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C^*-algebras arising from substitutions

MASARU FUJINO

Graduate School of Mathematics, Kyushu University, Hakozaki, Fukuoka, 812-8581, Japan
(e-mail: fujino@math.kyushu-u.ac.jp)

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Abstract. In this paper, we introduce a C^*-algebra associated with a primitive substitution. We show that when σ is proper, the C^*-algebra is simple and purely infinite and contains the associated Cuntz–Krieger algebra and the crossed product C^*-algebra of the corresponding Cantor minimal system. We calculate the K-groups.

1. *Introduction*

Cuntz constructed in his paper [**2**] a new C^*-algebra naturally associated with the $ax + b$-semigroup over $\mathbb{N}$. The C^*-algebra $\mathcal{Q}_{\mathbb{N}}$ is generated by a unitary u and isometries $\{s_n\}_{n\in\mathbb{N}^*}$ with range projections $e_n = s_n s_n^*$ satisfying the relations

$$s_n s_m = s_{nm}, \quad s_n u = u^n s_n, \quad \sum_{k=0}^{n-1} u^k e_n u^{-k} = 1$$

for $n, m \in \mathbb{N}^*$.

Cuntz showed in [**2**] that $\mathcal{Q}_{\mathbb{N}}$ is simple and purely infinite. Furthermore he showed $\mathcal{Q}_{\mathbb{N}}$ contains the Bunce–Deddens algebra and is also generated by the Bost–Connes algebra by adding one unitary which corresponds to addition.

In this paper, motivated by the above construction of the C^*-algebra, we introduce a C^*-algebra B_σ associated with a primitive substitution σ. The C^*-algebra B_σ is generated by the Cuntz–Krieger algebra [**3**] $\mathcal{O}_{E_\sigma^{(2)}} = C^*(\{s_f\}_{f\in E_\sigma^{(2)}})$, where $E_\sigma^{(2)}$ is the set of edges with partial order arising from σ, and a unitary u corresponding to addition (Vershik map) satisfying the following relations:

$$u\Bigg(\sum_{f \text{ is maximal in } E_\sigma^{(2)}} s_f\Bigg) = \Bigg(\sum_{f \text{ is minimal in } E_\sigma^{(2)}} s_f\Bigg)u$$

and for $f \in E_\sigma^{(2)}$ which is not maximal in $E_\sigma^{(2)}$,

$$us_f = s_g \quad \text{where } g \text{ is the successor of } f.$$

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Let $f_1, f_2, \ldots, f_{k-1}$ be maximal elements in $E_\sigma^{(2)}$ and f_k not a maximal element in $E_\sigma^{(2)}$. Let g_k be the successor of f_k and $(g_1, g_2, \ldots, g_k)$ be the unique path such that $g_1, \ldots, g_{k-1}$ are minimal. By using the relations above, we can calculate

$$\begin{aligned} us_{f_1}s_{f_2}\cdots s_{f_k} &= u\Bigg(\sum_{g \text{ is maximal in } E_\sigma^{(2)}} s_g\Bigg)s_{f_2}\cdots s_{f_k} \\ &= \Bigg(\sum_{g \text{ is minimal in } E_\sigma^{(2)}} s_g\Bigg)us_{f_2}\cdots s_{f_k} \\ &= \cdots = \Bigg(\sum_{g \text{ is minimal in } E_\sigma^{(2)}} s_g\Bigg)^{k-1} s_{g_k} = s_{g_1}s_{g_2}\cdots s_{g_k}. \end{aligned}$$

Our notation, $E_\sigma^{(2)}$, for the set of edges is chosen because this set can be identified with the (ordered) set of edges connecting label 1 and label 2 of the associated (stationary) Bratteli diagram (in §2).

We show that when σ is proper, B_σ is simple and purely infinite and contains the crossed product C^*-algebra of the corresponding Cantor minimal system, studied by Durand–Host–Skau [**5**], Giordano–Putnam–Skau [**6**] and Herman–Putnam–Skau [**7**]. Furthermore we show that a certain B_σ has a unique Kubo–Martin–Schwinger (KMS)-state and determine the K-groups of B_σ.

2. *C^*-algebras associated with substitutions*

In this paper, we study C^*-algebras associated with primitive substitutions and we introduce the definition of stationary ordered Bratteli diagrams following [**5**].

Let A be an alphabet consisting of at least two letters. We denote by A^+ the set of finite words of A. Then a map $\sigma : A \to A^+$ is said to be a substitution on A. By concatenation, for each $n > 0$, the nth iteration $\sigma^n : A \longrightarrow A^+$ is again a substitution. For any $a \in A$, let $\kappa(a, b) \in \mathbb{N}$ denote the number of occurrences of b in $\sigma(a)$. A substitution σ is said to be primitive if there exists $n > 0$ such that for every $a, b \in A$, b occurs in $\sigma^n(a)$.

Let σ be a primitive substitution on an alphabet A of N elements. Then we construct the stationary ordered Bratteli diagram $(V_\sigma, E_\sigma, \leq)$ associated to a substitution σ as follows: the vertex set V_σ is given as a disjoint union $V_\sigma^{(0)} \cup V_\sigma^{(1)} \cup V_\sigma^{(2)} \cup \cdots$ where $V_\sigma^{(0)}$ is one point set $\{x_0\}$ and each $V_\sigma^{(n)}$ $(n \geq 1)$ consists of N points. We denote the element of $V_\sigma^{(n)}$ by $x(a, n)$ $(a \in A)$, i.e. $V_\sigma^{(n)} = \{x(a, n) \mid a \in A\}$ $(n \geq 1)$. The vertex $x(a, n)$ is called the vertex of label a at level n.

The edge set E_σ is given as a disjoint union $E_\sigma^{(1)} \cup E_\sigma^{(2)} \cup E_\sigma^{(3)} \cup \cdots$ such that $s(E_\sigma^{(n)}) \subset V_\sigma^{(n-1)}$ and $r(E_\sigma^{(n)}) \subset V_\sigma^{(n)}$ where $r(e)$ denotes the range of the edge e and $s(e)$ the source of e. For each $a \in A$, $E_\sigma^{(1)}$ contains exactly one edge from x_0 to $x(a, 1)$. Each $E_\sigma^{(n)}$ consists of edges from $V_\sigma^{(n-1)}$ to $V_\sigma^{(n)}$. For an integer $n > 1$ and a, b in A, the number of edges from $x(b, n-1)$ to $x(a, n)$ is equal to $\kappa(a, b)$.

For the Bratteli diagram (V_σ, E_σ) constructed above, we have a range map $r : E_\sigma \longrightarrow V_\sigma$ and a source map $s : E_\sigma \longrightarrow V_\sigma$ so that $r(E_\sigma^{(n)}) \subset V_\sigma^{(n)}$ and $s(E_\sigma^{(n)}) \subset V_\sigma^{(n-1)}$.

For each $n \geq 1$, we define a partial order on $E_\sigma^{(n)}$ as follows.

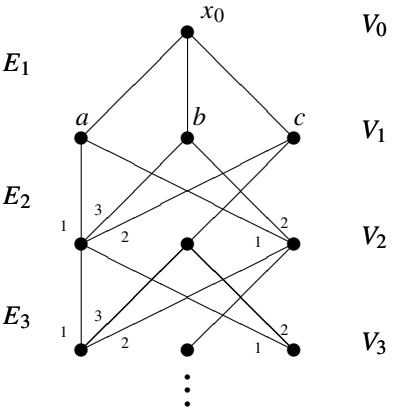
Two edges $e_1, e_2 \in E_\sigma$ are compatible if and only if $r(e_1) = r(e_2)$.

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For each $n \geq 2$ and $a \in V_\sigma^{(n)}$, we put $E_\sigma^{(n)}(a) = \{e \in E_\sigma^{(n)} \mid \text{the label of } r(e) \text{ is } a\}$. Let $\sigma(a) = a_1 a_2, \ldots, a_m$, then we denote the elements of $E_\sigma^{(n)}(a)$ by $e(a, k)$ $(1 \leq k \leq m)$ so that the label of $s(e(a, k))$ is a_k and $r(e(a, k)) = a$.

Let e_1 and e_2 be in $E_\sigma^{(n)}(n \geq 1)$ with $r(e_1) = r(e_2)$ and let a be the label of $r(e_1)$, then e_1 and e_2 are in $E_\sigma^{(n)}(a)$. When we write e_1 and e_2 as $e(a, i)$ and $e(a, j)$ $(1 \leq i, j \leq m)$, we put $e(a, i) < e(a, j)$ if $i < j$.

Example 2.1. Let σ be substitution on the alphabet $A = \{a, b, c\}$ defined by $\sigma(a) = acb$, $\sigma(b) = c$ and $\sigma(c) = ab$. Then the associated stationary ordered Bratteli diagram is given by the following diagram.



We basically follow the definition and property of ordered Bratteli diagrams in [**5–8**]. We introduce some necessary notations, definitions of ordered Bratteli diagrams.

Notation 2.2. Suppose that $(V_\sigma, E_\sigma, \leq)$ is the ordered Bratteli diagram associated to a primitive substitution σ.

(a) Let $x = x(a, n)$ be in $V_\sigma^{(n)}$. Then a is said to be the label of $x(a, n)$ and n the level of $x(a, n)$. Let $L(x)$ denote the label of x and $\ell(x)$ the level of x.

(b) Let $x = x(a, n) \in V_\sigma^{(n)}$. For $k \in \mathbb{Z}$ with $-(n-1) \leq k$, let x_{+k} denote the vertex $x(a, n+k) \in V_\sigma^{(n+k)}$. Let f be in $E_\sigma^{(n)}$ $(n \geq 1)$, then for $k \in \mathbb{Z}$ with $-(n-2) \leq k$, we denote by f_{+k} the edge in $E_\sigma^{(n+k)}$ such that $r(f_{+k}) = r(f)_{+k}$ and $s(f_{+k}) = s(f)_{+k}$, and having the same ordering label as f.

Definition 2.3. Let $(V_\sigma, E_\sigma, \leq)$ be the stationary ordered Bratteli diagram associated with a primitive substitution σ. Let $\mathcal{P}_\sigma$ denote the infinite path space associated with σ, i.e.

$$\mathcal{P}_\sigma = \{(f^{(1)}, f^{(2)}, \ldots) \mid f^{(k)} \in E_\sigma^{(k)},\ r(f^{(k)}) = s(f^{(k+1)}) \text{ for any } k \geq 1\}.$$

Let $\widetilde{V}_\sigma = V_\sigma \backslash (V_\sigma^{(0)} \cup V_\sigma^{(1)})$. For each $x \in \widetilde{V}_\sigma$, let $\mathcal{P}_\sigma(x)$ denote the set of finite paths connecting the top vertex x_0 and x, i.e.

$$\mathcal{P}_\sigma(x) = \{(f^{(1)}, f^{(2)}, \ldots, f^{(m)}) \mid f^{(k)} \in E_\sigma^{(k)}\ (1 \leq k \leq m),$$
$$r(f^{(k)}) = s(f^{(k+1)})\ (1 \leq k \leq m-1), r(f^{(m)}) = x\}.$$

Let $\mathcal{P}_\sigma^{(n)} = \bigcup_{x \in V_\sigma^{(n)}} P_\sigma(x)$ $(n \geq 2)$ and $\mathcal{P}_\sigma^{\text{fin}}$ denote the set $\bigcup_{n \geq 2} \mathcal{P}_\sigma^{(n)}$. The path $\mathcal{P}_\sigma^{\text{fin}}$ is called the finite path space arising from σ.

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Notation 2.4. Let σ be a primitive substitution.

(a) We let $E_\sigma^{\max}$ and $E_\sigma^{\min}$ denote the maximal and minimal edges in E_σ, i.e.

$$E_\sigma^{\max} = \{f \in E_\sigma \mid f \text{ is maximal in } r^{-1}(r(f))\}.$$

Let $\mathcal{P}_\sigma^{\max}$ and $\mathcal{P}_\sigma^{\min}$ denote the set of maximal and minimal infinite paths in $\mathcal{P}_\sigma$, i.e.

$$\mathcal{P}_\sigma^{\max} = \{(f^{(1)}, f^{(2)}, \ldots) \in \mathcal{P}_\sigma \mid f^{(k)} \in E_\sigma^{(k)} \cap E_\sigma^{\max}, r(f^{(k)}) = s(f^{(k+1)})\ (1 \le k)\}.$$

$\mathcal{P}_\sigma^{\min}$ is similarly defined.

Let $\mathcal{P}_\sigma^{\text{fin,max}}$ and $\mathcal{P}_\sigma^{\text{fin,min}}$ denote the set of maximal and minimal finite paths, i.e.

$$\mathcal{P}_\sigma^{\text{fin,max}} = \{(f^{(1)}, f^{(2)}, \ldots, f^{(m)}) \in \mathcal{P}_\sigma^{\text{fin}} \mid f^{(k)} \in E_\sigma^{(k)} \cap E_\sigma^{\max}\ (1 \le k \le m)\}.$$

$\mathcal{P}_\sigma^{\text{fin,min}}$ is similarly defined.

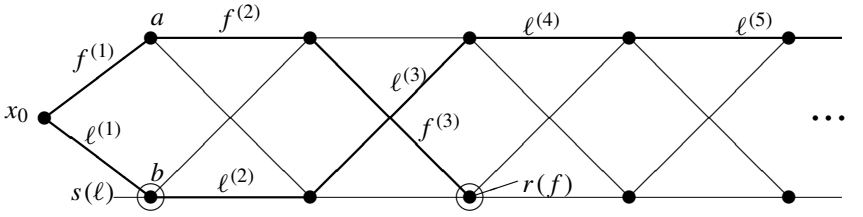
(b) For a finite path $f = (f^{(1)}, f^{(2)}, \ldots, f^{(n)}) \in \mathcal{P}_\sigma^{\text{fin}}$, let $s(f)$ and $r(f)$ denote the vertices $r(f^{(1)}) \in V_\sigma^{(2)}$ and $r(f^{(n)}) \in V_\sigma^{(n)}$ respectively and let $\ell(f)$ denote the length of finite path f, i.e. $\ell(f) = n$.

For an infinite path $\ell \in \mathcal{P}_\sigma$, let $s(\ell)$, the source of ℓ, denote the vertex $r(\ell^{(1)}) \in V_\sigma^{(1)}$.

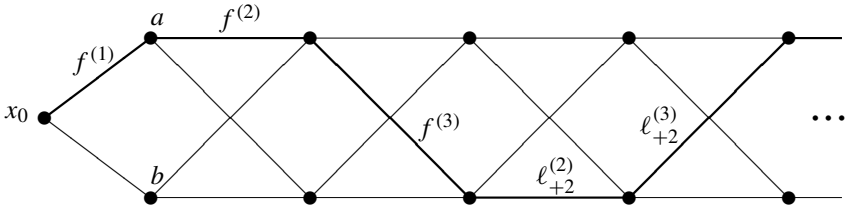
(c) Let $f, g \in \mathcal{P}_\sigma^{\text{fin}}$. If $L(r(f)) = L(s(g))$, then fg is the concatenation of the paths f and g, meaning that one starts with f and then follows with g, where g is appropriately translated. More precisely, for any $f = (f^{(1)}, f^{(2)}, \ldots, f^{(m)}) \in \mathcal{P}_\sigma^{(m)}$ and $g = (g^{(1)}, g^{(2)}, \ldots, g^{(n)}) \in \mathcal{P}_\sigma^{(n)}$ with $L(r(f)) = L(s(g))$, we define the finite path $fg \in \mathcal{P}_\sigma^{(m+n-1)}$ by $(f^{(1)}, f^{(2)}, \ldots, f^{(m)}, g^{(2)}_{+(m-1)}, g^{(3)}_{+(m-1)}, \ldots, g^{(n)}_{+(m-1)})$.

(d) Let $f \in \mathcal{P}_\sigma^{\text{fin}}$ and $\ell \in \mathcal{P}_\sigma$. For any $f = (f^{(1)}, f^{(2)}, \ldots, f^{(m)}) \in \mathcal{P}_\sigma^{(m)}$ and $\ell = (\ell^{(1)}, \ell^{(2)}, \ldots) \in \mathcal{P}_\sigma$ with $L(r(f)) = L(s(\ell))$, we define the infinite path $f\ell \in \mathcal{P}_\sigma$ by $(f^{(1)}, f^{(2)}, \ldots, f^{(m)}, \ell^{(2)}_{+(m-1)}, \ell^{(3)}_{+(m-1)}, \ldots)$.

Example 2.5. Let σ be substitution on the alphabet $\{a, b\}$ defined by $\sigma(a) = ab$ and $\sigma(b) = ab$ and let $f \in \mathcal{P}_\sigma^{(3)}$ and $\ell \in \mathcal{P}_\sigma$ be the following paths.



Then $f\ell = (f^{(1)}, f^{(2)}, f^{(3)}, \ell^{(2)}_{+2}, \ell^{(3)}_{+2}, \ldots) \in \mathcal{P}_\sigma$ becomes the following one.



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Definition 2.6. Let σ be a primitive substitution. Then $(V_\sigma, E_\sigma, \leq)$ is said to be properly ordered if ${}^{\#}\mathcal{P}_\sigma^{\min} = {}^{\#}\mathcal{P}_\sigma^{\max} = 1$. Then we write $\mathcal{P}_\sigma^{\min} = \{p_{\min}\}$ and $\mathcal{P}_\sigma^{\max} = \{p_{\max}\}$. Now we topologize the infinite path space $\mathcal{P}_\sigma$ by the family of cylinder sets $\{U(f)\}_{f \in \mathcal{P}_\sigma^{\text{fin}}}$ where

$$U(f) = \{g \in \mathcal{P}_\sigma \mid g^{(k)} = f^{(k)} \in E_\sigma^{(k)},\ 1 \leq k \leq \ell(f)\}.$$

Then each $U(f)$ is also closed and we note that $\mathcal{P}_\sigma$ is a compact Hausdorff space with a countable basis of clopen sets. Since σ is primitive, the cardinality of $\mathcal{P}_\sigma$ is infinite and $\mathcal{P}_\sigma$ has no isolated points, so we note that $\mathcal{P}_\sigma$ is a Cantor set.

Definition 2.7. A primitive substitution σ on an alphabet A is proper if there exist $b, c \in A$ such that the following hold.

(1) For each $a \in A$, b is the first letter of $\sigma(a)$.
(2) For each $a \in A$, c is the last letter of $\sigma(a)$.

It is easily seen that if a substitution σ is proper, then the stationary ordered Bratteli diagram $(V_\sigma, E_\sigma, \leq)$ is properly ordered.

We recall the definition of Vershik map in [**8**].

Definition 2.8. Let $(V_\sigma, E_\sigma, \geq)$ be an ordered Bratteli diagram associated with a primitive substitution σ. For each $f = (f^{(1)}, f^{(2)}, \ldots) \in \mathcal{P}_\sigma \backslash \mathcal{P}_\sigma^{\max}$, there exists $k \geq 1$ such that $f^{(1)}, \ldots, f^{(k-1)} \in E_\sigma^{\max}$ and $f^{(k)} \notin E_\sigma^{\max}$. Let $g^{(k)}$ be the successor of $f^{(k)}$ in $r^{-1}(r(f^{(k)}))$, then we write the minimal finite path from x_0 to $s(g^{(k)})$ as $(g^{(1)}, g^{(2)}, \ldots, g^{(k-1)})$. Then the infinite path $(g^{(1)}, g^{(2)}, \ldots, g^{(k)}, f^{(k+1)}, f^{(k+2)}, \ldots)$ is said to be the successor of f. We let λ_σ be a map defined by

$$\lambda_\sigma : \mathcal{P}_\sigma \backslash \mathcal{P}_\sigma^{\max} \to \mathcal{P}_\sigma \backslash \mathcal{P}_\sigma^{\min}, \quad \lambda_\sigma(f) = \text{the successor of } f.$$

The map λ_σ is said to be the Vershik map arising from the ordered Bratteli diagram $(V_\sigma, E_\sigma, \leq)$.

When the substitution σ is proper, we extend the Vershik map λ_σ to the whole infinite path space $\mathcal{P}_\sigma$ by defining $\lambda_\sigma(p_{\max}) = p_{\min}$. Then $\lambda_\sigma : \mathcal{P}_\sigma \longrightarrow \mathcal{P}_\sigma$ is a homeomorphism, thus $(\mathcal{P}_\sigma, \lambda_\sigma)$ is a topological dynamical system. This dynamical system is called the Bratteli–Vershik system associated to the ordered Bratteli diagram. It is known [**8**] that when σ is proper, λ_σ is minimal.

Definition 2.9. Let σ be a primitive substitution. For any $x \in \widetilde{V}_\sigma$, let $p_{\min}(x)$ (respectively $p_{\max}(x)$) denote the unique minimal (respectively maximal) path from the top vertex x_0 to x. Let $x \in \widetilde{V}_\sigma$, then there is a map:

$$\mathcal{P}_\sigma(x) \backslash \{p_{\max}(x)\} \ni f \longrightarrow (\text{the successor of } f) \in \mathcal{P}_\sigma(x) \backslash \{p_{\min}(x)\}.$$

We let the map be also expressed by the symbol λ_σ.

Let $(V_\sigma, E_\sigma, \geq)$ be a stationary ordered Bratteli diagram associated to a proper primitive substitution σ. Then we let T be the unitary operator defined by

$$T : \ell^2(\mathcal{P}_\sigma) \longrightarrow \ell^2(\mathcal{P}_\sigma), \quad T(\xi_\ell) = \xi_{\lambda_\sigma(\ell)},$$

where ξ_ℓ is the unit vector associated to $\ell \in \mathcal{P}_\sigma$. The canonical basis of $\ell^2(\mathcal{P}_\sigma)$ is $\{\xi_\ell \mid \ell \in \mathcal{P}_\sigma\}$.

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For each $f \in E_\sigma^{(2)}$, we let ef denote the finite path $(e, f) \in \mathcal{P}_\sigma^{(2)}$, where e is the unique edge in $E_\sigma^{(1)}$ connecting $x_0 \in V_\sigma^{(0)}$ to $s(f) \in V_\sigma^{(1)}$. Then we let V_f be the operator defined by

$$V_f : \ell^2(\mathcal{P}_\sigma) \longrightarrow \ell^2(\mathcal{P}_\sigma), \quad V_f \xi_\ell = \begin{cases} \xi_{ef\ell} & L(r(f)) = L(s(\ell)), \\ 0 & L(r(f)) \neq L(s(\ell)). \end{cases}$$

Then $\{V_f\}_{f \in E_\sigma^{(2)}}$ are partial isometries and the next lemma is known.

LEMMA 2.10. *The operators $\{V_f\}_{f \in E_\sigma^{(2)}}$ satisfy the Cuntz–Krieger relations, i.e.*

$$\sum_{f \in E_\sigma^{(2)}} V_f V_f^* = 1,$$

$$V_g^* V_g = \sum_{f \in E_\sigma^{(2)}, L(s(f)) = L(r(g))} V_f V_f^* (g \in E_\sigma^{(2)}).$$

LEMMA 2.11.

(a) *For any $f \in E_\sigma^{(2)} \backslash E_\sigma^{\max}$, $V_{\lambda_\sigma(f)} = T V_f$.*

(b) $T(\sum_{f \in E_\sigma^{(2)} \cap E_\sigma^{\max}} V_f) = (\sum_{f \in E_\sigma^{(2)} \cap E_\sigma^{\min}} V_f) T$.

Proof. (a) For any $\ell \in \mathcal{P}_\sigma$,

$$V_{\lambda_\sigma(f)} \xi_\ell = \begin{cases} \xi_{\lambda_\sigma(ef\ell)} & \text{if } L(r(f)) = L(s(\ell)) \\ 0 & \text{otherwise} \end{cases} = T V_f \xi_\ell.$$

(b) For any $\ell \in \mathcal{P}_\sigma$,

$$\begin{aligned} T\left(\sum_{f \in E_\sigma^{(2)} \cap E_\sigma^{\max}} V_f \right) \xi_\ell &= \xi_{\lambda_\sigma(eg\ell)} (g \in \mathcal{P}_\sigma^{(2)} \cap \mathcal{P}_\sigma^{\text{fin,max}} \quad \text{with } L(r(g)) = L(s(\ell))) \\ &= \xi_{eh\lambda_\sigma(\ell)} (h \in \mathcal{P}_\sigma^{(2)} \cap \mathcal{P}_\sigma^{\text{fin,min}} \quad \text{with } L(r(h)) = L(s(\lambda_\sigma(\ell)))) \\ &= \left(\sum_{f \in E_\sigma^{(2)} \cap E_\sigma^{\min}} V_f \right) T \xi_\ell. \end{aligned}$$

□

The above discussion leads to the following definition (where T becomes u and the V_f become the s_f).

Definition 2.12. Let σ be a proper primitive substitution. Then we define B_σ to be the universal unital C^*-algebra generated by a unitary u and partial isometries $\{s_f\}_{f \in E_\sigma^{(2)}}$ such that the following hold.

(a) (Cuntz–Krieger relations)

$$\sum_{f \in E_\sigma^{(2)}} s_f s_f^* = 1,$$

$$s_g^* s_g = \sum_{f \in E_\sigma^{(2)}, L(s(f)) = L(r(g))} s_f s_f^* \quad (g \in E_\sigma^{(2)}).$$

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(b) For any $f \in E_\sigma^{(2)} \setminus E_\sigma^{\max}$,

$$s_{\lambda_\sigma(f)} = u s_f.$$

(c)

$$u\left(\sum_{f \in E_\sigma^{(2)} \cap E_\sigma^{\max}} s_f\right) = \left(\sum_{f \in E_\sigma^{(2)} \cap E_\sigma^{\min}} s_f\right) u.$$

Example 2.13. In [**2**], Cuntz defined a simple C^*-algebra B_1 generated by a unitary u and an isometry s_2 satisfying the following relations.

$$s_2 u = u^2 s_2, \quad \sum_{k=0}^{2^n-1} u^k s_2^n s_2^{*n} u^{-k} = 1$$

for any $n \geq 2$.

The C^*-algebra B_1 is isomorphic to B_σ for a certain substitution σ. Let σ be the substitution on the alphabet $A = \{a, b\}$ defined by $\sigma(a) = ab$ and $\sigma(b) = ab$. Then $E_\sigma^{(2)} = \{f_1, \lambda_\sigma(f_1), f_2, \lambda_\sigma(f_2)\}$ where $f_1, f_2 \in E_\sigma^{(2)} \cap E_\sigma^{\min}$ with $L(r(f_1)) =$a and $L(r(f_2)) =$b. We put $s_{f_1} = s_2^2 s_2^*$, $s_{\lambda_\sigma(f_1)} = u s_2^2 s_2^*$, $s_{f_2} = u^2 s_2^2 s_2^* u^{-1}$ and $s_{\lambda_\sigma(f_2)} = u^3 s_2^2 s_2^* u^{-1}$. Then u, $\{s_f\}_{f \in E_\sigma^{(2)}}$ satisfy the relations (a)–(c) in Definition 2.12. According to Theorem 3.13 in §3, B_σ is simple. So B_σ is isomorphic to B_1.

Remark 2.14. The C^*-algebra associated with substitutions can be defined for general primitive substitutions (not necessarily proper) and we indicate how this is done.

So let σ be a primitive substitution. Then we let T be the operator defined by

$$T : \ell^2(\mathcal{P}_\sigma) \longrightarrow \ell^2(\mathcal{P}_\sigma), \quad T\xi_\ell = \begin{cases} \xi_{\lambda_\sigma(\ell)} & \ell \notin \mathcal{P}_\sigma^{\max}, \\ 0 & \ell \in \mathcal{P}_\sigma^{\max}. \end{cases}$$

Then clearly

$$T^*T + \chi_{\mathcal{P}_\sigma^{\max}} = TT^* + \chi_{\mathcal{P}_\sigma^{\min}} \quad \text{(we have that } \#(\mathcal{P}_\sigma^{\max}),\ \#(\mathcal{P}_\sigma^{\min}) < \infty).$$

Then T and $\{V_f\}_{f \in E_\sigma^{(2)}}$ satisfy the condition of Lemmas 2.10 and 2.11. Since σ is a primitive substitution, the cardinality of $\mathcal{P}_\sigma$ is infinite. Let J denote the ideal $C^*(T, \{V_f\}_{f \in E_\sigma^{(2)}}) \cap K(\ell^2(\mathcal{P}_\sigma))$ in $C^*(T, \{V_f\}_{f \in E_\sigma^{(2)}})$ (where $K(\ell^2(\mathcal{P}_\sigma))$ is the set of compact operators) and π be the natural map $C^*(T, \{V_f\}_{f \in E_\sigma^{(2)}}) \to C^*(T, \{V_f\}_{f \in E_\sigma^{(2)}})/J$. Then $\pi(T)$ is a unitary and $\{\pi(V_f)\}_{f \in E_\sigma^{(2)}}$ are partial isometries satisfying the equations, the Lemmas 2.10 and 2.11. When σ is primitive, we can realize thus the C^*-algebra B_σ generated by the quotient above.

3. *Simple and pure infiniteness for B_σ*

Notation 3.1. Let σ be a proper primitive substitution on an alphabet A.

(a) For any $f = (f^{(1)}, f^{(2)}, \ldots, f^{(n)}) \in \mathcal{P}_\sigma^{\text{fin}}$, we denote by s_f the partial isometry $s_{f^{(2)}} s_{f_{-1}^{(3)}} \ldots s_{f_{-(n-2)}^{(n)}}$, i.e. s_f maps ξ_ℓ ($\ell \in \mathcal{P}_\sigma$) to $\xi_{f\ell}$ if $L(r(f)) = L(s(f))$. In other words, one adds f as the initial path to ℓ. If $L(r(f)) \neq L(s(\ell))$, then s_f maps ξ_ℓ to 0.

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(b) For each $x \in \widetilde{V}_\sigma$, let $\nu(x) \in \mathbb{N}$ denote the cardinality of the set $\mathcal{P}_\sigma(x)$, i.e. $\nu(x)$ is the number of paths from the top vertex to x.

LEMMA 3.2. *Let σ be a proper primitive substitution.*

(a) $s_f s_g^* = 0$ *for any* $f, g \in \mathcal{P}_\sigma^{\mathrm{fin}}$ *with* $L(r(f)) \neq L(r(g))$.

(b) $\sum_{f \in E_\sigma^{(2)} \cap E_\sigma^{\max}} s_f s_f^* = (\sum_{g \in E_\sigma^{(2)} \cap E_\sigma^{\max}} s_g)(\sum_{h \in E_\sigma^{(2)} \cap E_\sigma^{\max}} s_h)^*$.

(c) *For any* $f \in \mathcal{P}_\sigma^{\mathrm{fin,min}}$ *and* $0 \leq k \leq \nu(r(f)) - 1$,

$$u^k s_f = s_{\lambda_\sigma^k(f)}.$$

(d) *Let* $f, g \in \mathcal{P}_\sigma^{\mathrm{fin,min}}$. *If* $-(\nu(r(f)) - 1) \leq k \leq \nu(r(g)) - 1$, *then*

$$s_f^* u^k s_g \in \{s_p^* s_q \mid p, q \in \mathcal{P}_\sigma^{\mathrm{fin}} \text{ with } r(p) = r(f) \text{ and } r(q) = r(g)\}.$$

(e) *We write* $\mathcal{P}_\sigma^{\min} = \{p_{\min}\}$. *Then we let* δ_σ *be the map defined by*

$$\delta_\sigma : \mathcal{P}_\sigma^{\mathrm{fin,min}} \longrightarrow \{0, 1\}, \quad \delta_\sigma(f) = \begin{cases} 1 & \text{if } L(r(f)) = L(s(p_{\min})), \\ 0 & \text{otherwise.} \end{cases}$$

For any $f \in \mathcal{P}_\sigma^{\mathrm{fin,min}}$,

$$s_f \Bigg(\sum_{g \in E_\sigma^{(2)} \cap E_\sigma^{\min}} s_g \Bigg) = \delta_\sigma(f) \sum_{h \in \mathcal{P}_\sigma^{(\ell(f)+1)} \cap \mathcal{P}_\sigma^{\mathrm{fin,min}}} s_h.$$

(f) *For any* $f, g \in \mathcal{P}_\sigma^{\mathrm{fin,min}}$ *with* $L(r(f)) = L(r(g))$, *we have*

$$\begin{aligned} s_f s_g^* = & \sum_{h \in \mathcal{P}_\sigma^{(2)} \setminus \mathcal{P}_\sigma^{\mathrm{fin,max}}} s_f s_{\lambda_\sigma(h)} s_{\lambda_\sigma(h)}^* s_g^* \\ & + \delta_\sigma(f) u \Bigg(\sum_{p \in \mathcal{P}_\sigma^{(\ell(f)+1)} \cap \mathcal{P}_\sigma^{\mathrm{fin,max}}} s_p \Bigg) \Bigg(\sum_{q \in \mathcal{P}_\sigma^{(\ell(g)+1)} \cap \mathcal{P}_\sigma^{\mathrm{fin,max}}} s_q \Bigg)^* u^{-1}. \end{aligned}$$

Proof. According to the Definition 2.12, (a)–(e) clearly hold.

(f) Let $f, g \in \mathcal{P}_\sigma^{\mathrm{fin,min}}$, then we have

$$\begin{aligned} s_f s_g^* &= s_f u \Bigg(\sum_{h \in \mathcal{P}_\sigma^{(2)}} s_h s_h^* \Bigg) u^{-1} s_g^* \\ &= \sum_{h \in \mathcal{P}_\sigma^{(2)} \setminus \mathcal{P}_\sigma^{\mathrm{fin,max}}} s_f s_{\lambda_\sigma(h)} s_{\lambda_\sigma(h)}^* s_g^* + s_f u \Bigg(\sum_{h \in \mathcal{P}_\sigma^{(2)} \cap \mathcal{P}_\sigma^{\mathrm{fin,max}}} s_h s_h^* \Bigg) u^{-1} s_g^*. \end{aligned}$$

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According to the relation (c) in Definition 2.12 and Lemma 3.2(b), (e), we have

$$\begin{aligned}
&s_f u\Bigg(\sum_{h\in\mathcal{P}_\sigma^{(2)}\cap\mathcal{P}_\sigma^{\mathrm{fin,max}}} s_h s_h^*\Bigg)u^{-1}s_g^* \\
&= s_f u\Bigg(\sum_{h\in\mathcal{P}_\sigma^{(2)}\cap\mathcal{P}_\sigma^{\mathrm{fin,max}}} s_h\Bigg)\Bigg(\sum_{i\in\mathcal{P}_\sigma^{(2)}\cap\mathcal{P}_\sigma^{\mathrm{fin,max}}} s_i\Bigg)^* u^{-1}s_g^* \\
&= s_f\Bigg(\sum_{h\in\mathcal{P}_\sigma^{(2)}\cap\mathcal{P}_\sigma^{\mathrm{fin,min}}} s_h\Bigg)uu^{-1}\Bigg(\sum_{i\in\mathcal{P}_\sigma^{(2)}\cap\mathcal{P}_\sigma^{\mathrm{fin,min}}} s_i\Bigg)^* s_g^* \\
&= \delta_\sigma(f)\Bigg(\sum_{h\in\mathcal{P}_\sigma^{(2)}\cap\mathcal{P}_\sigma^{\mathrm{fin,min}}} s_h\Bigg)^{\ell(f)} uu^{-1}\Bigg(\sum_{i\in\mathcal{P}_\sigma^{(2)}\cap\mathcal{P}_\sigma^{\mathrm{fin,min}}} s_i\Bigg)^{*\ell(g)} \\
&= \delta_\sigma(f)u\Bigg(\sum_{p\in\mathcal{P}_\sigma^{(\ell(f)+1)}\cap\mathcal{P}_\sigma^{\mathrm{fin,max}}} s_p\Bigg)\Bigg(\sum_{q\in\mathcal{P}_\sigma^{(\ell(g)+1)}\cap\mathcal{P}_\sigma^{\mathrm{fin,max}}} s_q\Bigg)u^{-1}
\end{aligned}$$

and this concludes the proof. □

For any $f\in\mathcal{P}_\sigma^{\mathrm{fin}}$, we let $\eta(f)$ denote the number $\min\{\nu(z)\mid z\in V_\sigma^{(\ell(f)-1)}\}$, i.e. $\eta(f)$ is the minimal number of paths from the top vertex to any vertex at level $\ell(f)-1$. The following lemma holds.

LEMMA 3.3. *For any $f, g\in\mathcal{P}_\sigma^{\mathrm{fin}}$ and $a, b\geq 0$, there exists $N\geq\ell(f)$ such that for any $n\geq N$, $u^a s_f s_g^* u^{-b}\in$ linear span$\{u^c s_p s_q^* u^{-d}\mid p\in\mathcal{P}_\sigma^{(n)}, q\in\mathcal{P}_\sigma^{(n-\ell(f)+\ell(g))}, 0\leq c<\eta(p)$ and $0\leq d<\eta(q)\}$.*

Proof. According to the relation $u^a s_f s_g^* u^{-b}=\sum_{h\in\mathcal{P}_\sigma^{(n)}} u^a s_f s_h s_h^* s_g^* u^{-b}$ ($a, b\geq 0$, $n\geq 2$ and $f, g\in\mathcal{P}_\sigma^{\mathrm{fin}}$) and $\lim_{n\to\infty}\min\{\nu(z)\mid z\in V_\sigma^{(n)}\}=\infty$ as σ is primitive, the lemma holds. □

For a proper and primitive substitution σ, we let

$$\mathcal{A}_\sigma=\text{linear span}\{u^a s_f s_g^* u^{-b}\mid a, b\geq 0 \text{ and } f, g\in\mathcal{P}_\sigma^{\mathrm{fin}}\}$$

and

$$\mathcal{L}_\sigma=\{u^a s_f s_g^* u^{-b}\mid f, g\in\mathcal{P}_\sigma^{\mathrm{fin}}, 0\leq a<\eta(f) \text{ and } 0\leq b<\eta(g)\}.$$

According to Lemma 3.3, the following lemma holds.

LEMMA 3.4.

$$\mathcal{A}_\sigma=\textit{linear span}\,\mathcal{L}_\sigma.$$

LEMMA 3.5. *Let σ be a proper primitive substitution, then*

$$*\text{-Alg}\{u, \{s_f\}_{f\in E_\sigma^{(2)}}\}=\mathcal{A}_\sigma.$$

Proof. Since $u, \{s_f\}_{f\in E_\sigma^{(2)}}$ are in $\mathcal{A}_\sigma$, to prove the lemma, it suffices to show that $\mathcal{A}_\sigma$ is an involutive algebra. We let $X=u^{a_1}s_{f_1}s_{g_1}^*u^{-b_1}$, $Y=u^{a_2}s_{f_2}s_{g_2}^*u^{-b_2}\in\mathcal{L}_\sigma$

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with $f_1, f_2, g_1, g_2 \in \mathcal{P}_\sigma^{\mathrm{fin,min}}$. According to Lemma 3.3, we may assume that $\ell(g_1) = \ell(f_2)$. We have

$$XY = u^{a_1} s_{f_1} s_{g_1}^* u^{a_2-b_1} s_{f_2} s_{g_2}^* u^{-b_2}.$$

We may assume that $a_2 - b_1 \geq 0$ without loss of generality. Then $0 \leq a_2 - b_1 < \nu(r(f_2)) + \eta(f_2)$.

If $a_2 - b_1 \leq \nu(r(f_2)) - 1$, then by using Lemma 3.2(c),

$$XY = u^{a_1} s_{f_1} s_{g_1}^* s_{\lambda_\sigma^{a_2-b_1}(f_2)} s_{g_2}^* u^{-b_2} \in \mathcal{A}_\sigma.$$

If $\nu(r(f_2)) \leq a_2 - b_1 < \nu(r(f_2)) + \eta(f_2)$, then we apply Lemma 3.2(f) to $s_{f_1} s_{g_1}^*$, and we have

$$\begin{aligned} XY = & \sum_{h \in \mathcal{P}_\sigma^{(2)} \setminus \mathcal{P}_\sigma^{\mathrm{fin,max}}} u^{a_1} s_{f_1} s_{\lambda_\sigma(h)} s_{\lambda_\sigma(h)}^* s_{g_1}^* u^{a_2-b_1} s_{f_2} s_{g_2}^* u^{-b_2} \\ & + \delta_\sigma(f_1) u^{a_1+1} \Bigg(\sum_{p \in \mathcal{P}_\sigma^{(\ell(f_1)+1)} \cap \mathcal{P}_\sigma^{\mathrm{fin,max}}} s_p \Bigg) \Bigg(\sum_{q \in \mathcal{P}_\sigma^{(\ell(g_1)+1)} \cap \mathcal{P}_\sigma^{\mathrm{fin,max}}} s_q \Bigg)^* \\ & \times u^{a_2-b_1-1} s_{f_2} s_{g_2}^* u^{-b_2}. \end{aligned}$$

It follows from Lemma 3.2(d) that

$$s_{\lambda_\sigma(h)}^* s_{g_1}^* u^{a_2-b_1} s_{f_2}, \quad s_q^* u^{a_2-b_1-1} s_{f_2} \in \{ s_l^* s_r \mid l, r \in \mathcal{P}_\sigma^{\mathrm{fin}} \}$$

for any $h \in \mathcal{P}_\sigma^{(2)} \setminus \mathcal{P}_\sigma^{\mathrm{fin,max}}$ and $q \in \mathcal{P}_\sigma^{(\ell(g_1)+1)} \cap \mathcal{P}_\sigma^{\mathrm{fin,max}}$ and that concludes the proof. □

We let D_σ denote the C^*-algebra generated by $\{u^k s_f s_f^* u^{-k} \mid f \in \mathcal{P}_\sigma^{\mathrm{fin}}, 0 \leq k\}$.

It is clear that any two projections of the form $s_f s_f^*$ ($f \in \mathcal{P}_\sigma^{\mathrm{fin}}$) commute. For $N \geq 2$, we let $D_\sigma^{(N)}$ denote the subalgebra of D_σ generated by $\{s_f s_f^* \mid f \in \mathcal{P}_\sigma^{(N)}\}$. It is easily seen that $D_\sigma^{(N+1)}$ contains $D_\sigma^{(N)}$.

Notation 3.6. Let σ be a proper primitive substitution. For any $f \in \mathcal{P}_\sigma^{\mathrm{fin}}$, we let $\widetilde{U}(f)$ denote the set defined by

$$\widetilde{U}(f) = \{ g \in \mathcal{P}_\sigma^{\mathrm{fin}} \mid g^{(1)} = f^{(1)}, g^{(2)} = f^{(2)}, \ldots, g^{(\ell(f))} = f^{(\ell(f))} \}.$$

LEMMA 3.7. *For any $f \in \mathcal{P}_\sigma^{fin}$ and $a \geq 0$, there exists $N \geq \ell(f)$ such that for any $n \geq N$,*

$$u^a s_f s_f^* u^{-a} \in \textit{linear span}\{ s_g s_g^* \mid g \in \mathcal{P}_\sigma^{(n)} \}.$$

Proof. According to Lemma 3.2(c), it suffices to show that for any $f \in \mathcal{P}_\sigma^{\mathrm{fin,max}}$, there exists $n \in \mathbb{N}$ such that $u s_f s_f^* u^{-1} \in$ linear span$\{s_g s_g^* \mid g \in \mathcal{P}_\sigma^{(n)}\}$. Since σ is proper, we write $\mathcal{P}_\sigma^{\max} = \{p_{\max}\}$. If $f \in \mathcal{P}_\sigma^{\mathrm{fin,max}}$ with $p_{\max} \notin \widetilde{U}(f)$, then we let $S(f)$ denote the set

$$S(f) = \{ g \in \widetilde{U}(f) \mid g^{(1)}, g^{(2)}, \ldots, g^{(\ell(g)-1)} \in E_\sigma^{\max} \text{ and } g^{(\ell(g))} \notin E_\sigma^{\max} \}.$$

Then by using the Cuntz–Krieger relations, we have

$$\begin{aligned} u s_f s_f^* u^{-1} = \sum_{g \in S(f)} u s_g s_g^* u^{-1} &= \sum_{g \in S(f)} s_{\lambda_\sigma(g)} s_{\lambda_\sigma(g)}^* \\ &\in \text{linear span}\{ s_h s_h^* \mid h \in \mathcal{P}_\sigma^{(N)} \} \text{ for some } N \geq \ell(f). \end{aligned}$$

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If $f \in \mathcal{P}_\sigma^{\mathrm{fin,max}}$ with $p_{\max} \in U(f)$, then

$$us_f s_f^* u^{-1} = u\Bigg(1 - \sum_{g \in \mathcal{P}_\sigma^{(\ell(f))}, f \neq g} s_g s_g^*\Bigg)u^{-1}$$
$$= 1 - \sum_{g \in \mathcal{P}_\sigma^{(\ell(f))}, f \neq g} s_{\lambda_\sigma(g)} s_{\lambda_\sigma(g)}^*$$

which concludes the proof. □

According to the Lemma 3.7, the following lemma holds.

LEMMA 3.8. *When σ is proper and primitive, D_σ is the inductive limit of the subalgebras $\{D_\sigma^{(N)}\}_{N \in \mathbb{N}}$.*

LEMMA 3.9. *Let σ be a proper primitive substitution. Then the C^*-algebra D_σ is isomorphic to $C(\mathcal{P}_\sigma)$.*

Proof. The fact that $\mathcal{P}_\sigma$ is totally disconnected implies that the linear span of $\{\chi_f \mid f \in \mathcal{P}_\sigma^{\mathrm{fin}}\}$ is dense in $C(\mathcal{P}_\sigma)$. It clearly holds that the linear span of $\{\chi_f \mid f \in \mathcal{P}_\sigma^{(n)}\}$ is isomorphic to $D_\sigma^{(n)}$. Since D_σ is the inductive limit of $D_\sigma^{(n)}$, D_σ is isomorphic to $C(\mathcal{P}_\sigma)$. □

By Definition 2.12, we get the following lemma.

LEMMA 3.10. *Let σ be a proper primitive substitution. If we put $e_f = s_f s_f^*$, then the unitary u and projections $\{e_f\}_{f \in \mathcal{P}_\sigma^{\mathrm{fin}}}$ satisfy the following relations.*

(i) $\sum_{f \in E_\sigma^{(2)}} e_f = 1$. *For any* $g \in \mathcal{P}_\sigma^{(n)}$, $e_g = \sum_{h \in E_\sigma^{(2)},\ L(r(g)) = L(s(h))} e_{gh}$.

(ii) $e_{\lambda_\sigma(f)} = u e_f u^{-1}$ *for any* $n \geq 2$ *and* $f \in \mathcal{P}_\sigma^{(n)} \setminus (\mathcal{P}_\sigma^{(n)} \cap \mathcal{P}_\sigma^{\mathrm{fin,max}})$.

(iii)
$$u\Bigg(\sum_{f \in \mathcal{P}_\sigma^{(n)} \cap \mathcal{P}_\sigma^{\mathrm{fin,max}}} e_f\Bigg)u^{-1} = \sum_{f \in \mathcal{P}_\sigma^{(n)} \cap \mathcal{P}_\sigma^{\mathrm{fin,min}}} e_f$$

for any $n \geq 2$.

Definition 3.11. Let σ be a proper primitive substitution. Then we define $\mathcal{F}_\sigma$ to be the universal unital C^*-algebra generated by a unitary u and projections $\{e_f\}_{f \in \mathcal{P}_\sigma^{\mathrm{fin}}}$ satisfying the relations (i), (ii) and (iii) in Lemma 3.10.

Let $(\mathcal{P}_\sigma, \lambda_\sigma)$ be a Bratteli–Vershik system associated with a proper primitive substitution σ. Then this gives a C^*-dynamical system $(C(\mathcal{P}_\sigma), \mathbb{Z}, \lambda_\sigma)$, where

$$\lambda_\sigma^n(f) = f \circ \lambda_\sigma^{-n} \ (n \in \mathbb{Z}).$$

PROPOSITION 3.12. *Let σ be a proper primitive substitution. Then $\mathcal{F}_\sigma$ is isomorphic to the crossed product $C(\mathcal{P}_\sigma) \rtimes_{\lambda_\sigma} \mathbb{Z}$.*

Proof. The crossed product is generated by projections χ_f $(f \in \mathcal{P}_\sigma^{\mathrm{fin}})$ and a unitary U implementing the automorphism λ_σ. It follows from Lemma 3.9 that there is a homomorphism of $C(\mathcal{P}_\sigma)$ into $\mathcal{F}_\sigma$ such that $\phi(\chi_f) = e_f$ for any $f \in \mathcal{P}_\sigma^{\mathrm{fin}}$. We check that

$$\phi(\lambda_\sigma(\chi_f)) = u e_f u^{-1}$$

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for any $f \in \mathcal{P}_\sigma^{\mathrm{fin}}$. If $f \in \mathcal{P}_\sigma^{\mathrm{fin}} \backslash \mathcal{P}_\sigma^{\mathrm{fin,max}}$, then $\phi(\lambda_\sigma(\chi_f)) = e_{\lambda_\sigma(f)} = ue_f u^{-1}$. We write $\mathcal{P}_\sigma^{\max} = \{p_{\max}\}$. If $f \in \mathcal{P}_\sigma^{\mathrm{fin,max}}$ is such that $p_{\max} \notin U(f)$, then we take the set $S(f) \subset \mathcal{P}_\sigma^{\mathrm{fin}}$ (the definition of $S(f)$ is given in the proof of Lemma 3.7). Then $\chi_f = \sum_{g \in S(f)} \chi_g$. Hence

$$\phi(\lambda_\sigma(\chi_f)) = \sum_{g \in S(f)} e_{\lambda_\sigma(g)} = u\left(\sum_{g \in S(f)} e_g\right)u^{-1} = ue_f u^{-1}.$$

If $f \in \mathcal{P}_\sigma^{\mathrm{fin,max}}$ such that $p_{\max} \in U(f)$, then $\chi_f = 1 - \sum_{g \in \mathcal{P}_\sigma^{(\ell(f))}, f \neq g} \chi_g$. Hence

$$\phi(\lambda_\sigma(\chi_f)) = \phi\left(\lambda_\sigma\left(1 - \sum_{g \in \mathcal{P}_\sigma^{(\ell(f))}, f \neq g} \chi_g\right)\right) = 1 - \sum_{g \in \mathcal{P}_\sigma^{(\ell(f))}, f \neq g} ue_g u^{-1} = ue_f u^{-1}.$$

It follows from the discussion above that $\mathcal{F}_\sigma$ provides a covariant representation of $(C(\mathcal{P}_\sigma), \mathbb{Z}, \lambda_\sigma)$. It follows from the universal property of the crossed product that there is a homomorphism of $C(\mathcal{P}_\sigma) \rtimes_{\lambda_\sigma} \mathbb{Z}$ onto $\mathcal{F}_\sigma$ taking U to u and χ_f to e_f. Since λ_σ is minimal, $C(\mathcal{P}_\sigma) \rtimes_{\lambda_\sigma} \mathbb{Z}$ is simple and we conclude the proof. □

We consider a unitary u and partial isometries $\{s_f\}_{f \in E_\sigma^{(2)}}$ satisfying the relations (a)–(c) of Definition 2.12 which generate B_σ. By the universal property of B_σ, for any $t \in \mathbb{T}$, there exists an automorphism α_t of B_σ such that $\alpha_t(u) = u$, $\alpha_t(s_f) = ts_f$ for each f in $E_\sigma^{(2)}$. Put

$$E(x) = \int_{\mathbb{T}} \alpha_t(x)\, dt, \quad x \in B_\sigma$$

where dt is the normalized Haar measure on $\mathbb{T}$. Then E is a faithful conditional expectation onto $\mathcal{F}_\sigma$.

We consider a unitary u and partial isometries $\{e_f\}_{f \in E_\sigma^{(2)}}$ satisfying the relations (i)–(iii) of Lemma 3.10 which generate $\mathcal{F}_\sigma$. By the universal property of $\mathcal{F}_\sigma$, for any $t \in \mathbb{T}$, there exists an automorphism β_t of $\mathcal{F}_\sigma$ such that $\beta_t(u) = e^{it}u$, $\beta_t(e_f) = e_f$ for each f in $E_\sigma^{(2)}$. Put

$$F(x) = \int_{\mathbb{T}} \beta_t(x)\, dt, \quad x \in \mathcal{F}_\sigma.$$

Then F is a faithful conditional expectation onto D_σ.

We put $G = F \circ E$, then G is a faithful conditional expectation of B_σ onto D_σ.

THEOREM 3.13. *Let σ be a proper primitive substitution. Then C^*-algebra B_σ associated with σ is simple and purely infinite.*

Proof. Let σ be a proper primitive substitution on an alphabet A with ${}^{\#}A = c$. To prove the theorem, it suffices to show that for any non-zero positive element $X \in B_\sigma$, there exists $Z \in B_\sigma$ such that Z^*XZ is invertible. Since G is faithful, $G(X) \neq 0$. We may assume that $\|G(X)\| = 1$. For arbitrary $0 < \varepsilon < (1/3c^2 + 1)$, there exists $Y \in \mathrm{Alg}_{\mathbb{C}}\{u, \{s_f\}_{f \in E_\sigma^{(2)}}\}$ such that $\|X - Y\| < \varepsilon$. Then we have

$$\|G(Y)\| = \|G(X) - G(X - Y)\| \geq \|G(X)\| - \|X - Y\| > \frac{3c^2}{3c^2 + 1}.$$

For any $N \geq 2$, we let

$$\mathcal{L}_\sigma^{(N)} = \{u^a s_f s_g^* u^{-b} \mid f, g \in \mathcal{P}_\sigma^{\mathrm{fin,min}}, \min\{\ell(f), \ell(g)\} = N,$$
$$\max\{\ell(f), \ell(g)\} \leq 2N \text{ and if } u^a s_f s_g^* u^{-b} \in D_\sigma \text{ then } u^a s_f s_g^* u^{-b} \in D_\sigma^{(N)}\}.$$

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and $\mathcal{A}_\sigma^{(N)} =$ linear span $\mathcal{L}_\sigma^{(N)}$. According to Lemmas 3.3 and 3.7, there exists $N > 0$ such that $Y \in \mathcal{A}_\sigma^{(N)}$. Now we choose paths $\{p(f, v) \mid f \in \mathcal{P}_\sigma^{(N)}, v \in V_\sigma^{(1)}\} \subset \mathcal{P}_\sigma^{\mathrm{fin}}$ as follows:

$$p(f, v) = p_1(f, v)p_2(f, v) \in \mathcal{P}_\sigma^{\mathrm{fin}}$$

where $p_1(f, v) \in \widetilde{U}(f)$ is such that when we write

$$p_1(f, v) = \lambda_\sigma^k(g) \quad (g \in \mathcal{P}_\sigma^{\mathrm{fin,min}} \text{ with } r(g) = r(p_1(f, v)) \text{ and } 0 \le k),$$

then $\max\{\nu(z) \mid z \in V_\sigma^{(2N)}\} \le k$.

Then $p_2(f, v) \in \mathcal{P}_\sigma^{\mathrm{fin}}$ has sufficiently large path length such that

$$L(s(p_2(f, v))) = L(r(p_1(f, v))), \quad L(r(p_2(f, v))) = L(v)$$

and $s^*_{p_2(f,v)} s_g s_{p_2(f,v)} = 0$ for any $g \in \bigcup_{m=2}^{N+1} \mathcal{P}_\sigma^{(m)}$.

We claim that if $x = u^a s_g s_h^* u^{-b} \in \mathcal{L}_\sigma^{(N)}$ $(g, h \in \mathcal{P}_\sigma^{\mathrm{fin,min}})$ with $G(x) = 0$, then $e_{p(f,v)} x e_{p(f,v)} = 0$ for any $f \in \mathcal{P}_\sigma^{(N)}$ and $v \in V_\sigma^{(1)}$, because

$$\begin{aligned} e_{p(f,v)} x e_{p(f,v)} &= s_{p(f,v)} s^*_{\lambda_\sigma^{-a}(p(f,v))} s_g s_h^* s_{\lambda_\sigma^{-b}(p(f,v))} s^*_{p(f,v)} \\ &= s_{p(f,v)} s^*_{p_2(f,v)} s^*_{\lambda_\sigma^{-a}(p_1(f,v))} s_g s_h^* s_{\lambda_\sigma^{-b}(p_1(f,v))} s_{p_2(f,v)} s^*_{p(f,v)} = 0. \end{aligned}$$

For any $v \in V_\sigma^{(1)}$, we let $\varphi_v^{(N)} \colon D_\sigma \longrightarrow D_\sigma$ be the map defined by

$$\varphi_v^{(N)}(x) = \sum_{f \in \mathcal{P}_\sigma^{(N)}} e_{p(f,v)} x e_{p(f,v)}.$$

Then the restricted map

$$\varphi_v^{(N)}|_{D_\sigma^{(N)}} : D_\sigma^{(N)} \longrightarrow C^*(\{e_{p(f,v)} \mid f \in \mathcal{P}_\sigma^{(N)}\})$$

is an isomorphism. According to the discussion above and the definition of $\mathcal{L}_\sigma^{(N)}$, we have

$$\varphi_v^{(N)}(G(Y)) = \sum_{f \in \mathcal{P}_\sigma^{(N)}} e_{p(f,v)} Y e_{p(f,v)}.$$

So there exists a path $p_v \in \{p(f, v) \mid f \in \mathcal{P}_\sigma^{(N)}\}$ such that

$$e_{p_v} Y e_{p_v} = \|G(Y)\| e_{p_v}$$

for each $v \in V_\sigma^{(1)}$.

Put $Z = \sum_{v \in V_\sigma^{(1)}} \|G(Y)\|^{-1/2} s_{p_v}$, then $\|Z\| \le \|G(Y)\|^{-1/2} c$ and we have

$$Z^* Y Z = \sum_{v \in V_\sigma^{(1)}} s^*_{p_v} s_{p_v} = 1.$$

Then $\|1 - Z^* X Z\| = \|Z^* Y Z - Z^* X Z\| \le \|Z\|^2 \|Y - X\| < 1/3$.

Thus $Z^* X Z$ is invertible and this concludes the proof. □

Remark 3.14. Let σ be a proper primitive substitution and $n \in \mathbb{N}$. Then $\{s_f\}_{f \in \mathcal{P}_\sigma^{(n+1)}} \subset B_\sigma$ satisfy the relations (a)–(c) in Definition 2.12 of B_{σ^n}. Thus B_{σ^n} is isomorphic to B_σ.

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4. *Actions on B_σ*

Let σ be a primitive substitution on an alphabet A. For $x \in A^{\mathbb{Z}}$, we denote by $x_{[j,\,k]}$ the word $x_j x_{j+1} \cdots x_k$. We define

$$X_\sigma = \{(x_i) \in A^{\mathbb{Z}} \mid \text{for any } j < k, \text{ there exists } n \in \mathbb{N} \text{ and } a \in A \text{ such that } x_{[j,k]} \text{ is a subword of } \sigma^n(a)\}$$

and let $T_\sigma : X_\sigma \longrightarrow X_\sigma$ be a map defined by $T_\sigma((x_k)_{k\in\mathbb{Z}}) = (x_{k+1})_{k\in\mathbb{Z}}$. It is known in [**10**] that for any primitive substitution σ, the substitution dynamical system (X_σ, T_σ) is minimal and uniquely ergodic.

A substitution $\sigma : A \longrightarrow \{\text{the set of words of } A\}$ naturally extends to a map $\sigma : A^{\mathbb{Z}} \longrightarrow A^{\mathbb{Z}}$:

$$\sigma(\cdots w_{-2}w_{-1}.w_0w_1\cdots) = \cdots\sigma(w_{-2})\sigma(w_{-1}).\sigma(w_0)\sigma(w_1)\cdots .$$

Let σ be a proper primitive substitution. An infinite word $w \in A^{\mathbb{Z}}$ is called a periodic point of σ if $\sigma(w) = w$ and is T_σ-periodic if there exists a positive integer k with $T_\sigma^k(w) = w$. A proper substitution σ is said to be periodic when there exists a periodic point of σ, which then necessarily is also T_σ-periodic. When σ is primitive, $|X_\sigma| < \infty$ if and only if σ is periodic.

If σ is periodic, proper and primitive, then we denote by p_σ the smallest period of its fixed point w and we set $u = w_{[0,\,p_\sigma-1]}$. As $\sigma(w) = w$, there exists $d_\sigma \geq 1$ such that $\sigma(u) = u \cdots u$ (d_σ times). We need the following fact from [**5**, Proposition 16].

PROPOSITION 4.1. *Let σ be a proper primitive substitution.*

(a) *If σ is non-periodic, then the Bratteli–Vershik system $(\mathcal{P}_\sigma, \lambda_\sigma)$ is isomorphic to the system (X_σ, T_σ).*

(b) *If σ is periodic, the system $(\mathcal{P}_\sigma, \lambda_\sigma)$ is isomorphic to an odometer with a stationary base $(p_\sigma, d_\sigma, d_\sigma, \ldots)$.*

The next lemma follows from [**4**, Theorem VIII.4.1].

LEMMA 4.2. *If σ is a periodic, proper and primitive substitution, then $\mathcal{F}_\sigma$ is isomorphic to the Bunce–Deddens algebra with supernatural number $p_\sigma d_\sigma^\infty$.*

Definition 4.3. Let σ be a primitive substitution on an alphabet A. For any $a, b \in A$, let $\kappa(a, b) \in \mathbb{N}$ denote the number of occurrences of b in $\sigma(a)$. Let M_σ denote its matrix: $M_\sigma = \{\kappa(a, b)\}_{a,b\in A}$. A primitive substitution σ is para-periodic if there exists $p \in \mathbb{N}$ such that $M_\sigma^2 = pM_\sigma$. Let σ be a para-periodic substitution, then we define a number $N_\sigma \in \mathbb{N}$ by $M_\sigma^2 = N_\sigma M_\sigma$.

Let σ be a proper primitive substitution. It is known [**4**, Corollary VIII.3.8] that there is a unique faithful trace $\varphi_\sigma^{(0)}$ on $\mathcal{F}_\sigma$. We let φ_σ be the state on B_σ defined by $\varphi_\sigma = \varphi_\sigma^{(0)} \circ E$, where as above E is the faithful conditional expectation from B_σ onto $\mathcal{F}_\sigma$.

LEMMA 4.4. *Let σ be a proper primitive substitution on an alphabet A.*

(a) *For any $f, g, h \in \mathcal{P}_\sigma^{\text{fin}}$ and $0 \neq k \in \mathbb{Z}$,*

$$\varphi_\sigma(u^k s_f s_f^*) = \varphi_\sigma(s_g s_h^* u^k s_h s_g^*) = 0.$$

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(b) *If σ is para-periodic, then for any $f, g \in \mathcal{P}_\sigma^{\mathrm{fin}}$ with $L(r(f)) = L(r(g))$, we have*

$$\varphi_\sigma(s_f s_f^*) = N_\sigma^{\ell(g)-\ell(f)} \varphi_\sigma(s_g s_g^*).$$

Proof. (a) Let $f, g, h \in \mathcal{P}_\sigma^{\mathrm{fin}}$. According to the relation

$$u^k s_f s_f^* = \sum_{g \in \mathcal{P}_\sigma^{(n)}} u^k s_f s_g s_g^* s_f^*$$

for any $n \geq 2$, we may first assume that $|k| < \nu(r(f))$ and $k \neq 0$. Then $s_f^* u^k s_f = 0$. Thus

$$\varphi_\sigma(u^k s_f s_f^*) = \varphi_\sigma(s_f s_f^* u^k s_f s_f^*) = 0,$$
$$\varphi_\sigma(s_g s_h^* u^k s_h s_g^*) = \sum_{p \in \mathcal{P}_\sigma^{(n)}} \varphi_\sigma(s_p s_p^* s_g s_h^* u^k s_h s_g^* s_p s_p^*) = 0 \quad (n \geq 2).$$

(b) We let $p_b^{(n)}$ denote the path in $\mathcal{P}_\sigma^{(n)} \cap \mathcal{P}_\sigma^{\mathrm{fin,min}}$ such that $L(r(p_b^{(n)})) = b$ for any $b \in A$ and $n \in \mathbb{N}$. Let $f, g \in \mathcal{P}_\sigma^{\mathrm{fin}}$ with $L(r(f)) = L(r(g))$ and $\ell(f) \leq \ell(g)$, then

$$\varphi_\sigma(s_f s_f^*) = \sum_{h \in \mathcal{P}_\sigma^{(\ell(g)-\ell(f)+2)}} \varphi_\sigma(s_f s_h s_h^* s_f^*).$$

Since $\varphi_\sigma(s_f s_f^*) = \varphi_\sigma(s_{f'} s_{f'}^*)$ for any $f, f' \in \mathcal{P}_\sigma^{\mathrm{fin}}$ with $r(f) = r(f')$ and

$$\#\{h \in \mathcal{P}^{(\ell(g)-\ell(f)+2)} \mid L(s(h)) = L(r(f)) \text{ and } L(r(h)) = b\} = N_\sigma^{\ell(g)-\ell(f)} \kappa(b, L(r(f)))$$

as σ is para-periodic, we have

$$\sum_{h \in \mathcal{P}_\sigma^{(\ell(g)-\ell(f)+2)}} \varphi_\sigma(s_f s_h s_h^* s_f^*) = N_\sigma^{\ell(g)-\ell(f)} \sum_{b \in A} \kappa(b, L(r(f))) \varphi_\sigma(s_{p_b^{(\ell(g)+1)}} s^*_{p_b^{(\ell(g)+1)}})$$
$$= N_\sigma^{\ell(g)-\ell(f)} \varphi_\sigma(s_g s_g^*). \qquad \square$$

LEMMA 4.5. *Let σ be a proper primitive substitution. Let $x = u^a s_f s_g^* u^{-b} \in \mathcal{L}_\sigma$, where $f, g \in \mathcal{P}_\sigma^{\mathrm{fin,min}}$. For any $n > \ell(f)$, x can be written as*

$$x = \sum_{\text{finite}} u^{a_k} s_{f_k} s_{g_k}^* u^{-b_k}$$

where $u^{a_k} s_{f_k} s_{g_k}^* u^{-b_k} \in \mathcal{L}_\sigma$, $f_k \in \mathcal{P}_\sigma^{(n)} \cap \mathcal{P}_\sigma^{\mathrm{fin,min}}$ and $\eta(f_k) \leq a_k < \nu(r(f_k)) + \eta(f_k)$.

Proof. Let $x = u^a s_f s_g^* u^{-b} \in \mathcal{L}_\sigma$ and $n > \ell(f)$. According to the relation $\sum_{h \in \mathcal{P}_\sigma^{(m)}} s_h s_h^* = 1$ $(m \geq 1)$, we may assume that $\ell(f) = n - 1$. If $a \geq \min\{\nu(z) \mid \ell(z) = n-1\}$, then

$$x = u^a s_f s_g^* u^{-b} = \sum_{h \in E_\sigma^{(2)}} u^a s_f s_h s_h^* s_g^* u^{-b}.$$

If $a < \min\{\nu(z) \mid \ell(z) = n-1\}$, we have, according to the Lemma 3.2(f),

$$x = \sum_{h \in E_\sigma^{(2)} \setminus E_\sigma^{\max}} u^a s_f s_{\lambda_\sigma(h)} s_{\lambda_\sigma(h)}^* s_g^* u^{-b}$$
$$+ u^{a+1} \Bigg(\sum_{h \in \mathcal{P}_\sigma^{(\ell(f)+1)} \cap \mathcal{P}_\sigma^{\mathrm{fin,max}}} s_h \Bigg) \Bigg(\sum_{i \in \mathcal{P}_\sigma^{(\ell(g)+1)} \cap \mathcal{P}_\sigma^{\mathrm{fin,max}}} s_i \Bigg)^* u^{-(b+1)}.$$

The above equations satisfy the conditions in Lemma 4.5. $\square$

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Definition 4.6. Let $\mathcal{A}$ be a C^*-algebra, θ an $\mathbb{R}$-action on $\mathcal{A}$, φ a state on $\mathcal{A}$ and $\beta > 0$. Then φ satisfies the β-KMS condition if for each $x, y \in \mathcal{A}$ there is a function f, bounded and continuous on $\{z \in \mathbb{C} \mid 0 \leq \mathrm{Im}(z) \leq \beta\}$ and analytic on $\{z \in \mathbb{C} \mid 0 < \mathrm{Im}(z) < \beta\}$, such that, for $t \in \mathbb{R}$,

$$f(t) = \varphi(\theta_t(x)y), \quad f(t + i\beta) = \varphi(y\theta_t(x)).$$

It is known that φ satisfies the β-KMS condition if and only if $\varphi(x\theta_{i\beta}(y)) = \varphi(yx)$ for any $x \in \mathcal{A}$ and y, an analytic element for $(\theta_t)_{t\in\mathbb{R}}$.

Let σ be a para-periodic proper substitution. Let $\theta : \mathbb{R} \longrightarrow \mathrm{Aut}(B_\sigma)$ be the map defined by

$$\theta_t(u) = u, \quad \theta_t(s_f) = N_\sigma^{it} s_f \quad (f \in E_\sigma^{(2)}).$$

Then θ is an $\mathbb{R}$-action on B_σ.

PROPOSITION 4.7. *Let σ be a para-periodic proper substitution. Then the state φ_σ is a 1-KMS state for $(\theta_t)_{t\in\mathbb{R}}$.*

Proof. To prove the proposition, it suffices, by Lemmas 3.5 and 4.5, to show that $\varphi_\sigma(xy) = \varphi_\sigma(y\theta_i(x))$ for any $x = u^{a_1} s_{f_1} s_{g_1}^* u^{-b_1}$, $y = u^{a_2} s_{f_2} s_{g_2}^* u^{-b_2} \in \mathcal{L}_\sigma$ with $f_1, f_2, g_1, g_2 \in \mathcal{P}_\sigma^{\mathrm{fin,min}}$, $\ell(g_1) = \ell(f_2)$, $\eta(g_1) \leq b_1 < \nu(r(g_1)) + \eta(g_1)$ and $\eta(f_2) \leq a_2 < \nu(r(f_2)) + \eta(f_2)$.

If $\ell(f_1) + \ell(f_2) \neq \ell(g_1) + \ell(g_2)$, then, from the calculation in the proof of Lemma 3.5, xy and yx can be described as the finite sum of $u^a s_f s_g^* u^{-b}$'s with $\ell(f) \neq \ell(g)$. Since $E(u^a s_f s_g^* u^{-b}) = 0$ when $\ell(f) \neq \ell(g)$, $\varphi_\sigma(xy) = \varphi_\sigma(yx) = 0$.

We let $\ell(f_1) + \ell(f_2) = \ell(g_1) + \ell(g_2)$, then $\ell(f_1) = \ell(g_2)$. If $b_1 \neq a_2$ or $f_2 \neq g_1$, then $s_{g_1}^* u^{a_2 - b_1} s_{f_2} = 0$. Thus $xy = 0$ and $\varphi_\sigma(yx) = \varphi_\sigma(s_{g_1} s_{g_1}^* u^{a_2 - b_1} s_{f_2} s_{g_2}^* u^{a_1 - b_2} s_{f_1} s_{g_1}^*) = 0$.

Next we let $b_1 = a_2$ and $g_1 = f_2$, then $f_1 = g_2$. If $a_1 \neq b_2$, then by using Lemma 4.4(a), $\varphi_\sigma(xy) = \varphi_\sigma(u^{a_1 - b_2} s_{f_1} s_{f_1}^*) = 0$ and $\varphi_\sigma(yx) = \varphi_\sigma(s_{f_1} s_{g_1}^* u^{a_1 - b_2} s_{g_1} s_{f_1}^*) = 0$.

We let $a_1 = b_2$. According to Lemma 4.4(b), we have

$$\begin{aligned} \varphi_\sigma(xy) &= \varphi_\sigma(s_{f_1} s_{f_1}^*) \\ &= N_\sigma^{\ell(g_1) - \ell(f_1)} \varphi_\sigma(s_{g_1} s_{g_1}^*) \\ &= \varphi_\sigma(y\theta_i(x)) \end{aligned}$$

and that concludes the proof. □

5. *The K-groups of B_σ*

We use the definition of semigroup crossed product C^*-algebras in [**11**].

Definition 5.1. Let B be a unital C^*-algebra and ρ be an endomorphism of B such that ρ is an isomorphism from B onto $\rho(1)B\rho(1)$. We define $B \rtimes_\rho \mathbb{N}$ to be the universal unital C^*-algebra generated by B and an isometry s such that $\rho(b) = sbs^*$ for any $b \in B$.

We denote by μ_1 the isometry $\sum_{f \in E_\sigma^{(2)} \cap E_\sigma^{\min}} s_f$ and let ρ be the map defined by

$$\rho : \mathcal{F}_\sigma \longrightarrow \mathcal{F}_\sigma, \quad \rho(x) = \mu_1 x \mu_1^*.$$

Then $\rho \in \mathrm{End}(\mathcal{F}_\sigma)$ and it is easily seen that B_σ is isomorphic to $\mathcal{F}_\sigma \rtimes_\rho \mathbb{N}$.

We recall some facts on the dimension group K_σ of a substitution dynamical system, cf. [**5**].

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Definition 5.2. Let σ be a primitive substitution on an alphabet A with $^{\#}A = d$, then we let

$$G_\sigma = \{x \in \mathbb{Q}^d \mid \text{there exists } n > 0 \text{ such that } M_\sigma^n x \in \mathbb{Z}^d\},$$
$$H_\sigma = \{x \in \mathbb{Q}^d \mid \text{there exists } n > 0 \text{ such that } M_\sigma^n x = 0\},$$
$$K_\sigma = G_\sigma / H_\sigma.$$

The group K_σ is called the dimension group of a substitution dynamical system.

We let ϕ be the endomorphism on $\mathbb{Z}^d$ defined by $\phi(x) = M_\sigma x$, then the direct limit of $(\mathbb{Z}^d, \phi)$ is isomorphic to K_σ.

Let σ be a proper primitive substitution on an alphabet $A = \{c_1, c_2 \dots c_d\}$. Let $C(\mathcal{P}_\sigma, \mathbb{Z})$ denote the set of continuous functions with values in $\mathbb{Z}$. It follows from the Pimsner–Voiculescu exact sequence that $K_0(\mathcal{F}_\sigma) = C(\mathcal{P}_\sigma, \mathbb{Z})/\partial_{\lambda_\sigma} C(\mathcal{P}_\sigma, \mathbb{Z})$ where $\partial_{\lambda_\sigma}(f) = f - f \circ \lambda_\sigma^{-1}$, for each $f \in C(\mathcal{P}_\sigma, \mathbb{Z})$ and $K_1(\mathcal{F}_\sigma) = \mathbb{Z}$.

For $n \geq 2$, we denote by C_n the subgroup of $C(\mathcal{P}_\sigma, \mathbb{Z})$ and H_n the subgroup of C_n defined by

$$C_n = \left\{ \sum_{f \in \mathcal{P}_\sigma^{(n)}} a_f \chi_f \;\middle|\; a_f \in \mathbb{Z} \right\},$$

$$H_n = \left\{ \sum_{f \in \mathcal{P}_\sigma^{(n)}} a_f \chi_f \in C_n \;\middle|\; \sum_{f \in \mathcal{P}_\sigma^{(n)},\ L(r(f))=c_k} a_f = 0 \text{ for any } 1 \leq k \leq d \right\}.$$

Then it clearly holds that $C_n \subset C_{n+1}$, $H_n \subset H_{n+1}$, $C(\mathcal{P}_\sigma, \mathbb{Z}) = \bigcup_{n=2}^\infty C_n$ and $\partial_{\lambda_\sigma} C(\mathcal{P}_\sigma, \mathbb{Z}) = \bigcup_{n=2}^\infty H_n$. We let $K_n = C_n/H_n$ and φ_n be the map on K_n defined by

$$\varphi_n : K_n \longrightarrow K_{n+1}, \quad \varphi_n(x + H_n) = x + H_{n+1}$$

for $n \geq 2$.

The group C_n/H_n is isomorphic to $\mathbb{Z}^d$ for any $n \geq 2$ with isomorphism ψ_n defined by

$$\psi_n\left(\sum_{f \in \mathcal{P}_\sigma^{(n)}} a_f \chi_f + H_n \right) = \left(\sum_{g \in \mathcal{P}_\sigma^{(n)}, L(r(g))=c_k} a_g \right)_{1 \leq k \leq d}.$$

Then we have $\phi \circ \psi_n = \psi_{n+1} \circ \varphi_n$ on K_n $(n \geq 2)$. Since $C(\mathcal{P}_\sigma, \mathbb{Z})/\partial_{\lambda_\sigma} C(\mathcal{P}_\sigma, \mathbb{Z})$ is the direct limit of $(C_n/H_n, \phi_n)$ and K_σ is the direct limit of $(\mathbb{Z}^d, \phi)$ respectively, $K_0(\mathcal{F}_\sigma)$ is isomorphic to K_σ.

Let σ be a proper primitive substitution. For any $x + H_\sigma \in K_\sigma$, there exists $y + H_\sigma \in K_\sigma$ such that $x - M_\sigma y \in H_\sigma$. Then we let $\tau_\sigma : K_\sigma \longrightarrow K_\sigma$ be a map defined by $\tau_\sigma(x + H_\sigma) = y + H_\sigma$.

THEOREM 5.3. *Let σ be a proper primitive substitution. Then the K-groups of B_σ are given by*

$$K_0(B_\sigma) \simeq \mathbb{Z} \oplus \mathrm{Coker}(1 - \tau_\sigma), \quad K_1(B_\sigma) \simeq \mathbb{Z} \oplus \mathrm{Ker}(1 - \tau_\sigma).$$

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Proof. It follows from [**1**] and the Pimsner–Voiculescu sequence in [**9**] that there is an exact sequence

$$\begin{array}{ccccccc} K_\sigma & \xrightarrow{i_*-\rho_*} & K_\sigma & \xrightarrow{i_*} & K_0(B_\sigma) & & \\ \uparrow & & & & \downarrow & & \\ K_1(B_\sigma) & \xleftarrow[i_*]{} & \mathbb{Z} & \xleftarrow[i_*-\rho_*]{} & \mathbb{Z} & & \end{array}$$

where ρ_* is the induced map of ρ on $K_0(\mathcal{F}_\sigma)$ and $K_1(\mathcal{F}_\sigma)$, respectively. Now $\rho_* = \tau_\sigma$ on $K_0(\mathcal{F}_\sigma)$ and $\rho_* = 1$ on $K_1(\mathcal{F}_\sigma)$ and this concludes the proof. □

Example 5.4. Let σ be a para-periodic proper substitution. Since the map τ_σ on K_σ is given by $\tau_\sigma(x + H_\sigma) = (x/N_\sigma) + H_\sigma$, then $K_0(B_\sigma) \simeq \mathbb{Z}$ and $K_1(B_\sigma) \simeq \mathbb{Z}$.

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References

[1] J. Cuntz. Simple C^*-algebras generated by isometries. *Comm. Math. Phys.* **57** (1977), 173–185.

[2] J. Cuntz. C^*-algebras associated with the $ax + b$-semigroup over $\mathbb{N}$. *K-theory and Noncommutative Geometry (Valladolid, 2006)*. European Mathematical Society, 2008, pp. 201–215.

[3] J. Cuntz and W. Krieger. A class of C^*-algebras and topological Markov chains. *Invent. Math.* **56** (1980), 251–268.

[4] K. R. Davidson. *C^*-Algebras by Example (Fields Institute Monographs, 6)*. American Mathematical Society, Providence, RI, 1996.

[5] F. Durand, B. Host and C. F. Skau. Substitutional dynamical systems, Bratteli diagrams and dimension groups. *Ergod. Th. & Dynam. Sys.* **19** (1999), 953–993.

[6] T. Giordano, I. F. Putnam and C. F. Skau. Topological orbit equivalence and C^*-crossed product. *J. Reine Angew. Math.* **469** (1995), 51–111.

[7] R. H. Herman, I. P. Putnam and C. F. Skau. Ordered Bratteli diagrams, dimension groups, and topological dynamics. *Int. J. Math.* **3** (1992), 827–864.

[8] B. Host. Substitution subshifts and Bratteli diagrams. *Topics in Symbolic Dynamics and Applications (London Mathematical Society Lecture Note Series, 279)*. Eds. F. Blanchard, A. Maass and A. Nogueira. Cambridge University Press, Cambridge, 1999.

[9] M. Pimsner and D. V. Voiculescu. K-groups of reduced crossed products by free groups. *J. Operator Theory* **4** (1982), 131–156.

[10] M. Queffelec. *Substitution Dynamical Systems (Lecture Notes in Mathematics, 1924)*. Springer, Berlin, 1987.

[11] M. Rørdam. Classification of certain infinite simple C^*-algebras. *J. Funct. Anal.* **131** (1995), 415–458.

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