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On two summability methods

By HÜSEYİN BOR†
Department of Mathematics, Erciyes University, Kayseri, Turkey

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1. *Introduction.* Let Σa_n be a given infinite series with partial sums s_n, and $r_n = na_n$. By u_n^α and t_n^α we denote the nth Cesàro means of order $\alpha\,(\alpha > -1)$ of the sequences (s_n) and (r_n), respectively. The series Σa_n is said to be absolutely summable (C,α) with index k, or simply summable $|C,\alpha|_k$, $k \geqslant 1$, if

$$\sum_{n=1}^{\infty} n^{k-1}|u_n^\alpha - u_{n-1}^\alpha|^k < \infty [\mathbf{2}]. \tag{1·1}$$

Since $t_n^\alpha = n(u_n^\alpha - u_{n-1}^\alpha)$ [**3**], condition (1·1) can also be written as

$$\sum_{n=1}^{\infty} \frac{1}{n}|t_n^\alpha|^k < \infty. \tag{1·2}$$

Let (p_n) be a sequence of positive real constants such that

$$P_n = \sum_{\nu=0}^{n} p_\nu \to \infty \quad \text{as } n \to \infty \quad (P_{-1} = p_{-1} = 0). \tag{1·3}$$

A series Σa_n is said to be summable $|\bar{N}, p_n|_k$, $k \geqslant 1$, if

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} |T_n - T_{n-1}|^k < \infty [\mathbf{1}], \tag{1·4}$$

where

$$T_n = (P_n)^{-1} \sum_{\nu=0}^{n} p_\nu s_\nu.$$

It should be noted that, if we take $p_n = 1$, then $|\bar{N}, p_n|_k$ summability is identical with $|C,1|_k$ summability.

2. The purpose of this paper is to establish a relation between the $|\bar{N}, p_n|_k$ and $|C,1|_k$ summability methods. In particular, we have pointed out that $|C,1|_k$ summability method can be obtained from $|\bar{N}, p_n|_k$ summability method by taking $p_n = 1$ for $n \in N$, where $k \geqslant 1$. However, it should be remarked that one can find a sequence (p_n) so that the methods $|\bar{N}, p_n|_k$ and $|C,1|_k$ are different from each other.

Now, we would like to clarify the following question.

When a series is summable $|C,1|_k$, what conditions should the sequence (p_n) satisfy in order that same series is summable $|\bar{N}, p_n|_k$ (provided that p_n must differ from one for at least one $n \in N$)?

3. To answer this question we give the following theorem.

THEOREM. *Let (p_n) be a sequence of positive real constants such that as $n \to \infty$*

$$\text{(i) } np_n = O(P_n), \quad \text{(ii) } P_n = O(np_n). \tag{3·1}$$

If Σa_n is summable $|C,1|_k$, then it is also summable $|\bar{N}, p_n|_k$, $k \geqslant 1$.

† Mail address: P.K. 213, Kayseri, Turkey.

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It should be noticed that the necessity of these conditions cannot be readily justified.

4. *Proof of the Theorem.* We have

$$T_n = \frac{1}{P_n}\sum_{\nu=0}^{n} p_\nu s_\nu = \frac{1}{P_n}\sum_{\nu=0}^{n} (P_\nu - P_{\nu-1})\, a_\nu. \tag{4·1}$$

Then

$$T_n - T_{n-1} = \frac{p_n}{P_n P_{n-1}}\sum_{\nu=1}^{n} P_{\nu-1} a_\nu = \frac{p_n}{P_n P_{n-1}}\sum_{\nu=1}^{n} \nu a_\nu P_{\nu-1}(\nu)^{-1}. \tag{4·2}$$

Now, applying Abel's partial summation formula (see [4]) to the sum

$$\sum_{\nu=1}^{n} \nu a_\nu P_{\nu-1}(\nu)^{-1},$$

$$\begin{aligned} T_n - T_{n-1} &= -p_n(P_n P_{n-1})^{-1}\sum_{\nu=1}^{n-1} (\nu+1)\,\nu^{-1} p_\nu t_\nu \\ &\quad + p_n(P_n P_{n-1})^{-1}\sum_{\nu=1}^{n-1} P_\nu t_\nu \nu^{-1} + (n+1)\, p_n t_n (nP_n)^{-1} \\ &= T_{n,1} + T_{n,2} + T_{n,3}, \quad \text{say.} \end{aligned}$$

To prove the theorem, by Minkowski's inequality, it is sufficient to show that

$$\sum_{r=1}^{\infty}\left(\frac{P_n}{p_n}\right)^{k-1} |T_{n,r}|^k < \infty, \quad \text{for } r = 1, 2, 3. \tag{4·3}$$

Now, applying Hölder's inequality, we have

$$\begin{aligned} \sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1} |T_{n,1}|^k &\leqslant \sum_{n=2}^{m+1} \frac{p_n}{P_n P_{n-1}^k}\left\{\sum_{\nu=1}^{n=1} \frac{\nu+1}{\nu} p_\nu |t_\nu|\right\}^k \\ &= O(1)\sum_{n=2}^{m+1} \frac{p_n}{P_n P_{n-1}} \sum_{\nu=1}^{n-1} p_\nu |t_\nu|^k \times \left\{\frac{1}{P_{n-1}}\sum_{\nu=1}^{n-1} p_\nu\right\}^{k-1} \\ &= O(1)\sum_{\nu=1}^{m} p_\nu |t_\nu|^k \sum_{n=\nu+1}^{m+1} \frac{p_n}{P_n P_{n-1}} = O(1)\sum_{\nu=1}^{m} \frac{p_\nu}{P_\nu}|t_\nu|^k \\ &= O(1)\sum_{\nu=1}^{m} \frac{1}{\nu}|t_\nu|^k = O(1) \quad \text{as} \quad m \to \infty. \end{aligned}$$

Using the fact that $P_\nu = O(\nu p_\nu)$, also we obtain

$$\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1} |T_{n,2}|^k = O(1)\sum_{\nu=1}^{m} \frac{1}{\nu}|t_\nu|^k = O(1) \quad \text{as } m \to \infty.$$

Finally, we have

$$\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{k-1} |T_{n,3}|^k = O(1)\sum_{n=1}^{m} \frac{1}{n}|t_n|^k = O(1) \quad \text{as } m \to \infty.$$

Therefore, we get

$$\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{k-1} |T_{n,r}|^k = O(1) \quad \text{as } m \to \infty, \quad \text{for } r = 1, 2, 3.$$

which completes the proof of the theorem.

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Remark. To see the hypotheses (i) and (ii) in the theorem are satisfied by at least one $p_n \neq 1$, it is sufficient to take $p_n = n$ for all $n \in N$.

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