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Hamiltonian form of the modified nonlinear Schrödinger equation for gravity waves on arbitrary depth

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The commonly used forms of the modified nonlinear Schrödinger equations for deep water (Dysthe, *Proc. R. Soc. Lond.* A, vol. 369, 1979, p. 105) and arbitrary depth (Brinch–Nielsen & Jonsson, *Wave Motion*, vol. 8, 1986, p. 455) do not conserve momentum and are not Hamiltonian. We show how these equations can be brought into Hamiltonian form, with the action, momentum and Hamiltonian being conserved. We derive the new fourth-order nonlinear Schrödinger equation for arbitrary depth, starting from the Zakharov equation enhanced with the new kernel of Krasitskii (*J. Fluid Mech.*, vol. 272, 1994, p. 1).

Key words: Hamiltonian theory, surface gravity waves, variational methods

1. Introduction

The first explicit formulation of a variational principle for water waves was presented by Luke (1967), who noted that the appropriate boundary conditions for waves on the free surface of a fluid can be obtained from a Lagrangian variational description. The Hamiltonian structure of the surface wave problem was discovered by Zakharov (1968) and rediscovered independently by Broer (1974) and Miles (1977). The Hamiltonian formulation states that the wave motion is governed by a pair of Hamilton's equations where the Hamiltonian is the total energy of the waves and where the canonically conjugate variables are the free-surface elevation and the velocity potential at the free surface.

One application of the Hamiltonian formulation is the derivation of the Zakharov integro-differential equation (Zakharov 1968), which describes nonlinear four-wave interactions within a spectrum of surface waves (see also Crawford, Saffman & Yuen 1980; Yuen & Lake 1982). Stiassnie & Shemer (1984) extended the Zakharov equation to include five-wave interactions. In the derivation of the Zakharov equation a separation between resonant and non-resonant interactions is employed in order to eliminate the 'trivial' non-resonant interactions from the evolution equation. In the earlier works (Zakharov 1968; Crawford *et al.* 1980; Yuen & Lake 1982; Stiassnie & Shemer 1984) a multiple-scale technique was employed to remove these fast, non-resonant interactions. It turns out however that this technique does not preserve the Hamiltonian structure of the original problem. This shortcoming of the original form of the Zakharov equation was first pointed out by Caponi, Saffman

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& Yuen (1982). In order to fix this problem Krasitskii (1994) used the technique of canonical transformation and was able to obtain the four- and five-wave Zakharov equations in forms that preserve the Hamiltonian, as well as the wave action and wave momentum. Some differences between the Hamiltonian and non-Hamiltonian forms of the Zakharov equation were discussed in Krasitskii & Kalmykov (1993).

An important class of models within the field of weakly nonlinear waves is the nonlinear Schrödinger (NLS) equations with higher-order modifications. In addition to weak nonlinearity, the NLS equations also require that the energy of free waves is limited to a narrow band around a characteristic wave. The cubic NLS equation for waves on deep water was first derived by Zakharov (1968) as a narrow-band limit of the more general Zakharov equation. It is natural to distinguish between two different regimes with respect to the water depth. If the depth is not smaller than the horizontal modulation length scale, i.e. $(k_c h)^{-1} = O(\Delta k/k_c)$, where Δk is the modulation wavenumber and k_c is the characteristic wavenumber, the waves can be considered deep-water waves. In the shallower case, i.e. $(k_c h)^{-1} = O(1)$, the waves can feel the bottom, and additional effects must be taken into account. In the following we refer to the two depth regimes, $(k_c h)^{-1} = O(\Delta k/k_c)$ and $(k_c h)^{-1} = O(1)$, as 'deep water' and 'finite depth' respectively. The most important effect related to finite depth is the wave-induced mean flow which enters the NLS equations to lower order on finite depth than on deep water. For three-dimensional waves on finite depth, the cubic NLS equations were derived by Benney & Roskes (1969) and Davey & Stewartson (1974). In the case of finite depth, the effect of the wave-induced mean flow becomes important, resulting in two coupled equations describing the modulation of the waves and the wave-induced mean flow, often referred to as the Davey–Stewartson system. To allow higher-order nonlinear effects, Dysthe (1979) extended the deep-water cubic NLS equation to fourth order in steepness and bandwidth. This modified NLS equation (sometimes called the MNLS equation or the Dysthe equation) also includes the effect of the induced mean flow, which enters the problem to fourth order on deep water (Dysthe 1979) and to cubic order on finite depth (Benney & Roskes 1969; Davey & Stewartson 1974). It has been shown that both the Davey–Stewartson system and the Dysthe equation can be obtained as narrow-band limits of the Zakharov equation (Stiassnie 1984; Stiassnie & Shemer 1984). A fourth-order equation for waves on water of finite depth was presented by Brinch-Nielsen & Jonsson (1986). Their equation includes additional terms involving the induced mean flow. Several other slightly modified versions of the higher-order NLS equations have been used by various authors (e.g. Lo & Mei 1985; Trulsen & Dysthe 1996, 1997; Trulsen *et al.* 2000).

The mathematical properties of the cubic NLS equations have been investigated by many authors for several decades (e.g. Sulem & Sulem 1999 and the references therein). These equations are known to obey a number of conservation laws. Among others they conserve the quantities

$$N = \int |\xi|^2 \, \mathrm{d}\boldsymbol{x} \tag{1.1}$$

and

$$\boldsymbol{P} = \frac{\mathrm{i}}{2} \int (\xi \nabla \xi^* - \xi^* \nabla \xi) \, \mathrm{d}\boldsymbol{x}, \tag{1.2}$$

where $\xi(\boldsymbol{x}, t)$ is the complex amplitude in which the NLS equations are expressed; $*$ denotes the complex conjugate; and $\mathrm{d}\boldsymbol{x} = \mathrm{d}x\,\mathrm{d}y$ denotes a two-dimensional surface

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element in the horizontal plane. The integration is performed either over an infinite domain for $\boldsymbol{x}$ with $\xi(\boldsymbol{x}, t)$ and its derivatives vanishing sufficiently fast as $|\boldsymbol{x}| \to \infty$ or over a finite domain with periodic boundary conditions. In the following we will refer to N and $\boldsymbol{P}$ as the wave action and momentum respectively. In addition, the cubic NLS equations conserve a Hamiltonian $H(\xi, \xi^*)$ such that

$$\frac{\partial \xi}{\partial t} = \frac{\delta H}{\delta \xi^*}, \quad \frac{\partial \xi^*}{\partial t} = -\frac{\delta H}{\delta \xi}, \tag{1.3}$$

where δ denotes the variational derivative. Hence, for application to water waves the Hamiltonian structure of the original water wave problem is preserved in these approximate models. Note also that the standard cubic NLS equation is integrable by the inverse scattering transform, and there exists a number of exact soliton and breather solutions to the cubic NLS equation (see e.g. Zakharov & Shabat 1972; Kuznetsov 1977; Peregrine 1983).

Although the cubic NLS equations preserve the relevant conservation laws, this is usually not the case for their higher-order extensions. It has been noted by several authors that the Dysthe equation is not Hamiltonian (Carter 2001; Segur, Henderson & Hammack 2005; Zakharov & Dyachenko 2010). Trulsen & Dysthe (1997) showed that the fourth-order NLS equations do not in general conserve the momentum (1.2). The lack of conserved quantities also applies to the fourth-order equation on finite depth in the form given by Brinch-Nielsen & Jonsson (1986).

In this paper we explain and repair these shortcomings of the higher-order NLS equations by presenting new equations, corresponding to the ones given by Dysthe (1979) and Brinch-Nielsen & Jonsson (1986), that obey the conservation laws and Hamiltonian structure represented by (1.1)–(1.3).

There are two reasons why the higher-order NLS equations do not have the expected Hamiltonian form. The first reason is the choice of variable. Traditionally the NLS equations have been expressed in either the first-harmonic complex amplitude of the surface elevation, $A_\zeta(\boldsymbol{x}, t)$, or the first-harmonic complex amplitude of the velocity potential evaluated at the quiescent water surface, $A_\phi(\boldsymbol{x}, t)$. While these variables are natural from the viewpoint of physics, they are not optimal from a mathematical point of view. This reason for the lack of Hamiltonian structure of the higher-order NLS equations was mentioned in a recent paper (Zakharov & Dyachenko 2010). In fact, one can show that the transformations from the Zakharov action variable $b(\boldsymbol{k}, t)$ to either $A_\zeta(\boldsymbol{x}, t)$ or $A_\phi(\boldsymbol{x}, t)$ are not canonical. In this paper the higher-order NLS equations are derived starting from the Zakharov equation, choosing the basic variable in the NLS equations to be the direct Fourier transform of the Zakharov variable $b(\boldsymbol{k}, t)$. This choice corresponds to a canonical transformation.

The second reason for the lack of Hamiltonian structure in the higher-order NLS equations is due to the same reason that the Zakharov equation is rendered not Hamiltonian when derived using a standard multiple-scale method. When deriving the NLS equations from the Zakharov equation this difficulty can be avoided by employing the kernel of Krasitskii (1994). We show that when taking the contribution from the kernel of Krasitskii (1994) into account, new terms appear in the NLS equation which secure both the Hamiltonian structure and the conservation of wave action and momentum. Note however that in the present context of narrow-band theory, the difference between the two forms of the Zakharov kernel is apparent only to highest order (fourth order in wave steepness and bandwidth with both assumed to be equally small) on finite depth. Therefore, this second reason for the lack of Hamiltonian structure applies only to the fourth-order equation on finite

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depth (Brinch-Nielsen & Jonsson 1986). For the fourth-order equation on deep water (the Dysthe equation), the change of basic variable in which to express the equation is sufficient to bring the equation into a Hamiltonian form. We anticipate that the application of a more sophisticated perturbation method will make it possible to derive the desired higher-order NLS equation directly without the detour through the Zakharov equation with Krasitskii's kernel.

The paper is organized as follows. In § 2, an introduction is given to the Zakharov equation and the general procedure of deriving NLS-type equations by taking the narrow-band limit of the Zakharov equation. In § 3 the wave-induced mean flow generated by a narrow-band wave field is considered using the spectral formulation within the framework of the Zakharov equation. The analysis of the induced mean flow is crucial for the derivation of the Hamiltonian NLS equation. The details of this derivation and the resulting Hamiltonian form of the fourth-order NLS equation, along with the main result and goal of this paper, are presented in § 4. In § 5 the stability of a uniform wavetrain is considered. Concluding remarks are given in § 6.

The conserved quantities of the new equations are discussed in Appendix A, showing that the new higher-order NLS equation derived in § 4 does indeed satisfy the conservation laws associated with (1.1)–(1.3). The expressions for the bound waves, which are needed for a correct reconstruction of the surface elevation, are given in Appendix B. Finally, the deep-water limit of the new equation is summarized in Appendix C.

2. Background

In the following we give a brief introduction to the Zakharov equation and the procedure of deriving the NLS equation from the Zakharov equation, which is the procedure employed in this paper.

2.1. *The Zakharov equation*

The starting point of the derivation of the Zakharov equation is the Hamiltonian description of water waves (Zakharov 1968; Broer 1974; Miles 1977) which states that the wave motion is governed by the pair of Hamilton's equations

$$\frac{\partial \zeta(\boldsymbol{x},t)}{\partial t} = \frac{\delta H}{\delta \psi(\boldsymbol{x},t)}, \quad \frac{\partial \psi(\boldsymbol{x},t)}{\partial t} = -\frac{\delta H}{\delta \zeta(\boldsymbol{x},t)}, \tag{2.1}$$

where $\boldsymbol{x}=(x,y)$ are the horizontal space coordinates; $\zeta(\boldsymbol{x},t)$ is the vertical surface displacement; $\psi(\boldsymbol{x},t)=\phi(\boldsymbol{x},z=\zeta(\boldsymbol{x},t),t)$ is the velocity potential evaluated on the free surface; and H is the Hamiltonian (the total energy). In the following we sometimes avoid writing the explicit t-dependence of functions.

We introduce the Fourier transforms of $\zeta(\boldsymbol{x},t)$ and $\psi(\boldsymbol{x},t)$ by

$$\hat{\zeta}(\boldsymbol{k}) = \frac{1}{2\pi}\int \zeta(\boldsymbol{x})\exp(-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}, \quad \hat{\psi}(\boldsymbol{k}) = \frac{1}{2\pi}\int \psi(\boldsymbol{x})\exp(-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}, \tag{2.2}$$

where $\boldsymbol{k}=(k_x,k_y)$ is the wave vector. Further we introduce the action density variable $a(\boldsymbol{k},t)$ such that

$$\hat{\zeta}(\boldsymbol{k}) = \mathscr{M}(\boldsymbol{k})[a(\boldsymbol{k})+a^*(-\boldsymbol{k})], \quad \hat{\psi}(\boldsymbol{k}) = -\mathrm{i}\mathscr{N}(\boldsymbol{k})[a(\boldsymbol{k})-a^*(-\boldsymbol{k})], \tag{2.3}$$

where

$$\mathscr{M}(\boldsymbol{k}) = \sqrt{\frac{\omega(\boldsymbol{k})}{2g}}, \quad \mathscr{N}(\boldsymbol{k}) = \sqrt{\frac{g}{2\omega(\boldsymbol{k})}}, \tag{2.4}$$

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$\omega(\boldsymbol{k}) = \sqrt{gk\tanh(kh)}$ being the dispersion relation of the linear waves and $k = |\boldsymbol{k}|$. It can be shown that (2.1) is equivalent to the equation

$$\mathrm{i}\frac{\partial a(\boldsymbol{k})}{\partial t} = \frac{\delta H}{\delta a^*(\boldsymbol{k})}, \tag{2.5}$$

where the Hamiltonian H is expressed in terms of $a(\boldsymbol{k})$ and $a^*(\boldsymbol{k})$. Under the assumption of weak nonlinearity, the Hamiltonian can be expanded in an integral power series of $a(\boldsymbol{k})$ and $a^*(\boldsymbol{k})$. From (2.5) one then gets an equation describing the time evolution of the variable $a(\boldsymbol{k})$ (see e.g. Krasitskii 1994 for details). The variable $a(\boldsymbol{k})$ describes both bound and free waves, and the Hamiltonian $H(a, a^*)$ contains both resonant and non-resonant terms. However, the non-resonant terms in the Hamiltonian are in a sense unimportant and can be removed through a suitable canonical transformation from $a(\boldsymbol{k})$ to a new variable $b(\boldsymbol{k})$, leading to

$$\mathrm{i}\frac{\partial b(\boldsymbol{k})}{\partial t} = \frac{\delta \tilde{H}}{\delta b^*(\boldsymbol{k})}, \tag{2.6}$$

where $\tilde{H} = \tilde{H}(b, b^*)$ is the 'reduced' Hamiltonian expressed in b and b^* through the canonical transformation $a = a(b, b^*)$. Krasitskii (1994) postulated the transformation between $a(\boldsymbol{k})$ and $b(\boldsymbol{k})$ in terms of an integral power series in the form

$$\begin{aligned} a_0 = b_0 &+ \int A^{(1)}_{0,1,2} b_1 b_2 \delta_{0-1-2}\,\mathrm{d}\boldsymbol{k}_{1,2} + \int A^{(2)}_{0,1,2} b_1^* b_2 \delta_{0+1-2}\,\mathrm{d}\boldsymbol{k}_{1,2} + \int A^{(3)}_{0,1,2} b_1^* b_2^* \delta_{0+1+2}\,\mathrm{d}\boldsymbol{k}_{1,2} \\ &+ \int B^{(1)}_{0,1,2,3} b_1 b_2 b_3 \delta_{0-1-2-3}\,\mathrm{d}\boldsymbol{k}_{1,2,3} + \int B^{(2)}_{0,1,2,3} b_1^* b_2 b_3 \delta_{0+1-2-3}\,\mathrm{d}\boldsymbol{k}_{1,2,3} \\ &+ \int B^{(3)}_{0,1,2,3} b_1^* b_2^* b_3 \delta_{0+1+2-3}\,\mathrm{d}\boldsymbol{k}_{1,2,3} + \int B^{(4)}_{0,1,2,3} b_1^* b_2^* b_3^* \delta_{0+1+2+3}\,\mathrm{d}\boldsymbol{k}_{1,2,3}. \end{aligned} \tag{2.7}$$

Here, we have adopted a compact notation, e.g. $A^{(1)}_{0,1,2} = A^{(1)}(\boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_2)$, $b_1 = b(\boldsymbol{k}_1)$, $\mathrm{d}\boldsymbol{k}_{1,2,3} = \mathrm{d}\boldsymbol{k}_1\,\mathrm{d}\boldsymbol{k}_2\,\mathrm{d}\boldsymbol{k}_2$ and $\delta_{0+1-2-3} = \delta(\boldsymbol{k} + \boldsymbol{k}_1 - \boldsymbol{k}_2 - \boldsymbol{k}_3)$. However, the complete notation is also used whenever appropriate. The procedure of determining the functions $A^{(n)}$ and $B^{(n)}$ such that the non-resonant terms are removed from the Hamiltonian, and such that the transformation (2.7) is canonical, is given in Krasitskii (1994). The result is the reduced Hamiltonian

$$\tilde{H} = \int \omega_0 b_0^* b_0\,\mathrm{d}\boldsymbol{k}_0 + \frac{1}{2}\int T_{0,1,2,3} b_0^* b_1^* b_2 b_3 \delta_{0+1-2-3}\,\mathrm{d}\boldsymbol{k}_{0,1,2,3}, \tag{2.8}$$

which together with (2.6) gives the Zakharov equation

$$\mathrm{i}\frac{\partial b_0}{\partial t} = \omega_0 b_0 + \int T_{0,1,2,3} b_1^* b_2 b_3 \delta_{0+1-2-3}\,\mathrm{d}\boldsymbol{k}_{1,2,3}, \tag{2.9}$$

describing the weakly nonlinear time evolution of $b(\boldsymbol{k})$. The expression for $T_{0,1,2,3}$ is given in Krasitskii (1994). Note however that Krasitskii (1994) uses the notation $\tilde{V}^{(2)}_{0,1,2,3}$ for the kernel function.

The function $T_{0,1,2,3} = T(\boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3)$ is defined in the eight-dimensional space $(\boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3)$. However, it is uniquely defined only where $\boldsymbol{k} + \boldsymbol{k}_1 - \boldsymbol{k}_2 - \boldsymbol{k}_3 = 0$. Outside this domain it is restricted only by a number of symmetry conditions, necessary for the conservation of the relevant conservation laws. From (2.8) it is clear that $T_{0,1,2,3} = T_{2,3,0,1}$. Moreover, T satisfies $T_{0,1,2,3} = T_{1,0,2,3} = T_{0,1,3,2}$, necessary for the

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conservation of wave action and momentum given respectively by

$$I = \int b^*(\boldsymbol{k}) b(\boldsymbol{k})\, \mathrm{d}\boldsymbol{k} \quad \text{and} \quad \boldsymbol{J} = \int \boldsymbol{k} b^*(\boldsymbol{k}) b(\boldsymbol{k})\, \mathrm{d}\boldsymbol{k}. \tag{2.10}$$

In the original derivation of the Zakharov equation (Zakharov 1968) as well as in some subsequent related works (e.g. Crawford *et al.* 1980; Yuen & Lake 1982; Stiassnie & Shemer 1984) the relation between $a(\boldsymbol{k}, t)$ and $b(\boldsymbol{k}, t)$ was found using a multiple-scale technique. This led to a kernel function that is symmetric only with respect to transpositions of $\boldsymbol{k}_2$ and $\boldsymbol{k}_3$. As first pointed out by Caponi *et al.* (1982), that form of the Zakharov equation is not Hamiltonian and does not conserve the wave action and momentum.

2.2. *Derivation of NLS-type equations from the Zakharov equation*

In addition to assuming weak nonlinearity, the NLS-type equations are based on the assumption of narrow bandwidth. It was shown by Zakharov (1968) that the cubic NLS equation on deep water follows as a narrow-band limit of the Zakharov equation. Later Stiassnie (1984) and Stiassnie & Shemer (1984) showed that both the Dysthe equation and the cubic NLS equation on finite depth (the Davey–Stewartson system) can be obtained as narrow-band limits of the Zakharov equation.

In order to derive the NLS equations from the Zakharov equation, one assumes that the variable $b(\boldsymbol{k})$ in the Zakharov equation represents energy only in a narrow band around the characteristic wave vector $\boldsymbol{k}_c = (k_c, 0)$. This allows us to write the wave vectors in the form

$$\boldsymbol{k} = \boldsymbol{k}_c + \boldsymbol{\chi}, \quad \boldsymbol{\chi} = (\lambda, \mu), \tag{2.11}$$

where $|\boldsymbol{\chi}|/k_c \ll 1$. Usually one assumes that the bandwidth is of the same order as the wave steepness, i.e. $|\boldsymbol{\chi}|/k_c = O(\epsilon)$, where $\epsilon = a_c k_c$ with a_c being the characteristic amplitude of the waves.

The general procedure of obtaining an NLS equation as a narrow-band limit of the Zakharov equation is outlined below. First, one introduces the new variable

$$\hat{B}(\boldsymbol{\chi}, t) = b(\boldsymbol{k}, t) \exp(\mathrm{i}\omega_c t), \tag{2.12}$$

where $\omega_c = \omega(\boldsymbol{k}_c)$, and its inverse Fourier transform with respect to $\boldsymbol{\chi}$,

$$B(\boldsymbol{x}, t) = \frac{1}{2\pi} \int \hat{B}(\boldsymbol{\chi}, t) \exp(\mathrm{i}\boldsymbol{\chi} \cdot \boldsymbol{x})\, \mathrm{d}\boldsymbol{\chi}. \tag{2.13}$$

By substituting $\hat{B}(\boldsymbol{\chi}, t)$ into the Zakharov equation (2.9) and taking the Fourier transform with respect to $\boldsymbol{\chi}$, one gets

$$\frac{\partial B}{\partial t} + \frac{1}{2\pi} \int \mathrm{i}[\omega_{c+0} - \omega_c] \hat{B}(\boldsymbol{\chi}_0) \exp(\mathrm{i}\boldsymbol{\chi}_0 \cdot \boldsymbol{x})\, \mathrm{d}\boldsymbol{\chi}_0$$
$$+ \frac{1}{(2\pi)^3} \int 4\mathrm{i}\pi^2 T_{c+3+2-1,c+1,c+2,c+3} \hat{B}^*(\boldsymbol{\chi}_1) \hat{B}(\boldsymbol{\chi}_2) \hat{B}(\boldsymbol{\chi}_3) \exp(\mathrm{i}\boldsymbol{\chi}_{3+2-1} \cdot \boldsymbol{x})\, \mathrm{d}\boldsymbol{\chi}_{1,2,3} = 0, \tag{2.14}$$

where $\omega_{c+0} = \omega(\boldsymbol{k}_c + \boldsymbol{\chi}_0)$, $T_{c+3+2-1,c+1,c+2,c+3} = T(\boldsymbol{k}_c + \boldsymbol{\chi}_{3+2-1}, \boldsymbol{k}_c + \boldsymbol{\chi}_1, \boldsymbol{k}_c + \boldsymbol{\chi}_2, \boldsymbol{k}_c + \boldsymbol{\chi}_3)$ and $\boldsymbol{\chi}_{3+2-1} = \boldsymbol{\chi}_3 + \boldsymbol{\chi}_2 - \boldsymbol{\chi}_1$. Now, by expanding ω_{c+0} and $T_{c+3+2-1,c+1,c+2,c+3}$ around $\boldsymbol{\chi}_0 = \boldsymbol{\chi}_1 = \boldsymbol{\chi}_2 = \boldsymbol{\chi}_3 = 0$ the integrals in (2.14) can be evaluated to yield an equation in $B(\boldsymbol{x}, t)$. For example, to the lowest significant order on infinitely deep water one

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finds

$$\omega_{c+0} - \omega_c = \omega_c \left(\frac{\lambda}{2k_c} - \frac{\lambda^2}{8k_c^2} + \frac{\mu^2}{4k_c^2} \right), \tag{2.15a}$$

$$T_{c+3+2-1,c+1,c+2,c+3} = T_{c,c,c,c} = \frac{k_c^3}{4\pi^2}. \tag{2.15b}$$

This leads to

$$\frac{\partial B}{\partial t} + \frac{\omega_c}{2k_c}\frac{\partial B}{\partial x} + \frac{\mathrm{i}\omega_c}{8k_c^2}\frac{\partial^2 B}{\partial x^2} - \frac{\mathrm{i}\omega_c}{4k_c^2}\frac{\partial^2 B}{\partial y^2} + \mathrm{i}k_c^3 |B|^2 B = 0, \tag{2.16}$$

which is the cubic NLS equation for water on infinite depth, expressed in the variable $B(\boldsymbol{x}, t)$. Note that traditionally the NLS equations have been expressed in either the complex amplitude of the surface elevation or the complex amplitude of the velocity potential. To obtain these forms of the equations, a trivial modification of the procedure outlined above is required. However, as mentioned, the choice of variable is crucial in order to obtain higher-order modifications of (2.16) that obey the relevant conservation laws.

The extension of this procedure to include higher-order terms in (2.16), or to include the effect of finite depth, is in principle straightforward and involves expanding (2.15) to higher orders in λ and μ. However, to leading order on finite depth or to second order on deep water, the expansion (2.15*b*) contains terms that cannot be expressed as a polynomial in λ and μ. This problem arises because some terms have a vanishing denominator in the limit $\boldsymbol{k}_0 = \boldsymbol{k}_2$ and $\boldsymbol{k}_1 = \boldsymbol{k}_3$. More specifically, in the case of finite depth, the function $T(\boldsymbol{k}_0, \boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3)$ is not uniquely defined in the limit $\boldsymbol{k}_0 = \boldsymbol{k}_2$ and $\boldsymbol{k}_1 = \boldsymbol{k}_3$, and its value depends on the direction of approach towards the limit. On deep water, the singularities cancel out, and the kernel is a regular continuous function (Zakharov 1999). However, the derivatives of the kernel function still suffer from the same type of non-uniqueness on deep water, as also mentioned by Zakharov (1999). As shown by Stiassnie (1984) and Stiassnie & Shemer (1984) and recently discussed in more detail by Janssen & Onorato (2007) and Stiassnie & Gramstad (2009), the 'singular' terms in the kernel can be related to the effect of the induced mean flow. In the following sections it will be shown in detail how most of the singular parts of the kernel lead to terms involving the induced mean flow in the modified NLS equation. However, when using Krasitskii's kernel function, some terms cannot be directly related to the induced mean flow. These terms have been treated differently and are discussed in detail in § 4.

2.3. *Definitions of functions*

It is convenient to define the following quantities, which appear in the following sections:

$$\sigma = \tanh(k_c h), \tag{2.17a}$$

$$d = \frac{k_c h}{\sigma}, \tag{2.17b}$$

$$s_0 = \sigma^2 - 1, \tag{2.17c}$$

$$s_1 = 2\sigma^6 - 13\sigma^4 + 12\sigma^2 - 9, \tag{2.17d}$$

$$C_g = \frac{\omega_c}{2k_c}(1 - s_0 d), \tag{2.17e}$$

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$$D_1 = -3s_0 + s_0(3\sigma^2 + 1)d, \tag{2.17f}$$

$$D_2 = -\sigma^2 + 5 + s_0^2 d, \tag{2.17g}$$

$$D_3 = -\sigma^2 + 5 + 2s_0(2\sigma^2 - 1)d - s_0^2(3\sigma^2 + 1)d^2, \tag{2.17h}$$

$$D_4 = -\sigma^2 + 5 + s_0(2\sigma^2 + 1)d - s_0(\sigma^4 + 5\sigma^2 + 2)d^2, \tag{2.17i}$$

$$D_5 = -\sigma^2 + 5 + s_0(11\sigma^2 - 23)d - s_0(19\sigma^4 - 34\sigma^2 - 1)d^2 + 3s_0^3(3\sigma^2 + 1)d^3, \tag{2.17j}$$

$$D_6 = -3(\sigma^2 - 5) + s_0(9\sigma^2 - 5)d - s_0(9\sigma^4 + 2\sigma^2 + 5)d^2 + s_0^3(3\sigma^2 + 1)d^3, \tag{2.17k}$$

$$D_7 = 3s_1 - s_0(2\sigma^2 - 3)(3\sigma^4 - 2\sigma^2 - 9)d, \tag{2.17l}$$

$$D_8 = -1 - 2s_0 d + s_0(3\sigma^2 + 1)d^2. \tag{2.17m}$$

3. The induced mean flow of a narrow-band wave field

In the following we treat the induced mean flow generated by a narrow-band wave field. In § 3.1 we derive a set of differential equations describing the induced mean flow potential. These equations have the same form as the existing equations for the induced mean flow, derived using multiple-scale methods in the context of the NLS equations (Benney & Roskes 1969; Davey & Stewartson 1974; Dysthe 1979; Brinch-Nielsen & Jonsson 1986).

In § 3.2 we exploit the current formulation, provided through the framework of the Zakharov equation, to obtain an alternative operator-form solution to the induced mean flow problem. This representation of the induced mean flow may turn out to be useful with respect to numerical simulations of the higher-order NLS equations.

3.1. *Derivation of equations for the induced mean flow*

In the current context of the Zakharov equation, the induced mean flow and mean free surface generated by a narrow-band wave field can be found from the canonical transformation (2.7) which expresses the bound modes, generated by non-resonant nonlinear interactions, in terms of the free-wave action variable $b(\boldsymbol{k})$.

We start out by writing the velocity potential $\phi(\boldsymbol{x}, z, t)$ in the form

$$\phi(\boldsymbol{x}, z, t) = \frac{1}{2\pi}\int \frac{\cosh\left[k(z+h)\right]}{\sinh\left[kh\right]}\hat{\phi}(\boldsymbol{k}, t)\exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x})\,\mathrm{d}\boldsymbol{k}. \tag{3.1}$$

Further, we make use of the following relation between $\hat{\phi}(\boldsymbol{k})$ and the Fourier transforms of the surface displacement and surface potential, $\hat{\zeta}(\boldsymbol{k})$ and $\hat{\psi}(\boldsymbol{k})$ respectively (Krasitskii 1994):

$$\hat{\phi}(\boldsymbol{k}) = \tanh\left[kh\right]\left(\hat{\psi}(\boldsymbol{k}) - \frac{1}{2\pi}\int k_1 \tanh\left[k_1 h\right]\hat{\psi}(\boldsymbol{k}_1)\hat{\zeta}(\boldsymbol{k}_2)\delta(\boldsymbol{k}-\boldsymbol{k}_1-\boldsymbol{k}_2)\,\mathrm{d}\boldsymbol{k}_{1,2}\right) + \cdots. \tag{3.2}$$

By inserting (2.3) and (2.7) into (3.2) as well as applying the narrow-bandwidth assumption such that the wave vectors can be written in the form (2.11), one can show that the induced mean flow potential $\bar{\phi}(\boldsymbol{x}, z, t)$ can be written as follows:

$$\bar{\phi}(\boldsymbol{x}, z, t) = \frac{1}{(2\pi)^2}\int \frac{\cosh\left[|\boldsymbol{\chi}_{2-1}|(z+h)\right]}{\cosh\left[|\boldsymbol{\chi}_{2-1}|h\right]}K(\boldsymbol{\chi}_1, \boldsymbol{\chi}_2)\hat{B}^*(\boldsymbol{\chi}_1)\hat{B}(\boldsymbol{\chi}_2)\exp(\mathrm{i}\boldsymbol{\chi}_{2-1}\cdot\boldsymbol{x})\,\mathrm{d}\boldsymbol{\chi}_{1,2}, \tag{3.3}$$

where $\hat{B}(\boldsymbol{\chi})$ is defined in (2.12) and

$$K(\boldsymbol{\chi}_1, \boldsymbol{\chi}_2) = K_{1,2} = -2\mathrm{i}\pi\mathscr{N}_{2-1}\left(A^{(2a)}_{2-1,c+1,c+2} - A^{(2a)}_{1-2,c+2,c+1}\right), \tag{3.4}$$

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in which $\mathscr{N}_{2-1} = \mathscr{N}(\boldsymbol{\chi}_2 - \boldsymbol{\chi}_1)$. The function $A^{(2a)}_{0,1,2}$ has the form

$$A^{(2a)}_{0,1,2} = A^{(2)}_{0,1,2} + \frac{1}{\pi}\mathscr{M}_0\mathscr{N}_1\mathscr{M}_2 k_1 \tanh[k_1 h]. \tag{3.5}$$

Further, one can expand the function $K_{1,2}$ around $\boldsymbol{\chi}_1 = \boldsymbol{\chi}_2 = 0$. To the first two orders, denoted $K^{(1)}_{1,2}$ and $K^{(2)}_{1,2}$ respectively, we find

$$K^{(1)}_{1,2} = \frac{\mathrm{i}g k_c D_2}{\omega^2_{2-1} - \Delta^2_{2-1}} \frac{\lambda_{2-1}}{4}, \tag{3.6a}$$

$$K^{(2)}_{1,2} = \frac{\mathrm{i}g}{\omega^2_{2-1} - \Delta^2_{2-1}} \left[\frac{D_3}{8}\lambda_{2-1}\lambda_{2+1} + \frac{D_2}{8}\mu_{2-1}\mu_{2+1} \right]. \tag{3.6b}$$

Here we have defined $\Delta_{2-1} = \omega(\boldsymbol{k}_2) - \omega(\boldsymbol{k}_1) = \omega(\boldsymbol{k}_c + \boldsymbol{\chi}_2) - \omega(\boldsymbol{k}_c + \boldsymbol{\chi}_1)$ as well as using the subscript notation such that e.g. $\omega_{2-1} = \omega(\boldsymbol{\chi}_2 - \boldsymbol{\chi}_1)$ and $\lambda_{2\pm1} = \lambda_2 \pm \lambda_1$.

Because of the form of $K^{(1)}_{1,2}$ and $K^{(2)}_{1,2}$, the integral (3.3) cannot in general be evaluated to yield an explicit solution for the induced mean flow. One can however show that (3.3) can be alternatively expressed in terms of a set of differential equations in the form

$$\nabla^2\bar{\phi} + \frac{\partial^2\bar{\phi}}{\partial z^2} = 0, \quad -h \leqslant z \leqslant 0, \tag{3.7a}$$

$$\frac{\partial\bar{\phi}}{\partial z} = 0, \quad z = -h, \tag{3.7b}$$

$$\frac{\partial^2\bar{\phi}}{\partial t^2} + g\frac{\partial\bar{\phi}}{\partial z} = F(B(\boldsymbol{x})), \quad z = 0, \tag{3.7c}$$

where $\nabla = (\partial/\partial x, \partial/\partial y)$ is the horizontal gradient vector and g is the acceleration due to gravity. It is clear that $\bar{\phi}(\boldsymbol{x}, z, t)$ in the form (3.3) fulfils the Laplace equation (3.7*a*) and the sea-bottom condition (3.7*b*). In order to find the inhomogeneity F, we insert (3.3) into (3.7*c*) and obtain

$$\begin{aligned} F &= \frac{1}{(2\pi)^2}\int \left(-\Delta^2_{2-1} + g|\boldsymbol{\chi}_{2-1}|\tanh[|\boldsymbol{\chi}_{2-1}|h]\right)\left(K^{(1)}_{1,2} + K^{(2)}_{1,2}\right)\hat{B}^*_1\hat{B}_2\exp(\mathrm{i}\boldsymbol{\chi}_{2-1}\cdot\boldsymbol{x})\,\mathrm{d}\boldsymbol{\chi}_{1,2} \\ &= \frac{1}{(2\pi)^2}\int \left(\omega^2_{2-1} - \Delta^2_{2-1}\right)\left(K^{(1)}_{1,2} + K^{(2)}_{1,2}\right)\hat{B}^*_1\hat{B}_2\exp(\mathrm{i}\boldsymbol{\chi}_{2-1}\cdot\boldsymbol{x})\,\mathrm{d}\boldsymbol{\chi}_{1,2} \\ &= \frac{gk_cD_2}{4}\frac{\partial}{\partial x}|B|^2 + \frac{gD_3}{8}\left(\mathrm{i}B\frac{\partial^2 B^*}{\partial x^2} + *\right) + \frac{gD_2}{8}\left(\mathrm{i}B\frac{\partial^2 B^*}{\partial y^2} + *\right). \end{aligned} \tag{3.8}$$

We have used $\partial\hat{B}/\partial t = -\mathrm{i}(\omega - \omega_c)\hat{B}$ to the first two orders, which follows from (2.14). Equations (3.7) are equivalent to the equations for the induced mean flow given in Brinch-Nielsen & Jonsson (1986).

Note that the system (3.7) allows homogeneous solution in the form of both free waves and $\bar{\phi} = \boldsymbol{U}\cdot\boldsymbol{x} + ct$, corresponding to a constant background current and a constant change in the quiescent water level.

3.2. *Operator-form solution to the induced mean flow*

Another convenient representation for the induced mean flow may be obtained from (3.3) by using the cross-correlation theorem, which states that if a function $\hat{h}(\boldsymbol{k})$ is

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defined by

$$\hat{h}(\boldsymbol{k}) = \int \hat{f}^*(\boldsymbol{k}')\hat{g}(\boldsymbol{k}+\boldsymbol{k}')\,\mathrm{d}\boldsymbol{k}', \tag{3.9}$$

then the inverse Fourier transform of $\hat{h}(\boldsymbol{k})$,

$$h(\boldsymbol{x}) = \frac{1}{2\pi}\int \hat{h}(\boldsymbol{k})\exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x})\,\mathrm{d}\boldsymbol{k}, \tag{3.10}$$

is given by $h(\boldsymbol{x}) = 2\pi f^*(\boldsymbol{x})g(\boldsymbol{x})$, where f and g are the inverse Fourier transforms of $\hat{f}$ and $\hat{g}$ respectively.

To take advantage of (3.9) we rewrite the expansion of the function $K_{1,2}$, given in (3.6), in the form

$$K_{1,2} = \frac{\mathrm{i}gk_c}{4}\frac{D_2\lambda_{2-1}}{\omega_{2-1}^2 - C_g^2\lambda_{2-1}^2} + \frac{\mathrm{i}g}{8k_c}\frac{k_cD_3\omega_{2-1}^2 - \omega_cC_gD_4\lambda_{2-1}^2}{\left(\omega_{2-1}^2 - C_g^2\lambda_{2-1}^2\right)^2}\lambda_{2-1}\lambda_{2+1} + \frac{\mathrm{i}gD_2}{8}\frac{\omega_{2-1}^2 + C_g^2\lambda_{2-1}^2}{\left(\omega_{2-1}^2 - C_g^2\lambda_{2-1}^2\right)^2}\mu_{2-1}\mu_{2+1}. \tag{3.11}$$

We further note that (3.11) can be rewritten as a finite sum of the form

$$K(\boldsymbol{\chi}_1, \boldsymbol{\chi}_2) = \sum_j S_j(\boldsymbol{\chi}_2 - \boldsymbol{\chi}_1)T_j(\boldsymbol{\chi}_1)U_j(\boldsymbol{\chi}_2). \tag{3.12}$$

For example, letting j run from one to five we may write

$$S_1(\boldsymbol{k}) = \frac{\mathrm{i}gk_c}{4}\frac{D_2k_x}{\omega^2 - C_g^2k_x^2}, \quad T_1(\boldsymbol{k}) = 1, \quad U_1(\boldsymbol{k}) = 1, \tag{3.13a}$$

$$S_2(\boldsymbol{k}) = \frac{\mathrm{i}gk_x}{8k_c}\frac{k_cD_3\omega^2 - \omega_cC_gD_4k_x^2}{\left(\omega^2 - C_g^2k_x^2\right)^2}, \quad T_2(\boldsymbol{k}) = 1, \quad U_2(\boldsymbol{k}) = k_x, \tag{3.13b}$$

$$S_3(\boldsymbol{k}) = S_2(\boldsymbol{k}), \quad T_3(\boldsymbol{k}) = k_x, \quad U_3(\boldsymbol{k}) = 1, \tag{3.13c}$$

$$S_4(\boldsymbol{k}) = \frac{\mathrm{i}gD_2k_y}{8}\frac{\omega^2 + C_g^2k_x^2}{\left(\omega^2 - C_g^2k_x^2\right)^2}, \quad T_4(\boldsymbol{k}) = 1, \quad U_4(\boldsymbol{k}) = k_y, \tag{3.13d}$$

$$S_5(\boldsymbol{k}) = S_4(\boldsymbol{k}), \quad T_5(\boldsymbol{k}) = k_y, \quad U_5(\boldsymbol{k}) = 1. \tag{3.13e}$$

Now, by writing the induced mean flow potential in the form

$$\bar{\phi}(\boldsymbol{x}, z, t) = \frac{1}{2\pi}\int \frac{\cosh\left[k(z+h)\right]}{\cosh\left[kh\right]}\hat{\bar{\phi}}(\boldsymbol{k}, t)\exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x})\,\mathrm{d}\boldsymbol{k}, \tag{3.14}$$

substituting (3.12) into (3.3) and applying the substitution $\boldsymbol{k} = \boldsymbol{\chi}_2 - \boldsymbol{\chi}_1$, $\boldsymbol{k}_1 = \boldsymbol{\chi}_1$, one finds that

$$\hat{\bar{\phi}}(\boldsymbol{k}, t) = \sum_j S_j(\boldsymbol{k})\frac{1}{2\pi}\int T_j(\boldsymbol{k}_1)\hat{B}^*(\boldsymbol{k}_1)U_j(\boldsymbol{k}+\boldsymbol{k}_1)\hat{B}(\boldsymbol{k}+\boldsymbol{k}_1)\,\mathrm{d}\boldsymbol{k}_1. \tag{3.15}$$

By virtue of the cross-correlation theorem this can be written as

$$\hat{\bar{\phi}}(\boldsymbol{k}, t) = \sum_j S_j(\boldsymbol{k})\mathscr{F}\{(\mathscr{F}^{-1}\{T_j^*(\boldsymbol{k})\hat{B}(\boldsymbol{k})\})^*\mathscr{F}^{-1}\{U_j(\boldsymbol{k})\hat{B}(\boldsymbol{k})\}\}, \tag{3.16}$$

where $\mathscr{F}$ and $\mathscr{F}^{-1}$ denote the Fourier transform and the inverse Fourier transform respectively. By using (3.11) and the decomposition outlined in (3.13) one can show

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that

$$\hat{\bar{\phi}}(\boldsymbol{k},t)=\tilde{\mathscr{P}}(\boldsymbol{k})\mathscr{F}(|B|^2)+\tilde{\mathscr{Q}}_1(\boldsymbol{k})\mathscr{F}\left(\mathrm{i}B\frac{\partial B^*}{\partial x}+*\right)+\tilde{\mathscr{Q}}_2(\boldsymbol{k})\mathscr{F}\left(\mathrm{i}B\frac{\partial B^*}{\partial y}+*\right), \quad (3.17)$$

where

$$\tilde{\mathscr{P}}(\boldsymbol{k})=\frac{\mathrm{i}gk_c}{4}\frac{D_2k_x}{\omega^2-C_g^2k_x^2}, \quad (3.18a)$$

$$\tilde{\mathscr{Q}}_1(\boldsymbol{k})=\frac{\mathrm{i}gk_x}{8k_c}\frac{k_cD_3\omega^2-\omega_cC_gD_4k_x^2}{\left(\omega^2-C_g^2k_x^2\right)^2}, \quad (3.18b)$$

$$\tilde{\mathscr{Q}}_2(\boldsymbol{k})=\frac{\mathrm{i}gD_2k_y}{8}\frac{\omega^2+C_g^2k_x^2}{\left(\omega^2-C_g^2k_x^2\right)^2}. \quad (3.18c)$$

Substitution of (3.17) into (3.14) leads to the following operator representation of the induced mean flow at $z=0$:

$$\bar{\phi}(\boldsymbol{x},z=0,t)=\mathscr{P}(|B|^2)+\mathscr{Q}_1\left(\mathrm{i}B\frac{\partial B^*}{\partial x}+*\right)+\mathscr{Q}_2\left(\mathrm{i}B\frac{\partial B^*}{\partial y}+*\right), \quad (3.19)$$

where the operators $\mathscr{P}$, $\mathscr{Q}_1$ and $\mathscr{Q}_2$ are defined in terms of $\tilde{\mathscr{P}}$, $\tilde{\mathscr{Q}}_1$ and $\tilde{\mathscr{Q}}_2$ such that e.g. $\mathscr{F}\{\mathscr{P}(f)\}=\tilde{\mathscr{P}}(\boldsymbol{k})\hat{f}$, for any function f with Fourier transform $\hat{f}$.

By using the same procedure as outlined above, one can also express the time and space derivatives of the induced mean flow in a similar manner, e.g.

$$\left.\frac{\partial\bar{\phi}}{\partial x}\right|_{z=0}=\mathscr{R}^{(x)}(|B|^2)+\mathscr{S}_1^{(x)}\left(\mathrm{i}B\frac{\partial B^*}{\partial x}+*\right)+\mathscr{S}_2^{(x)}\left(\mathrm{i}B\frac{\partial B^*}{\partial y}+*\right), \quad (3.20a)$$

$$\left.\frac{\partial\bar{\phi}}{\partial y}\right|_{z=0}=\mathscr{R}^{(y)}(|B|^2)+\mathscr{S}_1^{(y)}\left(\mathrm{i}B\frac{\partial B^*}{\partial x}+*\right)+\mathscr{S}_2^{(y)}\left(\mathrm{i}B\frac{\partial B^*}{\partial y}+*\right), \quad (3.20b)$$

$$\left.\frac{\partial\bar{\phi}}{\partial t}\right|_{z=0}=\mathscr{R}^{(t)}(|B|^2)+\mathscr{S}_1^{(t)}\left(\mathrm{i}B\frac{\partial B^*}{\partial x}+*\right)+\mathscr{S}_2^{(t)}\left(\mathrm{i}B\frac{\partial B^*}{\partial y}+*\right), \quad (3.20c)$$

where the operators are defined by

$$\tilde{\mathscr{R}}^{(x)}(\boldsymbol{k})=\mathrm{i}k_x\tilde{\mathscr{P}}(\boldsymbol{k}),\quad \tilde{\mathscr{S}}_1^{(x)}(\boldsymbol{k})=\mathrm{i}k_x\tilde{\mathscr{Q}}_1(\boldsymbol{k}),\quad \tilde{\mathscr{S}}_2^{(x)}(\boldsymbol{k})=\mathrm{i}k_x\tilde{\mathscr{Q}}_2(\boldsymbol{k}), \quad (3.21a)$$

$$\tilde{\mathscr{R}}^{(y)}(\boldsymbol{k})=\mathrm{i}k_y\tilde{\mathscr{P}}(\boldsymbol{k}),\quad \tilde{\mathscr{S}}_1^{(y)}(\boldsymbol{k})=\mathrm{i}k_y\tilde{\mathscr{Q}}_1(\boldsymbol{k}),\quad \tilde{\mathscr{S}}_2^{(y)}(\boldsymbol{k})=\mathrm{i}k_y\tilde{\mathscr{Q}}_2(\boldsymbol{k}), \quad (3.21b)$$

$$\left.\begin{aligned}\tilde{\mathscr{R}}^{(t)}(\boldsymbol{k})&=-\mathrm{i}C_gk_x\tilde{\mathscr{P}}(\boldsymbol{k}),\quad \tilde{\mathscr{S}}_1^{(t)}(\boldsymbol{k})=\frac{g\omega_ck_x^2}{32k_c}\frac{D_5\omega^2-C_g^2D_6k_x^2}{\left(\omega^2-C_g^2k_x^2\right)^2},\\ \tilde{\mathscr{S}}_2^{(t)}(\boldsymbol{k})&=\frac{gk_xk_y}{4}\frac{C_gD_2\omega^2}{\left(\omega^2-C_g^2k_x^2\right)^2}.\end{aligned}\right\} \quad (3.21c)$$

One should note that the expressions for the induced mean flow (3.20) are not uniquely defined by (3.21). This is shown by the fact that e.g. $\tilde{\mathscr{R}}^{(x,y,t)}(\boldsymbol{k})$ are not unique in the limit $\boldsymbol{k}\to 0$; i.e. their values depend on the direction of approach towards this limit. The same type of singularity is also present in the representation (3.6) when $\boldsymbol{\chi}_2-\boldsymbol{\chi}_1=0$. It is easily realized that different choices for $\tilde{\mathscr{R}}^{(x,y,t)}(\boldsymbol{k}=0)$ correspond to

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different values of a constant background current and quiescent water level. Hence, the non-uniqueness could be resolved by making a specific choice of background current and quiescent water level. Recall that the equations for the induced mean flow (3.7) allow a homogeneous solution in the form $\bar{\phi} = \boldsymbol{U} \cdot \boldsymbol{x} + ct$, which indeed corresponds to a constant background current and change in the quiescent water level.

One can show that for purely one-dimensional modulations, i.e. $k_y = 0$ in (3.21), the limit $k_x \to 0$ leads to a unique value for $\tilde{\mathscr{R}}^{(x,y,t)}(\boldsymbol{k} = 0)$. This solution can be shown to yield expressions for the induced mean flow identical to (16.99) of Whitham (1974) (see also Janssen & Onorato 2007; Stiassnie & Gramstad 2009).

4. Derivation of the Hamiltonian form of the fourth-order NLS equation

In the following the derivation of the new Hamiltonian fourth-order NLS equation for waves on finite depth is presented. The procedure is the same as outlined in § 2.2; however, we employ the finite-depth version of the kernel function $T_{0,1,2,3}$ and take the expansions in (2.15) to higher orders in the bandwidth compared with § 2.2.

We start out with (2.14), and as in § 2.2 the integral kernels can be expanded around $\boldsymbol{\chi}_0 = \boldsymbol{\chi}_1 = \boldsymbol{\chi}_2 = \boldsymbol{\chi}_3 = 0$ so that the two integrals in (2.14) can be evaluated. The first integral in (2.14) describes linear dispersion and can easily be calculated by Taylor-expanding the linear dispersion relation $\omega(\boldsymbol{k}_c + \boldsymbol{\chi})$ around $\boldsymbol{k}_c$ up to the desired accuracy (as in § 2.2). Alternatively these terms can be given in an exact form in terms of the following differential operator:

$$\mathscr{D}(\partial_x, \partial_y) = \mathrm{i}\left[\sqrt{g\mathscr{L}(\partial_x, \partial_y)\tanh\left(h\mathscr{L}(\partial_x, \partial_y)\right)} - \sqrt{gk_c\tanh\left(k_c h\right)}\right], \tag{4.1}$$

where

$$\mathscr{L}(\partial_x, \partial_y) = \sqrt{(k_c - \mathrm{i}\partial_x)^2 - \partial_y^2}, \tag{4.2}$$

in which $\partial_x = \partial/\partial x$ and $\partial_y = \partial/\partial y$. That is,

$$\frac{1}{2\pi}\int \mathrm{i}[\omega_{c+0} - \omega_c]\hat{B}(\boldsymbol{\chi}_0)\exp(\mathrm{i}\boldsymbol{\chi}_0 \cdot \boldsymbol{x})\,\mathrm{d}\boldsymbol{\chi}_0 = \mathscr{D}(\partial_x, \partial_y)B(\boldsymbol{x}). \tag{4.3}$$

We now turn to the expansion of $T_{c+3+2-1,c+1,c+2,c+3}$. It is useful to distinguish between three types of terms in $T_{c+3+2-1,c+1,c+2,c+3}$, i.e.

$$T_{c+3+2-1,c+1,c+2,c+3} = T^{(R)} + T^{(S1)} + T^{(S2)}. \tag{4.4}$$

Here we let $T^{(R)}$ contain the so-called regular parts of the kernel, i.e. the parts which have a well-defined value when $\boldsymbol{\chi}_1 = \boldsymbol{\chi}_2 = \boldsymbol{\chi}_3 = 0$ and can therefore be expanded as a polynomial in $\boldsymbol{\chi}_1$, $\boldsymbol{\chi}_2$ and $\boldsymbol{\chi}_3$.

Further we let $T^{(S1)}$ contain the terms that are directly related to the effect of the induced mean flow. As mentioned, these terms are non-unique in the limit $\boldsymbol{\chi}_1 = \boldsymbol{\chi}_2 = \boldsymbol{\chi}_3 = 0$. Since the same type of non-uniqueness is present in the expressions for the induced mean flow obtained in § 3, this part of the kernel can be turned into terms involving the induced mean flow. In this way the non-uniqueness of the kernel is represented in terms of the ambiguity of the constant background current and quiescent water level, as explained in § 3.2. Hence, no special treatment of the non-uniqueness is required at this point in the derivation.

Finally we let $T^{(S2)}$ denote the terms which can neither be expanded as a polynomial in $\boldsymbol{\chi}_1$, $\boldsymbol{\chi}_2$ and $\boldsymbol{\chi}_3$ nor be expressed in terms of the induced mean flow. This contribution

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from the kernel arises as a result of the modifications of the kernel due to Krasitskii (1994) and is necessary for the final NLS equation to be Hamiltonian. By using the same procedure as in § 3.2, this contribution can be turned into 'operator-form' terms in the modified NLS equation.

The expansion leading to the regular part $T^{(R)}$ is straightforward and gives the following to the two first orders in $\boldsymbol{\chi}$:

$$4\pi^2 \mathrm{i} T^{(R)} = -\frac{\mathrm{i}k_c^3 s_1}{8\sigma^3} - \frac{k_c^2 D_7}{16\sigma^3}(\mathrm{i}\lambda_2 + \mathrm{i}\lambda_3). \tag{4.5}$$

Further, by straightforward, though laborious, calculations one finds that the contribution from $T^{(S1)}$ can be written as

$$\begin{aligned} 4\pi^2 \mathrm{i} T^{(S1)} = {} & \frac{\mathrm{i}k_c}{2}\left[\mathrm{i}\lambda_{2-1} + \frac{s_0 k_c}{2\omega_c}(-\mathrm{i}\Delta_{2-1})\right]\left(K_{1,2}^{(1)} + K_{1,2}^{(2)}\right) + \frac{1}{2}\Bigg[- \frac{k_c D_1}{8\omega_c}[(-\mathrm{i}\Delta_{2-1})(\mathrm{i}\lambda_{2-1}) \\ & + 2(-\mathrm{i}\Delta_{2-1})(\mathrm{i}\lambda_3)] + \frac{1}{2}[(\mathrm{i}\lambda_{2-1})^2 + (\mathrm{i}\mu_{2-1})^2] + (\mathrm{i}\lambda_{2-1})(\mathrm{i}\lambda_3) + (\mathrm{i}\mu_{2-1})(\mathrm{i}\mu_3)\Bigg] K_{1,2}^{(1)} + (2 \leftrightarrow 3). \end{aligned} \tag{4.6}$$

The notation $(2 \leftrightarrow 3)$ denotes that the preceding expressions should be repeated, but with the indices 2 and 3 interchanged. We recall that $K_{1,2}^{(1)}$ and $K_{1,2}^{(2)}$, given in (3.6), are expressions related to the induced mean flow potential (3.3). This is used in the following to turn (4.6) into terms involving the induced mean flow.

Finally the contribution from $T^{(S2)}$ takes the form

$$4\pi^2 \mathrm{i} T^{(S2)} = \frac{\mathrm{i}g D_2 \lambda_{2-1}}{64} \frac{k_c s_0 \omega_{2-1}^2 - 2\omega_c C_g \lambda_{2-1}^2}{\left(\omega_{2-1}^2 - C_g^2 \lambda_{2-1}^2\right)^2}\left(D_8 \lambda_{2-1}\lambda_{3-1} + \frac{4k_c C_g}{\omega_c}\mu_{2-1}\mu_{3-1}\right) + (2 \leftrightarrow 3). \tag{4.7}$$

Now, by substituting the expressions (4.5)–(4.7) into the last integral in (2.14), this integral can be evaluated. The contributions from $T^{(R)}$ and $T^{(S1)}$ are now straightforward to handle, and one finds that

$$\begin{aligned} & \frac{1}{(2\pi)^3}\int 4\mathrm{i}\pi^2 (T^{(R)} + T^{(S1)})\hat{B}_1^* \hat{B}_2 \hat{B}_3 \exp(\mathrm{i}\boldsymbol{\chi}_{3+2-1}\cdot\boldsymbol{x})\,\mathrm{d}\boldsymbol{\chi}_{1,2,3} \\ & \quad = -\frac{\mathrm{i}k_c^3 s_1}{8\sigma^3}|B|^2 B - \frac{k_c^2 D_7}{8\sigma^3}|B|^2 \frac{\partial B}{\partial x} + \mathrm{i}k_c\left[\frac{\partial \bar{\phi}}{\partial x} + \frac{s_0 k_c}{2\omega_c}\frac{\partial \bar{\phi}}{\partial t}\right] B + \nabla\bar{\phi}\cdot\nabla B \\ & \qquad + \frac{B}{2}\nabla^2\bar{\phi} - \frac{k_c D_1}{8\omega_c}\left[\frac{\partial^2 \bar{\phi}}{\partial x \partial t} B + 2\frac{\partial \bar{\phi}}{\partial t}\frac{\partial B}{\partial x}\right], \quad z = 0. \end{aligned} \tag{4.8}$$

As mentioned, the contribution from $T^{(S2)}$ is somewhat more complicated, since it can neither be related to derivatives of $B(\boldsymbol{x})$ nor be directly related to the induced mean flow. However, by using the cross-correlation theorem as in § 3.2, the contribution from $T^{(S2)}$ can be expressed in terms of pseudo-differential operators, i.e.

$$\begin{aligned} & \frac{1}{(2\pi)^3}\int 4\mathrm{i}\pi^2 T^{(S2)}\hat{B}_1^* \hat{B}_2 \hat{B}_3 \exp(\mathrm{i}\boldsymbol{\chi}_{3+2-1}\cdot\boldsymbol{x})\,\mathrm{d}\boldsymbol{\chi}_{1,2,3} \\ & \quad = \mathscr{B}^{(x)}\left(B\frac{\partial B^*}{\partial x}\right) B + \mathscr{B}^{(x)}(|B|^2)\frac{\partial B}{\partial x} + \mathscr{B}^{(y)}\left(B\frac{\partial B^*}{\partial y}\right) B + \mathscr{B}^{(y)}(|B|^2)\frac{\partial B}{\partial y}, \end{aligned} \tag{4.9}$$

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where $\mathscr{B}^{(x)}$ and $\mathscr{B}^{(y)}$ are operators defined in Fourier variables by

$$\mathscr{F}\{\mathscr{B}^{(x)}(f)\} = \tilde{\mathscr{B}}^{(x)}(\boldsymbol{k})\hat{f} = \frac{gD_8D_2k_x^2}{32}\frac{k_cs_0\omega^2 - 2\omega_cC_gk_x^2}{\left(\omega^2 - C_g^2k_x^2\right)^2}\hat{f}, \tag{4.10a}$$

$$\mathscr{F}\{\mathscr{B}^{(y)}(f)\} = \tilde{\mathscr{B}}^{(y)}(\boldsymbol{k})\hat{f} = \frac{gk_cC_gD_2k_xk_y}{8\omega_c}\frac{k_cs_0\omega^2 - 2\omega_cC_gk_x^2}{\left(\omega^2 - C_g^2k_x^2\right)^2}\hat{f}. \tag{4.10b}$$

We note that also the operators (4.10) are non-unique for $\boldsymbol{k}\to 0$. Unlike the non-uniqueness appearing in the expressions for the induced mean flow, where the problem could be resolved by a specific choice of constant background current and quiescent water level, we have no direct physical guidance for the choice of the value of $\tilde{\mathscr{B}}^{(x)}(\boldsymbol{k}=0)$ and $\tilde{\mathscr{B}}^{(y)}(\boldsymbol{k}=0)$. However, as is shown in Appendix A, $\tilde{\mathscr{B}}^{(x)}$ and $\tilde{\mathscr{B}}^{(y)}$ are related to the operators defining the induced mean flow through the relations (A 5). These relations may be used to determine $\tilde{\mathscr{B}}^{(x)}(\boldsymbol{k}=0)$ and $\tilde{\mathscr{B}}^{(y)}(\boldsymbol{k}=0)$ when the choice of constant background current and quiescent water level has been made.

Finally, by combining (4.3), (4.8) and (4.9) we obtain the equation

$$\begin{aligned}&\frac{\partial B}{\partial t} + \mathscr{D}(\partial_x,\partial_y)B - \frac{\mathrm{i}k_c^3s_1}{8\sigma^3}|B|^2B + \mathrm{i}k_c\left[\frac{\partial\bar{\phi}}{\partial x} + \frac{s_0k_c}{2\omega_c}\frac{\partial\bar{\phi}}{\partial t}\right]B - \frac{k_c^2D_7}{8\sigma^3}|B|^2\frac{\partial B}{\partial x} + \nabla\bar{\phi}\cdot\nabla B\\ &\quad + \frac{B}{2}\nabla^2\bar{\phi} - \frac{k_cD_1}{8\omega_c}\left[\frac{\partial^2\bar{\phi}}{\partial x\partial t}B + 2\frac{\partial\bar{\phi}}{\partial t}\frac{\partial B}{\partial x}\right] + \mathscr{B}^{(x)}\left(B\frac{\partial B^*}{\partial x}\right)B + \mathscr{B}^{(x)}(|B|^2)\frac{\partial B}{\partial x}\\ &\quad + \mathscr{B}^{(y)}\left(B\frac{\partial B^*}{\partial y}\right)B + \mathscr{B}^{(y)}(|B|^2)\frac{\partial B}{\partial y} = 0, \quad z=0,\end{aligned} \tag{4.11}$$

which is the Hamiltonian form of the fourth-order NLS equations for water of finite depth – the main result and goal of this paper.

The evolution equation (4.11) permits the application of a wider range of potentials $\bar{\phi}$ than the particular solutions represented by (3.3) or (3.19). Any homogeneous solution of the system (3.7) can be included, including free-wave solutions. This technique has sometimes been employed for studies of the influence of longer waves on a field of short waves; e.g. Gramstad & Trulsen (2010) did this for a study of the effect of a swell on a field of wind waves.

Restricting to the case that $\bar{\phi}$ does not include homogeneous solutions from (3.7) it can be shown (see Appendix A) that the Hamiltonian H corresponding to (4.11) has the form

$$\begin{aligned}H = \int &-B^*\mathscr{D}(\partial_x,\partial_y)B - \frac{\mathrm{i}k_c}{2}\left[\frac{\partial\bar{\phi}}{\partial x} + \frac{s_0k_c}{2\omega_c}\frac{\partial\bar{\phi}}{\partial t}\right]|B|^2 + \frac{\mathrm{i}k_c^3s_1}{16\sigma^3}|B|^4\\ &+ \frac{k_c^2D_7}{32\sigma^3}|B|^2\left(B^*\frac{\partial B}{\partial x} - B\frac{\partial B^*}{\partial x}\right) - \frac{1}{4}\nabla\bar{\phi}\cdot\left[B^*\nabla B - B\nabla B^*\right]\\ &+ \frac{k_cD_1}{16\omega_c}\frac{\partial\bar{\phi}}{\partial t}\left(B^*\frac{\partial B}{\partial x} - B\frac{\partial B^*}{\partial x}\right)\mathrm{d}\boldsymbol{x}, \quad z=0.\end{aligned} \tag{4.12}$$

It is interesting to remark that the extra contribution from Krasitskii's kernel function, compared with the original version of the kernel, is exactly that contained in $T^{(S2)}$ above, leading to the terms involving $\mathscr{B}^{(x)}$ and $\mathscr{B}^{(y)}$ in (4.11). Thus, a derivation using the original form of the kernel would give (4.11) with $\mathscr{B}^{(x)} = \mathscr{B}^{(y)} = 0$. It is shown in Appendix A that the presence of the terms involving $\mathscr{B}^{(x)}$ and $\mathscr{B}^{(y)}$ is essential for the equation to be Hamiltonian and to conserve momentum.

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5. Stability of a uniform wavetrain

A uniform Stokes wave solution to (4.11) with the equation for the induced mean flow (3.7) is given by

$$B_s = a\left(\frac{g}{2\omega_c}\right)^{1/2}\exp\left(\frac{\mathrm{i}k_c^2\omega_c s_1}{16\sigma^4}a^2t - \frac{\mathrm{i}k_c^2\omega_c s_0^2}{8\sigma^2}a^2t + \mathrm{i}\theta\right), \quad \bar{\phi}_s = \frac{\omega_c^2 s_0}{4\sigma^2}a^2t, \tag{5.1}$$

where a and θ are the constant amplitude and the initial phase of the Stokes wave, respectively. The solution for $\bar{\phi}$ is chosen so that the mean surface $\bar{\zeta}$, given in (B 3*e*), is zero. This assures that the mean water depth is really equal to the assumed depth h.

The stability of the Stokes wave to sideband perturbations is investigated by assuming small perturbations of the form

$$B = B_s[1 + \alpha(\boldsymbol{x}, t) + \mathrm{i}\beta(\boldsymbol{x}, t)], \quad \bar{\phi} = \bar{\phi}_s + \bar{\Phi}(\boldsymbol{x}, z, t), \tag{5.2}$$

where $\alpha(\boldsymbol{x}, t)$, $\beta(\boldsymbol{x}, t)$ and $\bar{\Phi}(\boldsymbol{x}, z, t)$ are real functions. By using (3.19) one can show that

$$\bar{\Phi}(\boldsymbol{x}, z = 0, t) = a^2\frac{\omega_c}{k_c\sigma}\left[\mathscr{P}(\alpha) + \mathscr{Q}_1\left(\frac{\partial\beta}{\partial x}\right) + \mathscr{Q}_2\left(\frac{\partial\beta}{\partial y}\right)\right]. \tag{5.3}$$

We now insert (5.2) and (5.3) into (4.11) and linearize with respect to α and β. Further, we assume plane wave solutions for α and β in the form

$$\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \begin{pmatrix}\tilde{\alpha}\\ \tilde{\beta}\end{pmatrix}\exp[\mathrm{i}(\boldsymbol{\chi}\cdot\boldsymbol{x} - \Omega t)] + *, \tag{5.4}$$

where $\boldsymbol{\chi} = (\lambda, \mu)$. This gives a set of two linear equations for the determination of $(\tilde{\alpha}, \tilde{\beta})$. The condition for a non-trivial solution to this system leads to an equation for the perturbation frequency Ω. In particular we find the initial growth rate of the perturbations in the form

$$\mathrm{Im}(\Omega) = \mathrm{Im}\sqrt{D_i^2 + D_i\left(-\frac{\omega_c k_c^2 s_1}{8\sigma^4} + \frac{\omega_c}{\sigma}\tilde{\mathscr{R}}^{(x)}(\boldsymbol{\chi}) + \frac{s_0 k_c}{2\sigma}\tilde{\mathscr{R}}^{(t)}(\boldsymbol{\chi})\right)a^2 + L^2a^4}, \tag{5.5}$$

where

$$D_i = \frac{-\mathrm{i}}{2}\left[\mathscr{D}(\mathrm{i}\lambda, \mathrm{i}\mu) - \mathscr{D}^*(\mathrm{i}\lambda, \mathrm{i}\mu)\right] \tag{5.6}$$

and

$$L = \frac{\omega_c}{2k_c\sigma}\left(k_c\tilde{\mathscr{S}}_1^{(x)}(\boldsymbol{\chi}) - \frac{1}{2}\tilde{\mathscr{R}}^{(x)}(\boldsymbol{\chi}) + \frac{s_0k_c^2}{2\omega_c}\tilde{\mathscr{S}}_1^{(t)}(\boldsymbol{\chi}) + \frac{k_cD_1}{8\omega_c}\tilde{\mathscr{R}}^{(t)}(\boldsymbol{\chi}) - \tilde{\mathscr{B}}^{(x)}(\boldsymbol{\chi})\right)\lambda$$
$$+ \frac{\omega_c}{2k_c\sigma}\left(k_c\tilde{\mathscr{S}}_2^{(x)}(\boldsymbol{\chi}) - \frac{1}{2}\tilde{\mathscr{R}}^{(y)}(\boldsymbol{\chi}) + \frac{s_0k_c^2}{2\omega_c}\tilde{\mathscr{S}}_2^{(t)}(\boldsymbol{\chi}) - \tilde{\mathscr{B}}^{(y)}(\boldsymbol{\chi})\right)\mu. \tag{5.7}$$

Here, the function $\mathscr{D}^*(\cdot,\cdot)$ is the complex conjugate of the function $\mathscr{D}(\cdot,\cdot)$, given in (4.1).

Now, by using the exact expressions for the functions appearing in (5.7), given in (3.21) and (4.10), one actually finds that the expressions inside the parentheses in (5.7) cancel so that $L = 0$, and hence

$$\mathrm{Im}(\Omega) = \mathrm{Im}\sqrt{D_i^2 + D_i\left(-\frac{\omega_c k_c^2 s_1}{8\sigma^4} + \frac{\omega_c}{\sigma}\tilde{\mathscr{R}}^{(x)}(\boldsymbol{\chi}) + \frac{s_0k_c}{2\sigma}\tilde{\mathscr{R}}^{(t)}(\boldsymbol{\chi})\right)a^2}. \tag{5.8}$$

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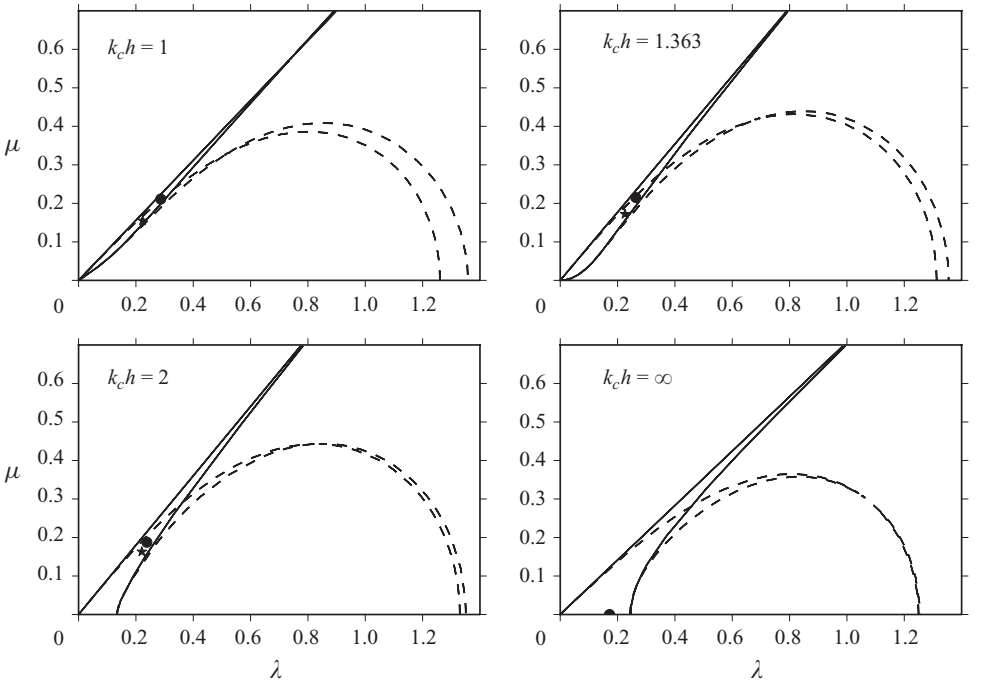


FIGURE 1. Instability regions and points of maximum growth rate for $ak_c = 0.1$ and different values for k_ch: – – – – – and ⋆, exact linear dispersion; ——— and ●, truncated linear dispersion.

Since all terms proportional to a^4 in the radicand in (5.5) disappear, we recover in principle the stability analysis obtained from the cubic NLS equation on finite depth (Benney & Roskes 1969), except for the possibility of exact linear dispersion in our formulation. However, Benney & Roskes (1969) assumed a solution to the induced mean flow problem that is only valid for relatively small k_ch. The present analysis contains no such assumption and is also valid in the limit $k_ch \to \infty$. Therefore we should expect discrepancies between Benney & Roskes (1969) and the present analysis for large k_ch. The growth rate (5.8) is identical to the one obtained by Brinch-Nielsen & Jonsson (1986) through their fourth-order equation because they chose to ignore terms proportional to a^4 in the radicand; those terms arose because of the non-Hamiltonian form of their equation. It is interesting to note that if we use the original form of the Zakharov kernel, i.e. excluding the extra terms appearing from Krasitskii's kernel, in the derivation, then $L \neq 0$ and (5.5) must be used with $\tilde{\mathscr{B}}^{(x)} = \tilde{\mathscr{B}}^{(y)} = 0$ in (5.7).

The regions of instability in the χ-plane obtained from (5.8), using both exact and truncated linear dispersion, are shown in figure 1. The locations of maximum growth rate are also indicated in the figure. We notice that when taking exact linear dispersion into account the boundaries of the instability regions resemble the shape of the exact resonance curve (Phillips 1960). This feature is also consistent with the exact results of McLean (1982). The condition for vanishing unstable growth rate along the λ-axis is given by setting the expression in parenthesis in (5.8) to zero for $\chi = 0$, which gives the well-known depth $k_ch = 1.363$.

With decreasing k_ch the part of the instability regions approaching the λ-axis at $\lambda \approx 1.3$ becomes wider. This is not consistent with McLean (1982), where this part of

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the instability regions is very narrow. Note that also in the case of truncated linear dispersion a widening of the instability region could have been seen for $|\boldsymbol{\chi}| > O(1)$ if the domain of the figures had been extended. While these 'outer' regions of instability are unphysical and outside the domain of validity of the equations, they may be a concern with regard to the numerical implementation of the equations.

6. Conclusion

We have enhanced the modified NLS equations for deep water (Dysthe 1979) and arbitrary depth (Brinch-Nielsen & Jonsson 1986) so that they conserve momentum and are Hamiltonian. The resolution in bandwidth has been improved by including exact linear dispersion. Explicit expressions have been given for the Hamiltonian, the solution for the induced flow and the third-order reconstruction of the surface elevation. We have shown that the modulational instability growth rate for a Stokes wave is well reproduced as long as the asymptotic bandwidth constraint is strictly observed. We anticipate that the new equations will facilitate analytical and numerical treatments that rely on the conservative and Hamiltonian form.

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Appendix A. Conservation laws

In the following we show that the new fourth-order NLS equation (4.11) does indeed conserve the wave action and momentum and has a Hamiltonian structure. Thus we consider the quantities N, $\boldsymbol{P} = (P_x, P_y)$ and H defined by (1.1)–(1.3).

A.1. *Wave action*

It is straightforward to show that (4.11) conserves the wave action

$$N = \int |B|^2 \,\mathrm{d}\boldsymbol{x}. \tag{A 1}$$

A.2. *Momentum*

We write the two components of the momentum as

$$P_x = \frac{1}{2}\int \left(\mathrm{i}B\frac{\partial B^*}{\partial x} + *\right)\mathrm{d}\boldsymbol{x}, \quad P_y = \frac{1}{2}\int \left(\mathrm{i}B\frac{\partial B^*}{\partial y} + *\right)\mathrm{d}\boldsymbol{x}. \tag{A 2}$$

From (4.11), one can derive the following equations for the momentum:

$$\begin{aligned}\frac{\mathrm{d}P_x}{\mathrm{d}t} = \int \frac{\partial\bar{\phi}}{\partial x}\left[k_c\frac{\partial}{\partial x}|B|^2 + \frac{1}{2}\frac{\partial}{\partial x}\left(\mathrm{i}B\frac{\partial B^*}{\partial x} + *\right) + \frac{1}{2}\frac{\partial}{\partial y}\left(\mathrm{i}B\frac{\partial B^*}{\partial y} + *\right)\right] \\ - \frac{\partial^2\bar{\phi}}{\partial x\partial t}\left[\frac{s_0k_c^2}{2\omega_c}|B|^2 - \frac{k_cD_1}{8\omega_c}\left(\mathrm{i}B\frac{\partial B^*}{\partial x} + *\right)\right] + \mathscr{B}^{(x)}(|B|^2)\frac{\partial}{\partial x}\left(\mathrm{i}B\frac{\partial B^*}{\partial x} + *\right) \\ + \mathscr{B}^{(y)}(|B|^2)\frac{\partial}{\partial x}\left(\mathrm{i}B\frac{\partial B^*}{\partial y} + *\right)\mathrm{d}\boldsymbol{x}, \quad z = 0,\end{aligned} \tag{A 3a}$$

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$$\frac{\mathrm{d}P_y}{\mathrm{d}t} = \int \frac{\partial\bar{\phi}}{\partial y}\left[k_c\frac{\partial}{\partial x}|B|^2 + \frac{1}{2}\frac{\partial}{\partial x}\left(\mathrm{i}B\frac{\partial B^*}{\partial x} + *\right) + \frac{1}{2}\frac{\partial}{\partial y}\left(\mathrm{i}B\frac{\partial B^*}{\partial y} + *\right)\right] - \frac{\partial^2\bar{\phi}}{\partial y\partial t}\left[\frac{s_0k_c^2}{2\omega_c}|B|^2 - \frac{k_cD_1}{8\omega_c}\left(\mathrm{i}B\frac{\partial B^*}{\partial x} + *\right)\right] + \mathscr{B}^{(y)}(|B|^2)\frac{\partial}{\partial y}\left(\mathrm{i}B\frac{\partial B^*}{\partial y} + *\right) + \mathscr{B}^{(x)}(|B|^2)\frac{\partial}{\partial y}\left(\mathrm{i}B\frac{\partial B^*}{\partial x} + *\right)\mathrm{d}\boldsymbol{x}, \quad z = 0. \tag{A 3b}$$

We now make use of the representations (3.20) and use the fact that the operators $\mathscr{R}^{(x,y,t)}$, $\mathscr{S}_1^{(x,y,t)}$ and $\mathscr{S}_2^{(x,y,t)}$ all are self-adjoint. One can then show that (A 3*a*) and (A 3*b*) can be written as

$$\frac{\mathrm{d}P_x}{\mathrm{d}t} = \int \left(\mathrm{i}B\frac{\partial B^*}{\partial x} + *\right)\left[k_c\mathscr{S}_1^{(x)} - \frac{1}{2}\mathscr{R}^{(x)} + \frac{s_0k_c^2}{2\omega_c}\mathscr{S}_1^{(t)} + \frac{k_cD_1}{8\omega_c}\mathscr{R}^{(t)} - \mathscr{B}^{(x)}\right]\frac{\partial}{\partial x}|B|^2 + \left(\mathrm{i}B\frac{\partial B^*}{\partial y} + *\right)\left[k_c\mathscr{S}_2^{(x)} - \frac{1}{2}\mathscr{R}^{(y)} + \frac{s_0k_c^2}{2\omega_c}\mathscr{S}_2^{(t)} - \mathscr{B}^{(y)}\right]\frac{\partial}{\partial x}|B|^2\,\mathrm{d}\boldsymbol{x}, \tag{A 4a}$$

$$\frac{\mathrm{d}P_y}{\mathrm{d}t} = \int \left(\mathrm{i}B\frac{\partial B^*}{\partial x} + *\right)\left[k_c\mathscr{S}_1^{(x)} - \frac{1}{2}\mathscr{R}^{(x)} + \frac{s_0k_c^2}{2\omega_c}\mathscr{S}_1^{(t)} + \frac{k_cD_1}{8\omega_c}\mathscr{R}^{(t)} - \mathscr{B}^{(x)}\right]\frac{\partial}{\partial y}|B|^2 + \left(\mathrm{i}B\frac{\partial B^*}{\partial y} + *\right)\left[k_c\mathscr{S}_2^{(x)} - \frac{1}{2}\mathscr{R}^{(y)} + \frac{s_0k_c^2}{2\omega_c}\mathscr{S}_2^{(t)} - \mathscr{B}^{(y)}\right]\frac{\partial}{\partial y}|B|^2\,\mathrm{d}\boldsymbol{x}. \tag{A 4b}$$

Now by using the exact expressions for the operators, (3.21) and (4.10), one actually finds that

$$k_c\mathscr{S}_1^{(x)} - \frac{1}{2}\mathscr{R}^{(x)} + \frac{s_0k_c^2}{2\omega_c}\mathscr{S}_1^{(t)} + \frac{k_cD_1}{8\omega_c}\mathscr{R}^{(t)} - \mathscr{B}^{(x)} = 0, \tag{A 5a}$$

$$k_c\mathscr{S}_2^{(x)} - \frac{1}{2}\mathscr{R}^{(y)} + \frac{s_0k_c^2}{2\omega_c}\mathscr{S}_2^{(t)} - \mathscr{B}^{(y)} = 0. \tag{A 5b}$$

Hence, we obtain $\mathrm{d}P_x/\mathrm{d}t = \mathrm{d}P_y/\mathrm{d}t = 0$, and (4.11) thus conserves the momentum. Note that the relations (A 5) have also appeared in the stability analysis in § 5 and led to the cancellation of the terms proportional to a^4.

A.3. *Hamiltonian*

We propose a Hamiltonian for (4.11) in the form

$$H = \int -B^*\mathscr{D}(\partial_x, \partial_y)B - \frac{\mathrm{i}k_c}{2}\left[\frac{\partial\bar{\phi}}{\partial x} + \frac{s_0k_c}{2\omega_c}\frac{\partial\bar{\phi}}{\partial t}\right]|B|^2 + \frac{\mathrm{i}k_c^3s_1}{16\sigma^3}|B|^4 + \frac{k_c^2D_7}{32\sigma^3}|B|^2\left(B^*\frac{\partial B}{\partial x} - B\frac{\partial B^*}{\partial x}\right) - \frac{1}{4}\nabla\bar{\phi}\cdot\left[B^*\nabla B - B\nabla B^*\right] + \frac{k_cD_1}{16\omega_c}\frac{\partial\bar{\phi}}{\partial t}\left(B^*\frac{\partial B}{\partial x} - B\frac{\partial B^*}{\partial x}\right)\mathrm{d}\boldsymbol{x}, \quad z = 0. \tag{A 6}$$

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Now, by taking the variational derivative with respect to B^* and again using the relation (A 5), we find that

$$\frac{\delta H}{\delta B^*} = -\mathscr{D}(\partial_x, \partial_y)B - \mathrm{i}k_c\left[\frac{\partial\bar{\phi}}{\partial x} + \frac{s_0 k_c}{2\omega_c}\frac{\partial\bar{\phi}}{\partial t}\right]B + \frac{\mathrm{i}k_c^3 s_1}{8\sigma^3}|B|^2 B + \frac{k_c^2 D_7}{8\sigma^3}|B|^2\frac{\partial B}{\partial x}$$
$$-\nabla\bar{\phi}\cdot\nabla B - \frac{B}{2}\nabla^2\bar{\phi} + \frac{k_c D_1}{8\omega_c}\left[\frac{\partial^2\bar{\phi}}{\partial x\partial t}B + 2\frac{\partial\bar{\phi}}{\partial t}\frac{\partial B}{\partial x}\right] - \mathscr{B}^{(x)}\left(B\frac{\partial B^*}{\partial x}\right)B$$
$$-\mathscr{B}^{(x)}(|B|^2)\frac{\partial B}{\partial x} - \mathscr{B}^{(y)}\left(B\frac{\partial B^*}{\partial y}\right)B - \mathscr{B}^{(y)}(|B|^2)\frac{\partial B}{\partial y}, \quad z = 0. \tag{A 7}$$

To obtain (A 7) we have used the representations for the induced mean flow (3.20). Note also that the operator $\mathscr{D}(\partial_x, \partial_y)$ satisfies the relation

$$\int -B^*\mathscr{D}(\partial_x, \partial_y)B\,\mathrm{d}\boldsymbol{x} = \int B\mathscr{D}^*(\partial_x, \partial_y)B^*\,\mathrm{d}\boldsymbol{x}. \tag{A 8}$$

Thus, (A 7) together with a corresponding calculation for $\delta H/\delta B$ shows that (4.11) satisfies the pair of equations

$$\frac{\delta H}{\delta B^*} = \frac{\partial B}{\partial t}, \qquad \frac{\delta H}{\delta B} = -\frac{\partial B^*}{\partial t} \tag{A 9}$$

and hence conserves the Hamiltonian (4.12).

Appendix B. Reconstruction of the surface elevation and expressions for bound waves

To find e.g. the surface elevation from knowledge of $B(\boldsymbol{x}, t)$ one needs formulae that relate B to physical variables and express bound wave contributions in terms of B. These relations can be found from the canonical transformation (2.7) in the same manner as that employed for finding the induced mean flow potential in § 3.

It is convenient to express the surface elevation in the form

$$\zeta(\boldsymbol{x}, t) = \bar{\zeta} + \tfrac{1}{2}\left[(A_\zeta + A_{\zeta 1})\mathrm{e}^{\mathrm{i}\theta} + A_{\zeta 2}\mathrm{e}^{2\mathrm{i}\theta} + A_{\zeta 3}\mathrm{e}^{3\mathrm{i}\theta} + \cdots + *\right], \tag{B 1}$$

where $\theta = \boldsymbol{k}_c\cdot\boldsymbol{x} - \omega_c t$. Here, $A_\zeta(\boldsymbol{x}, t)$ is the free-wave, first-harmonic complex amplitude of the surface elevation and depends linearly on $B(\boldsymbol{x}, t)$.

Furthermore, $A_{\zeta 1}(\boldsymbol{x}, t)$ contains the first-harmonic contributions from the canonical transformation and is a nonlinear function of $B(\boldsymbol{x}, t)$. This term was called 'quasi-linear' and discussed in some detail in a recent paper by Janssen (2009). He remarks that while the contribution from $A_{\zeta 1}(\boldsymbol{x}, t)$ has been overlooked in many previous studies, it is essential for the convergent behavior of the second-order spectrum.

Moreover, $A_{\zeta 2}(\boldsymbol{x}, t)$ and $A_{\zeta 3}(\boldsymbol{x}, t)$ represent bound waves due to quadratic and cubic nonlinear interactions and $\bar{\zeta}(\boldsymbol{x}, t)$ corresponds to the wave-induced mean surface. Under the narrow-bandwidth assumption, one can show that

$$A_\zeta = \frac{1}{2\pi}\int 2\mathscr{M}_{c+0}\hat{B}_0\exp(\mathrm{i}\boldsymbol{\chi}_0\cdot\boldsymbol{x})\,\mathrm{d}\boldsymbol{\chi}_0, \tag{B 2a}$$

$$A_{\zeta 1} = \frac{1}{(2\pi)^3}\int 8\pi^2\mathscr{M}_{c+3+2-1}\left[B^{(2)}_{c+3+2-1,c+1,c+2,c+3} + B^{(3)}_{-c+1-2-3,c+3,c+2,c+1}\right]$$
$$\times\,\hat{B}_1^*\hat{B}_2\hat{B}_3\exp(\mathrm{i}\boldsymbol{\chi}_{3+2-1}\cdot\boldsymbol{x})\,\mathrm{d}\boldsymbol{\chi}_{1,2,3}, \tag{B 2b}$$

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$$A_{\zeta 2} = \frac{1}{(2\pi)^2}\int 4\pi \mathscr{M}_{2c+1+2}\left[A^{(1)}_{2c+1+2,c+1,c+2} + A^{(3)}_{-2c-1-2,c+1,c+2}\right] \times \hat{B}_1\hat{B}_2\exp(\mathrm{i}\boldsymbol{\chi}_{1+2}\cdot\boldsymbol{x})\,\mathrm{d}\boldsymbol{\chi}_{1,2}, \quad (\mathrm{B}\,2c)$$

$$A_{\zeta 3} = \frac{1}{(2\pi)^3}\int 8\pi^2 \mathscr{M}_{3c+1+2+3}\left[B^{(1)}_{3c+1+2+3,c+1,c+2,c+3} + B^{(4)}_{-3c-1-2-3,c+1,c+2,c+3}\right] \times \hat{B}_1\hat{B}_2\hat{B}_3\exp(\mathrm{i}\boldsymbol{\chi}_{1+2+3}\cdot\boldsymbol{x})\,\mathrm{d}\boldsymbol{\chi}_{1,2,3}, \quad (\mathrm{B}\,2d)$$

$$\bar{\zeta} = \frac{1}{(2\pi)^2}\int 2\pi \mathscr{M}_{2-1}\left[A^{(2)}_{2-1,c+1,c+2} + A^{(2)}_{1-2,c+2,c+1}\right]\hat{B}_1^*\hat{B}_2\exp(\mathrm{i}\boldsymbol{\chi}_{2-1}\cdot\boldsymbol{x})\,\mathrm{d}\boldsymbol{\chi}_{1,2}. \quad (\mathrm{B}\,2e)$$

Further, by Taylor-expanding the kernels and calculating the integrals one finds

$$A_\zeta = \sqrt{\frac{2\omega_c}{g}}\left[\frac{1}{k_c\sigma}\mathscr{L}(\partial_x,\partial_y)\tanh\left[h\mathscr{L}(\partial_x,\partial_y)\right]\right]^{1/4} B = \sqrt{\frac{2\omega_c}{g}}\left[B - \frac{\mathrm{i}C_g}{2\omega_c}\frac{\partial B}{\partial x} + \cdots\right], \quad (\mathrm{B}\,3a)$$

$$A_{\zeta 1} = \sqrt{\frac{2\omega_c}{g}}\left[-\frac{k_c^3 R_1}{16\omega_c\sigma^5}|B|^2 B + \frac{k_c^2 s_0}{4\omega_c^2}\left.\frac{\partial\bar{\phi}}{\partial t}\right|_{z=0} B + \mathscr{C}(|B|^2)B\right], \quad (\mathrm{B}\,3b)$$

$$A_{\zeta 2} = -\frac{k_c^2(\sigma^2-3)}{2\omega_c\sigma^2}B^2 + \frac{\mathrm{i}k_c R_2}{4\omega_c\sigma^2}B\frac{\partial B}{\partial x}, \quad (\mathrm{B}\,3c)$$

$$A_{\zeta 3} = -\sqrt{\frac{2\omega_c}{g}}\frac{3k_c^3}{32\omega_c\sigma^5}(\sigma^2-3)(\sigma^4+3)B^3, \quad (\mathrm{B}\,3d)$$

$$\bar{\zeta} = -\frac{1}{g}\left.\frac{\partial\bar{\phi}}{\partial t}\right|_{z=0} + \frac{k_c^2 s_0}{2\omega_c}|B|^2 - \frac{k_c D_1}{8\omega_c}\left(\mathrm{i}B\frac{\partial B^*}{\partial x} + *\right), \quad (\mathrm{B}\,3e)$$

where

$$R_1 = 2\sigma^8 - 4\sigma^6 + 3\sigma^4 - 6\sigma^2 + 9, \quad (\mathrm{B}\,4a)$$

$$R_2 = s_0(\sigma^2-15)d + 3(\sigma^2-3), \quad (\mathrm{B}\,4b)$$

and the operator $\mathscr{C}$ is defined by

$$\mathscr{F}\{\mathscr{C}(f)\} = \tilde{\mathscr{C}}(\boldsymbol{k})\hat{f} = \frac{gk_c^2 D_2 k_x}{8\omega_c}\frac{k_c s_0\omega^2 - 2\omega_c C_g k_x^2}{\left(\omega^2 - C_g^2 k_x^2\right)^2}\hat{f}. \quad (\mathrm{B}\,5)$$

In the case of vanishing bandwidth $|B| = (g/2\omega_c)^{1/2}a$, where a is the same as in (5.1), the expressions (B 3*c*) and (B 3*d*) are in agreement with the Stokes wave expansion of Fenton (1985). The mean water level $\bar{\zeta}$ is ambiguous for a Stokes wave; however the narrow-band limit of (B 3*e*) agrees with Janssen & Onorato (2007) and Stiassnie & Gramstad (2009), as well as with (16.99) in Whitham (1974). The quasi-linear term (B 3*b*) agrees in the deep-water limit with Janssen (2009).

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Appendix C. Deep-water limit

If the water depth is sufficiently large, i.e. $(k_c h)^{-1} = O(\epsilon)$, the induced mean flow potential is one order of magnitude smaller. Hence, in this case the results in the preceding sections simplify greatly.

Formally, the deep-water limit is obtained by letting $k_c h \to \infty$. For the coefficients of the previous sections this means letting $\sigma = \tanh(k_c h) \to 1$ and $d = k_c h/\sigma \to \infty$. It is however clear that as $k_c h \to \infty$, σ approaches 1 much faster than d approaches ∞; actually for any $N < \infty$, $s_0 d^N = (\sigma^2 - 1)d^N \to 0$ for $k_c h \to \infty$. By using this one finds that the deep-water limit of (4.11) is

$$\frac{\partial B}{\partial t} + \mathscr{D}(\partial_x, \partial_y)B + \mathrm{i}k_c^3|B|^2 B + 3k_c^2|B|^2\frac{\partial B}{\partial x} + \mathrm{i}k_c\left.\frac{\partial \bar{\phi}}{\partial x}\right|_{z=0} B = 0, \tag{C 1}$$

where the deep-water limit of the operator $\mathscr{D}$ is

$$\mathscr{D}(\partial_x, \partial_y) = \mathrm{i}\sqrt{g}\left[\left[(k_c - \mathrm{i}\partial_x)^2 - \partial_y^2\right]^{1/4} - \sqrt{k_c}\right]. \tag{C 2}$$

Furthermore, (C 1) conserves the deep-water limit of the Hamiltonian (4.12):

$$H = \int -B^*\mathscr{D}(\partial_x, \partial_y)B - \frac{\mathrm{i}k_c^3}{2}|B|^4 - \frac{3k_c^2}{4}|B|^2\left(B\frac{\partial B^*}{\partial x} - B^*\frac{\partial B}{\partial x}\right) - \frac{\mathrm{i}k_c}{2}|B|^2\left.\frac{\partial \bar{\phi}}{\partial x}\right|_{z=0} \mathrm{d}\boldsymbol{x}. \tag{C 3}$$

The equations for the induced mean flow, namely (3.7), are now given by

$$\nabla^2\bar{\phi} + \frac{\partial^2\bar{\phi}}{\partial z^2} = 0, \quad -h \leqslant z \leqslant 0 \tag{C 4a}$$

$$\frac{\partial \bar{\phi}}{\partial z} = 0, \quad z = -h, \tag{C 4b}$$

$$\frac{\partial \bar{\phi}}{\partial z} = k_c\frac{\partial}{\partial x}|B|^2, \quad z = 0, \tag{C 4c}$$

or alternatively

$$\bar{\phi}(\boldsymbol{x}, z, t) = \frac{1}{2\pi}\int \frac{\cosh[k(z+h)]}{\sinh[kh]}\frac{\mathrm{i}k_c k_x}{k}\mathscr{F}(|B|^2)\exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x})\,\mathrm{d}\boldsymbol{k}. \tag{C 5}$$

The deep-water reconstruction formulae for the surface displacement become

$$A_\zeta = \sqrt{\frac{2\omega_c}{g}}\left[\frac{\mathscr{L}(\partial_x, \partial_y)}{k_c}\right]^{1/4} B = \sqrt{\frac{2\omega_c}{g}}\left[B - \frac{\mathrm{i}}{4k_c}\frac{\partial B}{\partial x} + \cdots\right], \tag{C 6a}$$

$$A_{\zeta 1} = -\sqrt{\frac{\omega_c}{8g}}\frac{k_c^3}{\omega_c}|B|^2 B, \tag{C 6b}$$

$$A_{\zeta 2} = \frac{k_c^2}{\omega_c}B^2 - \frac{3\mathrm{i}k_c}{2\omega_c}B\frac{\partial B}{\partial x}, \tag{C 6c}$$

$$A_{\zeta 3} = \sqrt{\frac{2\omega_c}{g}}\frac{3k_c^3}{4\omega_c}B^3, \tag{C 6d}$$

$$\bar{\zeta} = -\frac{1}{g}\left.\frac{\partial \bar{\phi}}{\partial t}\right|_{z=0}. \tag{C 6e}$$

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