Problem Corner

Solutions are invited to the following problems. They should be addressed to **Nick Lord** at **Tonbridge School, Tonbridge, Kent TN9 1JP** (e-mail: **njl@tonbridge-school.org**) and should arrive not later than 10 December 2019.

Proposals for problems are equally welcome. They should also be sent to Nick Lord at the above address and should be accompanied by solutions and any relevant background information.

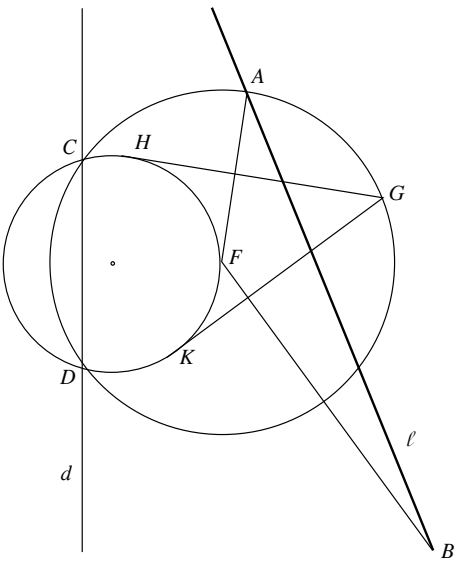
103.E (M. G. Elliott)

Let N be a positive integer and let S_N be the set of points in the plane with coordinates $(m^2 - n^2,\ 2mn)$ for positive integers m, n with $n < m \leqslant N$. Find, in terms of N, the radius of the circle with centre $(0,\ 0)$ which minimises the sum of the squares of the perpendicular distances from the points of S_N to the circle.

103.F (Geoff Strickland)

In the diagram d is a fixed straight line and F a fixed point. The point G is the reflection of F in a variable straight line ℓ which is drawn in such a way that the circle centre F and radius FG intersects d at two points C and D. The points H and K are the points of contact of the tangents from G to the circle CFD. The points A and B are points on ℓ such that FA and FB are perpendicular to GH and GK respectively.

Prove that A and B are the points of intersection of ℓ with the parabola having directrix d and focus F.



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103.G (Stan Dolan)

The quadratic $f(x)$ has integer coefficients and discriminant D. Given that $\dfrac{2f(-1)}{f'(0)}$ is an integer greater than or equal to $|D|$, prove that $f(x)$ has rational roots.

103.H (Peter Shiu)

Find positive integers a, b, c and A, B, C such that

$$\int_0^1 \frac{a(x(1-x))^4 + b(x(1-x))^{12}}{c(x^2+1)}\,dx = \frac{355}{113} - \pi$$

and

$$\int_0^1 \frac{A(x(1-x))^{12} + B(x(1-x))^{20}}{C(x^2+1)}\,dx = \frac{104348}{33215} - \pi.$$

Solutions and comments on **102.I**, **102.J**, **102.K**, **102.L** (November 2018).

102.I (Isaac Sofair)

In the 'Four T Puzzle ©', four T-shaped pieces fit into the large square shown in Figure 1; they also fit into the smaller square shown in Figure 2 (where each T-shape touches two adjacent sides of the square). Find (in surd form) the ratio of the side length of the smaller square to that of the larger square.

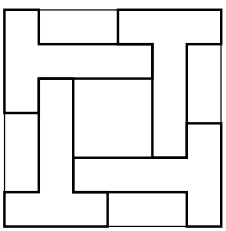


FIGURE 1

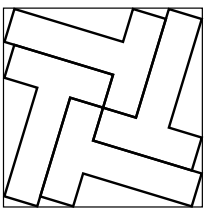


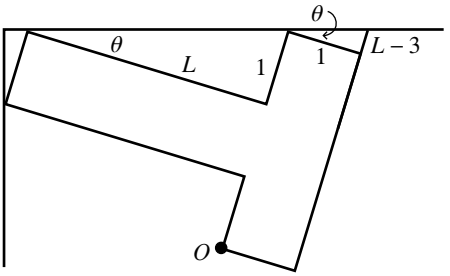
FIGURE 2

Answer: The ratio of side lengths is $6 : \sqrt{65 - 6\sqrt{13}}$.

This unusual geometrical problem attracted a variety of approaches. The shortest solution, which was given by several solvers, goes as follows.

With lengths allocated as in the diagram below showing the top left-hand corner of Figure 2, we have $\tan\theta = \dfrac{1}{L} = \dfrac{L-3}{1}$ from which $L^2 - 3L - 1 = 0$ and $L = \frac{1}{2}(3 + \sqrt{13})$, $\tan\theta = \frac{1}{2}(-3 + \sqrt{13})$.

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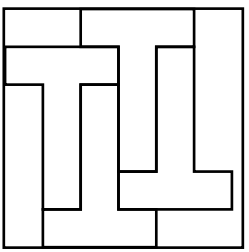


The vertical line through O shows that the side length of the smaller square is $2 \times 3 \cos\theta = 6 \cos\theta$; the side length of the larger square (in Figure 1) is $L + 3$.

The required ratio is thus $6 : (L + 3) \sec\theta$ which, on substituting the surd forms for L and $\tan\theta$ above, is equal to

$$6 : \frac{(9 + \sqrt{13})}{2}\sqrt{\frac{13 - 3\sqrt{13}}{2}} = 6 : \sqrt{65 - 6\sqrt{13}}.$$

A number of respondents gave the equivalent form $\frac{6}{17}\sqrt{5 + \frac{6}{\sqrt{13}}} : 1$ and the GCHQ Problem Solving Group observed that this ratio is close to $41 : 45$. As the proposer, Isaac Sofair, noted, because $L + 3 > 6$, there is a second way of fitting the four T-shapes into the larger square. This is shown in the diagram: there is a small gap on the right-hand side.



Correct solutions were received from: N. Curwen, S. Dolan, M. G. Elliott, GCHQ Problem Solving Group, A. P. Harrison, G. Howlett, P. M. King, J. A. Mundie, A. Ng, C. Starr, A. Tee and the proposer Isaac Sofair.

102.J (Zoltán Retkes)

Let P_n be a set of $n \geqslant 4$ points in space with the property that every choice of four points from P_n are non-coplanar and form a tetrahedron with volume not greater than 0.037. Show that P_n lies within a tetrahedron of unit volume.

Stan Dolan, Andrew Ng and the proposer, Zoltán Retkes, gave the succinct argument below; Brian Trustrum used transformations of a regular tetrahedron.

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Among the $\binom{n}{4}$ tetrahedra with vertices in P_n, let $ABCD$ be one of maximum volume, and let G be the centroid of $ABCD$. The enlargement scale factor -3, centre G maps $ABCD$ to a tetrahedron $A'B'C'D'$ with volume $\leqslant 27 \times 0.037 = 0.999$. An equivalent description of $A'B'C'D'$ is that it is the tetrahedron formed by planes through each vertex of $ABCD$ parallel to the corresponding base. This means that the points of P_n lie within or on the surface of $A'B'C'D'$; for if P is outside $A'B'C'D'$ then, for one if vertices of $ABCD$, say A, the tetrahedron $PBCD$ has greater height and hence greater volume than that of the maximising tetrahedron $ABCD$.

A further enlargement of $A'B'C'D'$ scale factor $1/\sqrt[3]{\text{vol}\,(A'B'C'D')}$, centre G gives a tetrahedron of volume 1 which contains inside it all the points of P_n.

Correct solutions were received from: S. Dolan, A. Ng, G. B. Trustrum and the proposer Zoltán Retkes.

102.K (Finbarr Holland)

Suppose t is a complex number. Prove that the solutions of the equation $\dfrac{t^3}{z^2} + \dfrac{(1-t)^3}{(1-z)^2} = 1$ are unimodular if, and only if, $|1-t| \leqslant 1 = |t|$.

This unusual problem attracted several appreciative comments from solvers.

The equation may be rewritten as

$$t^3(1-z)^2 + (1-t)^3 z^2 = z^2(1-z)^2$$

or

$$z^4 - 2z^3 + 3t(1-t)z^2 + 2t^3 z - t^3 = 0.$$

This factorises as $(z-t)^2\left(z^2 - 2(1-t)z - t\right) = 0$, where some solvers identified the double root from the fact that $z = t$ is an obvious root of both the original equation and its derivative. The quartic thus has roots $z_1 = z_2 = t$ and z_3, z_4 satisfying $z^2 - 2(1-t)z - t = 0$ (*).

- If z_1, z_2, z_3, z_4 are unimodular, then $|z_1| = |t| = 1$ and, from $z_3 + z_4 = 2(1-t)$, $\quad 2|1-t| \leqslant |z_3| + |z_4| = 2 \quad$ whence $|1-t| \leqslant 1$.
- If $|1-t| \leqslant 1 = |t|$, then z_1, z_2 are unimodular.

To show that z_3, z_4 are unimodular, a variety of approaches was used: the following is based on Neil Curwen's.

From (*), $\dfrac{-t}{\bar{z}^2}\left(\bar{z}^2 - 2(1-\bar{t})\bar{z} - \bar{t}\right) = 0$ or, since $t\bar{t} = 1$,

$$\frac{1}{\bar{z}^2} - 2(1-t)\frac{1}{\bar{z}} - t = 0.$$

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Thus if z is a root of (*), so too is $1/\bar{z}$.

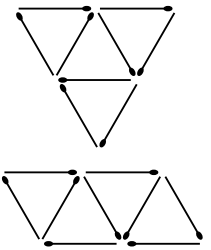
Case 1: $z_3 = \dfrac{1}{\bar{z}_3}$. Then z_3 and $z_4 = \dfrac{-t}{z_3}$ are both unimodular.

Case 2: $z_4 = \dfrac{1}{\bar{z}_3}$. Then, writing $z_3 = re^{i\theta}$, we have $z_4 = \dfrac{1}{r}e^{i\theta}$ and, from $z_3 + z_4 = 2(1-t)$, $r + \dfrac{1}{r} = 2|1-t| \leqslant 2$ forcing $r = 1$ and $z_3 = z_4 = e^{i\theta}$.

Correct solutions were received from: M. Bataille, N. Curwen, S. Dolan, M. G. Elliott, A. P. Harrison, G. Howlett, J. A. Mundie, A. Tee, G. B. Trustrum and the proposer Finbarr Holland.

102.L (Stan Dolan)

The diagram illustrates two ways that 4 congruent equilateral triangles can be made with a planar arrangement of 9 matches.



Given that T congruent equilateral triangles can be made with a planar arrangement of M matches, prove that

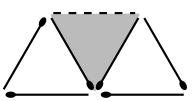
$$M \geqslant \frac{3T}{2} + \sqrt{\frac{3T}{2}}.$$

There were some impressively full and detailed analyses of this intriguing problem. The proposer, Stan Dolan, dealt neatly with the various issues that arise in his solution which follows.

Consider each match to have unit length. In any given arrangement of matches we can discard matches which are not edges of a triangle. We can also assume the arrangement is connected because otherwise we could reduce M but not T by joining components together.

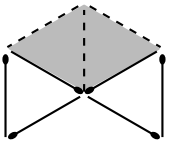
The arrangement is thus one of equilateral triangles on an isometric grid with a single external perimeter and, possibly, one or more internal boundaries. But any concavities on a boundary (either internal or external) can be filled in by repeated application of the following processes, each of which increases the value of $\dfrac{3T}{2} + \sqrt{\dfrac{3T}{2}} - M$.

- If the angle in the shaded region to be filled is 60°, then add an edge, increasing T by 1.

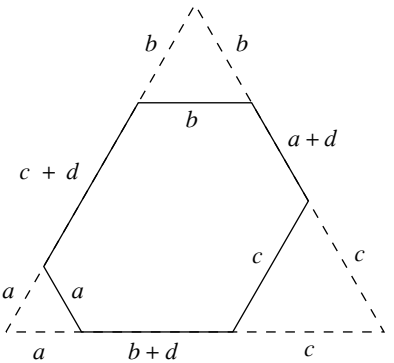


- If the angle is 120°, then add 3 edges, increasing T by 2.

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We now have an arrangement of equilateral triangles with no internal boundaries and external angles of either 240° or 300°. We can consider an external angle of 300° to be two external angles of 240° separated by an edge of zero length and, with this convention, we have an equiangular hexagon. This is an equilateral triangle from which equilateral triangles have been removed from each vertex and we may allocate non-negative integers a, b, c, d so that the lengths of edges are as in the diagram.



Then $T = (a + b + c + d)^2 - a^2 - b^2 - c^2$ and P, the perimeter of the hexagon, is given by $P = 2a + 2b + 2c + 3d$. Thus

$$P^2 = 6T + 2(a - b)^2 + 2(b - c)^2 + 2(c - a)^2 + 3d^2 \geqslant 6T.$$

Since internal matches are shared between two triangles and and those on the perimeter only belong to one,

$$M = \frac{3T + P}{2} \geqslant \frac{3T}{2} + \sqrt{\frac{3T}{2}}.$$

As several other solvers also observed, we have equality here when the equiangular hexagon is regular.

Graham Howlett further showed that the minimum number of matches required to construct T triangles is given by $\left\lceil \frac{3T}{2} + \sqrt{\frac{3T}{2}} \right\rceil$.

Correct solutions were received from: M. G. Elliott, G. Howlett, C. Starr, A. Tee, G. B. Trustrum and the proposer Stan Dolan.

Finally, my apologies to Graham Howlett for inadvertently omitting his name from the list of solvers of Problem **102.F** in the March 2019 *Gazette*.

10.1017/mag.2019.94 N.J.L.

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