Canad. Math. Bull. 2026, pp. 1–23
http://dx.doi.org/10.4153/S0008439526102136
© The Author(s), 2026. Published by Cambridge University Press on behalf of Canadian Mathematical Society

Well-posedness of second-order integro-differential equations

Shangquan Bu and Gang Cai

Abstract. In this article, we study well-posedness of the second-order integro-differential equations (P_5): $\mu * (Mu)''(t) + \nu * (Mu)'(t) + \eta * (Mu)(t) = \tau * Au(t) + f(t)$, $(t \in \mathbb{T} := [0, 2\pi])$ with periodic boundary conditions $Mu(0) = Mu(2\pi)$, $(Mu)'(0) = (Mu)'(2\pi)$, where A, M are closed operators in a Banach space X such that $D(A) \subset D(M)$, μ, ν, η, and τ are finite scalar measures on $\mathbb{R}$. Using known results on Fourier multipliers, we find necessary and sufficient conditions on the measures μ, ν, η, τ and the operators A, M to ensure well-posedness of (P_5) in $L^p(\mathbb{T}; X)$, periodic Besov spaces $B^s_{p,q}(\mathbb{T}; X)$, and Triebel–Lizorkin spaces $F^s_{p,q}(\mathbb{T}; X)$.

1 Introduction

In a series of recent publications, operator-valued Fourier multipliers on vector-valued function spaces are studied (see, e.g., [1–3, 5, 6, 10, 14, 17–21]). They are needed to study the existence and uniqueness of differential equations on Banach spaces. In [3, 5, 6, 8–10], the authors study well-posedness of the classical second-order problem (P_1) on L^p spaces, Besov spaces, and Triebel–Lizorkin spaces using operator-valued Fourier multipliers, where

$$(P_1) \quad \begin{cases} u''(t) = Au(t) + f(t), & (t \in \mathbb{T}) \\ u(0) = u(2\pi),\ u'(0) = u'(2\pi), \end{cases}$$

here X is a Banach space, A is a closed linear operator defined in X, and f is an X-valued function defined on $[0, 2\pi]$. (P_1) is L^p-well-posed if and only if $-k^2 \in \rho(A)$ for all $k \in \mathbb{Z}$ and $(k^2 R(-k^2, A))_{k\in\mathbb{Z}}$ is Rademacher bounded whenever X is a UMD Banach space and $1 < p < \infty$[5]. In Besov spaces $B^s_{p,q}(\mathbb{T}; X)$ and Triebel–Lizorkin spaces $F^s_{p,q}(\mathbb{T}; X)$ setting, well-posedness of (P_1) is equivalent to the assertion that $-k^2 \in \rho(A)$ for all $k \in \mathbb{Z}$ and $(k^2 R(-k^2, A))_{k\in\mathbb{Z}}$ is bounded [6, 10]. Recently, well-posedness of the following more general second-order integro-differential equations:

Received by the editors December 12, 2025 UTC; revised May 26, 2026 UTC; accepted June 1, 2026 UTC.

Published online on Cambridge Core June 8, 2026.

This work was supported by the Natural Science Foundation of China (Grant No. 12571141) and the Chongqing Talent Support Program (Grant No. cstc2024ycjh-bgzxm0121).

AMS subject classification: **45N05**, **43A15**, **45D05**, **47D06**.

Keywords: Fourier multiplier, well-posedness, integro-differential equation, Besov spaces, Triebel–Lizorkin spaces.

https://doi.org/10.4153/S0008439526102136 Published online by Cambridge University Press

$$(P_2)\quad \begin{cases} u''(t) + \alpha u'(t) + \frac{d}{dt}\left(\int_{-\infty}^{t} b(t-s)u(s)ds\right) \\ \qquad = Au(t) - \int_{-\infty}^{t} a(t-s)Au(s)ds + f(t), \quad (t \in \mathbb{T}) \\ u(0) = u(2\pi), u'(0) = u'(2\pi) \end{cases}$$

has been considered in [9], where α is a constant and $a, b \in L^1(\mathbb{R}_+)$. Much literature has devoted to the similar first-order integro-differential equations (P_3):

$$(P_3)\quad \begin{cases} \gamma_0 u'(t) + \frac{d}{dt}\left(\int_{-\infty}^{t} b(t-s)u(s)ds\right) + \gamma_\infty u(t) \\ \qquad = c_0 Au(t) - \int_{-\infty}^{t} a(t-s)Au(s)ds + f(t), \quad (t \in \mathbb{T}) \\ u(0) = u(2\pi), \end{cases}$$

where γ_0, γ_∞, and c_0 are constants, and $a, b \in L^1(\mathbb{R}_+)$. The class of equation of type (P_2) and (P_3) arises as models for nonlinear heat conduction in material of fading memory type and population dynamics. Such kind of integro-differential equations has been recently studied in an abstract way by Keyantuo, Lizama, and Poblete. More precisely, they have considered the following first-order problem:

$$(P_4)\quad \begin{cases} \mu * u'(t) + \nu * u(t) = \eta * Au(t) + f(t), \quad (t \in \mathbb{T}) \\ u(0) = u(2\pi), \end{cases}$$

here μ, ν, and η are finite scalar measures on $\mathbb{R}$. Under suitable conditions on the measures μ, ν, η and the operator A, they completely characterized well-posedness of (P_4) in Lebesgue–Bochner spaces $L^p(\mathbb{T}; X)$, periodic Besov spaces $B^s_{p,q}(\mathbb{T}; X)$, and periodic Triebel–Lizorkin spaces $F^s_{p,q}(\mathbb{T}; X)$ [15].

In this article, we study similar second-order degenerate integro-differential equations:

$$(P_5)\quad \begin{cases} \mu * (Mu)''(t) + \nu * (Mu)'(t) + \eta * (Mu)(t) = \tau * Au(t) + f(t), \quad (t \in \mathbb{T}) \\ Mu(0) = Mu(2\pi), (Mu)'(0) = (Mu)'(2\pi), \end{cases}$$

where μ, ν, η, and τ are finite scalar measures on $\mathbb{R}$, and A and M are closed linear operators in X such that $D(A) \subset D(M)$. It is clear that if we take $M = I_X$, $\mu = \delta_0$, $\nu = \alpha\delta_0 + b$, $\eta = 0$, and $\tau = \delta_0 - a$ in (P_5), then we recover (P_2). If we take $M = I_X$ and $\mu = 0$, then (P_5) becomes (P_4). Moreover, we take $M = I_X$, $\mu = \nu = \delta_0$, and $\tau = a$, then (P_5) becomes (3) in [18] (or (1.3) in [17]) when $b = 0$. Here, we identify a finite scalar measure with its density in $L^1(\mathbb{R})$ with respect to Lebesgue measure.

We notice that (P_5) has been studied by several authors in a more simple form and different boundary conditions. For instance, Chill and Srivastava [11] have considered L^p-well-posedness on a finite interval [0, T) for the abstract second-order problem

$$(P_6)\quad \begin{cases} u''(t) + Bu'(t) + Au(t) = f(t), \quad (t \in \mathbb{T}) \\ u(0) = x_0, \ u'(0) = x_1. \end{cases}$$

The semigroup theory and the trace spaces played important roles in their discussion. Under a suitable condition on the operators A and B, they gave a necessary and sufficient condition for (P_6) to be L^p-well-posed.

We will study well-posedness of (P_5) on three function spaces: $L^p(\mathbb{T}; X)$ for $1 < p < \infty$, periodic Besov spaces $B^s_{p,q}(\mathbb{T}; X)$ for $1 \le p, q \le \infty$, $s > 0$, and periodic Triebel–Lizorkin spaces $F^s_{p,q}(\mathbb{T}; X)$ for $1 \le p < \infty$, $1 \le q \le \infty$, $s > 0$, where

https://doi.org/10.4153/S0008439526102136 Published online by Cambridge University Press

$\mathbb{T} = [0, 2\pi]$. The main tools will be operator-valued Fourier multiplier theorems obtained in [5, Theorem 1.3], [6, Theorem 4.5], and [10, Theorem 3.2] on these function spaces. We completely characterize well-posedness of (P_5) using the Fourier transform of the measures μ, ν, η, τ and the closed operators A and M. For instance, when X is a UMD space and $1 < p < \infty$, $(-k^2\tilde{\mu}(k) + ik\tilde{\nu}(k) + \tilde{\eta}(k))_{k\in\mathbb{Z}}$ and $(\tilde{\tau}(k))_{k\in\mathbb{Z}}$ are 1-regular, then (P_5) is L^p-well-posed if and only if $\rho_M(A; \mu, \nu, \eta, \tau) = \mathbb{Z}$ (the resolvent set of (P_5)) and $\{k^2MN_k : k \in \mathbb{Z}\}$, $\{AN_k : k \in \mathbb{Z}\}$, and $\{N_k : k \in \mathbb{Z}\}$ are R-bounded (Theorem 3.4), where

$$N_k = [(-k^2\tilde{\mu}(k) + ik\tilde{\nu}(k) + \tilde{\eta}(k))M - \tilde{\tau}(k)A]^{-1}, (k \in \mathbb{Z}),$$

$\tilde{\mu}, \tilde{\tau}, \tilde{\nu}, \tilde{\eta}$ are the Fourier transforms of μ, τ, ν, η. Similar characterizations have also been obtained in $B^s_{p,q}$-well-posedness and $F^s_{p,q}$-well-posedness setting. The conditions that we impose on the measures μ, τ, ν, η are satisfied by a class of measures, which correspond to the most common kernels encountered in the applications. Furthermore, it is easy to see that our results are in accordance with the known results for (P_1) [5, 6, 10] and (P_2) [9].

The article is organized as follows: in Section 2, we give some preliminaries concerning M-boundedness and R-boundedness for operator sequences. In Section 3, we study L^p-well-posedness of (P_5). $B^s_{p,q}$-well-posedness and $F^s_{p,q}$-well-posedness of (P_5) will be discussed in Section 4. Section 5 gives some examples to which our abstract results may be applied.

2 Preliminaries

Let X be a Banach space, we denote by $\mathcal{L}(X)$ the set of all bounded linear operators on X. If A is a closed operator in X, we will denote by $\rho(A)$ its resolvent set. When $\lambda \in \rho(A)$, $R(\lambda, A) = (\lambda - A)^{-1}$ will be the resolvent operator. In what follows, we consider $D(A)$, $D(M)$ as Banach spaces equipped with their respective graph norms. Now, we study the following problems:

$$(P_5) \quad \begin{cases} \mu * (Mu)''(t) + \nu * (Mu)'(t) + \eta * (Mu)(t) = \tau * Au(t) + f(t), & (t \in \mathbb{T}) \\ Mu(0) = Mu(2\pi), (Mu)'(0) = (Mu)'(2\pi), \end{cases}$$

where μ, ν, η, and τ are finite scalar measures on the real line, A, M are closed linear operators in X such that $D(A) \subset D(M)$, and f is an X-valued function defined on $\mathbb{T} := [0, 2\pi]$. The solution u of (P_5) will be an X-valued function defined on $\mathbb{T}$ (extended by periodicity to $\mathbb{R}$). The resolvent set of (P_5) is defined by

$$\rho_M(A; \mu, \nu, \eta, \tau) := \big\{k \in \mathbb{Z} : (-k^2\tilde{\mu}(k) + ik\tilde{\nu}(k) + \tilde{\eta}(k))M - \tilde{\tau}(k)A \text{ is invertible from}$$

$$D(A) \text{ onto } X \text{ and } [(-k^2\tilde{\mu}(k) + ik\tilde{\nu}(k) + \tilde{\eta}(k))M - \tilde{\tau}(k)A]^{-1} \in \mathcal{L}(X)\big\}.$$

For the sake of simplicity, when $k \in \rho_M(A; \mu, \nu, \eta, \tau)$, we will use the following notation:

$$\begin{aligned} N_k &= [(-k^2\tilde{\mu}(k) + ik\tilde{\nu}(k) + \tilde{\eta}(k))M - \tilde{\tau}(k)A]^{-1}, \ (k \in \mathbb{Z}) \\ &:= [\alpha_k M - \beta_k A]^{-1}, \ (k \in \mathbb{Z}). \end{aligned} \tag{2.1}$$

https://doi.org/10.4153/S0008439526102136 Published online by Cambridge University Press

If $k \in \rho_M(A; \mu, \nu, \eta, \tau)$, then MN_k and AN_k make sense as $D(A) \subset D(M)$ by assumption, and they belong to $\mathcal{L}(X)$ by the Closed Graph Theorem and the closedness of A and M.

To study well-posedness of (P_5), we need to introduce some notions and notations. Let $f \in L^1(\mathbb{T}; X)$, we denote by

$$\hat{f}(k) = \frac{1}{2\pi} \int_0^{2\pi} e_{-k}(t) f(t) dt$$

the k-th Fourier coefficient of f, where $k \in \mathbb{Z}$ and $e_k(t) = e^{ikt}$ for $t \in \mathbb{R}$. For $x \in X$, we let $e_k \otimes x$ be the X-valued function given by $t \to e_k(t)x$. If μ is a finite scalar measure on $\mathbb{R}$, we denote by $\tilde{\mu}$ its Fourier transform, that is,

$$\tilde{\mu}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t} \mu(dt), \quad (\omega \in \mathbb{R}).$$

We notice that when μ has a density $\phi \in L^1(\mathbb{R})$, then

$$\tilde{\mu}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t} \phi(t) dt = \mathcal{F}\phi(\omega), \quad (\omega \in \mathbb{R})$$

coincides with the usual Fourier transform of ϕ.

Let $f \in L^1(\mathbb{T}; X)$ (extended by periodicity to $\mathbb{R}$) and μ be a finite scalar measure on $\mathbb{R}$. The convolution of μ and f is defined by

$$\mu * f(t) = \int_{-\infty}^{\infty} f(t-s) \mu(ds), \quad (t \in \mathbb{R}).$$

It is clear that this definition makes sense and $\mu * f \in L^1(\mathbb{T}; X)$. By Fubini's theorem, we have

$$\widehat{\mu * f}(k) = \tilde{\mu}(k) \hat{f}(k), \quad (k \in \mathbb{Z}). \tag{2.2}$$

Definition 2.1 [5] Let X be a Banach space and let $1 \le p < \infty$. We say that a sequence $(M_k)_{k \in \mathbb{Z}} \subset \mathcal{L}(X)$ is an L^p-multiplier, if for each $f \in L^p(\mathbb{T}; X)$, there exists a unique $g \in L^p(\mathbb{T}; X)$ such that $\hat{g}(k) = M_k \hat{f}(k)$ for all $k \in \mathbb{Z}$.

It follows from the Closed Graph Theorem that if $(M_k)_{k \in \mathbb{Z}}$ is an L^p-multiplier, then there exists a constant $C > 0$ such that for $f \in L^p(\mathbb{T}; X)$, we have $\| \sum_{k \in \mathbb{Z}} e_k \otimes M_k \hat{f}(k) \|_{L^p} \le C \|f\|_{L^p}$. This implies that when $(M_k)_{k \in \mathbb{Z}}$ is an L^p-multiplier, then $(M_k)_{k \in \mathbb{Z}}$ must be norm bounded. It is also clear from the definition that if $(M_k)_{k \in \mathbb{Z}}$ and $(N_k)_{k \in \mathbb{Z}}$ are L^p-multipliers, then $(N_k M_k)_{k \in \mathbb{Z}}$ is also an L^p-multiplier.

In the study of L^p-well-posedness of (P_5), we need the notion of Rademacher boundedness for sets of bounded linear operators. We refer to Bourgain [7], Weis [20, 21], and Arendt and Bu [5] for more information about this property. We merely recall the definition and some basic properties.

Let r_j be the j-th Rademacher function on [0,1] given by $r_j(t) = \operatorname{sgn}(\sin(2^{j-1}\pi t))$. For $x \in X$, we denote by $r_j \otimes x$ the vector-valued function $t \to r_j(t)x$.

https://doi.org/10.4153/S0008439526102136 Published online by Cambridge University Press

Definition 2.2 Let X be a Banach space. A family $\mathbf{T} \subset \mathcal{L}(X)$ is called R-bounded (R stands for Rademacher) if there exists $C \geq 0$ such that

$$\Big\| \sum_{j=1}^{n} r_j \otimes T_j x_j \Big\|_{L^1(0,1;X)} \leq C \Big\| \sum_{j=1}^{n} r_j \otimes x_j \Big\|_{L^1(0,1;X)} \tag{2.3}$$

for all $T_1, \ldots, T_n \in \mathbf{T}$, $x_1, \ldots, x_n \in X$ and $n \in \mathbb{N}$.

Remark 2.3

(a) Let $\mathbf{S}$,$\mathbf{T} \subset \mathcal{L}(X)$ be R-bounded sets. Then it is clear from the definition that $\mathbf{ST} := \{ST : S \in \mathbf{S}, T \in \mathbf{T}\}$ and $\mathbf{S} + \mathbf{T} := \{S + T : S \in \mathbf{S}, T \in \mathbf{T}\}$ are R-bounded.

(b) Each subset $\mathbf{M} \subset \mathcal{L}(X)$ of the form $\mathbf{M} = \{\lambda I : \lambda \in \Omega\}$ is R-bounded whenever $\Omega \subset \mathbb{C}$ is bounded. This follows from Kahane's contraction principle [16, Section 3.5.4].

Let X be a Banach space and $(M_k)_{k\in\mathbb{Z}} \subset \mathcal{L}(X)$. We let

$$\Delta^0 M_k = M_k, \quad \Delta M_k = \Delta^1 M_k = M_{k+1} - M_k, \quad (k \in \mathbb{Z}),$$

and

$$\Delta^n M_k = \Delta(\Delta^{n-1} M_k), \quad (k \in \mathbb{Z})$$

for $n = 2, 3, \ldots$. The sequence $(\Delta^n M_k)_{k\in\mathbb{Z}}$ is the n-th derivative of $(M_k)_{k\in\mathbb{Z}}$. Following [15], we say that $(M_k)_{k\in\mathbb{Z}}$ is M-bounded (M stands for Marcinkiewicz) of order n for some $n \in \mathbb{N} \cup \{0\}$, if

$$\sup_{0\leq l\leq n} \sup_{k\in\mathbb{Z}} \left\| k^l \Delta^l M_k \right\| < \infty. \tag{2.4}$$

Analogously, we define M-boundedness of order n for sequences $(a_k)_{k\in\mathbb{Z}}$ of complex numbers. This notion is very convenient when studying operator-valued Fourier multipliers on vector-valued function spaces on $\mathbb{T}$. We also need the notion of regularity for scalar sequences.

Definition 2.4 [15] Let $(a_k)_{k\in\mathbb{Z}}$ be a scalar sequence, we say that $(a_k)_{k\in\mathbb{Z}}$ is n-regular for some $n \geq 1$, if there exists $k_0 > 0$, such that $a_k \neq 0$ for all $|k| \geq k_0$ and

$$\sup_{0\leq l\leq n} \sup_{|k|\geq k_0} \left| k^l (\Delta^l a_k)/a_k \right| < \infty. \tag{2.5}$$

Note that if $(a_k)_{k\in\mathbb{Z}}$ is 1-regular, then $\lim_{|k|\to\infty} \frac{a_{k+1}}{a_k} = 1$. It is clear from the definitions that when $(a_k)_{k\in\mathbb{Z}}$ is bounded and n-regular, then it is M-bounded of order n. Therefore, by Remark 2.3(b), we obtain that $(a_k I_X)$ is L^p-Fourier multiplier when $(a_k)_{k\in\mathbb{Z}}$ is bounded and 1-regular. It can be seen easily that $(k^2)_{k\in\mathbb{Z}}$ is 3-regular. We will use the following results established in [15].

Theorem 2.5 [15, Theorem 4.6] *Let $(a_k)_{k\in\mathbb{Z}}$ and $(b_k)_{k\in\mathbb{Z}}$ be given sequences of complex numbers and let $1 \leq n \leq 3$. Then*

(i) *$(a_k)_{k\in\mathbb{Z}}$ is n-regular if and only if $(1/a_k)_{k\in\mathbb{Z}}$ is n-regular;*

(ii) *if $(a_k)_{k\in\mathbb{Z}}$ and $(b_k)_{k\in\mathbb{Z}}$ are n-regular, then (a_k/b_k) is n-regular.*

https://doi.org/10.4153/S0008439526102136 Published online by Cambridge University Press

3 L^p-well-posedness of (P_5)

In this section, we consider L^p-well-posedness of P_5 when $1 < p < \infty$. First, we introduce the periodic Sobolev spaces $W^{m,p}_{\mathrm{per}}(\mathbb{T};X)$ when $m = 1, 2$. We let $W^{1,p}_{\mathrm{per}}(\mathbb{T};X)$ be the space of all $u \in L^p(\mathbb{T};X)$ such that u is a.e. differentiable, $u' \in L^p(\mathbb{T};X)$ and $u(0) = u(2\pi)$. It is shown in [5] that

$$W^{1,p}_{\mathrm{per}}(\mathbb{T};X) = \Big\{u \in L^p(\mathbb{T};X) : \text{there exists } v \in L^p(\mathbb{T};X), \text{ such that } \hat{v}(k) = ik\hat{u}(k) \text{ for all } k \in \mathbb{Z}\Big\}. \tag{3.1}$$

If u and v are as in (3.1), then $v = u'$ a.e.and $u(0) = u(2\pi)$. We define $W^{2,p}_{\mathrm{per}}(\mathbb{T};X)$ as the space of all $u \in W^{1,p}_{\mathrm{per}}(\mathbb{T};X)$ such that $u' \in W^{1,p}_{\mathrm{per}}(\mathbb{T};X)$. Thus, by (3.1),

$$W^{2,p}_{\mathrm{per}}(\mathbb{T};X) = \Big\{u \in L^p(\mathbb{T};X) : \text{there exists } w \in L^p(\mathbb{T};X), \text{ such that } \hat{w}(k) = -k^2\hat{u}(k) \text{ for all } k \in \mathbb{Z}\Big\}. \tag{3.2}$$

If u and w are as in (3.2), then $w = u''$ a.e.and $u(0) = u(2\pi),\ u'(0) = u'(2\pi)$.

Let $1 \le p < \infty$, the solution space of the L^p-well-posedness of (P_5) is defined by

$$S_p(A,M) := \big\{u \in L^p(\mathbb{T};D(A)) :\ Mu \in W^{2,p}_{\mathrm{per}}(\mathbb{T};X)\big\},$$

here we consider $D(A)$ and $D(M)$ as Banach spaces equipped with their graph norms. $S_p(A,M)$ is a Banach space equipped with the norm

$$\|u\|_{S_p(A,M)} := \|u\|_{L^p} + \|Au\|_{L^p} + \|(Mu)'\|_{L^p} + \|(Mu)''\|_{L^p}.$$

If $u \in S_p(A,M)$, then Mu and $(Mu)'$ are X-valued continuous functions on $\mathbb{T}$, and $Mu(0) = Mu(2\pi)$, $(Mu)'(0) = (Mu)'(2\pi)$ by [5, Lemma 2.1].

We say that (P_5) is L^p-well-posed, if for each $f \in L^p(\mathbb{T};X)$, there exists a unique $u \in S_p(A,M)$ such that (P_5) is satisfied a.e. on $\mathbb{T}$. We notice that when (P_5) is L^p-well-posed, then each term appeared in (P_5) is in $L^p(\mathbb{T};X)$, this follows from Young's inequality. The following result is fundamental in our investigation. For the concept of UMD Banach spaces, we refer to [7].

If (P_5) is L^p-well-posed and $u \in S_p(A,M)$ is the unique strong L^p-solution of (P_5), then there exists a constant $C > 0$ such that for each $f \in L^p(\mathbb{T};X)$, we have

$$\|u\|_{S_p(A,M)} \le C\|f\|_{L^p}. \tag{3.3}$$

This can be obtained easily from the Closed Graph Theorem. In order to prove our main result of this section, we need the following preparations.

Theorem 3.1 [5, Theorem 1.3] *Let X be a UMD Banach space, $(M_k)_{k\in\mathbb{Z}} \subset \mathcal{L}(X)$, and let $1 < p < \infty$. We assume that the sets $\{M_k :\ k \in \mathbb{Z}\}$ and $\{k\Delta M_k :\ k \in \mathbb{Z}\}$ are R-bounded, then (M_k) is an L^p-multiplier.*

Proposition 3.2 [5, Proposition 1.11] *Let X, Y be Banach spaces, $1 \le p < \infty$, and let $(M_k)_{k\in\mathbb{Z}} \subset \mathcal{L}(X,Y)$ be an L^p-Fourier multiplier, then the set $\{M_k :\ k \in \mathbb{Z}\}$ is R-bounded.*

https://doi.org/10.4153/S0008439526102136 Published online by Cambridge University Press

We will use the following result due to Conejero, Lizama, Murillo-Arcila, and Seoane-Sepulveda.

Theorem 3.3 [12, Theorem 1.1] *Let $T : D(T) \subset X \to X$ be a closed linear operator defined in Banach space X. For each $k \in \mathbb{Z}$, let $H_k : X \to D(T)$ be a sequence of bounded and linear operators such that $0 \in \rho(H_k)$ for all $k \in \mathbb{Z}$. Suppose that $(c_k)_{k\in\mathbb{Z}} \subset \mathbb{C}$ is a 1-regular sequence and denote $M_k := c_k T H_k$ and $L_k := (H_k^{-1} - H_{k+1}^{-1})H_k$. If $\{M_k : k \in \mathbb{Z}\}$ and $\{kL_k : k \in \mathbb{Z}\}$ are R-bounded sets, then $\{M_k : k \in \mathbb{Z}\}$ is MR-bounded of order 1. If, in addition, $(c_k)_{k\in\mathbb{Z}}$ is 2-regular and the set $\{k^2 \Delta L_k : k \in \mathbb{Z}\}$ is R-bounded, then $\{M_k : k \in \mathbb{Z}\}$ is MR-bounded of order 2.*

The following is the main result of this section.

Theorem 3.4 *Let X be a UMD Banach space and let $1 < p < \infty$. Assume that A and M are closed operators in X such that $D(A) \subset D(M)$, μ, ν, η, and τ are bounded scalar measures on $\mathbb{R}$ such that $(\alpha_k)_{k\in\mathbb{Z}}$ and $(\beta_k)_{k\in\mathbb{Z}}$ are 1-regular, where α_k and β_k are defined by (2.1). Then the following assertions are equivalent:*

(i) *(P_5) is L^p-well-posed;*

(ii) *$\rho_M(A; \mu, \nu, \eta, \tau) = \mathbb{Z}$ and $\{k^2 MN_k : k \in \mathbb{Z}\}$, $\{AN_k : k \in \mathbb{Z}\}$, $\{N_k : k \in \mathbb{Z}\}$ are R-bounded, where N_k is defined by (2.1).*

Proof Now we assume that (ii) is true and we are going to show that (P_5) is L^p-well-posed for $1 < p < \infty$. We first show the uniqueness. Let $u \in S_p(A, M)$ such that

$$\mu * (Mu)''(t) + \nu * (Mu)'(t) + \eta * (Mu)(t) - \tau * Au(t) = 0$$

a.e. on $\mathbb{T}$ and $Mu(0) = Mu(2\pi)$, $(Mu)'(0) = (Mu)'(2\pi)$. Taking Fourier coefficient on both sides, we find that $\big[(-k^2\tilde{\mu}(k) + ik\tilde{\nu}(k) + \tilde{\eta}(k))M - \tilde{\tau}(k)A\big]\hat{u}(k) = 0$ for $k \in \mathbb{Z}$. Here, we have used (2.2) and the fact that when $h \in L^p(\mathbb{T}; D(A))$, then $\hat{h}(k) \in D(A)$ for all $k \in \mathbb{Z}$ and $A\hat{h}(k) = \widehat{Ah}(k)$ [5, Lemma 3.1]. It follows that $\hat{u}(k) = 0$ for $k \in \mathbb{Z}$ as $\rho_M(A; \mu, \nu, \eta, \tau) = \mathbb{Z}$ by assumption. Thus, $u = 0$ by the Uniqueness Theorem [5, p. 314]. We have shown the uniqueness.

Now we show the existence. Let $L_k = (N_k^{-1} - N_{k+1}^{-1})N_k$, we observe

$$\begin{aligned} L_k &= [\alpha_k M - \beta_k A - \alpha_{k+1} M + \beta_{k+1} A]N_k \\ &= -(\Delta\alpha_k)MN_k + (\Delta\beta_k)AN_k \\ &= \frac{\Delta\alpha_k}{\alpha_k}(\alpha_k MN_k) + \frac{\Delta\beta_k}{\beta_k}(\beta_k AN_k) \end{aligned}$$

when $|k| \geq k_0$. We also notice

$$\beta_k AN_k = \alpha_k MN_k - I_X. \tag{3.4}$$

Since $\{k^2 MN_k : k \in \mathbb{Z}\}$ is R-bounded and μ, ν, η, and τ are bounded scalar measures on $\mathbb{R}$, we have that $\{\alpha_k MN_k : k \in \mathbb{Z}\}$, $\{kMN_k : k \in \mathbb{Z}\}$, and $\{MN_k : k \in \mathbb{Z}\}$ are also R-bounded. Here, we have used the fact that the sums and products of R-bounded sets

https://doi.org/10.4153/S0008439526102136 Published online by Cambridge University Press

are also still R-bounded. We deduce by (3.4) that $\{\beta_k AN_k : k \in \mathbb{Z}\}$ is also R-bounded. Therefore, we obtain that $\{kL_k : k \in \mathbb{Z}\}$ is R-bounded, here we have use the fact that $(\alpha_k)_{k\in\mathbb{Z}}$ and $(\beta_k)_{k\in\mathbb{Z}}$ are 1-regular. If we put $c_k = 1$, $T = M$(or A, I_X), and $H_k = N_k$ in Theorem 3.3, then by Theorem 3.1, we deduce that $(MN_k)_{k\in\mathbb{Z}}$, $(AN_k)_{k\in\mathbb{Z}}$, and $(N_k)_{k\in\mathbb{Z}}$ are L^p-multipliers. Similarly, let $c_k = k^2$(or β_k), $T = M$(or A), and $H_k = N_k$ in Theorem 3.3, we obtain by Theorem 3.1 that $(k^2MN_k)_{k\in\mathbb{Z}}$ and $(\beta_k AN_k)_{k\in\mathbb{Z}}$ are L^p-multipliers, here we have used the fact that $(k^2)_{k\in\mathbb{Z}}$ and $(\beta_k)_{k\in\mathbb{Z}}$ are 1-regular. Let $f \in L^p(\mathbb{T}; X)$ be fixed. Then there exists $u_1, u_2, u_3, u \in L^p(\mathbb{T}; X)$ such that

$$\hat{u}(k) = N_k\hat{f}(k),\ \widehat{u_1}(k) = AN_k\hat{f}(k) \tag{3.5}$$

and

$$\widehat{u_2}(k) = MN_k\hat{f}(k),\ \widehat{u_3}(k) = -k^2MN_k\hat{f}(k) \tag{3.6}$$

for $k \in \mathbb{Z}$. We deduce from (3.4) that $\hat{u}(k) \in D(A)$ and $\widehat{u_1}(k) = AN_k\hat{f}(k) = A\hat{u}(k)$ for $k \in \mathbb{Z}$. Hence, $u \in L^p(\mathbb{T}; D(A))$ by [5, Lemma 3.1]. From (4.15) that $\widehat{u_2}(k) = MN_k\hat{f}(k) = M\hat{u}(k)$ for $k \in \mathbb{Z}$. Hence, $Mu \in L^p(\mathbb{T}; X)$ by [5, Lemma 3.1]. Again by (3.5), we have $\widehat{u_3}(k) = -k^2M\hat{u}(k)$ for $k \in \mathbb{Z}$. This implies that $Mu \in W^{2,p}_{\mathrm{per}}(\mathbb{T}; X)$ and $Mu(0) = Mu(2\pi)$, $(Mu)'(0) = (Mu)'(2\pi)$, $(Mu)'' = u_3$ by (3.2).

From the equality $\hat{u}(k) = N_k\hat{f}(k)$, we deduce that

$$\big[(-k^2\tilde{\mu}(k) + ik\tilde{\nu}(k) + \tilde{\eta}(k))M - \tilde{\tau}(k)A\big]\hat{u}(k) = \hat{f}(k)$$

for all $k \in \mathbb{Z}$. This is equivalent to say that

$$\mu * (Mu)''(t) + \nu * (Mu)'(t) + \eta * (Mu)(t) - \tau * Au(t) = f(t)$$

a.e. on $\mathbb{T}$ by the Uniqueness Theorem [5, p. 314]. Here, we have used the fact that $\widehat{u'}(k) = ik\hat{u}(k)$ when $k \in \mathbb{Z}$. This shows the existence. Thus, the implication (ii) $\Rightarrow$ (i) is true.

Now we give the proof for the implication (i) $\Rightarrow$ (ii). Assume that (P_5) is L^p-well-posed for $1 < p < \infty$. Let $k \in \mathbb{Z}$ and $y \in X$ be fixed, we consider the function f defined by $f(t) = e^{ikt}y$ when $t \in \mathbb{T}$. Then it is clear that $f \in L^p(\mathbb{T}; X)$, $\hat{f}(k) = y$, and $\hat{f}(n) = 0$ when $n \neq k$. Since (P_5) is L^p-well-posed, there exists a unique $u \in S_p(A, M)$ such that

$$\mu * (Mu)''(t) + \nu * (Mu)'(t) + \eta * (Mu)(t) - \tau * Au(t) = f(t) \tag{3.7}$$

a.e. on $\mathbb{T}$. We have $\hat{u}(n) \in D(A)$ when $n \in \mathbb{Z}$ by [5, Lemma 3.1] as $u \in L^p(\mathbb{T}; D(A))$. Taking Fourier transforms on both sides of (3.6), we obtain

$$\big[(-k^2\tilde{\mu}(k) + ik\tilde{\nu}(k) + \tilde{\eta}(k))M - \tilde{\tau}(k)A\big]\hat{u}(k) = y \tag{3.8}$$

and $\big[(-n^2\tilde{\mu}(n) + in\tilde{\nu}(n) + \tilde{\eta}(n))M - \tilde{\tau}(n)A\big]\hat{u}(n) = 0$ when $n \neq k$. This implies that the operator $\big[(-k^2\tilde{\mu}(k) + ik\tilde{\nu}(k) + \tilde{\eta}(k))M - \tilde{\tau}(k)A\big]$ defined on $D(A)$ with values in X is surjective. To show that it is also injective, we let $x \in D(A)$ be such that

$$\big[(-k^2\tilde{\mu}(k) + ik\tilde{\nu}(k) + \tilde{\eta}(k))M - \tilde{\tau}(k)A\big]x = 0.$$

https://doi.org/10.4153/S0008439526102136 Published online by Cambridge University Press

Let u be the function given by $u(t) = e^{ikt}x$ when $t \in \mathbb{T}$, then it is clear that $u \in S_p(A, M)$ and (P_5) is satisfied a.e. on $\mathbb{T}$ when $f = 0$. Thus, u is a strong L^p-solution of (P_5) when taking $f = 0$. We get $x = 0$ by the uniqueness assumption. We have shown that the operator $\left[(-k^2\tilde{\mu}(k) + ik\tilde{\nu}(k) + \tilde{\eta}(k))M - \tilde{\tau}(k)A\right]$ from $D(A)$ into X is injective. Therefore, $\left[(-k^2\tilde{\mu}(k) + ik\tilde{\nu}(k) + \tilde{\eta}(k))M - \tilde{\tau}(k)A\right]$ is bijective from $D(A)$ onto X.

Next, we show that $\left[(-k^2\tilde{\mu}(k) + ik\tilde{\nu}(k) + \tilde{\eta}(k))M - \tilde{\tau}(k)A\right]^{-1} \in \mathcal{L}(X)$. For $f(t) = e^{ikt}y$, we let $u \in S_p(A, M)$ such that u is the unique strong L^p-solution of (P_5). Then

$$\hat{u}(n) = \begin{cases} 0 & n \neq k, \\ \left[(-k^2\tilde{\mu}(k) + ik\tilde{\nu}(k) + \tilde{\eta}(k))M - \tilde{\tau}(k)A\right]^{-1} y & n = k, \end{cases} \tag{3.9}$$

by (3.7). This implies that u is given by

$$u(t) = e^{ikt}\left[(-k^2\tilde{\mu}(k) + ik\tilde{\nu}(k) + \tilde{\eta}(k))M - \tilde{\tau}(k)A\right]^{-1} y$$

when $t \in \mathbb{T}$. By (3.3), there exists a constant $C > 0$ independent from y and k, such that $\|u\|_{S_p(A,M)} \leq C\|f\|_{L^p}$. This implies that

$$\left\|\left[(-k^2\tilde{\mu}(k) + ik\tilde{\nu}(k) + \tilde{\eta}(k))M - \tilde{\tau}(k)A\right]^{-1} y\right\| \leq C\|y\|$$

when $y \in X$, or equivalently,

$$\left\|\left[(-k^2\tilde{\mu}(k) + ik\tilde{\nu}(k) + \tilde{\eta}(k))M - \tilde{\tau}(k)A\right]^{-1}\right\| \leq C.$$

Therefore, we have shown that $k \in \rho_M(A; \mu, \nu, \eta, \tau)$ for all $k \in \mathbb{Z}$. Thus, $\rho_M(A; \mu, \nu, \eta, \tau) = \mathbb{Z}$.

Finally, we prove that $(k^2MN_k)_{k\in\mathbb{Z}}$ and $(N_k)_{k\in\mathbb{Z}}$ define L^p-Fourier multipliers. Let $f \in L^p(\mathbb{T}; X)$, then there exists $u \in S_p(A, , M)$, a strong L^p-solution of (P_5) by assumption. Taking Fourier transforms on both sides of (P_5), we get that $\hat{u}(k) \in D(A)$ by [5, Lemma 3.1] and

$$[(-k^2\tilde{\mu}(k) + ik\tilde{\nu}(k) + \tilde{\eta}(k))M - \tilde{\tau}(k)A]\hat{u}(k) = \hat{f}(k)$$

for $k \in \mathbb{Z}$. Since $(-k^2\tilde{\mu}(k) + ik\tilde{\nu}(k) + \tilde{\eta}(k))M - \tilde{\tau}(k)A$ is invertible, we have

$$\hat{u}(k) = [(-k^2\tilde{\mu}(k) + ik\tilde{\nu}(k) + \tilde{\eta}(k))M - \tilde{\tau}(k)A]^{-1}\hat{f}(k) = N_k\hat{f}(k)$$

when $k \in \mathbb{Z}$. We conclude from $u \in S_p(A, M)$ that $Mu \in W^{2,p}_{\text{per}}(\mathbb{T}; X)$. We have

$$\widehat{(Mu)''}(k) = -k^2M\hat{u}(k),$$

when $k \in \mathbb{Z}$. We obtain that $(k^2MN_k)_{k\in\mathbb{Z}}$ and $(N_k)_{k\in\mathbb{Z}}$ define L^p-Fourier multipliers. It follows from Proposition 3.2 that the sets $\{k^2MN_k : k \in \mathbb{Z}\}$ and $\{N_k : k \in \mathbb{Z}\}$ are R-bounded. We have shown that the implication (i) $\Rightarrow$ (ii) is true. This completes the proof. ∎

We notice that, without the hypothesis that X is a UMD Banach space, the implication (i) $\Rightarrow$ (ii) remains valid under the same assumptions on A, M and the

https://doi.org/10.4153/S0008439526102136 Published online by Cambridge University Press

measures μ, ν, η, and τ. Furthermore, since the second assertion in Theorem 3.4 does not depend on the parameter $1 < p < \infty$, we have the following easy but useful consequence.

Corollary 3.5 *Let X be a UMD Banach space. Assume that A and M are closed operators in X such that $D(A) \subset D(M)$, μ, ν, η, and τ are bounded scalar measures on $\mathbb{R}$ such that $(\alpha_k)_{k\in\mathbb{Z}}$ and $(\beta_k)_{k\in\mathbb{Z}}$ are 1-regular. Then if (P_5) is L^p-well-posed for one $1 < p < \infty$, then it is L^p-well-posed for all $1 < p < \infty$.*

4 $B^s_{p,q}$-well-posedness and $F^s_{p,q}$-well-posedness of (P_5)

In this section, we study $B^s_{p,q}$-well-posedness and $F^s_{p,q}$-well-posedness of (P_5). Firstly, we briefly recall the definition of periodic Besov spaces in the vector-valued case introduced in [6]. Let $\mathcal{S}(\mathbb{R})$ be the Schwartz space of all rapidly decreasing smooth functions on $\mathbb{R}$. Let $\mathcal{D}(\mathbb{T})$ be the space of all infinitely differentiable functions on $\mathbb{T}$ equipped with the locally convex topology given by the seminorms $\|f\|_\alpha = sup_{x\in\mathbb{T}}|f^{(\alpha)}(x)|$ for $\alpha \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let $\mathcal{D}'(\mathbb{T}; X) = \mathcal{L}(\mathcal{D}(\mathbb{T}), X)$ be the space of all bounded linear operators from $\mathcal{D}(\mathbb{T})$ to X. In order to define Besov spaces, we consider the dyadic-like subsets of $\mathbb{R}$:

$$I_0 = \left\{t \in \mathbb{R} : \ |t| \le 2\right\}, \ I_k = \left\{t \in \mathbb{R} : \ 2^{k-1} < |t| \le 2^{k+1}\right\}$$

for $k \in \mathbb{N}$. Let $\phi(\mathbb{R})$ be the set of all systems $\phi = (\phi_k)_{k\in\mathbb{N}_0} \subset \mathcal{S}(\mathbb{R})$ satisfying $supp(\phi_k) \subset \bar{I_k}$ for each $k \in \mathbb{N}_0$,

$$\sum_{k\in\mathbb{N}_0} \phi_k(x) = 1 \qquad \text{for} \quad x \in \mathbb{R},$$

and for each $\alpha \in \mathbb{N}_0$,

$$sup_{\substack{x\in\mathbb{R}\\ k\in\mathbb{N}_0}} 2^{k\alpha}|\phi_k^{(\alpha)}(x)| < \infty.$$

Let $\phi = (\phi_k)_{k\in\mathbb{N}_0} \in \phi(\mathbb{R})$ be fixed. For $1 \le p, q \le \infty$, $s \in \mathbb{R}$, the X-valued periodic Besov space is defined by

$$B^s_{p,q}(\mathbb{T}; X) = \Bigg\{ f \in \mathcal{D}'(\mathbb{T}; X) : \ \|f\|_{B^s_{p,q}} = \left(\sum_{j\ge 0} 2^{sjq} \left\| \sum_{k\in\mathbb{Z}} e_k \otimes \phi_j(k)\hat{f}(k) \right\|_p^q \right)^{1/q} < \infty \Bigg\}$$

with the usual modification if $q = \infty$. The space $B^s_{p,q}(\mathbb{T}; X)$ is independent from the choice of ϕ and different choices of ϕ lead to equivalent norms $\|\cdot\|_{B^s_{p,q}}$ on $B^s_{p,q}(\mathbb{T}; X)$. $B^s_{p,q}(\mathbb{T}; X)$ equipped with the norm $\|\cdot\|_{B^s_{p,q}}$ is a Banach space. When $1 \le p < \infty$ and $s > 0$, we have $B^s_{p,q}(\mathbb{T}; X) \subset L^p(\mathbb{T}; X)$ and the embedding is continuous. See [6, Section 2] for more information about the space $B^s_{p,q}(\mathbb{T}; X)$.

Let $1 \le p, q \le \infty$, and $s > 0$. We say that (P_5) is $B^s_{p,q}$-well-posed, if for each $f \in B^s_{p,q}(\mathbb{T}; X)$, there exists a unique $u \in B^s_{p,q}(\mathbb{T}; D(A))$ such that $Mu \in B^{s+2}_{p,q}(\mathbb{T}; X)$ and

https://doi.org/10.4153/S0008439526102136 Published online by Cambridge University Press

(P_5) is satisfied a.e. on $\mathbb{T}$. We recall that when $u \in B^{s+2}_{p,q}(\mathbb{T}; X)$ for some $s > 0$, then u is twice differentiable a.e. on $\mathbb{T}$ and $u' \in B^{s+1}_{p,q}(\mathbb{T}; X)$, $u'' \in B^{s}_{p,q}(\mathbb{T}; X)$ [6]. We remark that when (P_5) is $B^s_{p,q}$-well-posed, then each term appeared in (P_5) is in $B^s_{p,q}(\mathbb{T}; X)$, this follows from Young's inequality.

The solution space of the $B^s_{p,q}$-well-posedness of (P_5) is defined by

$$S_{p,q,s}(A, M) := \left\{u \in B^s_{p,q}(\mathbb{T}; D(A)) : \ Mu \in B^{s+2}_{p,q}(\mathbb{T}; X)\right\},$$

here we consider $D(A)$ and $D(M)$ as Banach spaces equipped with their graph norms. $S_{p,q,s}(A, M)$ is a Banach space equipped with the norm

$$\|u\|_{S_{p,q,s}(A,M)} := \|u\|_{B^s_{p,q}} + \|Au\|_{B^s_{p,q}} + \|(Mu)'\|_{B^s_{p,q}} + \|(Mu)''\|_{B^s_{p,q}}.$$

If $u \in S_{p,q,s}(A, M)$, then Mu and $(Mu)'$ are X-valued continuous functions on $\mathbb{T}$, and $Mu(0) = Mu(2\pi)$, $(Mu)'(0) = (Mu)'(2\pi)$ by [5, Lemma 2.1].

If (P_5) is $B^s_{p,q}$-well-posed and $u \in S_{p,q,s}(A, M)$ is the unique strong $B^s_{p,q}$-solution of (P_5), then there exists a constant $C > 0$ such that for each $f \in B^s_{p,q}(\mathbb{T}; X)$, we have

$$\|u\|_{S_{p,q,s}(A,M)} \le C\|f\|_{B^s_{p,q}}. \tag{4.1}$$

This can be obtained easily from the Closed Graph Theorem.

The main tool in the study of $B^s_{p,q}$-well-posedness of (P_5) is the operator-valued Fourier multiplier result established in [6].

Definition 4.1 Let X be a Banach space, $1 \le p, q \le \infty$, $s \in \mathbb{R}$, and let $(M_k)_{k\in\mathbb{Z}} \subset \mathcal{L}(X)$. We say that $(M_k)_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$-multiplier, if for each $f \in B^s_{p,q}(\mathbb{T}; X)$, there exists a (unique) $u \in B^s_{p,q}(\mathbb{T}; X)$, such that $\hat{u}(k) = M_k\hat{f}(k)$ for all $k \in \mathbb{Z}$.

The following result has been obtained in [6].

Theorem 4.2 *Let X be a Banach space, $1 \le p, q \le \infty$, $s \in \mathbb{R}$, and let $(M_k)_{k\in\mathbb{Z}} \subset \mathcal{L}(X)$.*

(i) *If $(M_k)_{k\in\mathbb{Z}}$ is M-bounded of order 2, then $(M_k)_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$-multiplier.*
(ii) *If X is B-convex and $(M_k)_{k\in\mathbb{Z}}$ is M-bounded of order 1, then $(M_k)_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$-multiplier.*

Recall that a Banach space X is B-convex if it does not contain l^n_1 uniformly. This is equivalent to say that X has Fourier type $1 < p \le 2$, that is, the Fourier transform is a bounded linear operator from $L^p(\mathbb{R}; X)$ to $\ell^q(\mathbb{Z}; X)$, where $1/p + 1/q = 1$. It is well known that when $1 < p < \infty$, then $L^p(\mu)$ has Fourier type $\min\{p, \frac{p}{p-1}\}$.

In order to prove our main result, we need the following facts.

Proposition 4.3 *Let A and M be closed linear operators defined on a Banach space X satisfying $D(A) \subset D(M)$, where $1 \le p, q \le \infty$, and $s > 0$. Assume that $\rho_M(A; \mu, \nu, \eta, \tau) = \mathbb{Z}$ and the sets $\{k^2MN_k : \ k \in \mathbb{Z}\}$, $\{AN_k : \ k \in \mathbb{Z}\}$, and $\{N_k : \ k \in \mathbb{Z}\}$ are norm bounded. If μ, ν, η, and τ are bounded scalar measures on $\mathbb{R}$ such that $(\alpha_k)_{k\in\mathbb{Z}}$ and $(\beta_k)_{k\in\mathbb{Z}}$ are 2-regular, where N_k, α_k, and β_k are defined by (2.1). Then $(k^2MN_k)_{k\in\mathbb{Z}}$, $(N_k)_{k\in\mathbb{Z}}$, $(MN_k)_{k\in\mathbb{Z}}$, and $(AN_k)_{k\in\mathbb{Z}}$ are $B^s_{p,q}$-Fourier multipliers.*

https://doi.org/10.4153/S0008439526102136 Published online by Cambridge University Press

Proof Let $L_k = (N_k^{-1} - N_{k+1}^{-1})N_k$ when $k \in \mathbb{Z}$. Then the set $\{kL_k : k \in \mathbb{Z}\}$ is norm bounded by (3.4). To show that $(k^2MN_k)_{k\in\mathbb{Z}}$, $(N_k)_{k\in\mathbb{Z}}$, $(MN_k)_{k\in\mathbb{Z}}$, and $(AN_k)_{k\in\mathbb{Z}}$ are $B^s_{p,q}$-Fourier multipliers, we only need to prove that the set $\{k^2\Delta L_k : k \in \mathbb{Z}\}$ is norm bounded by Theorems 3.3 and 4.2. We observe

$$\begin{aligned}
\Delta L_k &= L_{k+1} - L_k \\
&= -\Delta\alpha_{k+1}MN_{k+1} + \Delta\beta_{k+1}AN_{k+1} + \Delta\alpha_k MN_k - \Delta\beta_k AN_k \\
&= -\Delta^2\alpha_k MN_{k+1} - \Delta\alpha_k M(N_{k+1} - N_k) + \Delta^2\beta_k AN_{k+1} \\
&\quad + \Delta\beta_k A(N_{k+1} - N_k) \\
&= -\frac{\Delta^2\alpha_k}{\alpha_k}\frac{\alpha_k}{\alpha_{k+1}}(\alpha_{k+1}MN_{k+1}) - \frac{\Delta\alpha_k}{\alpha_k}\frac{\alpha_k}{\alpha_{k+1}}(\alpha_{k+1}MN_{k+1})L_k \\
&\quad + \frac{\Delta^2\beta_k}{\beta_k}\frac{\beta_k}{\beta_{k+1}}(\beta_{k+1}AN_{k+1}) + \frac{\Delta\beta_k}{\beta_k}\frac{\beta_k}{\beta_{k+1}}(\beta_{k+1}AN_{k+1})L_k
\end{aligned} \tag{4.2}$$

when $|k| \geq k_0$. Since $\{k^2MN_k : k \in \mathbb{Z}\}$ is norm bounded, we have that $\{\alpha_k MN_k : k \in \mathbb{Z}\}$, $\{kMN_k : k \in \mathbb{Z}\}$, and $\{MN_k : k \in \mathbb{Z}\}$ are also norm bounded. Here, we have used the fact that μ, ν, η, and τ are bounded scalar measures on $\mathbb{R}$. Moreover, we deduce by (3.4) that $\{\beta_k AN_k : k \in \mathbb{Z}\}$ is also norm bounded. It follows from (3.4) that $\{kL_k : k \in \mathbb{Z}\}$ is norm bounded, here we have used the assumption that $(\alpha_k)_{k\in\mathbb{Z}}$ and $(\beta_k)_{k\in\mathbb{Z}}$ are 2-regular. Therefore, we obtain by (4.2) that the set $\{k^2\Delta L_k : k \in \mathbb{Z}\}$ is norm bounded. The proof is completed. ∎

Now we are ready to state our main result of this section.

Theorem 4.4 *Let X be a Banach space and let $1 \leq p, q \leq \infty$, and $s > 0$. Assume that A and M are closed operators in X such that $D(A) \subset D(M)$, μ, ν, η, and τ are bounded scalar measures on $\mathbb{R}$ such that $(\alpha_k)_{k\in\mathbb{Z}}$ and $(\beta_k)_{k\in\mathbb{Z}}$ are 2-regular. Then the following assertions are equivalent:*

(i) (P_5) *is $B^s_{p,q}$-well-posed;*

(ii) $\rho_M(A; \mu, \nu, \eta, \tau) = \mathbb{Z}$ *and* $\{k^2MN_k : k \in \mathbb{Z}\}$, $\{AN_k : k \in \mathbb{Z}\}$, $\{N_k : k \in \mathbb{Z}\}$ *are norm bounded, where N_k is defined by (2.1).*

Proof Now we assume that (ii) is true and we are going to show that (P_5) is $B^s_{p,q}$-well-posed. We first show the uniqueness. Let $u \in S_{p,q,s}(A, M)$ such that

$$\mu * (Mu)''(t) + \nu * (Mu)'(t) + \eta * (Mu)(t) - \tau * Au(t) = 0$$

a.e. on $\mathbb{T}$ and $Mu(0) = Mu(2\pi)$, $(Mu)'(0) = (Mu)'(2\pi)$. Taking Fourier coefficient on both sides, we find that $\big[(-k^2\tilde{\mu}(k) + ik\tilde{\nu}(k) + \tilde{\eta}(k))M - \tilde{\tau}(k)A\big]\hat{u}(k) = 0$ for $k \in \mathbb{Z}$. Here, we have used (2.2) and the fact that when $h \in B^s_{p,q}(\mathbb{T}; D(A))$, then $\hat{h}(k) \in D(A)$ for all $k \in \mathbb{Z}$ and $A\hat{h}(k) = \widehat{Ah}(k)$ [5, Lemma 3.1]. It follows that $\hat{u}(k) = 0$ for $k \in \mathbb{Z}$ as $\rho_M(A; \mu, \nu, \eta, \tau) = \mathbb{Z}$ by assumption. Thus, $u = 0$ by the Uniqueness Theorem [5, p. 314]. We have shown the uniqueness.

Now we show the existence. By Proposition 4.3, we obtain that $(MN_k)_{k\in\mathbb{Z}}$, $(k^2MN_k)_{k\in\mathbb{Z}}$, $(N_k)_{k\in\mathbb{Z}}$, and $(AN_k)_{k\in\mathbb{Z}}$ are $B^s_{p,q}$-multipliers. Then there exists $u_1, u_2, u_3, u \in B^s_{p,q}(\mathbb{T}; X)$ such that

https://doi.org/10.4153/S0008439526102136 Published online by Cambridge University Press

$$\hat{u}(k) = N_k \hat{f}(k),\ \widehat{u_1}(k) = AN_k \hat{f}(k) \tag{4.3}$$

and

$$\widehat{u_2}(k) = MN_k \hat{f}(k),\ \widehat{u_3}(k) = -k^2 MN_k \hat{f}(k) \tag{4.4}$$

for $k \in \mathbb{Z}$. We deduce from (4.3) that $\hat{u}(k) \in D(A)$ and $\widehat{u_1}(k) = AN_k \hat{f}(k) = A\hat{u}(k)$ for $k \in \mathbb{Z}$. Hence, $u \in B^s_{p,q}(\mathbb{T}; D(A))$ by [5, Lemma 3.1]. From (4.4) that $\widehat{u_2}(k) = MN_k \hat{f}(k) = M\hat{u}(k)$ for $k \in \mathbb{Z}$. Hence, $Mu \in B^s_{p,q}(\mathbb{T}; X)$ by [5, Lemma 3.1]. Again by (4.4), we have $\widehat{u_3}(k) = -k^2 M\hat{u}(k)$ for $k \in \mathbb{Z}$. This implies that $Mu \in B^{s+2}_{p,q}(\mathbb{T}; X)$ and $Mu(0) = Mu(2\pi)$, $(Mu)'(0) = (Mu)'(2\pi)$, $(Mu)'' = u_3$.

From the equality $\hat{u}(k) = N_k \hat{f}(k)$, we deduce that

$$\left[(-k^2\tilde{\mu}(k) + ik\tilde{\nu}(k) + \tilde{\eta}(k))M - \tilde{\tau}(k)A\right]\hat{u}(k) = \hat{f}(k)$$

for all $k \in \mathbb{Z}$. This is equivalent to say that

$$\mu * (Mu)''(t) + \nu * (Mu)'(t) + \eta * (Mu)(t) - \tau * Au(t) = f(t)$$

a.e. on $\mathbb{T}$ by the Uniqueness Theorem [5, p. 314]. Here, we have used the fact that $\widehat{u'}(k) = ik\hat{u}(k)$ when $k \in \mathbb{Z}$. This shows the existence. Thus, the implication (ii) $\Rightarrow$ (i) is true.

Now we give the proof for the implication (i) $\Rightarrow$ (ii). Assume that (P_5) is $B^s_{p,q}$-well-posed. Let $k \in \mathbb{Z}$ and $y \in X$ be fixed, we consider the function f defined by $f(t) = e^{ikt}y$ when $t \in \mathbb{T}$. Then it is clear that $f \in B^s_{p,q}(\mathbb{T}; X)$, $\hat{f}(k) = y$, and $\hat{f}(n) = 0$ when $n \neq k$. Since (P_5) is $B^s_{p,q}$-well-posed, there exists a unique $u \in S_{p,q,s}(A, M)$ such that

$$\mu * (Mu)''(t) + \nu * (Mu)'(t) + \eta * (Mu)(t) - \tau * Au(t) = f(t) \tag{4.5}$$

a.e. on $\mathbb{T}$. We have $\hat{u}(n) \in D(A)$ when $n \in \mathbb{Z}$ by [5, Lemma 3.1] as $u \in B^s_{p,q}(\mathbb{T}; D(A))$. Taking Fourier transforms on both sides of (3.6), we obtain

$$\left[(-k^2\tilde{\mu}(k) + ik\tilde{\nu}(k) + \tilde{\eta}(k))M - \tilde{\tau}(k)A\right]\hat{u}(k) = y \tag{4.6}$$

and $\left[(-n^2\tilde{\mu}(n) + in\tilde{\nu}(n) + \tilde{\eta}(n))M - \tilde{\tau}(n)A\right]\hat{u}(n) = 0$ when $n \neq k$. This implies that the operator $\left[(-k^2\tilde{\mu}(k) + ik\tilde{\nu}(k) + \tilde{\eta}(k))M - \tilde{\tau}(k)A\right]$ defined on $D(A)$ with values in X is surjective. To show that it is also injective, we let $x \in D(A)$ be such that

$$\left[(-k^2\tilde{\mu}(k) + ik\tilde{\nu}(k) + \tilde{\eta}(k))M - \tilde{\tau}(k)A\right]x = 0.$$

Let u be the function given by $u(t) = e^{ikt}x$ when $t \in \mathbb{T}$, then it is clear that $u \in S_{p,q,s}(A, M)$ and (P_5) is satisfied a.e. on $\mathbb{T}$ when $f = 0$. Thus, u is a strong $B^s_{p,q}$-solution of (P_5) when taking $f = 0$. We get $x = 0$ by the uniqueness assumption. We have shown that the operator $\left[(-k^2\tilde{\mu}(k) + ik\tilde{\nu}(k) + \tilde{\eta}(k))M - \tilde{\tau}(k)A\right]$ from $D(A)$ into X is injective. Therefore, $\left[(-k^2\tilde{\mu}(k) + ik\tilde{\nu}(k) + \tilde{\eta}(k))M - \tilde{\tau}(k)A\right]$ is bijective from $D(A)$ onto X.

Next, we show that $\left[(-k^2\tilde{\mu}(k) + ik\tilde{\nu}(k) + \tilde{\eta}(k))M - \tilde{\tau}(k)A\right]^{-1} \in \mathcal{L}(X)$. For $f(t) = e^{ikt}y$, we let $u \in S_{p,q,s}(A, M)$ such that u is the unique strong $B^s_{p,q}$-solution of (P_5). Then

https://doi.org/10.4153/S0008439526102136 Published online by Cambridge University Press

$$\hat{u}(n) = \begin{cases} 0 & n \neq k, \\ \left[(-k^2\tilde{\mu}(k) + ik\tilde{\nu}(k) + \tilde{\eta}(k))M - \tilde{\tau}(k)A\right]^{-1} y & n = k, \end{cases} \tag{4.7}$$

by (3.7). This implies that u is given by

$$u(t) = e^{ikt}\left[(-k^2\tilde{\mu}(k) + ik\tilde{\nu}(k) + \tilde{\eta}(k))M - \tilde{\tau}(k)A\right]^{-1} y$$

when $t \in \mathbb{T}$. By (3.5), there exists a constant $C > 0$ independent from y and k, such that $\|u\|_{S_{p,q,s}(A,M)} \leq C\|f\|_{B^s_{p,q}}$. This implies that

$$\left\|\left[N(k)y\right\| \leq C\|y\|, \left\| - k^2 MN(k)y\right\| \leq C\|y\|$$

when $y \in X$, or equivalently,

$$\left\|\left[N(k)\right\| \leq C, \left\|k^2 MN(k)\right\| \leq C.$$

Therefore, we obtain that $k \in \rho_M(A; \mu, \nu, \eta, \tau)$ for all $k \in \mathbb{Z}$. We have shown that the implication (i) $\Rightarrow$ (ii) is true. This finishes the proof. ∎

Since the second assertion of Theorem 4.4 does not depend on the choice of $1 \leq p, q \leq \infty$, and $s > 0$, we deduce that (P_5) is $B^s_{p,q}$-well-posed for some $1 \leq p, q \leq \infty$, and $s > 0$ if and only if it is $B^s_{p,q}$-well-posed for all $1 \leq p, q \leq \infty$, and $s > 0$. Furthermore, by Theorem 4.2, if the underlying Banach space X is B-convex, then a first-order Marcinkiewicz-type condition is already sufficient for $(M_k)_{k\in\mathbb{Z}}$ to be a $B^s_{p,q}$-multiplier. This observation and the proof of Theorem 3.4 show that when X is B-convex, then Theorem 4.4 remains valid under a weaker condition on the measures μ, ν, η, and τ: we only need to assume that the scalar sequences involved are 1-regular. We recall that, in the proof of Theorem 3.4, we only use Theorem 3.1 in which only a Marcinkiewicz-type condition of order 1 is involved.

Corollary 4.5 *Let X be a B-convex Banach space and let $1 \leq p, q \leq \infty$, and $s > 0$. Assume that A and M are closed operators in X such that $D(A) \subset D(M)$, μ, ν, η, and τ are bounded scalar measures on $\mathbb{R}$ such that $(\alpha_k)_{k\in\mathbb{Z}}$ and $(\beta_k)_{k\in\mathbb{Z}}$ are 1-regular. Then the following assertions are equivalent:*

(i) *(P_5) is $B^s_{p,q}$-well-posed;*

(ii) *$\rho_M(A; \mu, \nu, \eta, \tau) = \mathbb{Z}$ and $\{k^2 MN_k : k \in \mathbb{Z}\}$, $\{AN_k : k \in \mathbb{Z}\}$, $\{N_k : k \in \mathbb{Z}\}$ are norm bounded, where N_k is defined by (2.1).*

Periodic Hölder continuous function space on $\mathbb{T}$ is a particular case of periodic Besov space $B^s_{p,q}(\mathbb{T}; X)$. It follows from [6, Theorem 3.1] that $B^\alpha_{\infty,\infty}(\mathbb{T}; X) = C^\alpha_{\text{per}}(\mathbb{T}; X)$ whenever $0 < \alpha < 1$, where $C^\alpha_{\text{per}}(\mathbb{T}; X)$ is the space of all X-valued functions f defined on $\mathbb{T}$ satisfying $f(0) = f(2\pi)$ and $\sup_{x\neq y} \frac{\|f(x)-f(y)\|}{|x-y|^\alpha} < \infty$. Moreover, $\|f\|_{C^\alpha_{\text{per}}} = \max_{t\in\mathbb{T}} \|f(t)\| + \sup_{x\neq y} \frac{\|f(x)-f(y)\|}{|x-y|^\alpha}$ on $C^\alpha_{\text{per}}(\mathbb{T}; X)$ defines an equivalent norm of $B^\alpha_{\infty,\infty}(\mathbb{T}; X)$. If $0 < \alpha < 1$, we say that (P_5) is C^α_{per}-well-posed, if for every $f \in C^\alpha_{\text{per}}(\mathbb{T}; X)$, there exists a unique $u(t) \in D(A)$ such that $Mu \in C^{\alpha+2}_{\text{per}}(\mathbb{T}; X)$ and equation (P_5) holds true for all $t \in \mathbb{T}$, and $u \in C^\alpha_{\text{per}}(\mathbb{T}; D(A))$, where $C^{\alpha+2}_{\text{per}}(\mathbb{T}; X)$ is the

https://doi.org/10.4153/S0008439526102136 Published online by Cambridge University Press

space of all functions $u \in C^2(\mathbb{T}; X)$ such that $u',\ u'' \in C^{\alpha}_{\mathrm{per}}(\mathbb{T}; X)$, $u(0) = u(2\pi)$, and $u'(0) = u'(2\pi)$. Theorem 4.4 and Corollary 4.5 have the following useful corollary.

Corollary 4.6 *Let X be a Banach space and let $0 < \alpha < 1$. Assume that A and M are closed operators in X such that $D(A) \subset D(M)$, μ, ν, η, and τ are bounded scalar measures on $\mathbb{R}$ such that $(\alpha_k)_{k\in\mathbb{Z}}$ and $(\beta_k)_{k\in\mathbb{Z}}$ are 2-regular. Then the following assertions are equivalent:*

(i) (P_5) *is $C^{\alpha}_{\mathrm{per}}$-well-posed;*

(ii) $\rho_M(A; \mu, \nu, \eta, \tau) = \mathbb{Z}$ *and* $\{k^2 M N_k : k \in \mathbb{Z}\}$, $\{AN_k :\ k \in \mathbb{Z}\}$, $\{N_k : k \in \mathbb{Z}\}$ *are norm bounded, where N_k is defined by (2.1).*

If furthermore X is B-convex, then the two assertions above are still equivalent if we replace the assumptions that $(\alpha_k)_{k\in\mathbb{Z}}$ and $(\beta_k)_{k\in\mathbb{Z}}$ are 2-regular by the weaker assumptions that they are 1-regular.

In the rest of this section, we study $F^s_{p,q}$-well-posedness of (P_5). We briefly recall the definition of periodic Triebel–Lizorkin spaces in the vector-valued case introduced in [10]. Let $\phi = (\phi_k)_{k\in\mathbb{N}_0} \in \phi(\mathbb{R})$ be fixed. For $1 \le p < \infty$, $1 \le q \le \infty$, and $s \in \mathbb{R}$, X-valued periodic Triebel–Lizorkin space is defined by

$$F^s_{p,q}(\mathbb{T}; X) = \Bigg\{ f \in \mathcal{D}'(\mathbb{T}; X) :\ \|f\|_{F^s_{p,q}} = \left\| \left(\sum_{j\ge 0} 2^{sjq} \left\| \sum_{k\in\mathbb{Z}} e_k \otimes \phi_j(k) \hat{f}(k) \right\|^q_X \right)^{1/q} \right\|_{L_p} < \infty \Bigg\}$$

with the usual modification if $q = \infty$. The space $F^s_{p,q}(\mathbb{T}; X)$ is independent from the choice of ϕ and different choices of ϕ lead to equivalent norms on $F^s_{p,q}(\mathbb{T}; X)$. $F^s_{p,q}(\mathbb{T}; X)$ equipped with the norm $\|\cdot\|_{F^s_{p,q}}$ is a Banach space. See [10, Section 2] for more information about the spaces $F^s_{p,q}(\mathbb{T}; X)$.

Let $1 \le p < \infty, 1 \le q \le \infty, s \in \mathbb{R}$, and let $(M_k)_{k\in\mathbb{Z}} \subset \mathcal{L}(X)$. We will say that $(M_k)_{k\in\mathbb{Z}}$ is an $F^s_{p,q}$-multiplier, if for each $f \in F^s_{p,q}(\mathbb{T}; X)$, there exists $g \in F^s_{p,q}(\mathbb{T}; X)$, such that $\hat{g}(k) = M_k \hat{f}(k)$ for all $k \in \mathbb{Z}$. In this case, it follows from the Closed Graph Theorem that there exists a constant $C > 0$ such that for $f \in F^s_{p,q}(\mathbb{T}; X)$, we have $\| \sum_{k\in\mathbb{Z}} e_k \otimes M_k \hat{f}(k) \|_{F^s_{p,q}} \le C \|f\|_{F^s_{p,q}}$.

Let $1 \le p < \infty$, $1 \le q \le \infty$, and $s > 0$. We say that (P_5) is $F^s_{p,q}$-well-posed, if for each $f \in F^s_{p,q}(\mathbb{T}; X)$, there exists a unique $u \in F^s_{p,q}(\mathbb{T}; D(A))$ such that $Mu \in F^{s+2}_{p,q}(\mathbb{T}; X)$ and (P_5) is satisfied a.e. on $\mathbb{T}$. We recall that when $u \in F^{s+2}_{p,q}(\mathbb{T}; X)$ for some $s > 0$, then u is twice differentiable a.e. on $\mathbb{T}$ and $u' \in F^{s+1}_{p,q}(\mathbb{T}; X),\ u'' \in F^s_{p,q}(\mathbb{T}; X)$ [10]. We remark that when (P_5) is $F^s_{p,q}$-well-posed, then each term appeared in (P_5) is in $F^s_{p,q}(\mathbb{T}; X)$, this follows from Young's inequality.

The following $F^s_{p,q}$-multiplier theorem is due to Bu and Kim [10, Theorem 3.2].

Theorem 4.7 *Let X be a Banach space and let $(M_k)_{k\in\mathbb{Z}} \subset \mathcal{L}(X)$. Assume that $(M_k)_{k\in\mathbb{Z}}$ is M-bounded of order 3, then it is an $F^s_{p,q}$-multiplier whenever $1 \le p < \infty$,*

https://doi.org/10.4153/S0008439526102136 Published online by Cambridge University Press

$1 \le q \le \infty$, and $s \in \mathbb{R}$. If $(M_k)_{k\in\mathbb{Z}}$ is M-bounded of order 2, then it is an $F^s_{p,q}$-multiplier whenever $1 < p < \infty$, $1 < q \le \infty$, and $s \in \mathbb{R}$.

In order to prove $F^s_{p,q}$-well-posedness of (P_5), we need the following preparations.

Lemma 4.8 *Let $(c_k)_{\in\mathbb{Z}} \subset \mathbb{C}$ be a 3-regular sequence and denote $d_k = \frac{\Delta c_k}{c_k}$, then the sequences $(k^2\Delta d_k)_{\in\mathbb{Z}}$ and $(k^3\Delta^2 d_k)_{\in\mathbb{Z}}$ are bounded.*

Proof By the definition of d_n, we have

$$\begin{aligned}\Delta d_k &= \frac{\Delta c_{k+1}}{c_{k+1}} - \frac{\Delta c_k}{c_k}\\ &= \frac{\Delta c_{k+1} - \Delta c_k}{c_{k+1}} + \Big(\frac{1}{c_{k+1}} - \frac{1}{c_k}\Big)\Delta c_k\\ &= \Delta\Big(\frac{1}{c_k}\Big)\Delta c_k + \frac{\Delta^2 c_k}{c_{k+1}}, \qquad (4.8)\end{aligned}$$

which implies

$$\begin{aligned}\Delta^2 d_k &= \Delta\Big(\frac{1}{c_{k+1}}\Big)\Delta c_{k+1} + \frac{\Delta^2 c_{k+1}}{c_{k+2}} - \Delta\Big(\frac{1}{c_k}\Big)\Delta c_k - \frac{\Delta^2 c_k}{c_{k+1}}\\ &= \Delta^2\Big(\frac{1}{c_k}\Big)(\Delta c_{k+1}) + \Delta\Big(\frac{1}{c_k}\Big)(\Delta^2 c_k) + \Delta\Big(\frac{1}{c_{k+1}}\Big)\Delta^2 c_{k+1} + \frac{1}{c_{k+1}}(\Delta^3 c_k) \qquad (4.9)\end{aligned}$$

when $|k| \ge k_0$. Therefore, we obtain that the sequences $(k^2\Delta d_k)_{\in\mathbb{Z}}$ and $(k^3\Delta^2 d_k)_{\in\mathbb{Z}}$ are bounded as $(c_k)_{\in\mathbb{Z}}$ is 3-regular. This completes the proof. ∎

Theorem 4.9 *Let $T : D(T) \subset X \to X$ be a closed linear operator defined in Banach space X. For each $k \in \mathbb{Z}$, let $H_k : X \to D(T)$ be a sequence of bounded and linear operators such that $0 \in \rho(H_k)$ for all $k \in \mathbb{Z}$. Let $M_k := c_k T H_k$ and $L_k := (H_k^{-1} - H_{k+1}^{-1})H_k$. If $(c_k)_{k\in\mathbb{Z}}$ is 3-regular and the sets $\{M_k : k \in \mathbb{Z}\}$, $\{k^3\Delta^2 L_k : k \in \mathbb{Z}\}$, $\{k^2\Delta L_k : k \in \mathbb{Z}\}$, and $\{kL_k : k \in \mathbb{Z}\}$ are norm bounded, then $\{M_k : k \in \mathbb{Z}\}$ is M-bounded of order 3.*

Proof By virtue of the definition of M_k, we observe

$$\begin{aligned}\Delta M_k &= c_{k+1} T H_{k+1} - c_k T H_k\\ &= c_{k+1} T \Delta H_k + (\Delta c_k) T H_k\\ &= c_{k+1} T H_{k+1}(H_k^{-1} - H_{k+1}^{-1})H_k + \frac{\Delta c_k}{c_k} c_k T H_k.\\ &= d_k M_k + M_{k+1} L_k. \qquad (4.10)\end{aligned}$$

It follows that

$$\begin{aligned}\Delta^2 M_k &= d_{k+1} M_{k+1} + M_{k+2} L_{k+1} - d_k M_k - M_{k+1} L_k\\ &= d_{k+1}\Delta M_k + (\Delta d_k) M_k + M_{k+2}(\Delta L_k) + (\Delta M_{k+1}) L_k. \qquad (4.11)\end{aligned}$$

https://doi.org/10.4153/S0008439526102136 Published online by Cambridge University Press

Moreover, we obtain

$$\begin{aligned}\Delta^3 M_k &= d_{k+2}\Delta M_{k+1} + (\Delta d_{k+1})M_{k+1} + M_{k+3}(\Delta L_{k+1}) + (\Delta M_{k+2})L_{k+1}\\ &\quad - d_{k+1}\Delta M_k - (\Delta d_k)M_k - M_{k+2}(\Delta L_k) - (\Delta M_{k+1})L_k\\ &= d_{k+2}(\Delta^2 M_k) + (\Delta d_{k+1})(\Delta M_k) + (\Delta d_{k+1})(\Delta M_k) + (\Delta^2 d_k)M_k\\ &\quad + M_{k+3}(\Delta^2 L_k) + (\Delta M_{k+2})(\Delta L_k) + (\Delta M_{k+2})(\Delta L_k) + (\Delta^2 M_{k+1})L_k. \qquad (4.12)\end{aligned}$$

By Lemma 4.8, we obtain that sequences $(k^2\Delta d_k)_{\epsilon\mathbb{Z}}$ and $(k^3\Delta^2 d_k)_{\epsilon\mathbb{Z}}$ are bounded. Thus, we have from (4.10) that $\{k\Delta M_k : k \in \mathbb{Z}\}$ is norm bounded as $(kd_k)_{\epsilon\mathbb{Z}}$ and $\{kL_k : k \in \mathbb{Z}\}$ are bounded. Furthermore, we conclude from (4.11) that $\{k^2\Delta^2 M_k : k \in \mathbb{Z}\}$ is norm bounded as $(kd_k)_{\epsilon\mathbb{Z}}$, $(k^2\Delta d_k)_{\epsilon\mathbb{Z}}$, $\{kL_k : k \in \mathbb{Z}\}$, and $\{k^2\Delta L_k : k \in \mathbb{Z}\}$ are bounded. Therefore, we get from (4.12) that $\{M_k : k \in \mathbb{Z}\}$ is M-bounded of order 3. The proof is completed. ∎

Proposition 4.10 *Let A and M be closed linear operators defined on a Banach space X satisfying $D(A) \subset D(M)$, where $1 \le p < \infty$, $1 \le q \le \infty$, and $s > 0$. Assume that $\rho_M(A) = \mathbb{Z}$ and the sets $\{k^2 MN_k : k \in \mathbb{Z}\}$, $\{AN_k : k \in \mathbb{Z}\}$, and $\{N_k : k \in \mathbb{Z}\}$ are norm bounded. If μ, ν, η, and τ are bounded scalar measures on $\mathbb{R}$ such that $(\alpha_k)_{k\in\mathbb{Z}}$ and $(\beta_k)_{k\in\mathbb{Z}}$ are 3-regular, where N_k, α_k, and β_k are defined by (2.1). Then $(k^2 MN_k)_{k\in\mathbb{Z}}$, $(N_k)_{k\in\mathbb{Z}}$, $(MN_k)_{k\in\mathbb{Z}}$, and $(AN_k)_{k\in\mathbb{Z}}$ are $F^s_{p,q}$-Fourier multipliers.*

Proof Let $L_k = (N_k^{-1} - N_{k+1}^{-1})N_k$ when $k \in \mathbb{Z}$. By Proposition 4.3, we obtain that the set $\{k^2\Delta L_k : k \in \mathbb{Z}\}$ is norm bounded. In order to finish the proof, we only show that the set $\{k^3\Delta^2 L_k : k \in \mathbb{Z}\}$ is norm bounded by Theorem 4.9. In fact, it follows from (4.2) that

$$\Delta L_k = -(\Delta^2\alpha_k)MN_{k+1} - (\Delta\alpha_k)MN_{k+1}L_k + (\Delta^2\beta_k)AN_{k+1} + (\Delta\beta_k)AN_{k+1}L_k,$$

which implies

$$\begin{aligned}&\Delta^2 L_k\\ &= -(\Delta^2\alpha_{k+1})MN_{k+2} - (\Delta\alpha_{k+1})MN_{k+2}L_{k+1} + (\Delta^2\beta_{k+1})AN_{k+2} + (\Delta\beta_{k+1})AN_{k+2}L_{k+1}\\ &\quad + (\Delta^2\alpha_k)MN_{k+1} + (\Delta\alpha_k)MN_{k+1}L_k - (\Delta^2\beta_k)AN_{k+1} - (\Delta\beta_k)AN_{k+1}L_k\\ &= -(\Delta^3\alpha_k)MN_{k+2} - (\Delta^2\alpha_k)MN_{k+2}L_{k+1} - (\Delta^2\alpha_k)MN_{k+2}L_{k+1}\\ &\quad - (\Delta\alpha_k)M(N_{k+2}L_{k+1} - N_{k+1}L_k) + (\Delta^3\beta_k)AN_{k+2} + (\Delta^2\beta_k)AN_{k+2}L_{k+1}\\ &\quad + (\Delta^2\beta_k)AN_{k+2}L_{k+1} + (\Delta\beta_k)A(N_{k+2}L_{k+1} - N_{k+1}L_k)\\ &= -(\Delta^3\alpha_k)MN_{k+2} - (\Delta^2\alpha_k)MN_{k+2}L_{k+1} - (\Delta^2\alpha_k)MN_{k+2}L_{k+1}\\ &\quad - (\Delta\alpha_k)MN_{k+2}L_{k+1}^2 - (\Delta\alpha_k)MN_{k+1}(\Delta L_k) + (\Delta^3\beta_k)AN_{k+2} + (\Delta^2\beta_k)AN_{k+2}L_{k+1}\\ &\quad + (\Delta^2\beta_k)AN_{k+2}L_{k+1} + (\Delta\beta_k)AN_{k+2}L_{k+1}^2 + (\Delta\beta_k)AN_{k+1}(\Delta L_k)\end{aligned}$$

when $|k| \ge k_0$. Therefore, the set $\{k^3\Delta^2 L_k : k \in \mathbb{Z}\}$ is norm bounded as the sets $\{kL_k : k \in \mathbb{Z}\}$, $\{k^2\Delta L_k : k \in \mathbb{Z}\}$ and $\{\alpha_k MN_k : k \in \mathbb{Z}\}$, $\{\beta_k AN_k : k \in \mathbb{Z}\}$ are norm bounded. This finishes the proof. ∎

https://doi.org/10.4153/S0008439526102136 Published online by Cambridge University Press

The solution space of the $F^s_{p,q}$-well-posedness of (P_5) is defined by

$$S'_{p,q,s}(A,M) := \{u \in F^s_{p,q}(\mathbb{T};D(A)) : \ Mu \in F^{s+2}_{p,q}(\mathbb{T};X)\},$$

here we consider $D(A)$ and $D(M)$ as Banach spaces equipped with their graph norms. $S'_{p,q,s}(A,M)$ is a Banach space equipped with the norm

$$\|u\|_{S'_{p,q,s}(A,M)} := \|u\|_{F^s_{p,q}} + \|Au\|_{F^s_{p,q}} + \|(Mu)'\|_{F^s_{p,q}} + \|(Mu)''\|_{F^s_{p,q}}.$$

If $u \in S'_{p,q,s}(A,M)$, then Mu and $(Mu)'$ are X-valued continuous functions on $\mathbb{T}$, and $Mu(0) = Mu(2\pi)$, $(Mu)'(0) = (Mu)'(2\pi)$ by [5, Lemma 2.1].

If (P_5) is $F^s_{p,q}$-well-posed and $u \in S'_{p,q,s}(A,M)$ is the unique strong $F^s_{p,q}$-solution of (P_5), then there exists a constant $C > 0$ such that for each $f \in F^s_{p,q}(\mathbb{T};X)$, we have

$$\|u\|_{S'_{p,q,s}(A,M)} \le C\|f\|_{F^s_{p,q}}. \tag{4.13}$$

This can be obtained easily from the Closed Graph Theorem.

Theorem 4.11 *Let X be a Banach space and let $1 \le p < \infty$, $1 \le q \le \infty$, and $s > 0$. Assume that A and M are closed operators in X such that $D(A) \subset D(M)$, μ, ν, η, and τ are bounded scalar measures on $\mathbb{R}$ such that $(\alpha_k)_{k\in\mathbb{Z}}$ and $(\beta_k)_{k\in\mathbb{Z}}$ are 3-regular, where α_k and β_k are defined by (2.1). Then the following assertions are equivalent:*

(i) *(P_5) is $F^s_{p,q}$-well-posed.*

(ii) *$\rho_M(A;\mu,\nu,\eta,\tau) = \mathbb{Z}$ and $\{k^2MN_k : k \in \mathbb{Z}\}$, $\{AN_k : \ k \in \mathbb{Z}\}$, $\{N_k : k \in \mathbb{Z}\}$ are norm bounded, where N_k is defined by (2.1).*

Proof Now we assume that (ii) is true and we are going to show that (P_5) is $F^s_{p,q}$-well-posed. We first show the uniqueness. Let $u \in S'_{p,q,s}(A,M)$ such that

$$\mu * (Mu)''(t) + \nu * (Mu)'(t) + \eta * (Mu)(t) - \tau * Au(t) = 0$$

a.e. on $\mathbb{T}$ and $Mu(0) = Mu(2\pi)$, $(Mu)'(0) = (Mu)'(2\pi)$. Taking Fourier coefficient on both sides, we find that $\big[(-k^2\tilde{\mu}(k) + ik\tilde{\nu}(k) + \tilde{\eta}(k))M - \tilde{\tau}(k)A\big]\hat{u}(k) = 0$ for $k \in \mathbb{Z}$. Here, we have used (2.2) and the fact that when $h \in F^s_{p,q}(\mathbb{T};D(A))$, then $\hat{h}(k) \in D(A)$ for all $k \in \mathbb{Z}$ and $A\hat{h}(k) = \widehat{Ah}(k)$ [5, Lemma 3.1]. It follows that $\hat{u}(k) = 0$ for $k \in \mathbb{Z}$ as $\rho_M(A;\mu,\nu,\eta,\tau) = \mathbb{Z}$ by assumption. Thus, $u = 0$ by the Uniqueness Theorem [5, p. 314]. We have shown the uniqueness.

Now we show the existence. By Proposition 4.10, we obtain that $(MN_k)_{k\in\mathbb{Z}}$, $(k^2MN_k)_{k\in\mathbb{Z}}$, $(N_k)_{k\in\mathbb{Z}}$, and $(AN_k)_{k\in\mathbb{Z}}$ are $F^s_{p,q}$-multipliers. Then there exists $u_1, u_2, u_3, u \in F^s_{p,q}(\mathbb{T};X)$ such that

$$\hat{u}(k) = N_k\hat{f}(k), \ \widehat{u_1}(k) = AN_k\hat{f}(k) \tag{4.14}$$

and

$$\widehat{u_2}(k) = MN_k\hat{f}(k), \ \widehat{u_3}(k) = -k^2MN_k\hat{f}(k) \tag{4.15}$$

for $k \in \mathbb{Z}$. We deduce from (4.14) that $\hat{u}(k) \in D(A)$ and $\widehat{u_1}(k) = AN_k\hat{f}(k) = A\hat{u}(k)$ for $k \in \mathbb{Z}$. Hence, $u \in F^s_{p,q}(\mathbb{T};D(A))$ by [5, Lemma 3.1]. From (4.15) that $\widehat{u_2}(k) =$

https://doi.org/10.4153/S0008439526102136 Published online by Cambridge University Press

$MN_k\hat{f}(k) = M\hat{u}(k)$ for $k \in \mathbb{Z}$. Hence, $Mu \in F^s_{p,q}(\mathbb{T}; X)$ by [5, Lemma 3.1]. Again by (4.15), we have $\widehat{u_3}(k) = -k^2 M\hat{u}(k)$ for $k \in \mathbb{Z}$. This implies that $Mu \in F^{s+2}_{p,q}(\mathbb{T}; X)$ and $Mu(0) = Mu(2\pi)$, $(Mu)'(0) = (Mu)'(2\pi)$, $(Mu)'' = u_3$.

From the equality $\hat{u}(k) = N_k\hat{f}(k)$, we deduce that

$$\big[(-k^2\tilde{\mu}(k) + ik\tilde{\nu}(k) + \tilde{\eta}(k))M - \tilde{\tau}(k)A\big]\hat{u}(k) = \hat{f}(k)$$

for all $k \in \mathbb{Z}$. This is equivalent to say that

$$\mu * (Mu)''(t) + \nu * (Mu)'(t) + \eta * (Mu)(t) - \tau * Au(t) = f(t)$$

a.e. on $\mathbb{T}$ by the Uniqueness Theorem [5, p. 314]. Here, we have used the fact that $\widehat{u'}(k) = ik\hat{u}(k)$ when $k \in \mathbb{Z}$. This shows the existence. Thus, the implication (ii) $\Rightarrow$ (i) is true.

Now we give the proof for the implication (i) $\Rightarrow$ (ii). Assume that (P_5) is $F^s_{p,q}$-well-posed. Let $k \in \mathbb{Z}$ and $y \in X$ be fixed, we consider the function f defined by $f(t) = e^{ikt}y$ when $t \in \mathbb{T}$. Then it is clear that $f \in F^s_{p,q}(\mathbb{T}; X)$, $\hat{f}(k) = y$, and $\hat{f}(n) = 0$ when $n \neq k$. Since (P_5) is $F^s_{p,q}$-well-posed, there exists a unique $u \in S'_{p,q,s}(A, M)$ such that

$$\mu * (Mu)''(t) + \nu * (Mu)'(t) + \eta * (Mu)(t) - \tau * Au(t) = f(t) \tag{4.16}$$

a.e. on $\mathbb{T}$. We have $\hat{u}(n) \in D(A)$ when $n \in \mathbb{Z}$ by [5, Lemma 3.1] as $u \in F^s_{p,q}(\mathbb{T}; D(A))$. Taking Fourier transforms on both sides of (4.16), we obtain

$$\big[(-k^2\tilde{\mu}(k) + ik\tilde{\nu}(k) + \tilde{\eta}(k))M - \tilde{\tau}(k)A\big]\hat{u}(k) = y \tag{4.17}$$

and $\big[(-n^2\tilde{\mu}(n) + in\tilde{\nu}(n) + \tilde{\eta}(n))M - \tilde{\tau}(n)A\big]\hat{u}(n) = 0$ when $n \neq k$. This implies that the operator $\big[(-k^2\tilde{\mu}(k) + ik\tilde{\nu}(k) + \tilde{\eta}(k))M - \tilde{\tau}(k)A\big]$ defined on $D(A)$ with values in X is surjective. To show that it is also injective, we let $x \in D(A)$ be such that

$$\big[(-k^2\tilde{\mu}(k) + ik\tilde{\nu}(k) + \tilde{\eta}(k))M - \tilde{\tau}(k)A\big]x = 0.$$

Let u be the function given by $u(t) = e^{ikt}x$ when $t \in \mathbb{T}$, then it is clear that $u \in S'_{p,q,s}(A, M)$ and (P_5) is satisfied a.e. on $\mathbb{T}$ when $f = 0$. Thus, u is a strong $F^s_{p,q}$-solution of (P_5) when taking $f = 0$. We get $x = 0$ by the uniqueness assumption. We have shown that the operator $\big[(-k^2\tilde{\mu}(k) + ik\tilde{\nu}(k) + \tilde{\eta}(k))M - \tilde{\tau}(k)A\big]$ from $D(A)$ into X is injective. Therefore, $\big[(-k^2\tilde{\mu}(k) + ik\tilde{\nu}(k) + \tilde{\eta}(k))M - \tilde{\tau}(k)A\big]$ is bijective from $D(A)$ onto X.

Next, we show that $\big[(-k^2\tilde{\mu}(k) + ik\tilde{\nu}(k) + \tilde{\eta}(k))M - \tilde{\tau}(k)A\big]^{-1} \in \mathcal{L}(X)$. For $f(t) = e^{ikt}y$, we let $u \in S'_{p,q,s}(A, M)$ such that u is the unique strong $F^s_{p,q}$-solution of (P_5). Then

$$\hat{u}(n) = \begin{cases} 0 & n \neq k, \\ \big[(-k^2\tilde{\mu}(k) + ik\tilde{\nu}(k) + \tilde{\eta}(k))M - \tilde{\tau}(k)A\big]^{-1}y & n = k, \end{cases} \tag{4.18}$$

by (3.7). This implies that u is given by

$$u(t) = e^{ikt}\big[(-k^2\tilde{\mu}(k) + ik\tilde{\nu}(k) + \tilde{\eta}(k))M - \tilde{\tau}(k)A\big]^{-1}y$$

https://doi.org/10.4153/S0008439526102136 Published online by Cambridge University Press

when $t \in \mathbb{T}$. By (4.13), there exists a constant $C > 0$ independent from y and k, such that $\|u\|_{S^r_{p,q,s}(A,M)} \le C\|f\|_{F^s_{p,q}}$. This implies that

$$\|[N(k)y\| \le C\|y\|, \| - k^2MN(k)y\| \le C\|y\|$$

when $y \in X$, or equivalently,

$$\|[N(k)\| \le C, \|k^2MN(k)\| \le C.$$

Therefore, we obtain that $k \in \rho_M(A; \mu, \nu, \eta, \tau)$ for all $k \in \mathbb{Z}$. We have shown that the implication (i) $\Rightarrow$ (ii) is true. This finishes the proof. ∎

Concluding remark: (P_2) is a special case of (P_5) when taking $M = I_X, \mu = \delta_0, \nu = \alpha\delta_0 + b, \eta = 0$, and $\tau = \delta_0 - a$ for some $a, b \in L^1(\mathbb{R}_+)$. It is easy to verify that, when $\alpha \neq 0$, our hypotheses on μ, ν, η, and τ are weaker than the corresponding hypotheses imposed on $a, b \in L^1(\mathbb{R}_+)$ in [9]. Thus, we may consider the results obtained in this article as generalizations of results established in [9].

5 Applications

Example 5.1 Let Ω be a bounded domain in $\mathbb{R}^k$ with smooth boundary, m be a given nonnegative bounded measurable function on Ω, the kernel $a, b \in L^1(\mathbb{R}_+)$, and $1 < p < \infty$. We let X be the Hilbert space $H^{-1}(\Omega)$ and consider the second-order problem

$$\begin{cases} \frac{\partial^2}{\partial t^2}(m(x)u(t,x)) + \int_{-\infty}^0 a(t-s)\frac{\partial}{\partial s}(m(x)u(s,x))ds \\ \qquad + \int_{-\infty}^0 b(t-s)m(x)u(s,x))ds = Au(t,x) + f(t,x) \quad (t,x) \in \mathbb{T} \times \Omega, \\ \\ \qquad u(t,x) = 0, \qquad (t,x) \in \mathbb{T} \times \partial\Omega, \end{cases}$$

where f is defined on $\mathbb{T} \times \Omega$ and the Laplacian Δ only acts on the space variable $x \in \Omega$.

Let M be the multiplication operator on X by m with domain of definition $D(M) = \{u \in X : mu \in X\}$. Then there exist constants $C > 0, \rho > 0$, such that

$$\|M(zM + \Delta)^{-1}\| \le \frac{C}{1 + |z|} \tag{5.1}$$

whenever $Re(z) \le \rho(1 + |Im(z)|)$ by [13, Section 3.7], where Δ is the Laplacian on $H^{-1}(\Omega)$ with Dirichlet boundary condition. Let $A = \Delta$ and we assume that $D(A) \subset D(M)$. Then the above equation may be rewritten in the form

$$(Mu)''(t) + \int_{-\infty}^0 a(t-s)(Mu)'(s)ds + \int_{-\infty}^0 b(t-s)(Mu)(s)ds = Au(t) + f(t),$$

or equivalently,

$$\mu * (Mu)''(t) + \nu * (Mu)'(t) + \eta * (Mu)(t) = \tau * Au(t) + f(t) \tag{P}$$

with $\mu = \tau = \delta_0$ and $\nu = a\,dt, \eta = b\,dt$, which is an ordinary differential equation on $\mathbb{T}$ with values in X, where $f \in L^p(\mathbb{T}; X)$ is given, and the solution $u \in W^{1,p}_{\text{per}}(\mathbb{T}; D(A))$ satisfies $Mu \in W^{2,p}_{\text{per}}(\mathbb{T}; X)$.

https://doi.org/10.4153/S0008439526102136 Published online by Cambridge University Press

We notice that when $a \in L^1(\mathbb{R}_+)$, then a can be extended to $\mathbb{R}$ by defining $a(t) = 0$ when $t < 0$. With this convention, we have

$$\hat{a}(t) = \int_0^\infty e^{-ist} a(s)ds = \int_{-\infty}^\infty e^{-ist} a(s)dt = \tilde{\mu}(t)$$

for all $t \in \mathbb{R}$, where $\mu = adt$.

We assume that $\rho_M(A; \mu, \nu, \eta, \tau) = \mathbb{Z}$ and let $N_k = [(k^2 - ik\hat{a}(k) - \hat{b}(k))M + A]^{-1}$ when $k \in \mathbb{Z}$. Assume that $m > 0$ a.e. on Ω and m is regular enough so that the multiplication operator by m^{-1} is bounded on $H^{-1}(\Omega)$, then

$$\left\|(zM + \Delta)^{-1}\right\| \leq \frac{C}{1 + |z|} \tag{5.2}$$

whenever $Re(z) \leq \rho(1 + |Imz|)$ by (5.1). We assume furthermore that the sequence $k^2 - ik\hat{a}(k) - \hat{b}(k)$ is 1-regular and $\{AN_k : k \in \mathbb{Z}\}$ is norm bounded. Then (P) is L^p-well-posed. Indeed, the operator $(k^2 - ik\hat{a}(k) - \hat{b}(k))M + A : D(A) \to X$ is bijective and $[(k^2 - ik\hat{a}(k) - \hat{b}(k))M + A]^{-1} \in \mathcal{L}(X)$ whenever $k \in \mathbb{Z}$ by the assumption $\rho_M(A; \mu, \nu, \eta, \tau) = \mathbb{Z}$. It follows that the sets

$$\{k^2 MN_k : k \in \mathbb{Z}\}, \quad \{N_k : k \in \mathbb{Z}\}$$

are norm bounded by (5.1) and (5.2). We conclude that the problem (P) is L^p-well-posed by Theorem 3.4. Here, we have used the fact that $H^{-1}(\Omega)$ is a Hilbert space and the fact that every norm bounded subset of $\mathcal{L}(X)$ is R-bounded when X is isomorphic to a Hilbert space [5].

If we assume furthermore that the sequence $k^2 - ik\hat{a}(k) - \hat{b}(k)$ is 2-regular and $\{AN_k : k \in \mathbb{Z}\}$ is norm bounded, then we obtain the $B^s_{p,q}$-well-posedness of (P) when $s > 0,\ 1 \leq p, q \leq \infty$ by Theorem 4.4.

Example 5.2 Let H be a Hilbert space, and P be a densely defined positive self-adjoint operator on H with $P \geq \delta > 0$. Let $M = P - \varepsilon$ with $\varepsilon < \delta$, and let $A = \sum_{i=0}^k a_i P^i$ with $a_i \geq 0,\ a_k > 0$, where k is an integer ≥ 2. Then there exist $C > 0$ and $\rho > 0$ such that

$$\left\|M(zM + A)^{-1}\right\| \leq \frac{C}{1 + |z|} \tag{5.3}$$

whenever $Re(z) \geq -\rho(1 + |Imz|)$ by [13, p. 73]. If M is regular enough so that $0 \in \rho(M)$, then

$$\left\|(zM + A)^{-1}\right\| \leq \frac{C}{1 + |z|} \tag{5.4}$$

whenever $Re(z) \geq -\rho(1 + |Imz|)$ by (5.3).

Let $\Omega = (0, 1)$ and let $H = L^2(\Omega)$. It is clear that the operator $\frac{d^2}{dx^2}$ with domain $H^2(\Omega) \cap H_0^1(\Omega)$ generates a contraction semigroup on H and $P = -\frac{d^2}{dx^2}$ is positive and self-adjoint in H [4, Example 3.4.7]. Hence, $M = 1 + P$ has a bounded inverse. Let

https://doi.org/10.4153/S0008439526102136 Published online by Cambridge University Press

$a, b \in L^1(\mathbb{R}_+)$ and let $1 < p < \infty$. We consider the following equation:

$$\begin{cases} \frac{\partial^2}{\partial t^2}(1-\frac{\partial^2}{\partial x^2})u(t,x) + \int_{-\infty}^0 a(t-s)\frac{\partial}{\partial t}(1-\frac{\partial^2}{\partial x^2})u(s,x)ds + \int_{-\infty}^0 b(t-s)(1-\frac{\partial^2}{\partial x^2})u(s,x)ds \\ \qquad = \frac{\partial^4}{\partial x^4}u(t,x) + f(t,x), \qquad (t,x) \in \mathbb{T}\times\Omega, \\ u(t,0) = u(t,1) = \frac{\partial^2}{\partial x^2}u(t,0) = \frac{\partial^2}{\partial x^2}u(t,1) = 0, \qquad t \in \mathbb{T}. \end{cases}$$

This equation can be rewritten in the more compact form:

$$(\mathrm{P}') \qquad (Mu)''(t) + a * (Mu)'(t) + b * (Mu)(t) = (Au)(t) + f(t), \quad (t \in \mathbb{T}),$$

which is an ordinary differential equation on $\mathbb{T}$ with values in H, where $f \in L^p(\mathbb{T}; H)$ and the solution $u \in W^{1,p}_{\mathrm{per}}(\mathbb{T}; D(A))$ satisfies $Mu \in W^{2,p}_{\mathrm{per}}(\mathbb{T}; H)$, where $M = 1 - \frac{\partial^2}{\partial x^2}$ and $A = \Delta^2$, here we consider Δ as the Laplacian on $L^2(\Omega)$ with Dirichlet boundary condition. If $\rho_M(A; \mu, \nu, \eta, \tau) = \mathbb{Z}$, the sequence $k^2 - ik\hat{a}(k) - \hat{b}(k)$ is 1-regular and $\{AN_k : k \in \mathbb{Z}\}$ is norm bounded, one can obtain the L^p-well-posedness of (P') by using (5.3), (5.4), and Theorem 3.4. Here again, we have used the fact that $L^2(\Omega)$ is a Hilbert space and the fact that every norm bounded subset of $\mathcal{L}(X)$ is R-bounded when X is isomorphic to a Hilbert space [5]. One can also obtain the $B^s_{p,q}$-well-posedness of (P_2) when $s > 0$, $1 \le p, q \le \infty$ and the sequence $k^2 - ik\hat{a}(k) - \hat{b}(k)$ is 2-regular and $\{AN_k : k \in \mathbb{Z}\}$ is norm bounded by using Theorem 4.4.

Acknowledgements The authors are ranked in alphabetic order of their names and should be treated equally.

References

[1] E. Alvarez and C. Lizama, *A characterization of L^p-maximal regularity for time-fractional systems in UMD spaces and applications*. J. Differential Equations **389**(2024), 257–284.
[2] E. Alvarez, C. Lizama, and M. Murillo-Arcila, *Maximal regularity of solutions for the tempered fractional Cauchy problem*. J. Funct. Anal. **290**(2026), 111196, 35 pp.
[3] W. Arendt, C. Batty, and S. Bu, *Fourier multipliers for Hölder continuous functions and maximal regularity*. Stud. Math. **160**(2004), 23–51.
[4] W. Arendt, C. Batty, M. Hieber, and F. Neubrander, *Vector-valued Laplace transforms and Cauchy problems*, Birkhäuser, Basel, 2001.
[5] W. Arendt and S. Bu, *The operator-valued Marcinkiewicz multiplier theorem and maximal regularity*. Math. Z. **240**(2002), no. 2, 311–343.
[6] W. Arendt and S. Bu, *Operator-valued Fourier multipliers on periodic Besov spaces and applications*. Proc. Edinb. Math. Soc. **47**(2004), 15–33.
[7] J. Bourgain, *Vector-valued singular integrals and the H^1-BMO duality*. In: Probability theory and harmonic analysis (Cleveland, OH, 1983), *Monographs and Textbooks in Pure and Applied Mathematics, 98*, Dekker, New York, NY, 1988, pp. 1–19.
[8] S. Bu and Y. Fang, *Maximal regularity for integro-differential equations in periodic Triebel-Lizorkin spaces*. Taiwanese J. Math. **12**(2008), 281–292.
[9] S. Bu and Y. Fang, *Periodic solutions for second order integro-differential equations with infinite delay in Banach spaces*. Stud. Math. **184**(2008), 103–119.
[10] S. Bu and J. Kim, *Operator-valued Fourier multipliers on periodic Triebel spaces*. Acta Math. Sin. (Engl. Ser.) **21**(2005), 1049–1056.
[11] R. Chill and S. Srivastava, *L^p-maximal regularity for second order Cauchy problems*. Math. Z. **251**(2005), 751–781.
[12] J. A. Conejero, C. Lizama, M. Murillo-Arcila, and J. B. Seoane-Sepúlveda, *Well-posedness degenerate third order equations with delay and applications to inverse problems*. Israel J. Math. **229**(2019), 219–254.

https://doi.org/10.4153/S0008439526102136 Published online by Cambridge University Press

[13] A. Favini and A. Yagi, *Degenerate differential equations in Banach spaces*, Pure Mathematics and Applied Mathematics, 215, Dekker, New York, NY, 1999.
[14] V. Keyantuo and C. Lizama, *Fourier multipliers and integro-differential equations in Banach spaces*. J. Lond. Math. Soc. **69**(2004), 737–750.
[15] V. Keyantuo, C. Lizama, and V. Poblete, *Periodic solutions of integro-differential equations in vector-valued function spaces*. J. Differential Equations **246**(2009), 1007–1037.
[16] A. Pietsch and J. Wenzel, *Orthonormal systems and Banach space geometry*, Encyclopedia of Mathematics and its Applications, 70, Cambridge University Press, 1998.
[17] R. Ponce, *Well-posedness of second order differential equations with memory*. Math. Nachr. **295**(2022), 2246–2264.
[18] R. Ponce, *Periodic solutions to second-order differential equations with fading memory*. Math. Z. **307**(2024), Article no. 5, 19 pp.
[19] R. Ponce, *Approximate controllability of abstract discrete fractional systems of order* $1 < \alpha < 2$ *via resolvent sequences*. J. Optim. Theory Appl. **203**(2024), 359–385.
[20] L. Weis, *Operator-valued Fourier multipliers and maximal* L_p*-regularity*. Math. Ann. **319**(2001), 735–758.
[21] L. Weis, *A new approach to maximal* L_p*-regularity*. In: Evolution equations and their applications in physical and life sciences (Bad Herrenalb, 1998), Lecture Notes in Pure and Applied Mathematics, 215, Dekker, New York, NY, 2001, pp. 195–214.

Tsinghua University, China
e-mail: bushangquan@tsinghua.edu.cn

Chongqing Normal University, China
e-mail: caigang-aaaa@163.com

https://doi.org/10.4153/S0008439526102136 Published online by Cambridge University Press