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Dynamics of an Innovation Diffusion Model with Time Delay

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Abstract. A nonlinear mathematical model for innovation diffusion is proposed. The system of ordinary differential equations incorporates variable external influences (the cumulative density of marketing efforts), variable internal influences (the cumulative density of word of mouth) and a logistically growing human population (the variable potential consumers). The change in population density is due to various demographic processes such as intrinsic growth rate, emigration, death rate etc. Thus the problem involves two dynamic variables viz. a non-adopter population density and an adopter population density. The model is analysed qualitatively using the stability theory of differential equations, with the help of the corresponding characteristic equation of the system. The interior equilibrium point can be stable for all time delays to a critical value, beyond which the system becomes unstable and a Hopf bifurcation occurs at a second critical value. Employing normal form theory and a centre manifold theorem applicable to functional differential equations, we derive some explicit formulas determining the stability, the direction and other properties of the bifurcating periodic solutions. Our numerical simulations show that the system behaviour can become extremely complicated as the time delay increases, with a stable interior equilibrium point leading to a limit cycle with one local maximum and minimum per cycle (Hopf bifurcation), then limit cycles with more local maxima and minima per cycle, and finally chaotic solutions.

AMS subject classifications: 34C23, 34D20, 37L10, 49Q12, 92D25

Key words: Innovation diffusion model, evaluation period, stability analysis, Hopf bifurcation, normal form theory, centre manifold theorem.

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1. Introduction

Since the earliest work in the 1960s [1,2], there have been many investigations on the spread of new consumer durables. Various aspects in the mathematical modelling have been considered, but the Bass marketing innovation diffusion model [3] is still the foundation for new hypotheses to gain insight into the marketing of innovations. The Bass model assumes that the market is fixed, and the market penetration is described by

$$\frac{dA(t)}{dt} = p\big[m - A(t)\big] + \frac{q}{m}A(t)\big[m - A(t)\big],$$

where A(t) is the number of adopters at time t. The parameter p accounts for new adopters entering the market due to external influences (e.g. the activities of firms in the market, advertising, and attractiveness of the innovation), and the parameter q is the coefficient of imitation refering to the rate of influence of adopters on non-adopters. Refs. [4–13] illustrate the wide use of the Bass model in understanding the spread of new products, and the underlying phenomena responsible for the diffusion of new technology (market innovation) or new products in the market.

In a book, Rogers [14] summarised previous research and concluded that innovation diffusion consists of five steps viz. awareness (an individual is exposed to innovation), interest (individual seeks more information), evaluation (individual applies innovation to his or her situation), trial (individual uses innovation on a small scale), and adoption (the individual makes full use of the innovation). Further, through behavioural research he found that these five steps can be simplified into a two-step flow process i.e. media influences innovative opinion leaders to adopt a new product, who in turn influence other people to adopt the product [14–18]. In particular, Mahajan *et al.* [16] discussed earlier contributions in the management and marketing literature to cumulative understanding of innovation diffusion dynamics.

Models with time delay to exhibit product evaluation stages have been proposed by many researchers [19–22]. One mathematical model describes the dynamics for users of one product in two different patches [23]. Stability of a competitive innovation diffusion model in a market [24,25] and global stability of an innovation diffusion model for n products [26] have been discussed. An innovation diffusion model with nonlinear acceptance describes the dynamics of three competing products in a market [27], and a binomial innovation diffusion model for a variable size market with demographic processes of entrance or exit from market compartments has been considered [28]. Shukla *et al.* [29] proposed that the innovation diffusion process is affected by variable external influences such as advertisements and the change of density of the non-adopter population due to intrinsic growth rate, emigration or death rate *inter alia*. The model analysis shows that the adopter population density increases as the parameters related to the growth rate of the non-adopter population as the rate of external influences increase, and the main effect of the cumulative density of the variable external influences is to make the equilibrium level of the adopter population density reach its equilibrium at a much faster rate. Wang *et al.* [22] considered delay to describe the process of evaluation and decision-making, whereas Shukla *et al.* [29]

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described how the innovation diffusion process is affected by variable external influences as mentioned, although no aspect of the evaluation period was discussed. Wang *et al.* [22] considered the rate of growth of the non-adopter population to be proportional to the constant population, which does not seem realistic but this shortcoming can be removed by assuming a logistically growing non-adopters population that converges to some fixed carrying capacity of the system. The diffusion of new innovations has important marketing issues for companies and industries, and yet there are few studies with that in mind. Indeed, it seems that no attention has been paid so far to a logistically growing non-adopters population with an evaluation period (time delay).

Applied mathematicians, scientists and engineers in various fields have long worked on the observation and interpretation of dynamical patterns in artificial and natural systems. The innovation diffusion patterns identified by stability and Hopf bifurcation analysis are often qualitative, with Hopf bifurcation theory used to classify diffusion patterns exploited in design or forecasting. Here we extend the work of Wang *et al.* [22] on innovation diffusion by including a logistically growing non-adopter cumulative population density, together with variable external as well as internal influences on the adoption of an innovation viz. the effect of the media (electronic and print), and the number of adopters and interpersonal communication between an already adopter and a potential adopter. With a time delay, both the evaluation stage and the decision-making stage are represented in the system. In Section 2, we describe the proposed mathematical model consisting of a system of delay differential equations. Section 3 deals with some preliminaries such as positivity, the boundedness of solutions, equilibria and the characteristic equation. The asymptotic stability analysis of various equilibria is discussed in Section 4, and we analyse the delayed innovation diffusion model for the existence of Hopf bifurcation around the positive interior equilibrium in Section 5. The stability of periodic solutions for the local Hopf bifurcation is established by a technique based on normal form theory and a centre manifold theorem in Section 6. Numerical simulations to illustrate the validity of the analytical results are described in Section 7, and our concluding discussion is in Section 8.

2. Mathematical Model

The model of the innovation diffusion system is formulated by taking into consideration two populations viz. the non-adopter population with density $N(t)$ and the adopter population with density $A(t)$, where t denotes the time.. To be more realistic, we incorporate the demographic structure of a population into the Bass model, and make the following assumptions.

- The non-adopter population grows logistically, with the intrinsic growth rate r and maximum environmental carrying capacity K.
- The emigration and death rate δ of the population, the cumulative density of external influences p, and the cumulative density of internal influences q are all assumed constant.

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- Variable external (advertisements) and internal (word of mouth) influences affect each individual's decision, so the rate of change of the adopters depends on external and internal influences, their natural deaths, and the rate they otherwise discontinue using the product. We let ν be the rate of adopters returning to the non-adopter population (who may of course revert later), and let α be the rate at which adopters discontinue using the product permanently and would never use it again.
- The shifting of non-adopters to the adopter population is not instantaneous, but takes some time defined by the average evaluation period τ. Moreover, an individual who has information of the product at time $t-\tau$ may leave the evaluation class in the interval $[t-\tau, t]$ due to death or losing interest in the product.
- The survival probability of an individual through the evaluation stage is $e^{-\delta\tau}$; and if ζ is the rate individuals leave the evaluation stage on deciding not to buy the product, then $e^{-\zeta\tau}$ defines individuals still interested in the product at the end of the test period. Thus the success probability through the evaluation stage is $e^{-(\delta+\zeta)\tau}$, the probability that an individual who is aware of the product at time $t-\tau$ neither dies nor loses interest in the product at time t.
- Given the variable external and internal influences, the rate of awareness at time $t-\tau$ is $(p+qA(t-\tau))N(t-\tau)$. Thus $(p+qA(t-\tau))N(t-\tau)e^{-(\delta+\zeta)\tau}$ is the number of individuals who will eventually pass the evaluation stage $[t-\tau, t]$. It is also assumed that the individuals who pass the evaluation stage will definitely move to the adopter population class, so the rate of transfer from the potential adopter class to the adopter class at time t is $(p+qA(t-\tau))N(t-\tau)e^{-(\delta+\zeta)\tau}$ i.e. the rate of transfer of those with knowledge (information) about the innovation at time $t-\tau$ due to variable external as well as internal influences, who join the adopter class at time t.

In our model, the evaluation period is the control parameter shaping the dynamics of the non-adopter and adopter population, and all parameters are assumed to be positive constants. Under the above assumptions, our governing equations are

$$\begin{aligned}\frac{dN(t)}{dt} =& r\left(N(t)-\frac{N^2(t)}{K}\right)-\big(p+qA(t-\tau)\big)N(t-\tau)e^{-(\delta+\zeta)\tau}+(\alpha+\nu)A(t)\\ &-\delta N(t), \qquad (2.1)\\ \frac{dA(t)}{dt} =& \big(p+qA(t-\tau)\big)N(t-\tau)e^{-(\delta+\zeta)\tau}-(\delta+\alpha+\nu)A(t)\,, \qquad (2.2)\end{aligned}$$

to be solved subject to the initial conditions

$$N(\theta)=\varphi_1(\theta), \quad A(\theta)=\varphi_2(\theta), \quad \varphi_1(0)>0\,, \quad \varphi_2(0)>0\,,$$

where $\varphi_1(\theta)$, $\varphi_2(\theta)$ are continuous bounded functions in the interval $[-\tau, 0]$. More precisely, $\varphi_1(\theta)$, $\varphi_2(\theta) \in C([-\tau,0], \Re_+^2)$, the Banach space of continuous functions mapping the interval $[-\tau, 0]$ into $\Re_+^2 = \{(x_1, x_2) : x_i > 0,\ i = 1, 2\}$. From the fundamental theory of functional differential equations [30], we know that the solutions $N(t)$, $A(t)$ to this initial value problem are unique and positive on $[0, +\infty)$.

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3. Preliminaries

Lemma 3.1. *The interior equilibrium E_* of the system* (2.1)-(2.2) *corresponding to initial conditions $\varphi_1(0) > 0, \varphi_2(0) > 0$ on $[0, +\infty)$ is in the positive quadrant for all $t \geq 0$.*

Proof. Let $(N(t), A(t))$ be a solution of the system (2.1)-(2.2) subject to the prescribed initial conditions, and initially consider $t \in [0, \tau]$. From Eq. (2.2),

$$\frac{dA(t)}{dt} = \big(p + qA(t-\tau)\big)N(t-\tau)e^{-(\delta+\zeta)\tau} - (\delta+\alpha+\nu)A(t) \geq -(\delta+\alpha+\nu)A(t)\,,$$

such that $A(t) = \varphi_2(0)e^{-(\delta+\alpha+\nu)t} > 0\,,\ t \in [0, \tau]$ since $\varphi_2(0) > 0$ for $\theta \in [-\tau, 0]$ i.e. $A(t)$ is positive for $t \in [0, \tau]$.

From Eq. (2.2), for $t \in [0, \tau]$ we then also have

$$N(t) = \varphi_1(0)\exp\left[\int_0^t \left(r - \frac{rN}{K} - qA(s-\tau)e^{-(\delta+\zeta)\tau} + (\alpha+\nu)\frac{A}{N} - pe^{-(\delta+\zeta)\tau} - \delta\right)ds\right] > 0\,.$$

Similarly, we can successively consider the intervals $[\tau, 2\tau], \cdots, [n\tau, (n+1)\tau]$, $n \in N$, and conclude that the solutions of the system (2.1)-(2.2) remain positive for all $t \geq 0$. □

Lemma 3.2. *The non-negative solutions of the system* (2.1)-(2.2) *subject to the prescribed initial conditions in $\Re^2_+$ are uniformly bounded.*

Proof. Let $(N(t), A(t))$ be any solution, and define W(t) = N(t) + A(t) for all $t \geq 0$. Then we have

$$\frac{dW}{dt} + \delta W \leq \Theta$$

for any $\delta > 0$, so invoking a theorem on differential inequalities [31] we obtain

$$0 \leq W\big(N, A\big) \leq \frac{\Theta}{\delta} + \frac{W\big(N(0), A(0)\big)}{e^{\delta t}}\,,$$

where $\Theta = rK/4$. In the limit $t \to \infty$, we therefore have $0 \leq W(N, A) \leq \Theta/\delta$, so all the solutions of the system (2.1)-(2.2) that start in $\Re^2_+$ enter into the region $\Omega = \{(N(t), A(t)) : 0 \leq W(t) \leq (\Theta/\delta) + \epsilon\}$, for any $\epsilon > 0$. Thus $N(t)$, $A(t)$ ultimately have an upper bound in the region Ω for all $t > 0$. □

3.1. Equilibria

Irrespective of the parameter values the system (2.1)-(2.2) possesses three feasible equilibrium points in the closed positive quadrant of $\Re^2_+ = \{(x_1, x_2) : x_i > 0, 1 \leq i \leq 2\}$:

(i) the trivial equilibrium $E_0(0, 0)$;

(ii) the adopter free equilibrium $E_1(N_1, 0)$ where $N_1 = \frac{K(r-p-\delta)}{r}e^{-(\delta+\zeta)\tau}$, such that $N_1 > 0$ when $r > p + \delta$;

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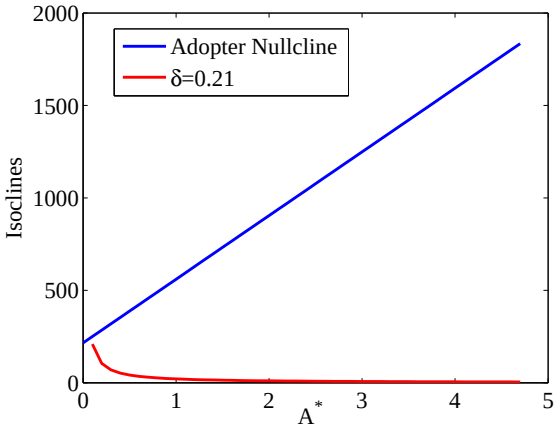


Figure 1: Non-existence of A^* when $\delta \geq 0.21$ for parametric values $K = 10000$, $r = 0.5251$, $q = 0.45$, $p = 0.32$, $\alpha = 0.001$, $\delta = 0.21$ and $\nu = 0.004$.

(iii) the interior equilibrium $E_*(N^*, A^*)$ where $N^* = \frac{(\delta+\alpha+\nu)A^*}{(p+qA^*)e^{-(\delta+\zeta)\tau}}$ and A^* are the roots of the equation

$$Kq^2\delta e^{-2(\delta+\zeta)\tau}A^{*2} - \left[Kq\kappa(r-\delta)e^{-(\delta+\zeta)\tau} - r\kappa^2 - 2Kpq\delta e^{-2(\delta+\zeta)\tau}\right]A^* - \left[Kp\kappa(r-\delta)e^{-(\delta+\zeta)\tau} - Kp^2\delta e^{-2(\delta+\zeta)\tau}\right] = 0,$$

such that $A^* = \frac{-\Phi_1 \pm \sqrt{\Phi_1^2+4\Phi_2}}{2\Phi_3}$ is positive provided $r > \delta$ and $\kappa > Max\{\frac{p\delta}{r-\delta}, Kq\}$ where

$$\Phi_1 = r\kappa^2 - Kq\kappa(r-\delta)e^{-(\delta+\zeta)\tau} + 2Kpq\delta e^{-2(\delta+\zeta)\tau},$$
$$\Phi_2 = K^2pq^2\delta\left\{r\kappa - \kappa\delta - p\delta e^{-(\delta+\zeta)\tau}\right\}e^{-3(\delta+\zeta)\tau},$$
$$\Phi_3 = Kq^2\delta e^{-2(\delta+\zeta)\tau},$$

and $\delta + \alpha + \nu = \kappa$.

It is quite difficult to find the exact unique interior equilibrium point in parametric form required for further investigation, but from the relative positions of the nullclines we can readily visualise the non-existence and existence of the positive equilibrium point (cf. Figs. 1 and 2). Here we only study the nonnegative equilibrium, so assume that $N^* > 0$, $A^* > 0$ for any time $t > 0$.

3.2. Characteristic equation

Assuming $\bar{E}(\bar{N}, \bar{A})$ denotes an arbitrary equilibrium, we linearise the system (2.1)-(2.2) about $\bar{E}$ to obtain the variational matrix of the delayed innovation diffusion model system:

$$J^* = \begin{bmatrix} (a_{10}+b_{10})e^{-\lambda\tau} & b_{01}e^{-\lambda\tau} + a_{01} \\ d_{10}e^{-\lambda\tau} & d_{01}e^{-\lambda\tau} + c_{01} \end{bmatrix},$$

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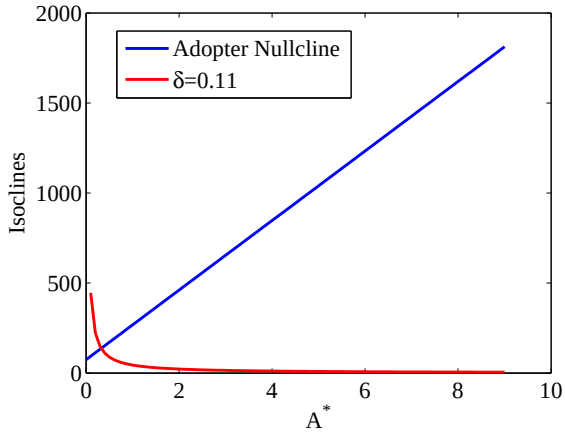


Figure 2: Existence of A^* when $\delta < 0.21$ for parametric values $K = 10000$, $r = 0.5251$, $q = 0.45$, $p = 0.32$, $\alpha = 0.001$, $\delta = 0.11$, and $\nu = 0.004$

where the coefficients of the linear terms are

$$a_{10} = r\left(1 - \frac{2\bar{N}}{K}\right) - \delta\,, \quad a_{01} = \alpha + \nu\,, \quad b_{10} = -\left(p + q\bar{A}\right)e^{-(\delta+\zeta)\tau}, \quad b_{01} = -q\bar{N}e^{-(\delta+\zeta)\tau},$$
$$c_{10} = 0\,, \quad c_{01} = -\left(\delta + \nu + \alpha\right), \quad d_{10} = \left(p + q\bar{A}\right)e^{-(\delta+\zeta)\tau}, \quad d_{01} = q\bar{N}e^{-(\delta+\zeta)\tau}.$$

On setting the determinant $|J^* - \lambda I| = 0$, we get the corresponding characteristic equation for J^* at $\bar{E}$:

$$D(\lambda, \tau) = P(\lambda, \tau) + Q(\lambda, \tau)e^{-\lambda\tau} = 0\,, \tag{3.1}$$

where

$$P(\lambda, \tau) = \lambda^2 - (a_{10} + c_{01})\lambda + a_{10}c_{01}\,,$$
$$Q(\lambda, \tau) = (a_{10}d_{01} + b_{10}c_{01} - a_{01}d_{10}) - (b_{10} + d_{01})\lambda\,.$$

4. Stability of the Equilibria

We now consider the stability of the three equilibria of the system (2.1)-(2.2) identified in the previous section.

4.1. Stability of equilibrium E_0

For the vanishing equilibrium $E_0(0, 0)$, Eq. (3.1) reduces to

$$\left(\lambda + (\delta + \nu + \alpha)\right)\left(\lambda + pe^{-(\delta+\zeta+\lambda)\tau} - r + \delta)\right) = 0\,,$$

so $-(\delta + \nu + \alpha)$ is one characteristic root and another satisfies

$$f_1(\lambda) = \lambda + pe^{-(\delta+\zeta+\lambda)\tau} - r + \delta = 0$$

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or $\lambda+\delta = r - pe^{-(\delta+\zeta+\lambda)\tau}$, such that if $Re(\lambda) \geq 0$ then

$$\delta \leq |\lambda+\delta| = \left| r - pe^{-(\delta+\zeta+\lambda)\tau} \right| \leq r\,.$$

Hence E_0 is locally asymptotically stable provided if $\delta > r$ i.e. provided the death rate of the population is more than the intrinsic growth rate of the non-adopter population, a condition that is to be expected.

4.2. Stability of equilibrium E_1

For the adopter free equilibrium $E_1(N_1, 0)$, Eq. (3.1) reduces to

$$\left(\lambda + (r-2p-\delta)e^{-(\delta+\zeta)\tau} + pe^{-(\delta+\zeta+\lambda)\tau}\right)\left(\lambda + (\delta+\nu+\alpha) - \frac{qK}{r}(r-p-\delta)e^{-(2\delta+2\zeta+\lambda)\tau}\right) = 0\,.$$

From the second factor

$$f_2(\lambda) = \lambda + (\delta+\nu+\alpha) - \frac{qK}{r}(r-p-\delta)e^{-(2\delta+\zeta+\lambda)\tau} = 0\,,$$

we have

$$\lambda + (\delta+\nu+\alpha) = \frac{qK}{r}(r-p-\delta)e^{-(2\delta+2\zeta+\lambda)\tau}$$

such that if $Re(\lambda) \geq 0$ then $\delta+\nu+\alpha \geq qKe^{-2(\delta+\zeta)\tau}$. Similarly, from the first factor we have $p+\delta \leq r$, and we obtain the following theorem.

Theorem 4.1. *If $\delta+\nu+\alpha > qKe^{-2(\delta+\zeta)\tau}$ and $p+\delta < r$ are satisfied, the equilibrium $E_1(N_1, 0)$ is globally asymptotically stable.*

Proof. From Eq. (2.1) we have

$$lim_{t\to\infty} sup N(t) \leq K\,,$$

such that for sufficiently large $t \geq t_1$

$$N(t) \leq K + \epsilon\,,$$

where $\epsilon > 0$ is sufficiently small. From Eq. (2.2), we have $t \geq t_2 + \tau$ and

$$\frac{dA(t)}{dt} \leq \left[q(K+\epsilon) - (\delta+\alpha+\nu)\right]A,$$

where $\epsilon > 0$ can be chosen small enough that $q(K+\epsilon) - (\delta+\alpha+\nu) < 0$. It follows that

$$lim_{t\to\infty} A(t) = 0\,,$$

so for sufficiently large $t \geq t_2$, $A(t) \leq \varrho$ where $\varrho > 0$ is sufficiently small. Also from Eq. (2.1)-(2.2) we also have

$$lim_{t\to\infty} N(t) = K,$$

so E_1 is a global attractor. Given the local stability, E_1 is globally asymptotically stable. □

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4.3. Stability of equilibrium E_*

We now discuss the stability analysis of equilibrium point E_* of the system (2.1)-(2.2). Let us first examine the model without any evaluation period. In the absence of delay (i.e. for $\tau = 0$), Eq. (3.1) becomes

$$\lambda^2 - tr(J^*)\lambda + det(J^*) = 0 \tag{4.1}$$

where $tr(J^*) = a_{10} + c_{01} + b_{10} + d_{01}$ and $det(J^*) = a_{10}c_{01} + a_{10}d_{01} + b_{10}c_{01} - a_{01}d_{10}$. From the Routh-Hurwitz criterion, all roots of Eq. (4.1) have negative real parts i.e. the positive equilibrium E_* is locally asymptotically stable (LAS) provided both $tr(J^*) < 0$ and $det(J^*) > 0$ hold, our assumption henceforth referred to as (H_1). We proceed to discuss the stability of the system (2.1)-(2.2) for some positive τ, and the Hopf bifurcation analysis of the model about the positive equilibrium point $E_*(N^*, A^*)$ in the next section.

5. Analysis of the Delayed Model

The stability of the system (2.1)-(2.2) with a time delay τ around E_* is first discussed, providing some analytical results seen later in the numerical simulation.

Lemma 5.1 (cf. Refs. [32, 33])**.**

(i) The positive equilibrium E_* of the system (2.1)-(2.2) *is absolutely stable if and only if the equilibrium $E_*(N^*, A^*)$ of the corresponding ODE system is asymptotically stable and the characteristic equation* (3.1) *has no purely imaginary roots for any $\tau > 0$.*

(ii) The positive equilibrium $E_*(N^*, A^*)$, of the system (2.1)-(2.2) *is conditionally stable if and only if all the roots of the characteristic equation* (3.1) *have negative real parts at $\tau = 0$ and there exists some positive value of τ such that the characteristic equation (3.1) has a pair of purely imaginary roots $\pm i\omega$.*

The positive equilibrium point is asymptotically stable if all solutions of $D(\lambda, \tau)$ have negative real parts. As stated above, all solutions of $D(\lambda, \tau)$ have negative real parts at $\tau = 0$ if (H_1) holds. From the continuity of the solutions $\lambda(\tau)$ of $D(\lambda, \tau)$, Eq. (3.1) can have solutions with positive real parts if and only if $D(\lambda, \tau)$ has solutions with zero real parts for some $\tau > 0$ i.e. there exist purely imaginary solutions of Eq. (3.1).

For the proof of second part, we state the following theorem:

Theorem 5.1 (cf. Ref. [33])**.** *The interior equilibrium point E_* is conditionally stable if (H_1) holds for the system* (2.1)-(2.2).

In order to determine how time delay affects the stability properties of interior equilibrium point, we consider τ as the bifurcation parameter. To consider instability caused by a time delay τ, let $\lambda = i\omega$ ($\omega > 0$, $i = \sqrt{-1}$) denote a purely imaginary root of the characteristic equation (3.1) for some $\tau > 0$. Then setting $\lambda = i\omega$ into Eq. (4.1) and using the process

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followed in Ref. [34,35] to separate real and imaginary parts of the characteristic equation (3.1), we have

$$(a_{10}d_{01}+b_{10}c_{01}-a_{01}d_{10})\cos\omega\tau-(b_{10}+d_{01})\,\omega\sin\omega\tau=\omega^2-a_{10}c_{01}\,,$$
$$-(b_{10}+d_{01})\,\omega\cos\omega\tau-(a_{10}d_{01}+b_{10}c_{01}-a_{01}d_{10})\sin\omega\tau=(a_{10}+c_{01})\omega\,.$$

On eliminating the trigonometric functions from these two equations, we obtain a fourth degree equation in ω viz.

$$\omega^4+\left[(a_{10}+c_{01})^2-2a_{10}c_{01}-(b_{10}+d_{01})^2\right]\omega^2+\left[a_{10}^2c_{01}^2-(a_{10}d_{01}+b_{10}c_{01}-a_{01}d_{10})^2\right]=0\,, \tag{5.1}$$

which on writing $y=\omega^2$ becomes

$$G(y)=y^2+G_1y+G_2=0\,, \tag{5.2}$$

where

$$G_1=(a_{10}+c_{01})^2-2a_{10}c_{01}-(b_{10}+d_{01})^2\,,\qquad G_2=a_{10}^2c_{01}^2-(a_{10}d_{01}+b_{10}c_{01}-a_{01}d_{10})^2\,.$$

The asymptotic stability conditions follow according to the Routh-Hurwitz criterion for a characteristic polynomial of a two-dimensional system [36]. Thus all the roots of Eq. (5.2) are negative if $G_1>0$ and $G_2>0$ (i.e. no positive root exists), so the characteristic equation (3.1) does not have purely imaginary roots. Since (H_1) ensures that all roots of Eq. (4.1) have negative real parts, by Rouche's Theorem it follows that all roots of (5.1) have negative real parts too. For any time delay $\tau>0$, the interior equilibrium E_* is thus locally asymptotically stable as stated in the following theorem.

Theorem 5.2. *Assume that $G_1>0$ and $G_2>0$, then the interior equilibrium point $E_*=(N^*,A^*)$ is locally asymptotically stable for any time delay $\tau>0$.*

5.1. Estimation of the length of delay to preserve stability

Similarly to Ref. [37], we can apply the Nyquist criterion to get estimates on the length of delay that preserves stability for our system (2.1)-(2.2).

Lemma 5.2 (Nyquist Criterion)**.** *If L is the arc length of a curve encircling the right half-plane, the curve $\bar{P}_L(L)$ will encircle the origin a number of times equal to the difference between the number of poles and the number of zeroes of $\bar{P}_L(L)$ in the right half-plane.*

We consider the space of all real valued continuous functions defined on $[-\tau,\infty)$ satisfying the initial conditions on $[-\tau,0]$. Leting $U(t)=(N(t)-\widetilde{N^*},A(t)-\widetilde{A^*})$ where $U(t)=(u_1(t),u_2(t))$, for the system (2.1)-(2.2) we obtain the linearisation about the equilibrium point $E_*(\widetilde{N^*},\widetilde{A^*})$ viz.

$$\frac{du_1}{dt}=\Big(r-\frac{2r\widetilde{N^*}}{K}-\delta\Big)u_1-\Big(p+q\widetilde{A^*}\Big)e^{-(\delta+\zeta)\tau}u_{1\tau}+(\alpha+\nu)u_2-q\widetilde{A^*}e^{-(\delta+\zeta)\tau}u_{2\tau}\,,$$
$$\frac{du_2}{dt}=\Big(p+q\widetilde{A^*}\Big)e^{-(\delta+\zeta)\tau}u_{1\tau}-(\delta+\alpha+\nu)u_2+q\widetilde{A^*}e^{-(\delta+\zeta)\tau}u_{2\tau}\,;$$

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and then taking the Laplace transform

$$sL[u_1]-u_1(0)=\Big(r-\frac{2r\widetilde{N^*}}{K}-\delta\Big)L[u_1]-\big(p+q\widetilde{A^*}\big)e^{-(\delta+\zeta)\tau}L[u_{1\tau}] + (\alpha+\nu)L[u_2]-q\widetilde{N^*}e^{-(\delta+\zeta)\tau}L[u_{2\tau}]\,, \tag{5.3}$$

$$sL[u_2]-u_2(0)=\big(p+q\widetilde{A^*}\big)e^{-(\delta+\zeta)\tau}L[u_{1\tau}] - (\delta+\alpha+\nu)L[u_2]+q\widetilde{N^*}e^{-(\delta+\zeta)\tau}L[u_{2\tau}]\,, \tag{5.4}$$

where in Eq. (5.3) we have

$$L[u_{1\tau}]=\int_0^\infty e^{-st}u_1(t-\tau)\,dt=\int_0^\tau e^{-st}u_1(t-\tau)\,dt+\int_\tau^\infty e^{-st}u_1(t-\tau)\,dt\,,$$

or on setting $t=t_1+\tau$

$$L[u_{1\tau}]=\int_{-\tau}^0 e^{-s(t_1+\tau)}u_1(t_1)\,dt_1+\int_0^\infty e^{-s(t_1+\tau)}u_1(t_1)\,dt_1=e^{-s\tau}R_1+e^{-s\tau}L[u_1],$$

where

$$R_1=\int_{-\tau}^0 e^{-st}u_1(t)\,dt\,.$$

Similarly, in Eq. (5.4) we have

$$L[u_{2\tau}]=\int_{-\tau}^0 e^{-s(t_1+\tau)}u_2(t_1)dt_1+\int_0^\infty e^{-s(t_1+\tau)}u_2(t_1)\,dt_1=e^{-s\tau}R_2+e^{-s\tau}L[u_2],$$

where

$$R_2=\int_{-\tau}^0 e^{-st}u_2(t)\,dt\,.$$

Thus the system (5.3)-(5.4) can be rewritten

$$(A_1-sI)\begin{pmatrix}L[u_1]\\L[u_2]\end{pmatrix}=A_2\,,$$

where

$$A_1=\begin{pmatrix}r-\frac{2r\widetilde{N^*}}{K}-\delta-\big(p+q\widetilde{A^*}\big)e^{-(\delta+\zeta)\tau}e^{-s\tau} & \alpha+\nu-q\widetilde{N^*}e^{-(\delta+\zeta)\tau}e^{-s\tau}\\ \big(p+q\widetilde{A^*}\big)e^{-(\delta+\zeta)\tau}e^{-s\tau} & q\widetilde{N^*}e^{-(\delta+\zeta)\tau}e^{-s\tau}-(\delta+\alpha+\nu)\end{pmatrix},$$

$$A_2=\begin{pmatrix}u_1(0)-(R_1+R_2)e^{-s\tau}\\ u_2(0)-(R_1+R_2)e^{-s\tau}\end{pmatrix}.$$

It is notable that if $L[u_1]$ has poles with positive real parts, then the inverse Laplace transform of $L[u_1]$ has terms which exponentially increase with time. Thus if $E_*(\widetilde{N^*},\widetilde{A^*})$ is locally

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asymptotically stable, then it is necessary and sufficient that all poles of $L[u_1]$ have negative real parts. Let

$$F(s) = s^2 + p_1 s + p_0 + (q_1 s + q_0)^{-s\tau} = 0\,,$$

where $p_1 = -(a_{10} + c_{01})$, $p_0 = a_{10}c_{01}$, $q_1 = -(b_{10} + d_{01})$, and $q_0 = a_{10}d_{01} + b_{10}c_{01} - a_{01}d_{10}$. Similarly to Ref. [38] once again, from the Nyquist criterion we find that the condition for the local asymptotic stability of $E_*(\widetilde{N^*}, \widetilde{A^*})$ is $Re\ F(iu_0) = 0$ and $Im\ F(iu_0) > 0$ i.e.

$$-u_0^2 + p_0 + q_0\cos(u_0\tau) + q_1 u_0 \sin(u_0\tau) = 0\,, \tag{5.5}$$

$$p_1 u_0 + q_1 u_0 \cos(u_0\tau) - q_0 \sin(u_0\tau) > 0\,, \tag{5.6}$$

where u_0 is the smallest positive root for which $Re[F(iu_0)] = 0$ and $Im[F(iu_0)] > 0$. To estimate the length of delay, we need the conditions (5.5) and (5.6) that are sufficient to guarantee stability i.e. to estimate τ, we seek an upper bound u^+ of u_0, independent of τ, such that (5.6) holds for all values of $u \in [0, u^+]$, and in particular at $u = u_0$. From Eq. (5.5) we have

$$u_0^2 = p_0 + q_0\cos(u_0\tau) + q_1 u_0 \sin(u_0\tau) \tag{5.7}$$

such that

$$u_0^2 - |q_1|\,u_0 - |p_0| + |q_0| \le 0\,, \tag{5.8}$$

so if

$$u^+ = \frac{|q_1| + \sqrt{|q_1|^2 + 4(|p_0| + |q_0|)}}{2} \tag{5.9}$$

then $u_0 \le u^+$. From the inequality (6.3) we have

$$u_0^2 > \frac{q_1 u_0^2}{p_1}\cos(u_0\tau) - \frac{q_0 u_0}{p_1}\sin(u_0\tau)\,, \tag{5.10}$$

and from Eq. (5.7) and inequality (5.10) we obtain

$$-\left(q_1 u_0 + \frac{q_0 u_0}{p_1}\right)\sin(u_0\tau) + \left(\frac{q_1 u_0^2 - q_0}{p_1}\right)\Big[\cos(u_0\tau) - 1\Big] < p_0 - \frac{q_1 u_0^2}{p_1} + q_0\,.$$

Using the inequalities $\sin(u_\tau) \le u_\tau$ and $1 - \cos(u_\tau) \le u^2\tau^2/2$, after simplification we get

$$\Lambda_1\tau^2 + \Lambda_2\tau < \Lambda_3$$

where

$$\Lambda_1 = \frac{\left|q_1 u_0^2 - q_0\right|}{2\,|p_1|}(u^+)^2, \qquad \Lambda_2 = \frac{|p_1 q_1 + q_0|}{|p_1|}u^+, \qquad \Lambda_3 = p_0 - \frac{q_1 u_0^2}{p_1} + q_0\,,$$

when (5.5) and (5.6) can be satisfied. The unique positive root is given by

$$\tau_+ = \frac{1}{2\Lambda_1}\left(-\Lambda_2 + \sqrt{\Lambda_1^2 + 4\Lambda_1\Lambda_3}\right), \tag{5.11}$$

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so for $0 \leq \tau \leq \tau_+$ the Nyquist criterion holds and τ_+ is the estimate for the length of delay for which stability is preserved, with τ_+ dependent on the parameters of the system. In summary, we have obtained the following theorem:

Theorem 5.3. *If there exists a time delay τ_+ given by Eq.* (5.11)*, then for any τ such that $0 \leq \tau \leq \tau_+$, the equilibrium $E_*(N^*, A^*)$ is locally asymptotically stable.*

5.2. Hopf bifurcation analysis

We now consider criteria for instability, and in particular criteria for Hopf bifurcation to occur using the time delay τ as the bifurcation parameter. Suppose that Eq. (5.2) has two positive roots (say y_1, y_2), so that Eq. (5.1) has two positive roots $\omega_k = \sqrt{y_k}$, $k = 1, 2$. Substituting the ω_k into Eq. (3.1), we solve for the critical value of τ at which E_* loses stability viz.

$$\tau_k = \frac{1}{\omega_0} \arcsin\left[\frac{\{a_{10}(a_{01}d_{10} - a_{10}d_{01}) + c_{01}(a_{01}d_{10} - b_{10}c_{01}) - \omega_0^2(b_{10} + d_{01})\}\omega_0}{(a_{10}d_{01} + b_{10}c_{01} - a_{01}d_{10})^2 + (b_{10} + d_{01})^2\omega_0^2}\right] + \frac{2k\pi}{\omega_0} \tag{5.12}$$

where $k = 0, 1, 2, 3, \cdots$, so at $\tau = \tau_k$ the characteristic equation (3.1) has a pair of purely imaginary roots $\pm i\omega_0$. Further, if the condition (H_1) holds then all roots of Eq. (3.1) have negative real parts when $\tau = 0$, hence the positive equilibrium E_* of system (2.1)-(2.2) is conditionally stable.

Now in order to check the transversality condition for the Hopf bifurcation to occur, we must have $d\xi/d\tau \neq 0$ at $\xi = 0$. By putting $\lambda(\tau) = \xi(\tau) + i\omega(\tau)$ into the characteristic equation (3.1) and taking derivative with respect to τ, on separating real and imaginary parts and setting $\xi = 0$, $\omega = \omega_0$ and $\tau = \tau_0$ we obtain

$$\frac{d\xi}{d\tau} = \frac{GE - HF}{E^2 + F^2} \neq 0\,,$$

where

$$\begin{aligned}
E =& -(a_{10} + c_{01}) - (a_{10}d_{01} + b_{10}c_{01} - a_{01}d_{10})\tau_0 cos(\omega_0\tau_0) \\
&+ (b_{10} + d_{01})\omega_0\tau_0 sin(\omega_0\tau_0) - (b_{10} + d_{01})cos(\omega_0\tau_0)\,, \\
F =& -2\omega_0 - (a_{10}d_{01} + b_{10}c_{01} - a_{01}d_{10})\tau_0 sin(\omega_0\tau_0) \\
&- (b_{10} + d_{01})sin(\omega_0\tau_0) - (b_{10} + d_{01})\omega_0\tau_0 cos(\omega_0\tau_0)\,, \\
G =& (a_{10}d_{01} + b_{10}c_{01} - a_{01}d_{10})\omega_0 sin(\omega_0\tau_0) + (b_{10} + d_{01})\omega_0^2 cos(\omega_0\tau_0)\,, \\
H =& -(b_{10} + d_{01})\omega_0^2 sin(\omega_0\tau_0) + (a_{10}d_{01} + b_{10}c_{01} - a_{01}d_{10})\omega_0 cos(\omega_0\tau_0)\,.
\end{aligned}$$

A delay induced small amplitude periodic solution bifurcates from a positive equilibrium point when the roots of the characteristic equation (3.1) cross the imaginary axis as τ varies from less than τ_k to greater than τ_k. The transversality condition therefore holds and the

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conditions for Hopf bifurcation [39] are then satisfied at $\tau = \tau_0$, the least positive value of τ_k given by Eq. (5.12). The stability of the zero solution of system (2.1)-(2.2) depends on the locations of the roots of the characteristic equation (3.1). When all of these roots fall in the left half of the complex plane the trivial solution is stable, and otherwise it is unstable. In order to investigate the distribution of roots of the transcendental equation (3.1), the following lemma is helpful.

Lemma 5.3 (cf. Ref. [40]). *For the transcendental equation*

$$\begin{aligned} P\left(\lambda, e^{-\lambda\tau_1}, \cdots, e^{-\lambda\tau_m}\right) =& \lambda^n + p_1^{(0)}\lambda^{n-1} + \cdots + p_{n-1}^{(0)}\lambda + p_n^{(0)} \\ &+ \left[p_1^{(1)}\lambda^{n-1} + \cdots + p_{n-1}^{(1)}\lambda + p_n^{(1)}\right]e^{-\lambda\tau_1} + \cdots \\ &+ \left[p_1^{(m)}\lambda^{n-1} + \cdots + p_{n-1}^{(m)}\lambda + p_n^{(m)}\right]e^{-\lambda\tau_m} = 0\,, \end{aligned}$$

as $(\tau_1, \tau_2, \tau_3, \cdots, \tau_m)$ *vary the sum of orders of the zeroes of*

$$P\left(\lambda, e^{-\lambda\tau_1}, \cdots, e^{-\lambda\tau_m}\right)$$

in the open right half-plane can change, and one zero appears only on or crosses the imaginary axis.

From this lemma and the above analysis of the Hopf bifurcation, we obtain the following theorem:

Theorem 5.4 (cf. Ref. [41]). *Suppose that* E_* *exists and* (H_1) *holds for the system* (2.1)-(2.2). *Then in the presence of an evaluation period* τ*:*

(i) if $\tau \in [0, \tau_k)$, *the positive equilibrium* E_* *is locally asymptotically stable;*
(ii) if $\tau > \tau_k$, *the positive equilibrium* E_* *is unstable; and*
(iii) at $\tau = \tau_k$, *an Hopf bifurcation occurs around* E_* *i.e. as* τ *increases through* τ_k , E_* *bifurcates into small periodic solutions where* τ_k *is given by*

$$\tau_k = \frac{1}{\omega_0}\arcsin\left[\frac{\left\{a_{10}(a_{01}d_{10} - a_{10}d_{01}) + c_{01}(a_{01}d_{10} - b_{10}c_{01}) - \omega_0^2(b_{10} + d_{01})\right\}\omega_0}{(a_{10}d_{01} + b_{10}c_{01} - a_{01}d_{10})^2 + (b_{10} + d_{01})^2\omega_0^2}\right].$$

6. Stability of the Periodic Solutions

According to the analysis in previous section, we obtain conditions that guarantee the system (2.1)-(2.2) undergoes Hopf bifurcation at the interior equilibrium point E_* for $\tau = \tau_0$, and $\omega = \pm i\omega_0$ are the corresponding purely imaginary roots of the characteristic equation at E_*. In this section, we consider the direction of the Hopf bifurcation and the stability of the bifurcating periodic solutions of the system (2.1)-(2.2) from the positive equilibrium E_* at the critical value τ_k $(k = 0, 1, 2, 3, \cdots)$, by applying normal form theory and the centre manifold theorem cf. Hassard *et al.* [39] and Liu *et al.* [42]. We use the method of computing normal forms for Functional Differential Equations introduced

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by Faria *et al.* [43], to compute the reduced system on the centre manifold for the pair of conjugate purely imaginary solutions of the characteristic equation (3.1).

We treat $\mu=\tau-\tau_0$ as the new bifurcation parameter, so $\mu=0$ is the Hopf bifurcation value. Let $U(t)=(N(t)-\widetilde{N^*},A(t)-\widetilde{A^*})^T$, $\tau=\tau_k+\mu$, $\overline{U_i}(\tau t)$ and $U_t(\theta)=U(t+\theta)$ where $\theta\in[-1,0]$. On dropping the bar and widetilde superscripts for simplicity, the parametric form of the system (2.1)-(2.2) is then equivalent to the following Functional Differential Equation (FDE) in $C=C([-\tau,0],\Re^2_+)$:

$$\dot{U}(t)=L_\mu(U_t)+f(\mu,U_t) \tag{6.1}$$

where $U(t)=(u_1(t),u_2(t))^T\in\Re^2_+$, $L_\mu:C\to\Re^2_+$ and $f:\Re\times C\to\Re$, in which we define for $\phi=(\phi_1,\phi_2)^T\in C([-1,0],\Re^2_+)$

$$\begin{aligned} L_\mu(\phi)&=(\tau_k+\mu)\Big[F_1\phi(0)+F_2\phi(-\tau)\Big],\\ f(\mu,\phi)&=(\tau_k+\mu)\begin{pmatrix}f_1\\ f_2\end{pmatrix}, \end{aligned} \tag{6.2}$$

where

$$F_1=\begin{pmatrix}a_{10} & a_{01}\\ c_{10} & c_{01}\end{pmatrix},\qquad F_2=\begin{pmatrix}b_{10} & b_{01}\\ d_{10} & d_{01}\end{pmatrix},$$

and the coefficients of the linear terms are given by

$$a_{10}=r\left(1-\frac{2N^*}{K}\right)-\delta,\quad a_{01}=\alpha+\nu,\quad b_{10}=-\big(p+qA^*\big)e^{-(\delta+\zeta)\tau},\quad b_{01}=-qN^*e^{-(\delta+\zeta)\tau},$$
$$c_{10}=0,\quad c_{01}=-\big(\delta+\nu+\alpha\big),\quad d_{10}=\big(p+qA^*\big)e^{-(\delta+\zeta)\tau},\quad d_{01}=qN^*e^{-(\delta+\zeta)\tau}.$$

We note that the coefficients of the second order terms are $a_{20}=-r/K$, $a_{11}=-qe^{-(\delta+\zeta)\tau}$, $b_{20}=0$ and $b_{11}=qe^{-(\delta+\zeta)\tau}$, and that

$$f_1=a_{20}\phi_1^2(0)+a_{11}\phi_1(-1)\phi_2(-1),\qquad f_2=b_{11}\phi_1(-1)\phi_2(-1).$$

From the Riesz representation theorem, there exists a 2×2 matrix function $\eta(\mu,\theta)$ with bounded variation for $-1\le\theta\le0$ such that

$$L_\mu(\phi)=\int_{-1}^{0}d\eta(\mu,\delta)\phi(\theta)\ \text{ for }\ \phi\in C\big([-1,0],\Re^2_+\big),$$

where we can choose

$$\eta(\theta,\mu)=F_1\delta(\theta)-F_2\delta(\theta+1)$$

with δ defined by

$$\delta(\theta)=\begin{cases}0, & \theta\neq0,\\ 1, & \theta=0.\end{cases}$$

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For $\phi \in C([-1,0],\Re_+^2)$, we define

$$A(\mu)\phi(\theta) = \begin{cases} d\phi(\theta)/d\theta\,, & \theta \in [-1,0)\,, \\ \int_{-1}^{0} d\eta(s,\mu)\,\phi(s)\,, & \theta = 0\,, \end{cases}$$

and

$$R(\mu)\phi(\theta) = \begin{cases} 0\,, & \theta \in [-1,0)\,, \\ f(\mu,\phi)\,, & \theta = 0\,, \end{cases}$$

so Eq. (6.1) is equivalent to

$$\dot{U}(t) = A(\mu)U_t(\theta) + R(\mu)U_t(\theta)\,. \tag{6.3}$$

For $\psi \in C^1([0,1],(\Re_+^2)^*)$, we also define

$$A^*(0)\psi(s) = \begin{cases} -d\psi(s)/ds\,, & \text{if } s \in (0,1]\,, \\ \int_{-1}^{0} d\eta(t,0)\,\psi(-t)\,, & \text{if } s = 0\,. \end{cases}$$

We shall simply write A for $A(0)$, A^* for $A^*(0)$, $\eta(\theta)$ for $\eta(\theta,0)$, and R for $R(0)$.

Let us now define an inner bilinear product

$$\langle \psi(s), \phi(\theta)\rangle = \bar{\psi}(0)\phi(0) - \int_{-1}^{0}\int_{\xi=0}^{\theta} \bar{\psi}(\xi-\theta)\,d\eta(\theta)\,\phi(\xi)\,d\xi\,, \tag{6.4}$$

where ϕ is the complex eigenfunction corresponding to the characteristic root $i\omega\theta$ of the operator A, ψ is the complex eigenfunction corresponding to the characteristic root $-i\omega s$ of the operator A^*, and $\bar{\psi}$ denotes the complex conjugate of ψ. We define $h(\theta)$ and $h^*(s)$ to be the eigenvectors of the respective adjoint operators $A(0)$ and A^* corresponding to the eigenvalue $i\omega_0\tau_k$ and $-i\omega_0\tau_k$ i.e. $Ah(\theta) = i\omega_0\tau_k h(\theta)$ and $A^*h^*(s) = -i\omega_0\tau_k h^*(\theta)$, where we choose $h(\theta) = (1,h_2)^T e^{i\omega_0\tau_k\theta}$ for $\theta \in [-1,0)$ and $h^*(s) = D(1,h_2^*)e^{i\omega_0\tau_k s}$ for $s \in (0,1]$.

We have $[(F_1+F_2e^{-i\omega_0\tau_k})-i\omega I]h(0) = 0$ from the definition of $A(0)$, $L_\mu(\phi)$ and $\eta(\theta,\mu)$, such that

$$h_2 = \frac{-d_{10}e^{-i\omega_0\tau_k}}{c_{01}+d_{01}e^{-i\omega_0\tau_k}-i\omega_0}\,;$$

and from the definition of A^* we obtain

$$A^*(\mu)\psi(s) = \int_{-1}^{0} d\eta(t,0)\psi(-t) = F_1^T\psi(0) + F_2^T\psi(-1)\,,$$

such that

$$h_2^* = -\frac{a_{10}+b_{10}e^{i\omega_0\tau_k}+i\omega_0}{d_{10}e^{i\omega_0\tau_k}}\,.$$

Using Eq. (6.4) we get

$$\bar{D} = \frac{1}{1+h_2\bar{h_2^*}+[(b_{10}+h_2b_{01})+\bar{h_2^*}(d_{10}+h_2d_{01})]\tau_k e^{-i\omega_0\tau_k}}\,,$$

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such that $\langle h^*(s), h(\theta)\rangle = 1$ and $\langle h^*(s), \bar{h}(\theta)\rangle = 0$. We now compute the coordinates to describe the centre manifold C_0 at $\mu = 0$.

Let U_t be the solution of Eq. (6.3) when $\mu = 0$. Define $z(t) = < h^*, U_t >$, and

$$W(t,\theta) = U_t(\theta) - 2Re\{z(t)h(\theta)\}. \tag{6.5}$$

On the centre manifold C_0, we have

$$W(t,\theta) = W\big(z(t), \bar{z}(t), \theta\big) = W_{20}(\theta)\frac{z^2}{2} + W_{11}(\theta)z\bar{z} + W_{02}(\theta)\frac{\bar{z}^2}{2} + \cdots \tag{6.6}$$

where z and $\bar{z}$ are local coordinates for C_0 in the h and h^* directions, respectively. Here we consider only real solutions as W is real if U_t is real. For the solution $U_t \in C_0$ of Eq. (6.3), since $\mu = 0$ we have

$$\dot{z}(t) = i\omega_0\tau_k z + \bar{h}^*(0) f\big(0, W(z,\bar{z},0) + 2Re(zh(\theta))\big) = i\omega_0\tau_k z + \bar{h}^*(0) f_0(z,\bar{z}),$$

or $\dot{z}(t) = i\omega_0\tau_k z + g(z,\bar{z})$ where

$$g(z,\bar{z}) = \bar{h}^*(0) f_0(z,\bar{z}) = g_{20}(\theta)\frac{z^2}{2} + g_{11}(\theta)z\bar{z} + g_{02}(\theta)\frac{\bar{z}^2}{2} + g_{21}(\theta)\frac{z^2\bar{z}}{2} + \dots; \tag{6.7}$$

and from Eqs. (6.5) and (6.6),

$$U_t(\theta) = W_{20}(\theta)\frac{z^2}{2} + W_{11}(\theta)z\bar{z} + W_{02}(\theta)\frac{\bar{z}^2}{2} + (1, h_2)^T e^{i\omega_0\tau_k\theta} z + (1, \bar{h}_2)^T e^{-i\omega_0\tau_k\theta}\bar{z}\cdots.$$

Now from Eqs. (6.2) and (6.7), it follows that

$$\begin{aligned} g(z,\bar{z}) = \tau_k \overline{D}\Big[&\big(\triangle_{11} + \triangle_{21}\bar{h}_2^*\big)z^2 + \big(\triangle_{12} + \triangle_{22}\bar{h}_2^*\big)z\bar{z} + \big(\triangle_{13} + \triangle_{23}\bar{h}_2^*\big)\bar{z}^2 \\ &+ \big(\triangle_{14} + \triangle_{24}\bar{h}_2^*\big)\frac{z^2}{2}\bar{z} + \cdots\Big], \end{aligned} \tag{6.8}$$

where

$$\begin{aligned} \triangle_{11} &= a_{20} + a_{11}\alpha e^{-2i\omega_0\tau_k}, \\ \triangle_{12} &= 2a_{20} + a_{11}(h_2 + \bar{h}_2), \\ \triangle_{13} &= a_{20} + a_{11}\bar{h}_2 e^{2i\omega_0\tau_k}, \\ \triangle_{14} &= a_{20}\Big(2W_{20}^1(0) + 4W_{11}^1(0)\Big) \\ &\quad + a_{11}\Big(2W_{11}^2(-1) + W_{20}^2(-1)e^{i\omega_0\tau_k} + 2W_{11}^1(-1)h_2 e^{-i\omega_0\tau_k} + W_{20}^1(-1)\bar{h}_2 e^{i\omega_0\tau_k}\Big), \\ \triangle_{21} &= b_{11}h_2 e^{-2i\omega_0\tau_k}, \\ \triangle_{22} &= b_{11}(h_2 + \bar{h}_2), \\ \triangle_{23} &= b_{11}\bar{h}_2 e^{2i\omega_0\tau_k}, \\ \triangle_{24} &= b_{11}\Big(2W_{11}^2(-1) + W_{20}^2(-1)e^{i\omega_0\tau_k} + 2W_{11}^1(-1)h_2 e^{-i\omega_0\tau_k} + W_{20}^1(-1)\bar{h}_2 e^{i\omega_0\tau_k}\Big). \end{aligned}$$

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Following the same algorithms and using a similar computation process as in Hassard *et al.* [39], we obtain the coefficients determining the important quantities by comparing coefficients of (6.7) and (6.8):

$$\left.\begin{array}{l} g_{20} = 2\overline{D}\tau_k\big(\triangle_{11} + \triangle_{21}\,\bar{h}_2^*\big) \\ g_{11} = \overline{D}\tau_k\big(\triangle_{12} + \triangle_{22}\,\bar{h}_2^*\big) \\ g_{02} = 2\overline{D}\tau_k\big(\triangle_{13} + \triangle_{23}\,\bar{h}_2^*\big) \\ g_{21} = \overline{D}\tau_k\big(\triangle_{14} + \triangle_{24}\,\bar{h}_2^*\big) \end{array}\right\}. \tag{6.9}$$

Since W_{20} and W_{11} occur in g_{21}, we need to to compute them. From Eqs. (6.3) and (6.5),

$$\begin{aligned} \dot{W} &= \dot{U}(t) - \dot{z}h - \dot{\bar{z}}\bar{h} \\ &= \begin{cases} A(0)W - 2Re\bar{h}^*(0)f_0h(\theta), & \theta \in [-1,0) \\ A(0)W - 2Re\bar{h}^*(0)f_0h(0) + f_0(z,\bar{z}), & \theta = 0 \end{cases} \\ &\triangleq A(0)W + H(z,\bar{z},\theta), \end{aligned} \tag{6.10}$$

where

$$H\big(z(t),\bar{z}(t),\theta\big) = H_{20}(\theta)\frac{z^2}{2} + H_{11}(\theta)z\bar{z} + H_{02}(\theta)\frac{\bar{z}^2}{2} + \dots . \tag{6.11}$$

Substituting Eq. (6.11) into Eq. (6.10) and equating coefficients we get

$$\big(A(0) - 2i\omega_0\tau_k I\big)W_{20}(\theta) = -H_{20}(\theta), A(0)W_{11}(\theta) = -H_{11}(\theta), \tag{6.12}$$

so from Eq. (6.10) for $\theta \in [-1,0)$ we have

$$H\big(z(t),\bar{z}(t),\theta\big) = -\bar{h}^*(0)f_0h(\theta) - h^*(0)\bar{f}_0\bar{h}(\theta) = -g(z,\bar{z})h(\theta) - \bar{g}(z,\bar{z})\bar{h}(\theta). \tag{6.13}$$

Then using Eq. (6.7) in (6.13) and comparing the coefficients with (6.11) we get

$$H_{20}(\theta) = -g_{20}h(\theta) - \bar{g}_{02}\bar{h}(\theta), \tag{6.14}$$

$$H_{11}(\theta) = -g_{11}h(\theta) - \bar{g}_{11}\bar{h}(\theta). \tag{6.15}$$

From the definition of $A(0)$, Eqs. (6.12) and (6.14), we obtain

$$\dot{W}_{20}(\theta) = 2i\omega_0\tau_k W_{20}(\theta) + g_{20}h(\theta) + \bar{g}_{02}\bar{h}(\theta),$$

so for $h(\theta) = (1,h_2)^T e^{i\omega_0\tau_k\theta}$ we have the solution

$$W_{20}(\theta) = \frac{ig_{20}}{\omega_0\tau_k}h(0)e^{i\omega_0\tau_k\theta} + \frac{i\bar{g}_{02}}{3\omega_0\tau_k}\bar{h}(0)e^{-i\omega_0\tau_k\theta} + E_1e^{2i\omega_0\tau_k\theta}. \tag{6.16}$$

Similarly, from Eqs. (6.12) and (6.15) we get

$$W_{11}(\theta) = -\frac{ig_{11}}{\omega_0\tau_k}h(0)e^{i\omega_0\tau_k\theta} + \frac{i\bar{g}_{11}}{\omega_0\tau_k}\bar{h}(0)e^{-i\omega_0\tau_k\theta} + E_2, \tag{6.17}$$

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where $E_1 = (E_1^{(1)}, E_1^{(2)})^T$ and $E_2 = (E_2^{(1)}, E_2^{(2)})^T$ are two-dimensional constant vectors that can be determined by setting $\theta = 0$ in $H(z, \bar{z}, \theta)$. Again, from the definition of $A(0)$ and Eq. (6.12) we have

$$\int_{-1}^{0} d\eta(\theta) W_{20}(\theta) = 2i\omega_0\tau_k W_{20}(0) - H_{20}(0), \tag{6.18}$$

$$\int_{-1}^{0} d\eta(\theta) W_{11}(\theta) = -H_{11}(0), \tag{6.19}$$

where $\eta(\theta) = \eta(0, \theta)$.

From Eq. (6.10), we know when $\theta = 0$,

$$H(z, \bar{z}, 0) = -2Re\big(\bar{h}^*(0) f_0 h(0)\big) + f_0(z, \bar{z}) = -\bar{h}^*(0) f_0 h(0) - h^*(0)\bar{f}_0 \bar{h}(0) + f_0(z, \bar{z})$$

i.e.

$$\begin{aligned} &H_{20}(\theta)\frac{z^2}{2} + H_{11}(\theta) z\bar{z} + H_{02}(\theta)\frac{\bar{z}^2}{2} + \cdots \\ &= -h(0)\left\{ g_{20}(\theta)\frac{z^2}{2} + g_{11}(\theta) z\bar{z} + g_{02}(\theta)\frac{\bar{z}^2}{2} + .. \right\} - \bar{h}(0)\left\{ \bar{g}_{20}\frac{\bar{z}^2}{2} + \bar{g}_{11} z\bar{z} + \bar{g}_{02}\frac{z^2}{2} + \cdots \right\} \\ &\quad + f_0(z, \bar{z}) \end{aligned} \tag{6.20}$$

From Eq. (6.5) we have

$$\begin{aligned} U_t(\theta) &= W(t, \theta) + 2Re\big\{ z(t) h(\theta) \big\} \\ &= W(t, \theta) + z(t) h(\theta) + \bar{z}(t)\bar{h}(\theta) \\ &= W_{20}(\theta)\frac{z^2}{2} + W_{11}(\theta) z\bar{z} + W_{02}(\theta)\frac{\bar{z}^2}{2} + \ldots, \end{aligned}$$

and we also obtain

$$f_0 = 2\tau_k \begin{pmatrix} \triangle_{11} \\ \triangle_{21} \end{pmatrix} \frac{z^2}{2} + \tau_n \begin{pmatrix} \triangle_{12} \\ \triangle_{22} \end{pmatrix} z\bar{z} + \ldots. \tag{6.21}$$

From Eqs. (6.20) and (6.21) we also have

$$H_{20}(0) = -g_{20} h(0) - \bar{g}_{02}\bar{h}(0) + 2\tau_k \begin{pmatrix} \triangle_{11} \\ \triangle_{21} \end{pmatrix}, \tag{6.22}$$

$$H_{11}(0) = -g_{11} q(0) - \bar{g}_{11}\bar{q}(0) + \tau_n \begin{pmatrix} \triangle_{12} \\ \triangle_{22} \end{pmatrix}. \tag{6.23}$$

Since $i\omega_0\tau_k$ is the eigenvalue of $A(0)$ corresponding to $h(0)$ and $-i\omega_0\tau_k$ is the eigenvalue of $A^*(0)$ corresponding to $\bar{h}(0)$, then

$$\left\{ i\omega_0\tau_k I - \int_{-1}^{0} e^{i\omega_0\tau_k\theta} d\eta(\theta) \right\} h(0) = 0,$$

$$\left\{ -i\omega_0\tau_k I - \int_{-1}^{0} e^{-i\omega_0\tau_k\theta} d\eta(\theta) \right\} \bar{h}(0) = 0.$$

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Substituting Eqs. (6.16) and (6.22) into Eq. (6.18) and simplifying, we find

$$E_1 = 2\Big(2i\omega_0\tau_k - F_1 - F_2 e^{-2i\omega_0\tau_k}\Big)^{-1}(\triangle_{11}, \triangle_{21})^T.$$

Similarly, substituting Eqs. (6.17) and (6.23) into Eq. (6.19) we have

$$E_2 = \big(-F_1 - F_2\big)^{-1}(\triangle_{12}, \triangle_{22})^T.$$

Thus we can find $W_{20}(\theta)$, $W_{11}(\theta)$ from Eqs. (6.16) and (6.17), and g_{21} can be computed from Eq. (6.9), so we can finally compute the following quantities:

$$\begin{cases} c_1(0) = \dfrac{i\Big\{g_{20}g_{11} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3}\Big\}}{2\omega_0\tau_k} + \dfrac{g_{21}}{2}, \\ \mu_2 = -Re\big\{c_1(0)\big\}/Re\Big\{\dfrac{d\lambda(\tau_k)}{d\tau}\Big\}, \\ \beta_2 = 2Re\big\{c_1(0)\big\}, \\ T_2 = -\dfrac{\Im\big\{c_1(0)\big\} + \mu_2\Im\Big\{\frac{d\lambda(\tau_k)}{d\tau}\Big\}}{\omega_0\tau_k}, \quad k = 0, 1, 2, \cdots, \end{cases}$$

where g_{ij} are given by Eq. (6.9), the sign of μ_2 determines the direction of the Hopf bifurcation, the sign of β_2 determines the stability of the bifurcating periodic solution, and the sign of T_2 determines the period of the bifurcating periodic solution in the centre manifold at the critical value τ_0. Consequently, by following Hassard *et al.* [39] we have the properties of the Hopf bifurcation at the critical value of $\tau = \tau_0$ as summarised in the following theorem.

Theorem 6.1 (cf. Hassard *et al.* [39])**.** *The basic properties of the Hopf bifurcation at the critical value of τ_0 are as follows:*

- *(a) the sign of μ_2 determines the direction of the Hopf bifurcation i.e. if $\mu_2 > 0\ (< 0)$ then the Hopf bifurcation is supercritical (subcritical) and the bifurcation periodic solutions exist for $\tau > \tau_0$ $(\tau < \tau_0)$;*
- *(b) the sign of β_2 determines the stability of the bifurcating periodic solution i.e. the bifurcation periodic solutions are stable (unstable) if $\beta_2 < 0$ (> 0); and*
- *(c) the sign of T_2 determines the period of the bifurcating periodic solutions i.e. the period increases (decreases) if $T_2 > 0$ (< 0).*

From Theorems (5.4) and (6.1), we can readily conclude that the existence and stability of the bifurcation periodic solutions are only determined by the sign of $Re(c_1(0))$. Specifically, if $Re(c_1(0)) < 0$ the system (2.1)-(2.2) has stable periodic solutions for $\tau > \tau_0$ in a τ_0-neighbourhood.

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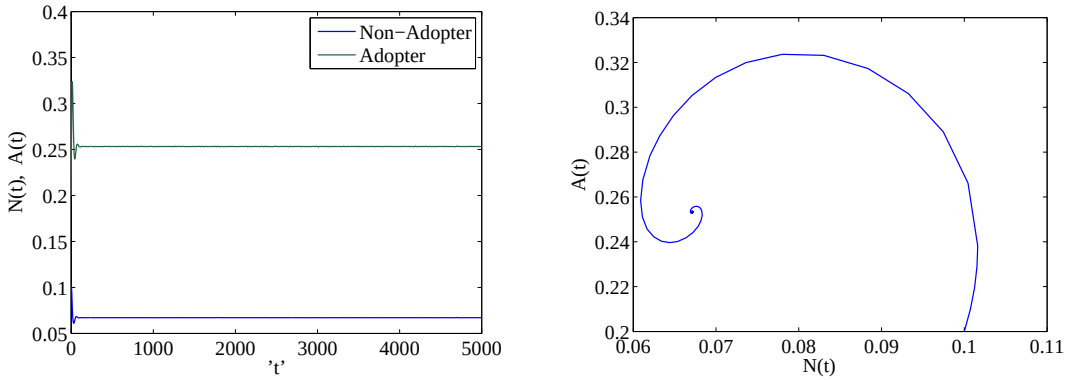


Figure 3: Solution trajectories predicting the local stability for the Non-Adopter and Adopter populations without any evaluation period τ.

7. Numerical Simulation

We illustrate our theoretical results for the system (2.1)-(2.2) by numerical simulation, adopting the parameters values $K = 10000$, $r = 0.5251$, $q = 0.45$, $p = 0.32$, $\alpha = 0.001$, $\delta = 0.11$, $\nu = 0.004$, $\zeta = 0.10$ i.e. we consider

$$\frac{dN(t)}{dt} = 0.5251\left(N(t) - \frac{N^2(t)}{10000}\right) - \big(0.32 + 0.45A(t-\tau)\big)N(t-\tau)e^{-(0.11+0.10)\tau} + (0.001 + 0.004)A(t) - 0.11N(t), \tag{7.1}$$

$$\frac{dA(t)}{dt} = \big(0.32 + 0.45A(t-\tau)\big)N(t-\tau)e^{-(0.11+0.10)\tau} - (0.11 + 0.001 + 0.004)A(t). \tag{7.2}$$

We used the Matlab package DDE23 to solve the system (2.1)-(2.2) subject to the initial data $N(t) = 0.1$, $A(t) = 0.2$. There is an interior equilibrium point $E_* = (0.0671, 0.2533)$, and we have the local asymptotic stability condition in the absence of time delay as H_1 : $tr(J^*) = -0.1499 < 0$, $det(J^*) = 0.0443 > 0$ as shown in Fig. 3. With the adopted parameter values, for the delayed innovation diffusion system there is a purely imaginary root $i\omega_0$ where $\omega_0 = 0.4629$ from Eq. (5.1), so for the model system (7.1)-(7.2) the critical value of the evaluation period from Eq. (5.12) is $\tau = \tau_0 = 1.0984$ such that E_* becomes unstable as τ passes through τ_0 (taking other parameters fixed). Moreover, at $\tau = \tau_0$, the transversality condition $\{d\xi/d\tau\}_{\tau=\tau_0,\omega=\omega_0} = 0.0513 \neq 0$ for the existence of Hopf bifurcation is also satisfied, so that the interior equilibrium E_* remains stable for $0 \leq \tau < 1.0984$ and becomes unstable for $\tau \geq 1.0984$. Fig. 4 shows the stable dynamics of the innovation diffusion system for $\tau = 1.0232$, and the Hopf bifurcation with limit cycle is shown for $\tau = 1.1333$ in Fig. 5. Thus it is evident that there is a range of values for the parameter $'\tau'$ such that the system produces asymptotic stability around an interior equilibrium E_* for $\tau < \tau_0$, and as τ increases beyond this critical value of evaluation period the system becomes unstable in the form of a limit cycle. There is a threshold limit of the evaluation period (τ, the time taken by a non-adopter to become a member of the adopter

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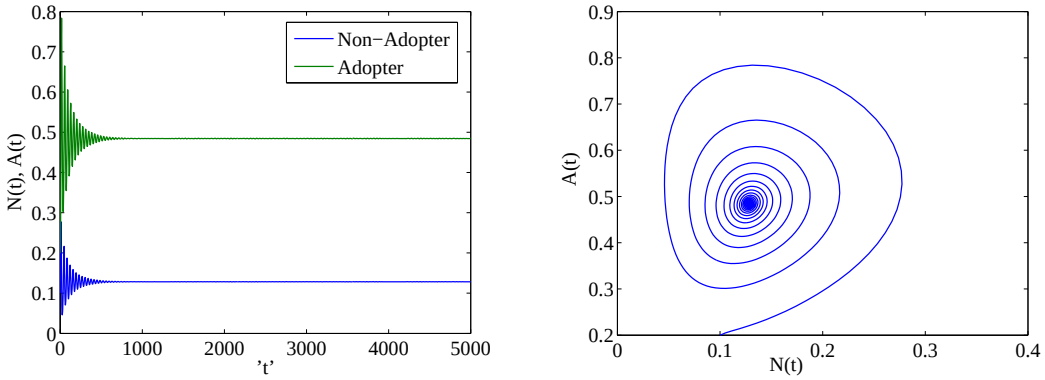


Figure 4: Convergence of the solution trajectories to $E_*(0.1284, 0.4844)$ for the system (2.1)-(2.2) at $\tau = 1.0232 < \tau_0^*$.

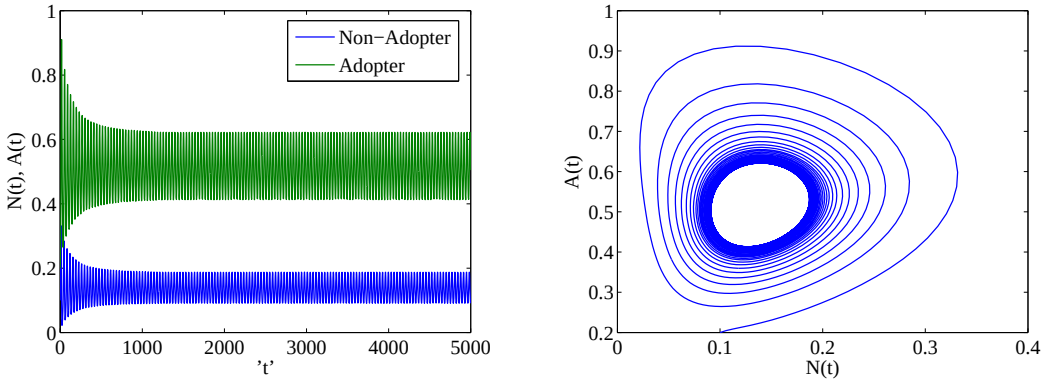


Figure 5: Bifurcating periodic solutions around $E_*(0.1351, 0.5091)$ for the Non-Adopter and Adopter populations at $\tau = 1.1333$ with increasing time.

population class), below which the system produces asymptotic stability and above it system shows excitability. We therefore conclude that the system around the interior equilibrium E_* enters into a Hopf bifurcation and exhibits that cyclic nature for a certain length of evaluation time.

Further, the eigenvectors as defined Section 6 are

$$h = \begin{pmatrix} 1 \\ 0.2765 - 2.0952i \end{pmatrix}, \qquad h^* = \begin{pmatrix} 1 \\ -0.0183 - 0.2236i \end{pmatrix},$$

and from Section 6 we computed the direction of the Hopf bifurcation where the stability determining quantities for the Hopf-bifurcating periodic solution at $\tau = \tau_0$ are

$$\begin{cases} c_1(0) = -0.6565 - 2.3526i, \\ \mu_2 = 22.1791, \\ \beta_2 = -1.3130, \\ T_2 = 7.4368. \end{cases}$$

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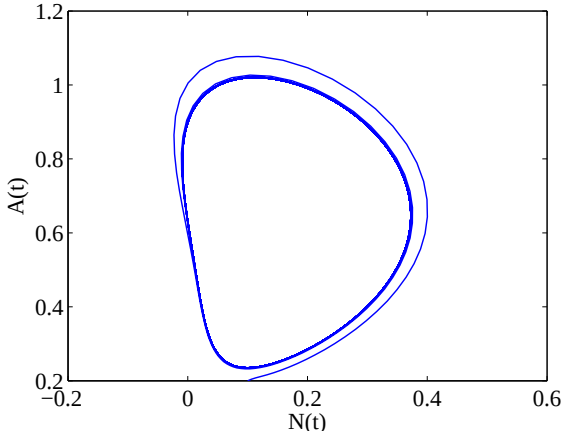


Figure 6: A stable limit cycle at $\tau = 1.2433$.

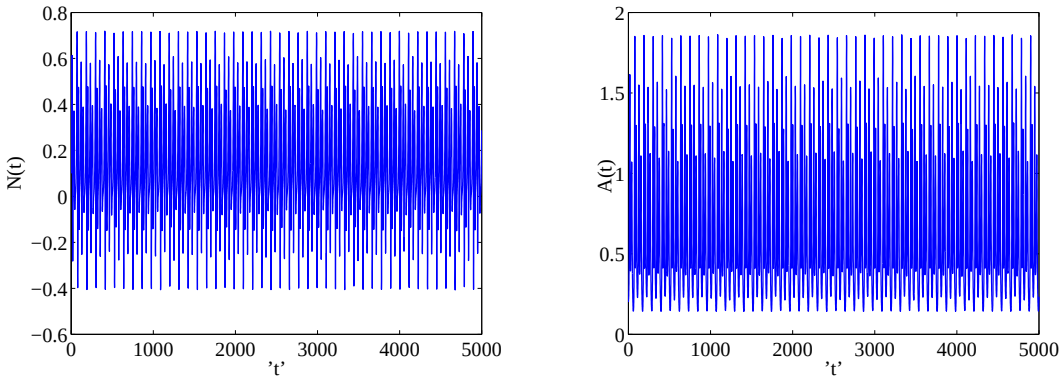


Figure 7: The attractive quasi-periodic oscillations of the solution curves for the Non-adopter and Adopter populations at $\tau = 1.4734$ with initial value $(0.1, 0.2)$.

Thus from Theorem (5.1) the Hopf bifurcation is supercritical, and the bifurcating periodic solutions are orbitally asymptotically stable, increasing as τ increases further through its critical value τ_0 (cf. Figs. 5-6).

Furthermore, when the system (2.1)-(2.2) is integrated with the same initial data for some longer delay beyond the critical value $'\tau_0'$, the periodic solutions occur in the interval $[1.0984, 1.3745)$; and when the value of τ is taken in the interval $[1.3745, 1.4833)$, quasi-periodic solutions and then the existence of chaotic attractors around interior equilibrium point E_* arise. In particular, banded chaos or quasi-periodic solutions occur for $\tau = 1.4734$ (cf. Figs. 7-8), and then chaotic attractors (cf. Figs. 9-10). The complex dynamic of the proposed model system from limit cycle to chaos is depicted by time series and phase plane diagram in Figs. 5-10. Our theoretical and numerical simulation demonstrate that after a small delay in the innovation diffusion process the complexity of the system increases Our results show that only a small time period is required for innovation to be diffused in the market, and the evaluation time period has a destabilising effect on the considered

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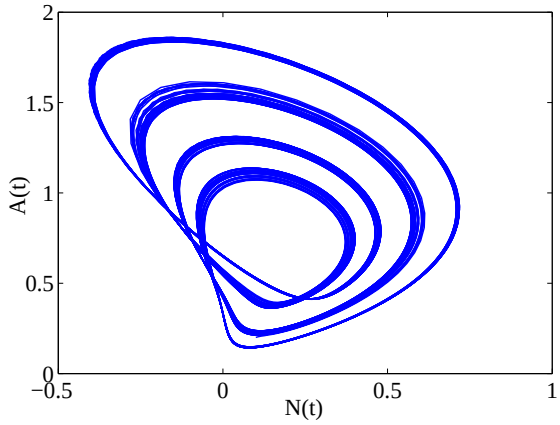


Figure 8: Phase plane depicting quasi-periodic oscillations around E_* for the Non-Adopter and Adopter populations at $\tau = 1.4734$.

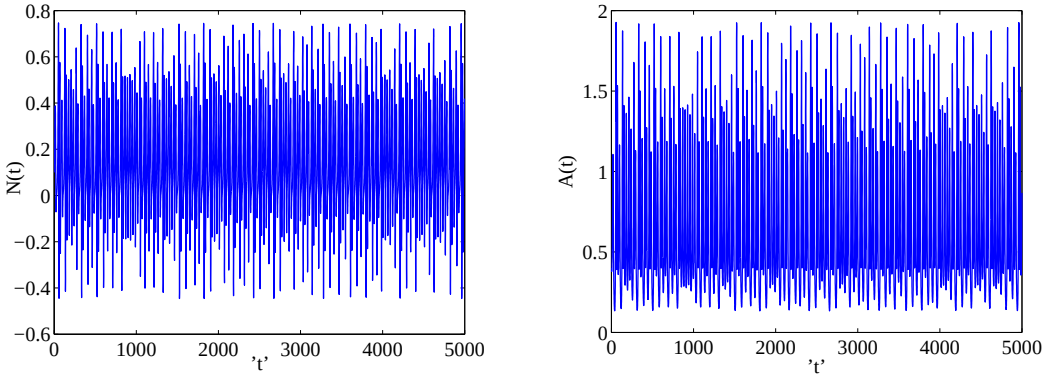


Figure 9: Time series depicting irregular oscillations of solution curves at $\tau = 1.4833$ with initial value $(0.1, 0.2)$.

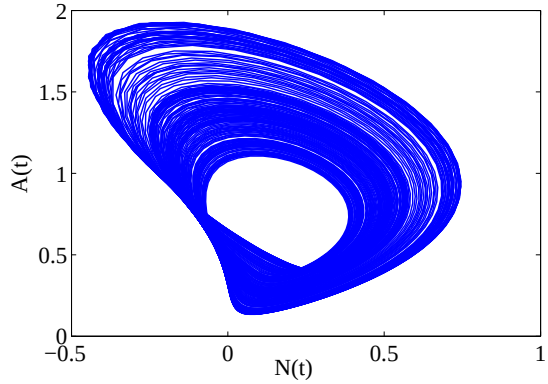


Figure 10: Phase plane giving chaotic attractors around E_* for Non-Adopter and Adopter populations at $\tau = 1.4833$.

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innovation diffusion model system, thus changing its state from a locally asymptotically stable point to a limit cycle, then to quasi-periodic solutions, and finally to chaos in the market.

8. Discussion

Our basic aim was to investigate the innovation diffusion process of the system (2.1)-(2.2), involving a logistically growing non-adopter population and incorporating an evaluation period (time delay τ) in the Bass model. While the system has local asymptotic stability without the evaluation period, with its inclusion there is Hopf bifurcation and a limit cycle. We found a threshold limit ($0 < \tau < 1.0984$) beyond which this occurs, with periodic oscillations around the interior equilibrium point E_* at $\tau = 1.1333$ (cf. Fig. 5), so the number of adopters can fluctuate in time. More stable periodic solutions of the innovation diffusion system around E_* at $\tau = 1.2433$ were also found (cf. Fig. 6), predicting the final level of adopters and the maximum market profit according to the state of the equilibrium point.

In most realistic situations, discrete sales patterns soon exhibit oscillatory behaviour [44, 45] as depicted by our model system, supporting our deterministic continuous model for a large population. Moreover, using the normal form theory and centre manifold theorem we determined the direction and stability of the bifurcating periodic solutions by calculating numerical values for key parameters ($c_1(0)$, μ_2, β_2 and T_2), and hence observed that the Hopf bifurcation is supercritical in nature and the bifurcating periodic solutions are orbitally asymptotically stable. Thus we have found that evaluation period has a vital role to play in establishing the periodic oscillations in the innovation diffusion system i.e. the long term usage of new innovation by the population. It was also shown that for some high value of delay, the system exhibits chaotic dynamic around E_* (cf. Figs. 7-10). Further, the value of the time delay beyond the bifurcation point causes complex dynamical behaviour of the given innovation diffusion system. In brief, the incorporation of an evaluation period together with a logistically growing non-adopter population has proved to be useful in capturing realistic scenarios.

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