarrow{A_0} \mathbb{Q}^{|V_1'|} \xrightarrow{A_1} \mathbb{Q}^{|V_2'|} \xrightarrow{A_2} \cdots.$$

Let $A_1(1, \dots, 1)^{\mathrm{tr}} = (\alpha_1, \alpha_2, \dots, \alpha_{|V_2'|})^{\mathrm{tr}}$, where tr denotes transposition. Then $J_2' A_1(1, 1, \dots, 1)^{\mathrm{tr}} = (1, 1, \dots, 1)^{\mathrm{tr}}$, where J_2' is the diagonal $(|V_2'| \times |V_2'|)$ matrix

$$\mathrm{diag}(\alpha_1^{-1}, \alpha_2^{-1}, \dots, \alpha_{|V_2'|}^{-1}).$$

Let m_2 be the least common multiple of the denominators of the entries of $J_2' A_1$, and let $k_2 = 2m_2 l_2$, where l_2 is chosen so that the conditions of Lemma 10 will be satisfied between levels 1 and 2, with the incidence matrix B_1 defined below. Set $J_2 = k_2 J_2'$. Then all entries of $J_2 A_1$ are in $\mathbb{N}$ and are divisible by $2l_2$. Also $J_2 A_1(1, \dots, 1)^{\mathrm{tr}} = (k_2, k_2, \dots, k_2)^{\mathrm{tr}}$. Assume we have constructed $J_2, J_3, \dots, J_{n-1}$. Let J_n' be the appropriate diagonal invertible matrix over $\mathbb{Q}_+$ such that $J_n' A_{n-1} J_{n-1}^{-1}(1, 1, \dots, 1)^{\mathrm{tr}} = (1, 1, \dots, 1)^{\mathrm{tr}}$. Let m_n be the least common multiple of the denominators of the entries of $J_n' A_{n-1} J_{n-1}^{-1}$, and let $k_n = n m_n l_n$, where l_n is chosen such that the conditions of Lemma 10 will be

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satisfied between levels $n-1$ and n, with the incidence matrix B_{n-1} defined below. Set $J_n = k_n J'_n$. Then all entries of $J_n A_{n-1} J_{n-1}^{-1}$ are in $\mathbb{N}$ and are divisible by nl_n. Also, $J_n A_{n-1} J_{n-1}^{-1}(1, 1, \ldots, 1)^{\mathrm{tr}} = (k_n, k_n, \ldots, k_n)^{\mathrm{tr}}$. Observe that n is a divisor of k_n. Letting $B_{n-1} = J_n A_{n-1} J_{n-1}^{-1}$ for $n \geq 1$, where we set $J_0 = \mathrm{id}$, $J_1 = \mathrm{id}$, we obtain a commutative diagram:

$$\begin{array}{ccccccc} G: \mathbb{Q} & \xrightarrow{A_0} & \mathbb{Q}^{|V'_1|} & \xrightarrow{A_1} & \mathbb{Q}^{|V'_2|} & \xrightarrow{A_2} & \cdots \\ {\scriptstyle J_0=\mathrm{id}}\downarrow & & {\scriptstyle J_1=\mathrm{id}}\downarrow & & {\scriptstyle J_2}\downarrow & & \\ H: \mathbb{Q} & \xrightarrow{B_0} & \mathbb{Q}^{|V'_1|} & \xrightarrow{B_1} & \mathbb{Q}^{|V'_2|} & \xrightarrow{B_2} & \cdots, \end{array}$$

which establishes an order isomorphism between the two ordered groups G and H associated to the two systems. Now H is order isomorphic to the dimension group associated to the system

$$\mathbb{Z} \xrightarrow{B_0} \mathbb{Z}^{|V'_1|} \xrightarrow{B_1} \mathbb{Z}^{|V'_2|} \xrightarrow{B_2} \cdots, \tag{$\dagger$}$$

since the latter system yields a divisible group because n is a divisor of each entry of B_n. Hence we conclude that $G \cong K_0(V, E)$, where (V, E) is the Bratteli diagram associated to ($\dagger$). Since $B_n(1, 1, \ldots, 1)^{\mathrm{tr}} = (k_n, k_n, \ldots, k_n)$, we see that (V, E) has the equal path number property.

By construction, we have ensured that (V, E) satisfies the conditions of Lemma 10. (In fact, this is achieved by letting $l_n > \max\{|V'_{n-1}|, 6|V'_n|\}$.) Combining this with Theorem 8, we conclude that there exists a proper ordering of (V, E) such that the associated Bratteli–Vershik system (Y, ψ) satisfies the first part of the theorem (cf. also §1.8).

Now the maximal equicontinuous factor (X, ϕ) of (Y, ψ) is the odometer associated to the $\mathfrak{a}$-adic group $G_{\mathfrak{a}}$, with $\mathfrak{a} = (k_n)_{n=1}^{\infty}$ (cf. the remark subsequent to Theorem 8). Recall that the odometers associated to the $\mathfrak{a}$-adic groups $G_{\mathfrak{a}_1}$ and $G_{\mathfrak{a}_2}$, where $\mathfrak{a}_1 = (p_i)_{i=1}^{\infty}$ and $\mathfrak{a}_2 = (\tilde{p}_i)_{i=1}^{\infty}$, are conjugate if and only if for each n there exists an m such that $\prod_{i=1}^{n} p_i$ is a divisor of $\prod_{i=1}^{m} \tilde{p}_i$, and the symmetrical condition (switching the p_i's and the $\tilde{p}_i$'s) also holds. Since n is a divisor of k_n for each n, we thus obtain the maximal equicontinuous factor as claimed.

The last part of the theorem is a consequence of the fact that the (topological) eigenvalues all come from the maximal equicontinuous factor. □

COROLLARY 13. *Let G be a simple dimension group which is divisible. Then there exists a Toeplitz flow (Y, ψ) of zero entropy over a two-symbol space such that $K^0(Y, \psi) \cong G$.*

Proof. This is an immediate consequence of the proof of the theorem. □

Remark. It is natural to ask whether a uniquely ergodic Toeplitz flow (Y, ψ) is orbit equivalent to its maximal equicontinuous factor (X, ϕ). For this to be the case, we must show that the set of values

$$\left\{ \int_Y f \, d\mu : f \in C(Y, \mathbb{Z}) \right\}, \tag{$\ddagger$}$$

where μ is the unique ψ-invariant probability measure, consists of rational numbers, cf. §1.8. However, this is not true in general. In fact, let $G = \mathbb{Q} + \mathbb{Q}\alpha (\subset \mathbb{R})$, where α is

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irrational. Then G is a linearly ordered and divisible dimension group with order unit 1. In particular, G has a unique state. By Corollary 13 there is a Toeplitz flow (Y, ψ) such that $K^0(Y, \psi) \cong G$ as ordered groups with distinguished order units. Clearly (Y, ψ) is uniquely ergodic and the set of values (‡) will be $\mathbb{Q} + \mathbb{Q}\alpha$, containing irrational numbers.

4. *Miscellaneous applications of dimension groups and Bratteli diagram techniques to Toeplitz flows*

4.1. *Toeplitz flows and their maximal equicontinuous factors.*

It is noteworthy that the maximal equicontinuous factor (X, ϕ) of a Toeplitz flow (Y, ψ) can be detected from $K^0(Y, \psi)$ (with distinguished order unit). In fact, the *rational subdimension group* $\mathbb{Q}(G)$ of $G = K^0(Y, \psi)$, where

$$\mathbb{Q}(G) = \{g \in G : ng = m\mathbf{1}, n \in \mathbb{N}, m \in \mathbb{Z}\},$$

is order isomorphic to a subgroup of $\mathbb{Q}$, via the map $g \in \mathbb{Q}(G) \to m/n \in \mathbb{Q}$, where $ng = m\mathbf{1}$. It turns out that $K^0(X, \phi) \cong \mathbb{Q}(G)$, and this determines (X, ϕ) uniquely, cf. [**11**, Remark to Theorem 2.4], [**20**, Theorem 2.2]. Furthermore, one can also show that $G/\mathbb{Q}(G)$ is torsion free [**12**].

PROPOSITION 14. *Two strongly orbit equivalent Toeplitz flows have the same maximal equicontinuous factor.*

Proof. As pointed out in §1.8, the K^0-group is a complete invariant for strong orbit equivalence. Since the maximal equicontinuous factor can be detected from the K^0-group, the result follows. □

We now state a theorem of which the first assertion is known [**21**, Theorem 3.1], [**7**, Section 3]. Our technique provides us with an easy proof—illustrating a feature of the Bratteli diagram approach.

THEOREM 15. *Let (X, ϕ) be the odometer associated to the $\mathfrak{a}$-adic group $G_\mathfrak{a}$, where $\mathfrak{a} = (q_i)_{i=1}^\infty$. There exists a uniquely ergodic* 0–1 *Toeplitz flow (Y, ψ) of zero entropy such that (X, ϕ) is the maximal equicontinuous factor of (Y, ψ).*

Furthermore, we may choose (Y, ψ) such that $K^0(Y, \psi) \cong K^0(X, \phi)$ as ordered groups with distinguished order units—hence (Y, ψ) and (X, ϕ) are strong orbit equivalent.

If (X, ϕ) is associated to a stationary *$\mathfrak{a}$-adic group, i.e. $q_k = q_{k+1} = q_{k+2} = \cdots$ from a certain k on, we may choose (Y, ψ) such that it is also a substitution minimal system.*

Proof. The Bratteli–Vershik model for the odometer is exhibited in Figure 2. By telescoping the diagram, we may assume that each q_i, $i \in \mathbb{N}$, can be written as $q_i = r_i + s_i$, where $r_i \to \infty$ and $s_i \to \infty$. Now we microscope and telescope (in that order) the Bratteli diagram (V, E) of the odometer as shown in Figure 5. We see that the right-most diagram $(\overline{V}, \overline{E})$ in Figure 5 satisfies the equal path number property. As $|\overline{V}_k| = 2$, by Lemma 10, its subsequent remark and Theorem 8 we may give $(\overline{V}, \overline{E})$ a proper ordering such that the associated Bratteli–Vershik system (Y, ψ) is a Toeplitz flow of zero entropy over a two-symbol alphabet. According to the remarks subsequent to Theorem 8, the maximal equicontinuous factor (X, ϕ) is as claimed. Now

$$K^0(Y, \psi) \cong K_0(\overline{V}, \overline{E}) \cong K_0(V, E) \cong K^0(X, \phi)$$

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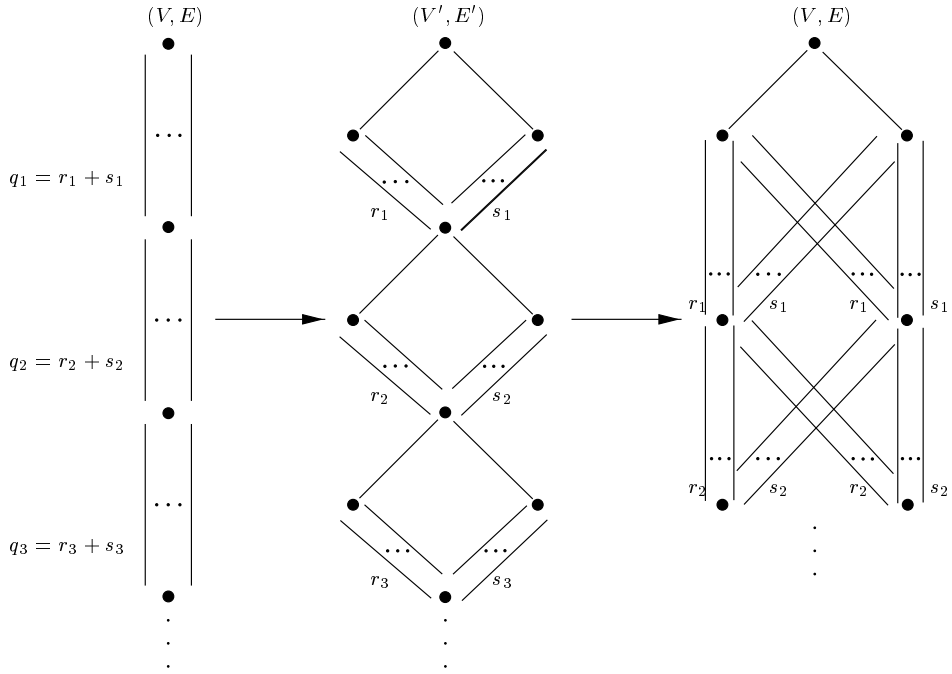


FIGURE 5. Diagram manipulations as described in the proof of Theorem 15.

as ordered groups with distinguished order units (cf. §§1.7 and 1.8). In particular, (Y, ψ) is uniquely ergodic since (X, ϕ) is so (cf. §1.8).

If the $\mathfrak{a}$-adic group is stationary, we may clearly do the construction in Figure 5 such that $(\overline{V}, \overline{E})$ with the imposed ordering is stationary, i.e. from a certain level on, the diagram with the ordering repeats. Then it is known that the associated Bratteli–Vershik system is substitution minimal [**8, 10**]. □

Remarks. It has recently been shown that there is an uncountable number of non-conjugate 0–1 Toeplitz flows of zero entropy with the same maximal equicontinuous factor [**7**]. (In fact, the result is even stronger, in that any entropy value $t \in [0, \log 2)$ can be preset. See also our Theorem 16.)

We would like to point out that by our approach we could obtain a similar result, since there is an algorithmic way to determine when two Bratteli–Vershik systems are *pointedly* conjugate [**11**, Theorem 3.6]. In our context, this means that we can decide when two Toeplitz flows are conjugate by a map which is over the zero element of the maximal equicontinuous factor (cf. [**7**] for terminology). The immense leverage we have in choosing the diagrams above, makes it possible to find a 'lot of' non-(pointedly)conjugate Toeplitz flows.

Let $\pi : (Y, \psi) \to (X, \phi)$ be the factor map, where (Y, ψ) and (X, ϕ) are as in Theorem 15. It may be shown [**13**, Proposition 3.1] that π induces a canonical embedding π^* of $K^0(X, \phi)$ as a subdimension group of $K^0(Y, \psi)$. In our case, it is easy to show that

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$\pi^*(K^0(X,\phi)) = K^0(Y,\psi)$. In fact, one shows that $\pi^*(K^0(X,\phi)) = \mathbb{Q}(K^0(Y,\psi)) = K^0(Y,\psi)$ $(= \mathbb{Z}[1/\mathfrak{a}])$.

Boyle and Handelman [**1**] showed, using Bratteli diagrams, that there exist Cantor minimal systems which are strong orbit equivalent to the 2-odometer, and which assume every entropy value in $[0,\infty]$. By examining their proof of this result, we realized that it could be adapted in order to obtain the following theorem.

THEOREM 16. *For any $0 \le t < \infty$ there exists a uniquely ergodic Toeplitz flow (Y,ψ) with entropy equal to t, and with maximal equicontinuous factor the 2-odometer. Furthermore, (Y,ψ) is strong orbit equivalent to the 2-odometer. In particular, if $0 \le t < \log 2$, we may choose (Y,ψ) to be a 0–1 flow.*

Proof. In the proof of [**1**, Theorem 2.8], Boyle and Handelman construct a Bratteli diagram (V,E) with the equal path number property in such a way that $K_0(V,E) \cong \mathbb{Z}[1/2]$ as ordered groups with distinguished order units. They introduce a proper ordering of (V,E) such that the associated Bratteli–Vershik system (X,ϕ) has entropy equal to t. It is not clear that (X,ϕ) is expansive (and so is a Toeplitz flow). However, they do show that the 'truncated system' (Y_1,S_1) obtained via the map $\tilde{\pi}_1$ (cf. §1.6) has entropy t (cf. the proof of [**1**, Lemma 2.6]). Barring the zero entropy case $t = 0$, which has already been treated in Theorem 15, we conclude that (Y_1,S_1) is an aperiodic minimal subshift. Hence (Y_1,S_1) is a Toeplitz flow (cf. the first part of the proof of Theorem 8). Set $(Y,\psi) = (Y_1,S_1)$. By the second remark subsequent to Theorem 15, there is an embedding $(\tilde{\pi}_1)_*$ of $K^0(Y,\psi)$ as a subdimension group of $K^0(X,\phi)$ preserving the distinguished order units. Now $K^0(X,\phi) \cong K_0(V,E) \cong \mathbb{Z}[1/2]$, and it is easily verified that $\mathbb{Z}[1/2]$ has no subdimension group (containing 1) except $\mathbb{Z}$ or $\mathbb{Z}[1/2]$. We must thus have $K^0(Y,\psi) \cong \mathbb{Z}[1/2]$ as ordered groups with distinguished order units. Hence the 2-odometer is the maximal equicontinuous factor of (Y,ψ), by the remarks at the very beginning of this section. Also, (Y,ψ) is strong orbit equivalent to the 2-odometer, since the associated K^0-groups are order isomorphic.

Finally, if $0 \le t < \log 2$, Boyle and Handelman show that one may choose the number of vertices at level 1 of the Bratteli diagram (V,E) to be two, with a single edge connecting each of these to the top vertex. Hence the (Y_1,S_1) above is a subshift over a two-symbol alphabet. □

Remark. If one substitutes p-odometer for 2-odometer in the theorem, where p is a prime number, one obtains analogous results.

4.2. *Kakutani equivalence—an example.* If (X,ϕ) is Kakutani equivalent to a substitution minimal system (Y,ψ) (arising from a primitive, aperiodic substitution) then (X,ϕ) is again substitution minimal [**8, 10**]. One may ask whether the analogous result is true for Toeplitz flows. The answer is a resounding no, as the following remarkable example will show. In fact, we shall exhibit an example of an induced system (X,ϕ) of a Cantor minimal system (Y,ψ), the latter being simultaneously Toeplitz and substitution minimal (associated to a constant length substitution), such that (X,ϕ) is *prime*, i.e. (X,ϕ) has no non-trivial factors. In particular, (X,ϕ) can neither be Toeplitz nor can (X,ϕ) be

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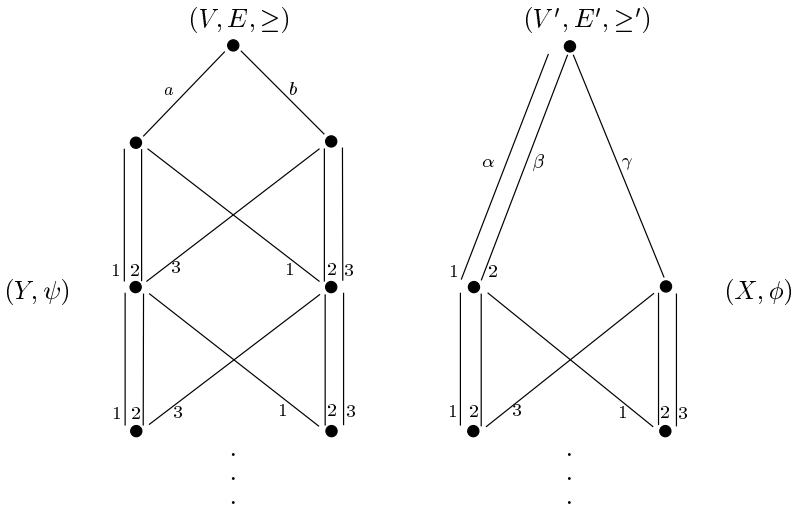


FIGURE 6. A Toeplitz system and a Kakutani equivalent non-Toeplitz system.

associated to a substitution of constant length, since such systems have odometer factors. In fact, we shall show that (X, ϕ) is conjugate to the *Chacon substitution system*, i.e. the substitution minimal system associated to the *Chacon substitution* $0 \to 0010, 1 \to 1$. The latter system is well known to be prime.

Specifically, let (Y, ψ) be the Bratteli–Vershik system associated to the properly ordered Bratteli diagram $(V, E, \geq)$ exhibited to the left in Figure 6. Firstly, (Y, ψ) is conjugate to a substitution minimal system associated to a proper (aperiodic) substitution θ of constant length, namely $a \to aab, b \to abb$ [**8**, Proposition 16]. (In fact, one shows that $\tilde{\pi}_1(x_{\min})$ is equal to the limit bisequence ω obtained by applying iterations of θ to $b.a$, i.e.

$$\omega = \tilde{\pi}_1(x_{\min}) = \lim_n \overleftarrow{\theta^n(b)}.\lim_n \overrightarrow{\theta^n(a)} = \ldots aababbabb.aabaababb\ldots,$$

where . is used to keep track of the zeroth coordinate. Note that $\theta(\omega) = \omega$. Here $x_{\min}$ is the unique minimal path of the diagram $(V, E, \geq)$ of Figure 6, and we obtain that (Y, ψ) is conjugate via the map $\tilde{\pi}_1$ to (Y_1, S_1), since ψ separates paths at level 1 (easily seen), cf. §1.6.)

Secondly, since (V, E) has the equal path number property, it is conjugate to a Toeplitz flow according to Theorem 8.

Consider now the system (X, ϕ) associated to the properly ordered Bratteli diagram $(V', E', \geq')$ exhibited to the right in Figure 6. Since (V', E') is obtained from (V, E) by first telescoping the latter between level 0 and level 2 and then 'erasing' some of the resulting new edges, we obtain that (X, ϕ) is an induced system of (Y, ψ), cf. §1.5.

We sketch the argument why (X, ϕ) is conjugate to the Chacon substitution system. In the first place, similarly to the above, (X, ϕ) is conjugate via the map $\tilde{\pi}_1$ to (Y_1', S_1'), where we use $'$ to refer to entities associated to the diagram $(V', E', \geq')$ in Figure 6. Clearly,

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$\tilde{\pi}'(x'_{\min})$ is obtained from $\tilde{\pi}_1(x_{\min})$ by substituting $\alpha\beta$ for a and γ for b. So

$$\omega' = \tilde{\pi}'_1(x'_{\min}) = \ldots\alpha\beta\alpha\beta\gamma\alpha\beta\gamma\gamma\alpha\beta\gamma\gamma.\alpha\beta\alpha\beta\gamma\alpha\beta\alpha\beta\gamma\alpha\beta\gamma\gamma\ldots.$$

Observe that in the bisequence ω' the symbol α appears only before β, and the symbol γ appears only before α or γ. This implies that the sliding 1-block code (cf. Lind and Marcus [**17**]) associated to

$$\alpha \to 0, \quad \beta \to 1, \quad \gamma \to 0$$

will be a one-to-one map F from Y'_1 ($\subseteq \{\alpha, \beta, \gamma\}^{\mathbb{Z}}$) onto $C = F(Y'_1)$ ($\subseteq \{0, 1\}^{\mathbb{Z}}$). Clearly, F intertwines the two shift maps S'_1 and S, respectively, and so (Y'_1, S'_1) is conjugate to (C, S).

We finish the argument by showing that (C, S) is the Chacon substitution system. It is clearly enough to show that the language $L(\overline{\omega})$, i.e. all subwords of $\overline{\omega}$, where

$$\overline{\omega} = F(\omega') = \ldots 0101001000100.01010010100100\ldots$$

is contained in the language $L(\omega_C)$ associated to ω_C, where

$$\omega_C = \lim_n \overleftarrow{\sigma^n}(0).\lim_n \overrightarrow{\sigma^n}(0) = \ldots 0010001010010.0010001010010\ldots$$

is the bisequence fixed under the Chacon substitution σ, where

$$\sigma = \begin{cases} 0 \to 0010 \\ 1 \to 1. \end{cases}$$

Observe that neither $\overline{\omega}$ nor ω_C contains the subword 11. Because of minimality (uniform recurrence), it is sufficient to show that any subword of the one-sided sequence $\overline{\omega}_+ = \overline{\omega}[0, \infty)$ is a subword of the one-sided sequence $(\omega_C)_+ = \omega_C[0, \infty)$.

For any substitution χ on an alphabet A there is a natural extension by concatenation to $A^{\mathbb{Z}_+}$, also denoted by χ. Here $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. Now we have the commutative diagram

$$\begin{array}{ccc} \{a, b\}^{\mathbb{Z}_+} & \xrightarrow{\theta} & \{a, b\}^{\mathbb{Z}_+} \\ {\scriptstyle G}\downarrow & & {\scriptstyle G}\downarrow \\ \{0, 1\}^{\mathbb{Z}_+} & \xrightarrow{\rho} & \{0, 1\}^{\mathbb{Z}_+}, \end{array}$$

where ρ is the word-to-word substitution

$$\rho = \begin{cases} 01 \to 01010 \\ 0 \to 0100 \end{cases}$$

and G is the map $G(a) = 01$, $G(b) = 0$. In fact, one checks that $\rho \circ G(a) = G \circ \theta(a)$, $\rho \circ G(b) = G \circ \theta(b)$. Since $\theta(\omega_+) = \omega_+$ and $G(\omega_+) = F(\omega'_+) = \overline{\omega}_+$, we conclude that $\rho(\overline{\omega}_+) = \overline{\omega}_+$. (Note that ρ is uniquely defined on sequences starting with zero and where 11 does not occur.) Now let γ be a non-empty word on $\{0, 1\}$ such that γ starts with zero and 11 does not occur in γ. We claim that

$$\sigma(\gamma)0 = 0\rho(\gamma). \qquad (\flat)$$

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It is true for $\gamma = 0$ and it is easily established by induction. With the same condition on γ, we also claim that

$$\sigma^n(\gamma 0) = u_1 \rho^n(\gamma) 0 u_2 \qquad (\sharp)$$

for some u_1, u_2. In fact, for $n = 0$, we obtain (using ($\flat$))

$$\sigma(\gamma 0) = \sigma(\gamma)\sigma(0) = \sigma(\gamma)0010 = 0\rho(\gamma)010 = u_1\rho(\gamma)0u_2,$$

where $u_1 = 0$, $u_2 = 10$. Assume ($\sharp$) is true for n. Applying σ to both sides of ($\sharp$) and using ($\flat$) we obtain

$$\begin{aligned}\sigma^{n+1}(\gamma 0) &= \sigma(u_1)\sigma(\rho^n(\gamma))\sigma(0)\sigma(u_2)\\ &= \sigma(u_1)\sigma(\rho^n(\gamma))0010\sigma(u_2)\\ &= \sigma(u_1)0\rho^{n+1}(\gamma)010\sigma(u_2)\\ &= u_1'\rho^{n+1}(\gamma)0u_2',\end{aligned}$$

where $u_1' = \sigma(u_1)0$, $u_2' = 10\sigma(u_2)$. So ($\sharp$) is proved by induction.

Now choose $\gamma = 01$ and use ($\sharp$) to obtain $\sigma^n(010) = u_1\rho^n(01)0u_2$. Since 010 occurs in $(\omega_C)_+$, $\sigma^n(010)$ occurs in $(\omega_C)_+$. Since any subword in $(\overline{\omega})_+$ is a subword of $\rho^n(01)$ for some $n \in \mathbb{N}$, the proof is complete.

Notice that ($\flat$) implies that $(\omega_C)_+ = \sigma((\omega_C)_+) = 0\rho((\omega_C)_+)$. This gives us the clue to find which path in the Bratteli diagram $(V', E', \geq')$ of Figure 6 corresponds to ω_C, i.e. which path that is mapped to ω_C by $\tilde{\pi}_1'$ when we set $\alpha = 0$, $\beta = 1$, $\gamma = 0$. In fact, it is the path going down vertically on the right-hand side of the diagram, starting with the single edge at the top and then the edge with ordinal number 3. All the other edges have ordinal numbers equal to two. (There is exactly one other path in $(V', E', \geq')$ that is mapped by $\tilde{\pi}_1$ to a sequence $\tilde{\omega}$ such that $(\tilde{\omega})_+$ is equal to $(\omega_C)_+$, namely the path going down vertically on the left-hand side of the diagram, from level 2 on, the edges all having ordinal number equal to two. From level 2 to level 1 the edge has ordinal number equal to three.)

It is easy to exhibit a primitive, aperiodic substitution τ such that the associated substitution minimal system is conjugate to the above example, i.e. to the Chacon system, using [**8**, Lemma 9]. In fact, τ may be chosen to be the substitution over the alphabet $A = \{d, e, f\}$

$$\tau = \begin{cases} d \to de \\ e \to def \\ f \to deff. \end{cases}$$

Remark. As was pointed out in §1.8, $K^0(Y, \psi)$ and $K^0(X, \phi)$ are order isomorphic, as (Y, ψ) and (X, ϕ) are Kakutani equivalent. However, the distinguished order units are not preserved under the isomorphism. This has dramatic consequences as far as the rational subdimension groups are concerned. In fact, it is easy to show directly from the two diagrams in Figure 6 that $\mathbb{Q}(K^0(Y, \psi)) \cong \mathbb{Z}[1/3]$ and $\mathbb{Q}(K^0(X, \phi)) \cong \mathbb{Z}$.

Open problem. Answering a question posed in an earlier version of this paper, Downarwicz and Durand [**4**] characterized completely the factors of a Toeplitz flow.

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It follows from earlier results [**7**] that if the Cantor minimal factor of a Toeplitz flow is expansive, then it is Toeplitz. (We are grateful to Lacroix for pointing this out to us.)

One may ask the even more ambitious question: is every factor (not necessarily assuming a Cantor system) of a Toeplitz flow of the above nature? (It can be shown that Toeplitz flows cannot have (topological) connected factors [**6**]. This is in contrast to substitution minimal systems, which may have such factors.)

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