Check for updates

*J. Appl. Probab.* **62,** 1475–1492 (2025)
*doi:*10.1017/jpr.2025.15

# PERCOLATION IN LATTICE *K*-NEIGHBOR GRAPHS

BENEDIKT JAHNEL,* *Technische Universität Braunschweig*
JONAS KÖPPL,** *Weierstrass Institute for Applied Analysis and Stochastics*
BAS LODEWIJKS,*** *Universität Augsburg*
ANDRÁS TÓBIÁS,**** *Budapest University of Technology and Economics, and Alfréd Rényi Institute of Mathematics*

**Abstract**

We define a random graph obtained by connecting each point of $\mathbb{Z}^d$ independently and uniformly to a fixed number $1 \leq k \leq 2d$ of its nearest neighbors via a directed edge. We call this graph the directed $k$-neighbor graph. Two natural associated undirected graphs are the undirected and the bidirectional $k$-neighbor graph, where we connect two vertices by an undirected edge whenever there is a directed edge in the directed $k$-neighbor graph between the vertices in at least one, respectively precisely two, directions. For these graphs we study the question of percolation, i.e. the existence of an infinite self-avoiding path. Using different kinds of proof techniques for different classes of cases, we show that for $k = 1$ even the undirected $k$-neighbor graph never percolates, while the directed $k$-neighbor graph percolates whenever $k \geq d + 1$, $k \geq 3$, and $d \geq 5$, or $k \geq 4$ and $d = 4$. We also show that the undirected 2-neighbor graph percolates for $d = 2$, the undirected 3-neighbor graph percolates for $d = 3$, and we provide some positive and negative percolation results regarding the bidirectional graph as well. A heuristic argument for high dimensions indicates that this class of models is a natural discrete analogue of the $k$-nearest-neighbor graphs studied in continuum percolation, and our results support this interpretation.

*Keywords:* Lattice $k$-neighbor graphs; directed $k$-neighbor graph; undirected $k$-neighbor graph; bidirectional $k$-neighbor graph; 1-dependent percolation; oriented percolation; negatively correlated percolation models; connective constant; planar duality; coexistence of phases

2010 Mathematics Subject Classification: Primary 60K35
Secondary 82B43

Received 13 April 2024; accepted 20 January 2025.

* Postal address: Institut für Mathematische Stochastik, Technische Universität Braunschweig Universitätsplatz 2, 38106 Braunschweig, Germany, and Weierstrass Institute for Applied Analysis and Stochastics, Mohrenstraße 39, 10117 Berlin, Germany. Email: benedikt.jahnel@tu-braunschweig.de
** Postal address: Weierstrass Institute for Applied Analysis and Stochastics, Mohrenstraße 39, 10117 Berlin, Germany. Email: jonas.koeppl@wias-berlin.de
*** Postal address: Mathematical Department, Universität Augsburg, Universitätsstraße 2, 86159 Augsburg, Germany. Email: bas.lodewijks@uni-a.de
**** Postal address: Department of Computer Science and Information Theory, Budapest University of Technology and Economics, Műegyetem rkp. 3., H-1111 Budapest, Hungary, and HUN-REN Alfréd Rényi Institute of Mathematics, Reáltanoda utca 13-15, H-1053 Budapest, Hungary. Email: tobias@cs.bme.hu

© The Author(s), 2025. Published by Cambridge University Press on behalf of Applied Probability Trust.



https://doi.org/10.1017/jpr.2025.15 Published online by Cambridge University Press

## 1. Introduction

In the last few decades, the study of $k$-nearest-neighbor-type models in continuum percolation has attracted significant attention. Häggström and Meester [7] introduced the concept of the *undirected* $k$-nearest-neighbor graph, where the vertex set consists of a homogeneous Poisson point process in $\mathbb{R}^d$. In this graph, two points are connected by an edge if at least one of them belongs to the $k$ nearest neighbors of the other. A *cluster* refers to a maximal connected component in this graph, and the graph is said to *percolate* if it contains an infinite (or unbounded) cluster. Since the percolation probability does not depend on the intensity of the Poisson point process, the only parameters affecting the percolation behavior are $k$ and $d$. [7] showed that for $k = 1$ the model does not percolate in any dimension, while for large enough $d$, percolation occurs when $k = 2$. Moreover, they established that for any $d \geq 2$ there exists a $k \in \mathbb{N}$ such that the graph percolates.

This initial study of continuum $k$-nearest-neighbor graphs was later extended by Balister and Bollobás [2], who investigated three possible senses of percolation in the model where each vertex is connected to its $k$ nearest neighbors by a *directed* edge: *in-percolation* (resp. *out-percolation*), which occurs by definition whenever some point of the point process exhibits an infinite incoming (resp. outgoing) path ending (resp. starting) at it, and *strong percolation*, which means that there exists a strongly connected infinite component in the graph, i.e., a component where from any point there exists a directed path to any other point. Additionally, they introduced another undirected graph, called the *bidirectional* $k$-nearest-neighbor graph, where two vertices are connected whenever they are mutually among each other's $k$ nearest neighbors. For the two-dimensional case, they verified percolation in the undirected graph (defined analogously to [7]) for $k \geq 11$, in the directed graph in all three senses (in-, out- and strong percolation) for $k \geq 13$, and in the bidirectional graph for $k \geq 15$.

Recently, Jahnel and Tóbiás [15] showed that in the bidirectional graph there is no percolation for $k = 2$ in any dimension, even if the underlying point process is not a Poisson point process but an arbitrary deletion-tolerant and stationary point process (satisfying some basic non-degeneracy conditions). Their proof exploits the simplicity of the structure of the bidirectional graph for $k = 2$, which has degrees bounded by two. Proving the absence of percolation for $k = 3, 4$ seems to be out of reach at the moment even in the Poisson case, and it is also not entirely clear whether these assertions hold in all dimensions or only for $d = 2$.

In this article, we introduce and analyze a discrete counterpart of the continuum $k$-nearest-neighbor model, aiming to gain a deeper understanding of its underlying structure and fundamental properties. By taking a step back and considering this discrete version we hope to shed some light on its behavior in a more controlled setting. For the directed $k$-neighbor graph ($k$-DnG) defined on the lattice $\mathbb{Z}^d$, each vertex is connected precisely to $k$ of its $2d$ nearest neighbors using the $\ell_1$-metric. The connections are chosen independently and uniformly (without replacement) for each vertex. By connecting nearest-neighbor pairs with undirected edges if they share at least one edge (or bidirectional edges if they share two directed edges) in the $k$-DnG, we obtain the undirected ($k$-UnG) or bidirectional ($k$-BnG) $k$-neighbor graph, respectively.

https://doi.org/10.1017/jpr.2025.15 Published online by Cambridge University Press

At least for high dimensions $d$ this discrete model can be expected to behave similarly to its continuum counterpart. Indeed, if we consider the continuum nearest-neighbor percolation model with $k = 2$, where the intensity of the underlying Poisson point process is such that the expected number of points in a unit ball equals one, and let $Y_i$ denote the position of the $i$th nearest neighbor for $i \in \{1, 2\}$, then [7, Lemma 3.2] showed that $|Y_i|$ converges in probability to 1 as $d$ tends to infinity, as well as that the conditional distribution of $Y_i$, given $|Y_i| = r$, is uniform on the sphere $\{x \in \mathbb{R}^d : |x| = r\}$. Moreover, the volume of the intersection of two spheres with radii $r_1, r_2 \in (0.9, 1.1)$, whose centers are at least 0.9 units apart, is negligible compared to the volume of either sphere [7, Lemma 3.3]. As a result, for large dimensions the continuum nearest-neighbor graph has connections at distance around 1, which are established almost independently among different pairs of vertices. As such, an approximation of this graph on $\mathbb{Z}^d$ should yield similar behavior when $d$ is large.

Let us note that our model is an element of a larger class of models introduced in [10] under the name *degenerate random environments* (see also [11–14] and the references therein for further work) and is also a key building block in the models presented in [1] that also include a reinforcement mechanism. Further, the $k$-BnG can be viewed as a 1-dependent Bernoulli bond percolation model, where each edge is 'open' (included in the $k$-BnG) with a probability of $p = k^2/(4d^2)$. In other words, any given edge is open with probability $p$, and the events of edges being open are independent when the edges are pairwise non-adjacent. Similarly, the $k$-UnG follows the same pattern with $p = (1 - k/(4d))k/d$. It is worth noting that these lattice $k$-neighbor graphs exhibit negative correlation properties, setting them apart from classical models such as the random cluster model [6]. The presence of an edge in the models under consideration here clearly decreases the likelihood of neighboring edges being present.

Throughout this paper, our main focus revolves around investigating percolation phenomena in all three variations of the model: $k$-DnG, $k$-UnG, and $k$-BnG. Specifically, we explore the presence and absence of percolation in each variant, unraveling the intricate behavior of these models.

The remainder of this article is organized as follows. We first collect the necessary notation and formulate our main results in Section 2. In Section 3 we then discuss our findings and mention some related conjectures and open questions. The proofs of our main results can all be found in Section 4.

## 2. Setting and main results

Consider the $d$-dimensional hypercubic lattice $\mathbb{Z}^d$. We are interested in the percolation behavior of the *k-neighbor graph* ($k$-nG) in which each vertex independently chooses uniformly $k \leq 2d$ of its $2d$ nearest neighbors. We distinguish three different types of $k$-nGs:

- The *directed k-nG* ($k$-DnG), in which we open a directed edge from the vertex to each of the $k$ chosen neighbors.
- The *undirected k-nG* ($k$-UnG), in which we open an undirected edge from the vertex to each of the $k$ chosen neighbors.
- The *bidirectional k-nG* ($k$-BnG), in which we open an undirected edge between two vertices whenever both choose the other one as one of their $k$ neighbors.

https://doi.org/10.1017/jpr.2025.15 Published online by Cambridge University Press

We are interested in the behavior of the percolation probabilities

$$\theta^{\mathrm{D}}(k, d) = \mathbb{P}(o \rightsquigarrow \infty \text{ in the } k\text{-DnG on } \mathbb{Z}^{\mathrm{d}}), \tag{2.1}$$

respectively $\theta^{\mathrm{U}}(k, d)$ and $\theta^{\mathrm{B}}(k, d)$, where $o \rightsquigarrow \infty$ represents the event that there exists a path along open (directed) edges from the origin to infinity. Let us note that the percolation probabilities are clearly non-decreasing functions in $k$. The dependence on the dimension $d$ is more subtle and – in contrast to many other percolation models – not monotone.

While the out-degree in the $k$-DnG is (almost surely) equal to $k$, the degree of a vertex in the $k$-BnG is a binomial random variable with parameters $k$ and $k/2d$, so that the expected degree is given by $k^2/2d$, which is less than $k$ unless $k = 2d$. In the case of the $k$-UnG, the degree is distributed according to $k + D$, where $D$ is a binomial random variable with parameters $2d - k$ and $k/2d$, and hence the expected degree is given by $k(4d - k)/(2d)$.

### 2.1. Results for the directed $k$-neighbor graph

As a first step, we study the cases where the occurrence of percolation is deterministic. On the one hand, choosing only one neighbor is never enough for percolation; on the other hand, choosing a sufficiently large number of neighbors leads to the almost-sure existence of a path connecting the origin to infinity.

**Remark 2.1.** The following assertions are easy to show and already partially mentioned in [10].

(i) For all $d \geq 1$, $\theta^{\mathrm{D}}(1, d) = 0$.

(ii) Whenever $k \geq d + 1$ we have $\theta^{\mathrm{D}}(k, d) = 1$.

Let us sketch why these statements are true.

(i) If there exists a path starting from the origin and reaching an $\ell_1$-distance $n$ from the origin, then this path is unique and has at least $n$ steps. But such a path exists with probability at most $2d \cdot (2d - 1)^{n-1} \cdot (2d)^{-n}$, which converges to zero as $n$ tends to infinity. Here, $2d \cdot (2d - 1)^{n-1}$ is an upper bound for the number of $n$-step paths in $\mathbb{Z}^d$.

(ii) The reason this holds is that the maximal distance to the origin is strictly increasing between generations. To be more precise, for $n \in \mathbb{N}_0$ we define $G_n$, the *nth generation* of vertices, inductively as follows. We initialize $G_0 = \{0\}$, and define $G_{n+1}$ as the set of those neighbors chosen by the vertices of $G_n$ (according to the definition of the $k$-DnG) that are not included in $\cup_{j=0}^{n} G_j$. Since $k \geq d + 1 = d$, among any $k$ neighbors of any vertex $x$ there exists one, say $y$, whose $\ell_1$-norm $\|y\|_1$ is larger than $\|x\|_1$. In this way, we can show via induction that, for all $n \in \mathbb{N}$, $\max_{x \in G_n} \|x\|_1 < \max_{x \in G_{n+1}} \|x\|_1$. This then implies that there exists an infinitely long directed path starting at the origin.

Let us now turn our attention towards the intermediate supercritical percolation phase. Here, our tools for the proof are restricted to sufficiently high dimensions $d$.

**Theorem 2.1.** *If $d \geq 4$ and $k \geq 4$, or if $d \geq 5$ and $k = 3$, we have $\theta^{\mathrm{D}}(k, d) > 0$.*

The proof can be found in Section 4.1 and mainly relies on a variation of the technique established in [5] plus precise estimates on the probability that two independent simple

https://doi.org/10.1017/jpr.2025.15 Published online by Cambridge University Press

oriented random walks meet each other and then take a common step. In fact the proof of Theorem 2.1 establishes positive probabilities for the existence of infinite, self-avoiding, and *monotone* paths, i.e. paths with strictly increasing $\ell_1$-norm.

Last, but not least, we come to the question of monotonicity with respect to the parameters. It is also easy to see that for $d \geq k$ we have $\theta^{\mathrm{D}}(k, d) < 1$ and, by the assertion of Remark 2.1, $\theta^{\mathrm{D}}(k, d) = 1$ for $d \leq k-1$, so $d \mapsto \theta^{\mathrm{D}}(k, d)$ can be decreasing. However, in view of similar results in the continuum [7], it seems reasonable to assume that $d \mapsto \theta^{\mathrm{D}}(k, d)$ is increasing for large $d$. At least, we can deduce the following monotonicity along diagonals of the parameter space.

**Theorem 2.2.** *For all* $k, d \geq 1$, $\theta^{\mathrm{D}}(k+1, d+1) \geq \theta^{\mathrm{D}}(k, d)$.

The proof of Theorem 2.2 can be found in Section 4.1 and is based on a coupling argument. We will see that it is essential that *both* $k$ and $d$ increase by one to make the coupling work. For example, issues arise when trying to deduce a similar coupling between the settings $k = d = 2$ and $k = 2$, $d = 3$. Though we expect percolation occurs in both settings, a similar coupling would give rise to a two-dimensional model where some vertices have only out-degree one. In such a case, it seems that the additional edges that appear in the 2-DnG in two dimensions are pivotal in the sense that they enable percolation to occur, whereas it may not occur without the presence of these edges. Still, we expect that if we restrict ourselves to parameters $k \leq d$ a similar relation between the $d$-dimensional and $(d+1)$-dimensional settings should exist. We state this in Conjecture 3.2 below.

### 2.2. Results for the undirected *k*-neighbor graph

To state our results and proofs for the $k$-UnG we first introduce the following notation. It is clear that $\theta^{\mathrm{U}}(k, d) \geq \theta^{\mathrm{D}}(k, d)$ for any $k, d \in \mathbb{N}$ because any directed edge in the $k$-DnG corresponds to an undirected edge in the $k$-UnG (and at most two directed edges can correspond to the same undirected edge). This way, Theorem 2.1 Remark 2.1(ii) also hold with $\theta^{\mathrm{D}}$ replaced by $\theta^{\mathrm{U}}$ everywhere. Moreover, we can now also deal with lower dimensions.

**Theorem 2.3.** $\theta^{\mathrm{U}}(2, 2) > 0$ *and* $\theta^{\mathrm{U}}(3, 3) > 0$.

However, even in the undirected sense, $k = 1$ is still not sufficient for percolation in any dimension $d \in \mathbb{N}$.

**Proposition 2.1.** $\theta^{\mathrm{U}}(1, d) = 0$ *for all* $d \geq 1$.

We recently learned that Proposition 2.1 directly follows from results in [9] studying a more general percolation model. Nevertheless, since the notation in [9] is quite different, we also present our proof for the reader's convenience and for completeness.

### 2.3. Results for the bidirectional *k*-neighbor graph

Clearly, $\theta^{\mathrm{B}}(k, d) \leq \theta^{\mathrm{D}}(k, d)$, so we already know that, if each vertex chooses a single neighbor, the bidirectional model will not percolate. A stronger result holds for this model, however. The following lemma presents an upper bound for $k$ in terms of $d$ for which we can verify that the bidirectional model does not percolate. To state this lemma, let us denote by $c(d)$ the *connective constant* of $\mathbb{Z}^d$ (see, e.g., [6]), which is defined by $c(d) := \lim_{n\to\infty} c_n(d)^{1/n}$, where $c_n(d)$ is the number of self-avoiding paths of length $n$ in the $d$-dimensional hypercubic lattice

https://doi.org/10.1017/jpr.2025.15 Published online by Cambridge University Press

TABLE 1. Percolation results and open cases for the BnG/DnG/UnG. For given $k$ and $d$, 'Y' means that the given graph percolates, 'N' means that it does not, while 'O' means that the given case is (at least partially) open. Percolation in the case $k = d = 2$ was verified in [4]; see Section 3.1 for a discussion on the case $k = 2$ for general $d$.

| | $k=1$ | $k=2$ | $k=3$ | $k=4$ | $k=5$ | $k=6$ | $k\geq 7$ |
|---|---|---|---|---|---|---|---|
| $d=1$ | N/N/N | Y/Y/Y | — | — | — | — | — |
| $d=2$ | N/N/N | N/Y/Y | O/Y/Y | Y/Y/Y | — | — | — |
| $d=3$ | N/N/N | N/O/O | N/O/Y | O/Y/Y | Y/Y/Y | Y/Y/Y | — |
| $d=4$ | N/N/N | N/O/O | N/O/O | O/Y/Y | O/Y/Y | O/Y/Y | Y/Y/Y |
| $d\geq 5$ | N/N/N | N/O/O | N/Y/Y | O/Y/Y | O/Y/Y | O/Y/Y | O/Y/Y |

that start at the origin. Via a quick subadditivity argument we can show that the limit $c(d)$ actually exists and satisfies $d < c(d) < 2d - 1$.

**Lemma 2.1.** *For all $k$ such that $k(k-1) < 2d(2d-1)/c(d)$, $\theta^{\mathrm{B}}(k, d) = 0$.*

Let us note that this result can be interpreted in two ways. First, for given $d$, it guarantees absence of percolation for all sufficiently small $k$ and, for example, $\theta^{\mathrm{B}}(2, d) = 0$ for any $d \geq 2$ (while it is clear that $\theta^{\mathrm{B}}(2, 1) = 1$), $\theta^{\mathrm{B}}(3, d) = 0$ for any $d \geq 3$, and $\theta^{\mathrm{B}}(4, d) = 0$ for any $d \geq 6$, as can be seen from the lower bounds on $c(d)$ in [8, Table 1] (e.g. $c(6) \geq 10.874\,038$). However, also for fixed $k$, since $2d(2d-1)/c(d) > 2d$, for sufficiently large $d$, there is no percolation thanks to Lemma 2.1. This is because, in high dimensions, it is unlikely for two neighboring vertices to pick the same connecting edge. This behavior is rather different from the case of the $k$-DnG and the $k$-UnG, where there is percolation for all $k \geq 3$ in all sufficiently high dimensions.

The proof strategy for Theorem 2.3 is to verify that percolation occurs when restricting ourselves to a two-dimensional plane, see Section 4.2. This approach is also applicable in the bidirectional model, and yields the following result.

**Proposition 2.2.** $\theta^{\mathrm{B}}(k, d) > 0$ *whenever* $k > d\sqrt{4(1 - 1/c(2))}$.

An application of the upper bound $c(2) \leq 2.679\,192\,495$ from [17] for the connective constant of $\mathbb{Z}^2$ immediately yields the following corollary.

**Corollary 2.1.** $\theta^{\mathrm{B}}(k, d) > 0$ *whenever* $k > d\sqrt{4(1 - 1/2.679\,192\,495)} \approx 1.583\,355d$.

This corollary allows us to verify percolation, e.g., for the $(2d-1)$-BnG for $d \geq 3$, for the $(2d-2)$-BnG for $d \geq 5$, for the $(2d-3)$-BnG for $d \geq 8$, and for the $(2d-4)$-BnG for $d \geq 10$. (Thus, the smallest-dimensional positive percolation result that we obtain from the corollary is that $\theta^{\mathrm{B}}(5, 3) > 0$.)

Note that, compared to the dimension $d$, we always need rather large $k$ to percolate. In high dimensions, we can improve the ratio slightly by using that the $k$-BnG model (just as the $k$-UnG model) in any dimension $d$ is a 1-dependent Bernoulli bond percolation model. Let

$$p_{\mathrm{sup}}(\mathbb{Z}^d) = \sup\{p \in (0, 1) \colon \text{some 1-dependent bond percolation model does not percolate}\}.$$

According to a recent result in [3], we have $\lim_{d\to\infty} p_{\mathrm{sup}}(\mathbb{Z}^d) \leq 0.5847$. This, together with the fact that the $k$-BnG model (just as the $k$-UnG model) in any dimension $d$ is a 1-dependent

https://doi.org/10.1017/jpr.2025.15 Published online by Cambridge University Press

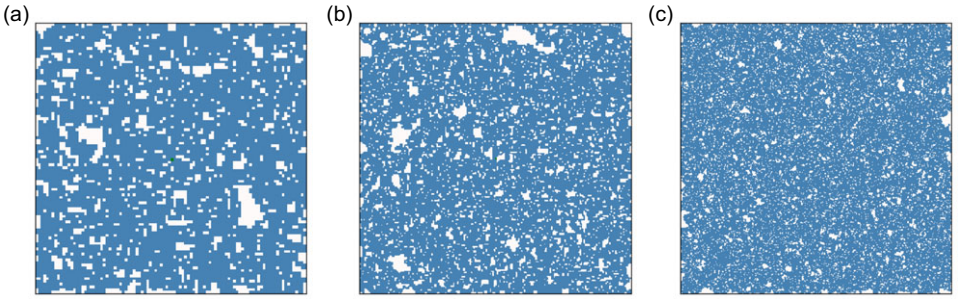



FIGURE 1. Samples of the 2-DnG in boxes with side length $n$. The colored vertices are those contained in the connected component of the origin.

bond percolation model where each edge is open with probability $p = k^2/(4d^2)$, immediately implies the following proposition.

**Proposition 2.3.** *For any* $\alpha > 2\sqrt{0.5847} \approx 1.5293$, $\theta^{\mathrm{B}}(\lfloor \alpha d \rfloor, d) > 0$ *for all* $d$ *sufficiently large.*

Let us remark that another result in [3] is that $p_{\sup}(\mathbb{Z}^2) \leq 0.8457$. This, together with the assertion that $p_{\sup}(\mathbb{Z}^d) \leq p_{\sup}(\mathbb{Z}^{d-1})$, yields that for $\alpha > 2\sqrt{0.8457}$ we have $\theta^{\mathrm{B}}(\lfloor \alpha d \rfloor, d) > 0$. However, this statement is weaker than Proposition 2.2 given that $2\sqrt{0.8457} \approx 1.8392 > \sqrt{4(1 - 1/2.679\,192\,495)}$.

Of course, the $k$-UnG model is also a 1-dependent bond-percolation model, but the same approach unfortunately does not yield any new results in this case. Indeed, here, each edge is open with probability $p = k(4d - k)/(4d^2)$. Thus, for $k = d$ it always holds that $p = 3/4 < 0.8457$, while for $k > d$ (and for $d \geq 2$ even for $k \geq 4$) we already know that $\theta^{\mathrm{D}}(k, d) = 1$.

We summarize our results in Table 1.

## 3. Outlook and open problems

Although we have managed to prove the occurrence or absence of percolation in a wide range of cases for the $k$-□nG model ($\square \in \{\mathrm{D}, \mathrm{U}, \mathrm{B}\}$), there are still many cases where the techniques used did not provide any conclusive results, especially in low dimensions. In this section we want to comment on a few of those that we deem to be interesting, and also discuss some further conjectures on the behavior of discrete $k$-neighbor graphs.

### 3.1. Directed $k$-neighbor graphs

Until very recently, for the $k$-DnG the most pressing open question was whether it percolates for $k = 2$ in $d = 2$. Very recently, the first two authors of this paper together with David Coupier and Benoît Henry published a preprint proving that it indeed percolates for $k = d = 2$ [4, Theorem 2.3]. According to simulations (see Figure 1), the proportion of vertices that can be reached from the origin is actually quite high. The heuristic that it is easier for the DnG to percolate in higher dimensions suggests the following conjecture.

**Conjecture 3.1.** *The 2-DnG percolates in all dimensions* $d \in \mathbb{N}$.

Note that the complement of the $k$-DnG in $d$ dimensions with respect to $\mathbb{Z}^d$, i.e. the graph with vertex set $\mathbb{Z}^d$ and the edge set formed by all nearest-neighbor edges of $\mathbb{Z}^d$ not contained in

https://doi.org/10.1017/jpr.2025.15 Published online by Cambridge University Press

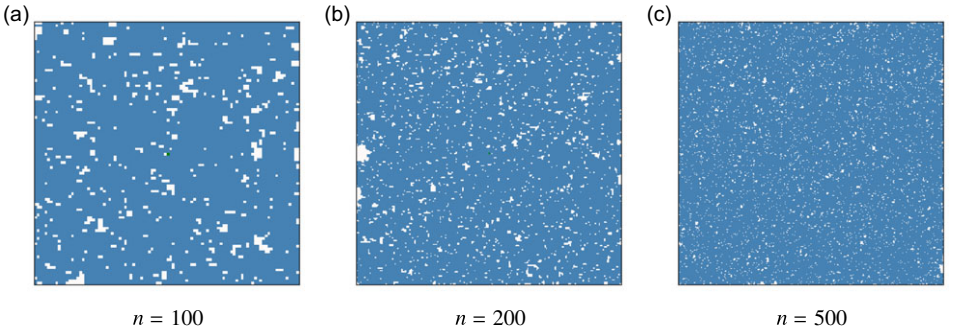



FIGURE 2. Samples of the 3-BnG in boxes with side length $n$. The colored vertices are those contained in the connected component of the origin.

the $k$-DnG, is in distribution a $(2d - k)$-DnG. So the 2-DnG is an example of phase coexistence in a self-complementary (directed) graph in two dimensions.

With regards to the monotonicity statement, Theorem 2.2, we expect a corresponding statement also to hold along the diagonal of the $(k, d)$-plane grid.

**Conjecture 3.2.** *For all $k, d \geq 2$ such that $k \leq d$, $\theta^{\mathrm{D}}(k, d+1) \geq \theta^{\mathrm{D}}(k, d)$.*

Given percolation in the 2-DnG for $d = 2$, one possible way to prove Conjecture 3.1 would be to verify Conjecture 3.2, but at least for higher dimensions $d$ there might be simpler arguments making use of intersection properties of random walks to show that $\theta^{\mathrm{D}}(2, d) > 0$. Of course, this would also imply that $\theta^{\mathrm{U}}(k, d) > 0$ for all $k, d \geq 2$.

### 3.2. Bidirectional $k$-neighbor graphs

For the bidirectional model the most intriguing question seems to be what the smallest $k = k(d)$ is such that $\theta^{\mathrm{B}}(k(d), d) > 0$ holds for all $d$ (or at least for all $d$ sufficiently large). Heuristically, we would expect that we do not percolate when the expected degree of a vertex is less than two, because the chance of backtracking, i.e. not getting a new edge, is then too large. However, as soon as we have an expected degree of at least two, we could imagine that this is sufficient to percolate, since we usually get at least one new edge in each step while walking along a path. At least in low dimensions (which are still accessible for numerical computations) the numerical tests seem to support the following conjecture – see Figures 2 and 3.

**Conjecture 3.3.** *The $k$-BnG percolates in dimension $d$ if and only if $k \geq 2\sqrt{d}$.*

In particular this would imply that the $d$-BnG does *not* percolate for dimensions $d = 2, 3$ but percolates for $d \geq 4$, and that the 3-BnG percolates in $d = 2$.

### 3.3. XOR percolation

Let us briefly consider another variant of lattice $k$-neighbor graphs, namely the *exclusively unidirectional $k$-nG* ($k$-XnG), in which we open an undirected edge between two vertices whenever exactly one of them chooses the other one as one of its $k$ chosen neighbors. The letter X refers to the 'XOR' (exclusive 'or') in the edge-drawing rule.

https://doi.org/10.1017/jpr.2025.15 Published online by Cambridge University Press

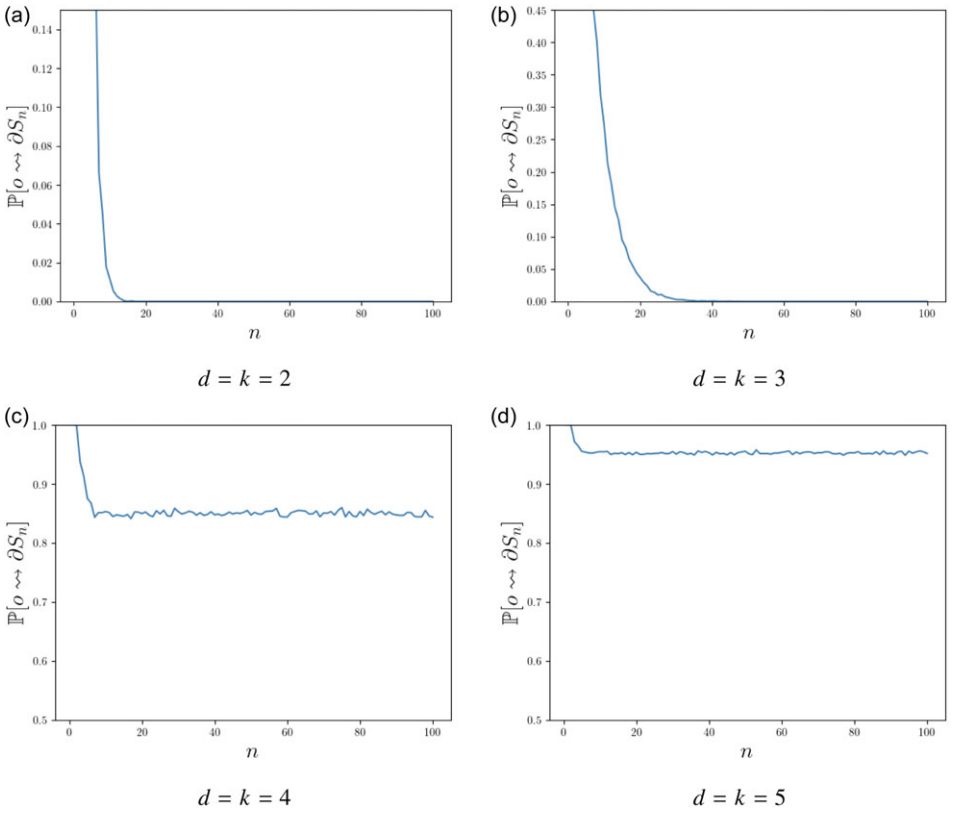



FIGURE 3. Probability of reaching the boundary of boxes $S_n$ of varying side length $n$ in the $d$-BnG for $d \in \{2, 3, 4, 5\}$ (sample size: 10 000).

Although the parameter $k$ of the $k$-XnG can range between 1 and $2d$ (just as for the $k$-BnG, $k$-DnG, and $k$-UnG), actually it suffices to consider $1 \leq k \leq d$, thanks to the following considerations.

**Remark 3.1.** For any $1 \leq k \leq 2d-1$, the $k$-XnG equals the $(2d-k)$-XnG in distribution. Indeed, the statement follows from the fact that the $k$-XnG contains precisely the undirected nearest-neighbor edges of $\mathbb{Z}^d$ that are included in the $k$-DnG in one direction and in the complementary $(2d-k)$-DnG in the other direction, and the same holds for the $(2d-k)$-DnG.

Of course, the assertion of Remark 3.1 also holds for $k = 2d$ if we define the 0-XnG as $\mathbb{Z}^d$ with no edges, from which it is clear that $\theta^{\mathrm{X}}(2d, d) = 0$ for all $d$, where we write $\theta^{\mathrm{X}}$ for the percolation probability of the XnG analogously to (2.1). Hence, we can limit our analysis to the cases $1 \leq k \leq d$.

It is also easy to see that, since the $k$-XnG is a subgraph of the $k$-UnG, we have $\theta^{\mathrm{X}}(1, d) = 0$ for all $d \geq 1$ and thus, by Remark 3.1, $\theta^{\mathrm{X}}(2d-1, d) = 0$ for all $d$. Moreover, the 2-XnG is also not percolating in one dimension, so the first non-trivial case is $k = d = 2$; see Figure 4 for some simulations.

Since the probability of an edge being open in the $k$-XnG is given by $k(2d-k)/(2d^2)$, which is maximized at $k = d$ with maximum value $\frac{1}{2}$, the $d$-XnG seems to be critical in dimension $d$.

https://doi.org/10.1017/jpr.2025.15 Published online by Cambridge University Press

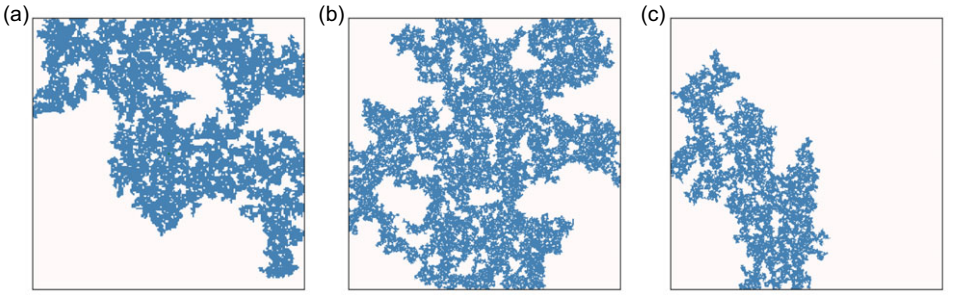



FIGURE 4. Samples of the 2-XnG in boxes with side length $n$. The colored vertices are those contained in the connected component of the origin.

Moreover, for $k \geq 2$, and $e_1$, $e_2$ edges that share a common vertex,

$$\mathbb{P}(e_1, e_2 \text{ open}) = \frac{(2d-k)(k(4k-1)(2d-k)-k^2)}{8d^3(2d-1)} \leq \frac{k^2(2d-k)^2}{4d^4} = \mathbb{P}(e_1 \text{ open})^2,$$

with equality if and only if $k = d$. Thus, the model features negative correlations that decrease in $k \in \{2, \ldots, d\}$ and in particular, for $k = d$, we even have pairwise independence. This, together with the fact that, for fixed $d$, the probability of an edge being open in the $k$-XnG increases, allows us to formulate the following conjecture.

**Conjecture 3.4.** *For fixed $d$, $k \mapsto \theta^{\mathrm{X}}(k, d)$ is strictly monotone increasing in $k \in \{2, \ldots, d\}$.*

Despite our heuristic justification, it is not clear to us how to verify this conjecture. In particular, we are not aware of a coupling between the $k$-XnG and the $l$-XnG for $1 \leq k < l \leq d$, while the $k$-DnG (respectively $k$-UnG, $k$-BnG) is a subgraph of the $l$-DnG (resp. $l$-UnG, $l$-BnG) whenever $1 \leq k \leq l \leq 2d$.

Let us finally mention that, in view of the first-moment method for the proof of existence of subcritical regimes (see, for example, Lemma 2.1) we would need to establish $\mathbb{P}(e_1 \text{ open} \mid e_2 \text{ open})c(d) < 1$, which is unfortunately not true for any $2 \leq k \leq d$. Also, none of the methods we have used to prove existence of supercritical percolation regimes seem to apply to the $k$-XnG, so determining whether it actually percolates for certain choices of $d$ and $k$ remains a goal for our future research.

Focusing on the case $k = 2$, we have performed simulations that suggest absence of percolation for $d = 2$ – see Figures 5(a) and 5(b).

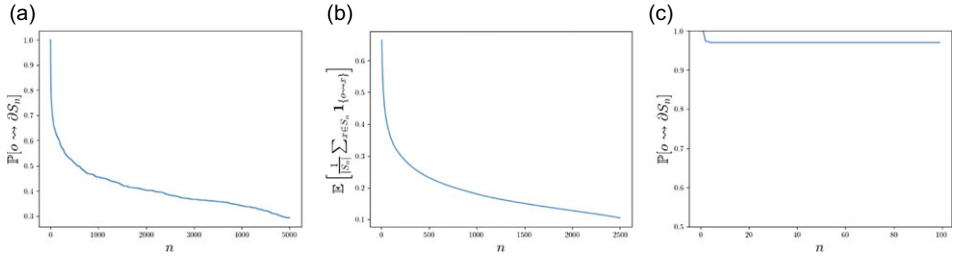
Based on this and Figure 5(c), we can at least formulate the following conjecture.

**Conjecture 3.5.** *The 2-XnG percolates for all $d \geq 3$ but does not percolate for $d = 2$.*

### 3.4. In-percolation and strong percolation

As already mentioned in the introduction, our notion of directed percolation (2.1) corresponds to out-percolation in [2]. In this paper, we have not developed any specific proof techniques for strong percolation or in-percolation. Nevertheless, it is clear that, for fixed $d$ and $k$, percolation in the $k$-BnG implies strong percolation, while strong percolation implies both in- and out-percolation in the $k$-DnG. Moreover, in- or out-percolation in the $k$-DnG implies percolation in the $k$-UnG, analogously to the continuum case (where these implications were

https://doi.org/10.1017/jpr.2025.15 Published online by Cambridge University Press




Probability of reaching the boundary of boxes with side length $n$ in the 2-XnG in $d = 2$.

Proportion of vertices reached from the origin in boxes with side length $n$ in the 2-XnG in $d = 2$.

Probability of reaching the boundary of boxes with side length $n$ in the 2-XnG in $d = 3$.

FIGURE 5. Numerical experiments for the XnG in dimensions $d = 2, 3$ (sample size=1000).

mentioned in [2]). Hence, our positive percolation results for the $k$-BnG yield similar results for strong percolation, and our negative (out-)percolation results for the $k$-DnG imply the absence of strong percolation. Furthermore, writing

$$\theta^{\mathrm{D}'}(k, d) = \mathbb{P}(\text{there exists an infinite directed path in the } k\text{-DnG ending at } o)$$

for the analogue of $\theta^{\mathrm{D}}(k, d)$ for in-percolation, we have $\theta^{\mathrm{D}'}(1, d) = 0$, since $\theta^{\mathrm{U}}(1, d) = 0$ holds thanks to Proposition 2.1.

**Conjecture 3.6.** *For all* $d \in \mathbb{N}$ *and* $1 \le k \le 2d$, $\theta^{\mathrm{D}'}(k, d) > 0$ *if and only if* $\theta^{\mathrm{D}}(k, d) > 0$.

**Remark 3.2.** Let us note that the same assertion is not true with 'with positive probability' replaced by 'almost surely' everywhere. Indeed, let $d + 1 \le k < 2d$, then we know from Remark 2.1(ii) that $\theta^{\mathrm{D}}(k, d) = 1$. However, $\theta^{\mathrm{D}'}(k, d) < 1$ because the probability of the event that all the $2d$ nearest neighbors of $o$ choose only neighbors different from $o$ as endpoints of their outgoing edges is positive.

One reason why we could be led to believe that Conjecture 3.6 is true is the following simple argument, which is based on the mass-transport principle (see, e.g., [16, Chapter 8]). Consider the function $F(x, y;\omega) := \mathbf{1}_{\{x\to y\}}(\omega)$, $x, y \in \mathbb{Z}^d$, where $\omega$ is a realization of the directed $k$-DnG on $\mathbb{Z}^d$. This function $F$ is clearly diagonally invariant with respect to lattice shifts, i.e. $F(x + z, y + z; \omega + z) = F(x, y; \omega)$. So, if we let $\overrightarrow{C_0}$ denote the *out-component* of the origin and $\overleftarrow{C_0}$ denote the *in-component*, then the classical mass-transport principle [16, p. 274] implies that

$$\mathbb{E}\big[|\overrightarrow{C_0}|\big] = \sum_{x\in\mathbb{Z}^d} \mathbb{E}[F(0, x; \cdot)] = \sum_{x\in\mathbb{Z}^d} \mathbb{E}[F(x, 0; \cdot)] = \mathbb{E}\big[|\overleftarrow{C_0}|\big].$$

Hence, if we have out-percolation at parameters $d$, $k$, then at least the expected size of the in-cluster is also infinite, and vice versa. A sharp-threshold result (including exponential tails of the cluster size distribution in the subcritical phase) would yield the conjecture.

Let us further remark that Conjecture 3.6 is not known either in the continuum case. Without such a result, it seems that in-percolation is in general more difficult to deal with than out-percolation, due to increased combinatorial complexity. For example, showing that the 1-DnG does not percolate was relatively straightforward (cf. the proof of Remark 2.1(i)), but proving the lack of in-percolation in the same graph (cf. the proof of Proposition 2.1) already presented

https://doi.org/10.1017/jpr.2025.15 Published online by Cambridge University Press

more challenges. Further, in the two-dimensional Poisson case in the proof of [2, Theorem 2], the authors showed directly that out-percolation occurs for $k = 13$, but for in-percolation for the same $k$ the same method did not work. They derived that their arguments for out-percolation actually imply strong percolation and therefore also in-percolation.

We have seen that, for $d$ large, out-percolation in the $k$-DnG already occurs for $k = 3$, while for the $k$-BnG percolation definitely requires $k = \Omega(\sqrt{d})$, and all the examples where we have verified percolation are such that $k > d$. It is an interesting open question whether strong percolation is closer to directed (out-)percolation than to bidirectional percolation in this respect. The aforementioned arguments of [2] suggest the former.

## 4. Proofs

### 4.1. Proofs for the $k$-DnG

In this section, we carry out the proofs of Theorems 2.1 and 2.2.

*Proof of Theorem* 2.1. We follow [5, Section 2]. Since we have already verified the assertion of Remark 2.1, without loss of generality we will assume that $k \leq d$. Let $\mathcal{R}_n$ denote the set of oriented paths from the origin to level $n$ (i.e. all vertices at $\ell_1$-distance $n$) with strictly increasing $\ell_1$-distance that stay within the first closed quadrant, in other words all vertices on the path have non-negative coordinates, and let $N_n$ be the number of open paths in $\mathcal{R}_n$. Then $W_n = (2/k)^n N_n$ is a martingale with respect to $\mathcal{F}_n$, the sigma-algebra generated by the first $n$ steps of a simple and symmetric oriented random walk $T$ with $T_0 = o$ and $T_{n+1} = T_n + e_i$, where $i$ is uniformly chosen from $\{1, \ldots, d\}$ for all $n \in \mathbb{N}_0$. Indeed, using the fact that the expected number of monotone (i.e. with respect to $\ell_1$-distance increasing) edges is given by the expectation of a hypergeometric random variable, i.e.,

$$\sum_{\ell=0}^{k} \ell \binom{d}{\ell} \binom{d}{k-\ell} \Big/ \binom{2d}{k} = \frac{k}{2},$$

and we can use independence to see that $\mathbb{E}[N_{n+1} \mid \mathcal{F}_n] = kN_n/2$.

Since $\mathbb{E}[W_1] = 1$, we can apply the martingale convergence theorem to ensure the existence of a random variable $W$ with $W_n \to W$ almost surely. Using the second-moment method, it now suffices to show that $\limsup_{n\uparrow\infty} \mathbb{E}[W_n^2] < \infty$, since by the Paley–Zygmund inequality we then have that $\mathbb{P}(W > 0) > 0$, which implies $\theta^{\mathrm{D}}(k, d) > 0$.

For this, note that

$$\mathbb{E}[N_n^2] = \sum_{s,t \in \mathcal{R}_n} \mathbb{P}(s, t \text{ open}) = \sum_{s,t \in \mathcal{R}_n} p^{K(s,t)} q^{L(s,t)} p^{2(n-K(s,t)-L(s,t))},$$

where $p = k/(2d)$ denotes the probability that a given edge is open, $q = k(k-1)/(2d(2d-1))$ denotes the probability that two different edges that emerge from the same vertex are open, $K(s, t)$ is the number of joint edges in two paths $s$ and $t$, and $L(s, t)$ is the number of vertices $x$ in $s$ and $t$ such that $s$ and $t$ do not have a joint edge after $x$. Now note that $q < p^2$, which implies that $\mathbb{E}[N_n^2] \leq \sum_{s,t \in \mathcal{R}_n} p^{2n-K(s,t)}$. The right-hand side is precisely the same as the right-hand side of the display below [5, (2.7)]. Thus, a verbatim application of the arguments of [5, Section 2] implies that $\lim_{n\to\infty} \mathbb{E}W_n^2 < \infty$ holds if and only if $p = k/(2d) > \varrho(d)$, where for two independent (simple, symmetric, nearest-neighbor, oriented) random walks $S = (S_n)_{n\in\mathbb{N}_0}$ and $S' = (S'_n)_{n\in\mathbb{N}_0}$ on the first quadrant of $\mathbb{Z}^d$ started from the origin, we define

$$\varrho(d) = \mathbb{P}(\text{there exists } m \geq 0 \colon S_m = S'_m \text{ and } S_{m+1} = S'_{m+1}).$$

https://doi.org/10.1017/jpr.2025.15 Published online by Cambridge University Press

TABLE 2. Values of the right-hand side of (4.3) and the smallest $k$ such that this right-hand side is below $k/(2d)$, for $4 \le d \le 7$.

| | $d=4$ | $d=5$ | $d=6$ | $d=7$ |
|---|---|---|---|---|
| $U(d)$, the right-hand side of (4.3) | 0.693 093 | 0.394 622 | 0.268 615 | 0.199 707 |
| Smallest $k$ such that $U(d) < k/(2d)$ | 6 | 4 | 4 | 3 |

Further, let $\tau_d = \inf\{m \ge 0\colon S_m = S'_m \text{ and } S_{m+1} = S'_{m+1}\}$. According to [5, p. 155], for all $d$,

$$\mathbb{P}(\tau_d = 0) = d^{-1}, \qquad \mathbb{P}(\tau_d = 1) = 0, \qquad \mathbb{P}(\tau_d = 2) = d^{-3} - d^{-4}, \tag{4.1}$$

and, for $3 \le \ell \le d$, $\mathbb{P}(\tau = \ell) \le d^{-\ell}\ell!$, while, for $j \ge 1$ and $\ell \in (jd, (j+1)d]$,

$$\mathbb{P}(\tau = \ell) \le d^{-1}(2\pi d)^{1/2}\left(\frac{\mathrm{e}^{-1/13}}{\sqrt{2\pi}}\right)^d j^{-(d-1)/2}, \tag{4.2}$$

so that

$$\begin{aligned}\varrho(d) &= \mathbb{P}(\tau_d < \infty)\\ &\le d^{-1} + d^{-3} - d^{-4} + \sum_{\ell=3}^{d} d^{-\ell}\ell! + (2\pi d)^{1/2}\left(\frac{\mathrm{e}^{-1/13}}{\sqrt{2\pi}}\right)^d \sum_{j=1}^{\infty} j^{-(d-1)/2}\\ &= d^{-1} + d^{-3} - d^{-4} + \sum_{\ell=3}^{d} d^{-\ell}\ell! + (2\pi d)^{1/2}\left(\frac{\mathrm{e}^{-1/13}}{\sqrt{2\pi}}\right)^d \zeta\left(\frac{d-1}{2}\right) =: U(d),\end{aligned} \tag{4.3}$$

where $\zeta$ denotes the Riemann zeta function. To conclude that we do indeed percolate, it suffices to verify that $U(d)$ is less than $k/(2d)$. In low dimensions we can compute the right-hand side numerically. More precisely, Table 2 shows the smallest values of $k$ such that $U(d)$ is less than $k/(2d)$, for $d = 4, 5, 6, 7$.

In particular, we see from Table 2 that $U(d) < k/(2d)$ holds for $k = 3$ and $d = 7$. Now, in order to lift this to higher dimensions $d > 7$ for the same $k = 3$, we rewrite

$$2dU(d) = 2\left(1 + \frac{1}{d^2} - \frac{1}{d^3} + \sum_{\ell=3}^{d} \ell!/d^{\ell-1} + (2\pi)^{1/2} g(d)\zeta\left(\frac{d-1}{2}\right)\right),$$

with $g(d) = d^{3/2}(\mathrm{e}^{-1/13}/\sqrt{2\pi})^d$. Now, observe that, for all $d \ge 4$, the terms $d^{-2} - d^{-3}$, $\sum_{\ell=3}^{d} \ell!/d^{\ell-1}$, $g(d)$, and $\zeta((d-1)/2)$ are all decreasing in $d$. Therefore, for a given $k$, if $2dU(d) \le k$, then $2d'U(d') \le k$ for all $d' > d$, which implies the claim.

Note that Table 2 also guarantees percolation for $k \ge 6$ for $d = 4$ and for $k \ge 4$ for $d = 5, 6$. Therefore, in order to finish the proof of the theorem, we have to treat three special low-dimensional cases separately: $(k, d) = (4, 4)$ (of course, percolation here also implies percolation for $(k, d) = (5, 4)$), $(k, d) = (3, 5)$, and $(k, d) = (3, 6)$. As for the first one, we use a computer to get the precise values of $\mathbb{P}(\tau_4 = k)$ for $k = 3, 4, 5$ by generating all length-six random walk trajectories and computing how many of them belong to these events respectively (for $\mathbb{P}(\tau_4 = l)$, $l = 0, 1, 2$, we of course obtain the same precise values as [5]). This yields

$$\mathbb{P}(\tau_4 = 3) = \frac{3}{512}, \qquad \mathbb{P}(\tau_4 = 4) = \frac{279}{65\,536}, \qquad \mathbb{P}(\tau_4 = 5) = \frac{831}{262\,144}.$$

https://doi.org/10.1017/jpr.2025.15 Published online by Cambridge University Press

Thus, using (4.1) and (4.2),

$$\begin{aligned}\varrho(4) &\le \frac{1}{4} + \frac{3}{256} + \frac{3}{512} + \frac{279}{65\,536} + \frac{831}{262\,144} \\ &\quad + \frac{3}{4}\sqrt{8\pi}\left(\frac{\mathrm{e}^{-1/13}}{\sqrt{2\pi}}\right)^4 + \sum_{j=2}^{\infty}\sqrt{8\pi}\left(\frac{\mathrm{e}^{-1/13}}{\sqrt{2\pi}}\right)^4 \sum_{j=1}^{\infty} j^{-3/2} \\ &\approx 0.495\,542 < 0.5 = k/(2d).\end{aligned}$$

It follows that $\theta^{\mathrm{D}}(4, 4) > 0$.

With similar methods we can show that $\theta^{\mathrm{D}}(3, 5) > 0$ and $\theta^{\mathrm{D}}(3, 6) > 0$. Indeed, for $d = 5$ we obtain, via explicit computer-assisted computations,

$$\mathbb{P}(\tau_5 = 3) = \frac{44}{15\,625}, \qquad \mathbb{P}(\tau_5 = 4) = \frac{712}{390\,625}, \qquad \mathbb{P}(\tau_5 = 5) = \frac{12\,136}{9\,765\,625}.$$

This already implies that $\theta^{\mathrm{D}}(3, 5) > 0$ because

$$\varrho(5) \le \frac{1}{5} + \frac{4}{625} + \frac{44}{15\,625} + \frac{712}{390\,625} + \frac{12\,136}{9\,765\,625} + \sqrt{10\pi}\left(\frac{\mathrm{e}^{-1/13}}{\sqrt{2\pi}}\right)^5 \zeta(2) \approx 0.275\,703,$$

which is less than $k/(2d) = 0.3$. For $d = 6$, determining the value of $\mathbb{P}(\tau_6 = k)$, $k = 3, \ldots, 6 = d$ numerically would be computationally quite demanding. However, we see from Table 2 that the right-hand side of (4.3) is just slightly larger than $k/(2d) = 1/4 = 0.25$. It turns out that in this case, it suffices to compute $\mathbb{P}(\tau_6 = 3)$ numerically, which equals $35/23\,328$. Estimating $\mathbb{P}(\tau_6 > 3)$ according to (4.3), we obtain

$$\varrho(6) \le \frac{1}{6} + \frac{5}{1296} + \frac{35}{23\,328} + \sum_{k=4}^{6} 6^{-k} k! + \sqrt{12\pi}\left(\frac{\mathrm{e}^{-1/13}}{\sqrt{2\pi}}\right)^6 \zeta\left(\frac{5}{2}\right) \approx 0.242\,338 < 0.25 = \frac{k}{2d},$$

which concludes the proof. □

**Remark 4.1** Let us briefly note that by (4.1) we directly have that $\varrho(d) > 1/d$. Therefore, $\varrho(d) < k/(2d)$ never holds for $k = 2$, whence for $k = 2$ the approach of the proof of Theorem 2.1 is not applicable. For $d \le 3$ (where the cases $k = 2$, $d = 3$ and $k = 3$, $d = 3$ are open), the issue is that the sum $\sum_{j=1}^{\infty} j^{-(d-1)/2}$ (cf. (4.3)) does not converge. Indeed, this technique, like many others in statistical physics, only works when we are in at least four dimensions. Finally, for $d = 4$ and $k = 3$ we could hope that a combination of the proof techniques of the theorem and some explicit computations can work, but we would need a sufficiently tight upper bound on $\mathbb{P}(\tau_4 = k)$ up to $k \approx 12$, which exceeds our available computing capacity.

Now we continue with the proof of Theorem 2.2.

***Proof of Theorem 2.2***. We use a randomized coupling approach that ensures that, if the origin is connected to infinity in the $d$-dimensional $k$-DnG, then the origin is also connected to infinity in the $(d + 1)$-dimensional $(k + 1)$-DnG.

For this, consider the following algorithm started at the origin $(o, o) \in \mathbb{Z}^d \times \mathbb{Z}$. Let $h_o \in \mathbb{N}_0$ be the smallest integer such that the edge $((o, h_o), (o, h_o + 1))$ is closed. Note that, since $k \le 2d$, we have $k + 1 < 2(d + 1)$, so that almost surely such an $h_o$ exists. Now, note that $(o, h_o)$ has

https://doi.org/10.1017/jpr.2025.15 Published online by Cambridge University Press

at least $k$ open directed edges towards neighbors in $\mathbb{Z}^d \times \{h_o\}$. In the case that there are even $k+1$ such edges, we sample $k$ of the associated neighbors uniformly and call them $x_1^o, \dots, x_k^o$. Next, for each of the neighbors $x_i^o$, we again find the first closed edge $((x_i^o, h_{x_i^o}), (x_i^o, h_{x_i^o}+1))$ and proceed as before. We iterate this algorithm further, but only for those nodes that have a $\mathbb{Z}^d$-component that has not been visited in previous steps. For those nodes with a $\mathbb{Z}^d$-component that is not visited at all, we simply draw $k$ of their neighbors in $\mathbb{Z}^d$ uniformly at random.

Hence, by symmetry and independence, the resulting configuration of open directed edges in $\mathbb{Z}^d$ follows the distribution of the $d$-dimensional $k$-DnG. Second, if there exists an open directed path connecting the origin to infinity in a configuration in $\mathbb{Z}^d$, necessarily, in any pre-image of that configuration in $\mathbb{Z}^{d+1}$, the origin is also connected to infinity via a directed open path. This gives the desired result. □

### 4.2. Proofs for the *k*-UnG

*Proof of Theorem 2.3.* We prove the positive percolation probability in the 2-UnG via a dual approach. That is, let $\mathbb{Z}'^2 := \left\{x + \left(\frac{1}{2}, \frac{1}{2}\right) : x \in \mathbb{Z}^2\right\}$ be the two-dimensional dual lattice of $\mathbb{Z}^2$. An edge $e' \in \mathbb{Z}'^2$ has exactly one edge $e \in \mathbb{Z}^2$ that crosses $e$', and we say the edge $e$' is open if and only if the edge $e$ is open. For the 2-UnG model in two dimensions it thus follows that, for any such edge $e' \in \mathbb{Z}'^2$, $\mathbb{P}(e' \text{ is closed}) = 1 - \mathbb{P}(e \text{ is open}) = 1 - \frac{3}{4} = \frac{1}{4}$. Moreover, the negative correlations between edges that are present in the 2-UnG model also appear in its dual. Namely, whenever, for edges $e_1', e_2' \in \mathbb{Z}'^2$, their unique associated edges $e_1, e_2 \in \mathbb{Z}^2$ are neighbors, then $\mathbb{P}(e_1', e_2' \text{ closed}) = \frac{1}{24} \le \frac{1}{16} = \mathbb{P}(e_1' \text{ closed})^2$. Otherwise, the status of the two dual edges is independent. This can be extended so that, for any $n \ge 2$ and any self-avoiding path $(e_1', \dots, e_n')$ of nearest-neighbor vertices in $\mathbb{Z}'^2$ with $\mathbb{P}(e_1', \dots, e_n' \text{ closed}) > 0$,

$$\mathbb{P}(e_1', \dots, e_n' \text{ closed}) = \prod_{i=0}^{n-1} \mathbb{P}(e_{i+1}' \text{ closed} \mid e_1', \dots, e_i' \text{ closed}) \le 4^{-n}.$$

As such, we can use the negative correlation to bound the probability that there exists a closed circuit in the dual that surrounds the origin. For any $n \ge 4$, the number of such circuits of length $n$ is at most $nc_{n-1}(2)$, where we recall that $c_n(2)$ denotes the number of self-avoiding paths of length $n$ in $\mathbb{Z}^2$. As $c_n(2) = (c(2) + o(1))^{n-1}$ by the definition of $c_n$ and $c$ (cf. Section 2.3), we thus arrive at

$$\begin{aligned} 1 - \theta^{\mathrm{U}}(2,2) &= \mathbb{P}(\text{there exists a closed circuit } \gamma \text{ in } \mathbb{Z}'^2 \text{ that surrounds the origin}) \\ &\le \sum_{n \ge 4} \sum_{\text{circuits } \gamma : |\gamma| = n} \mathbb{P}(\gamma \text{ closed in } \mathbb{Z}'^2) \le \sum_{n \ge 4} n(c(2) + o(1))^{n-1} 4^{-n}, \end{aligned} \tag{4.4}$$

which is finite, since $c(2) \le 3$. We now apply an argument from [18] (see also the references therein) to show that this probability is in fact smaller than one 1, which yields $\theta^{\mathrm{U}}(2,2) > 0$. Let $D_m = \{0, \dots, m\}^2$ and say that a vertex $i \in \mathbb{Z}^2$ is 'wet' when there exists a $j \in D_m$ such that $i \to j$, that is, when there exists a path of open edges from $i$ to $j$ in the primary lattice. Suppose the component of the origin is finite almost surely. Then the number of wet vertices is finite almost surely as well, and there exists a circuit of closed edges in the dual that surrounds all the wet sites. By the same argument as in (4.4),

$$\mathbb{P}(D_m \not\to \infty) \le \sum_{n \ge 4m} \sum_{\text{circuits } \gamma : |\gamma| = n} \mathbb{P}(\gamma \text{ closed in } \mathbb{Z}'^2) \le \sum_{n \ge 4m} n(c(2) + o(1)) 4^{-n},$$

https://doi.org/10.1017/jpr.2025.15 Published online by Cambridge University Press

where we can start the outer sum from $4m$ since a circuit that surrounds all wet vertices must surround $D_m$ and hence have length $4m$ at least. As a result, since the sum is finite for each $m \in \mathbb{N}$, choosing $m$ large enough yields that $\mathbb{P}(D_m \to \infty) > 0$ and hence $\mathbb{P}(i \to \infty) > 0$ for some $i \in D_m$. By translation invariance, it then follows that $\theta^{\mathrm{U}}(2, 2) > 0$, as desired.

To prove that $\theta^{\mathrm{U}}(3, 3) > 0$, it suffices to show that with positive probability $o$ is connected to infinity via any two-dimensional plane $S$ of $\mathbb{Z}^3$ including $o$. This can be done via the dual approach employed in the first part of the proof. Namely, each edge $e$ of the 3-UnG is open with probability $\mathbb{P}(e \text{ open}) = 1 - \mathbb{P}(e \text{ closed}) = 1 - \left(\binom{5}{3}/\binom{6}{3}\right)^2 = \frac{3}{4}$, which applies in particular to all edges contained in the restriction of the 3-UnG to $S$. Thus, for any edge $e'$ of the dual lattice corresponding to $S$, we have $\mathbb{P}(e' \text{ closed}) = 1/4$. Then, for any two dual edges for which their associated non-duals are not nearest neighbors, their status is independent, and otherwise,

$$\mathbb{P}(e_1', e_2' \text{ closed}) = \binom{5}{3}\binom{5}{3}\binom{4}{3}\Big/\binom{6}{3}^3 = \frac{1}{20} \le \frac{1}{16} = \mathbb{P}(e_1' \text{ closed})^2,$$

whence the proof can be finished analogously to the one for the 2-UnG. □

*Proof of Proposition* 2.1. Assume for a contradiction that we do have $\theta^{\mathrm{U}}(1, d) > 0$ for some $d \in \mathbb{N}$, and let $A$ then denote the event of positive probability that there exists an infinite path starting from $o$ consisting of edges of the 1-UnG. Let $o = X_0, X_1, X_2, \ldots$ be any infinite path of the 1-UnG starting from $o$. Now, if $X_0, X_1, X_2$ is the sequence of vertices in an infinite path (on the event $A$), we put $K = \inf\{k \ge 0 \colon (X_k, X_{k+1}) \text{ is not an edge of the 1-DnG}\}$. We know from the proof of Remark 2.1(i) that $K < \infty$ almost surely. Next, let us define

$$L = \inf\{k > K \colon (X_k, X_{k+1}) \text{ is an edge of the 1-DnG}\}.$$

Now we claim that $L = \infty$ on the event $\{K < \infty\}$ (where we put $K = \infty$ on the event $A^{\mathrm{c}}$). Indeed, for all $k = K, K+1, \ldots, L-1$, $(X_{k+1}, X_k)$ is an edge of the 1-DnG because it follows from the definition of $K$ that $(X_k, X_{k+1})$ is not an edge of the 1-DnG, but $(X_k, X_{k+1})$ is an edge of the 1-UnG, and this is only possible if $(X_{k+1}, X_k)$ is an edge of the 1-DnG. But now, if $L < \infty$, then $(X_L, X_{L-1})$ is an edge of the 1-DnG, so that $(X_L, X_{L+1})$ cannot be an edge of the 1-DnG, since there is only one edge going out of $X_L$. This implies the claim.

Let us denote the $\ell_1$-sphere of radius $n \in \mathbb{N}_0$ by $S_n^{(1)}$ (with $S_0^{(1)} = \{o\}$). Note that $\mathbb{P}(\kappa < \infty \mid A) = 1$, where $\kappa = \inf\{k \ge 0 \colon o \not\to S_k^{(1)} \text{in the 1-DnG}\}$ represents the event that there exists no path along open (directed) edges from the origin to some vertex in $S_k^{(1)}$. Since $o \in S_0^{(1)}$ and $o$ is always connected to one of its nearest neighbors by a directed edge, we have $\mathbb{P}(\kappa \ge 2) = 1$. Given that $\kappa$ is an $\mathbb{N}_0 \cup \{\infty\}$-valued random variable, we can find $k_0 \in \mathbb{N} \setminus \{1\}$ and $\varepsilon > 0$ such that $\mathbb{P}(\kappa = k_0 \mid A) \ge \varepsilon$. Now, by our previous observation that $L = \infty$ on $\{K < \infty\}$, since from $o$ to $S_{\kappa-1}^{(1)}$ there always exists a directed path in the 1-DnG, on the event $\{\kappa = k_0\} \cap A$, there must exist a self-avoiding directed path of length $n$ in the 1-DnG ending at some vertex of $S_{k_0-1}^{(1)}$ for all $n \in \mathbb{N}_0$.

Thus, writing $c_{n,k_0-1}(d)$ for the number of self-avoiding paths in the $d$-dimensional lattice of length $n$ starting from (or ending at) $S_{k_0-1}^{(1)}$, it is clear that

$$\lim_{n\to\infty} c_{n,k_0-1}(d)^{1/n} = \lim_{n\to\infty} c_{n,1}(d)^{1/n} = c(d) \le 2d - 1,$$

https://doi.org/10.1017/jpr.2025.15 Published online by Cambridge University Press

where $c(d)$ is the connective constant of $\mathbb{Z}^d$. Now, any self-avoiding path of length $n$ ending at $S^{(1)}_{k_0-1}$ is open with probability $(2d)^{-n}$. Hence,

$$\mathbb{P}\big(\text{there exists a self-avoiding path of length } n \text{ starting from } \mathrm{S}^{(1)}_{\mathrm{k}_0-1} \text{included in 1-DnG}\big) \leq \left(\frac{2d-1}{2d}\right)^n,$$

which decays exponentially fast in $n$. This contradicts the assertion that the left-hand side is bounded from below by $\mathbb{P}(A)\varepsilon > 0$ independently of $n$. Therefore, $\mathbb{P}(A) = 0$, as desired. □

### 4.3. Proofs for the *k*-BnG

*Proof of Lemma* 2.1. Note first that the system is negatively correlated in the sense that, for all directed edges $\vec{\ell}_1 = (x_1, x_2) \neq \vec{\ell}_2 = (y_1, y_2)$, $\mathbb{P}(\vec{\ell}_1 \text{ open and } \vec{\ell}_2 \text{ open}) \leq \mathbb{P}(\vec{\ell}_1 \text{ open})^2$. Indeed, if $x_1 \neq y_1$, then the inequality is an equality by independence. However, if $x_1 = y_1$ and $x_2 \neq y_2$, then $\mathbb{P}(\vec{\ell}_1 \text{ open} \mid \vec{\ell}_2 \text{ open}) = (k-1)/(2d-1) \leq k/(2d) = \mathbb{P}(\vec{\ell}_1 \text{ open})$. Note that the negative correlation also carries over to the bidirectional case since, for $\ell_1, \ell_2$ undirected with $x_1 = y_1$ and $x_2 \neq y_2$, $\mathbb{P}(\ell_1 \text{ open} \mid \ell_2 \text{ open}) = k(k-1)/(2d(2d-1)) \leq k^2/(2d)^2 = \mathbb{P}(\ell_1 \text{ open})$. Let us denote by $\ell_n$ the straight line starting at the origin up to the node $(ne_1)$. Then, using the first moment method and negative correlations, for all $n \geq 1$,

$$\begin{aligned}&\mathbb{P}(o \leftrightsquigarrow \infty \text{ in the } k\text{-BnG on } \mathbb{Z}^{\mathrm{d}})\\ &\leq \mathbb{P}(\text{there exists a self-avoiding open path starting at } o \text{ of length } n)\\ &\leq c_n(d)^n \mathbb{P}(\ell_n \text{ is open}).\end{aligned}$$

Finally, for a suitable constant $C > 0$, $\mathbb{P}(\ell_n \text{ is open}) \leq Cp^n$, where $p = k(k-1)/(2d(2d-1))$ is the probability that the origin samples two fixed edges to be open. This leads to the criterion for absence of percolation $c(d)p < 1$. □

*Proof of Proposition 2.2.* As in the proof of the assertion $\theta^{\mathrm{U}}(3, 3) > 0$ of Theorem 2.3, it suffices to verify that there is an infinite path starting from $o$ in the restriction of the $k$-BnG to a fixed two-dimensional plane $S$ including $o$ with positive probability. Using the duality approach of the proof of Theorem 2.3, dual edges of the two-dimensional lattice of $S$ are open with probability $(k/(2d))^2$ (which is the probability that the corresponding edge of the original graph is open in the bidirectional sense). Further, the indicators of two dual edges being open, i.e. the corresponding edges of the $k$-BnG restricted to $S$ being open, are non-positively correlated. Indeed, we saw already in the proof of Lemma 2.1 that the indicators of two edges of the $k$-BnG being open are either independent or strictly negatively correlated. Now, if the indicators of two events are independent or strictly negatively correlated, then so are the indicators of the complements of the two events. We see from the proof of Theorem 2.3 that, given these non-positive correlations, it suffices to choose $k$ and $d$ in such a way that a dual edge is closed with probability at most $1/c(2)$. This holds whenever $(k/(2d))^2 > 1 - 1/c(2)$ or, equivalently, $k > d\sqrt{4(1 - 1/c(2))}$, as asserted. □

**Acknowledgements**

The authors very much thank two anonymous referees for their valuable remarks that helped to substantially improve the manuscript. Additionally, the authors thank D. Mitsche, G. Pete, and B. Ráth for interesting discussions and comments.

https://doi.org/10.1017/jpr.2025.15 Published online by Cambridge University Press

## Funding information

BJ and JK acknowledge the financial support of the Leibniz Association within the Leibniz Junior Research Group on *Probabilistic Methods for Dynamic Communication Networks* as part of the Leibniz Competition. BL was supported by the grant GrHyDy ANR-20-CE40-0002. AT was partially supported by the ERC Consolidator Grant 772466 'NOISE'. Two further funding acknowledgements by AT: this paper was supported by the János Bolyai Research Scholarship of the Hungarian Academy of Sciences; project no. STARTING 149835 has been implemented with the support provided by the Ministry of Culture and Innovation of Hungary from the National Research, Development and Innovation Fund, financed under the STARTING_24 funding scheme.

## Competing interests

There were no competing interests to declare which arose during the preparation or publication process of this article.

## References

[1] AMIR, G., HEYDENREICH, M. AND HIRSCH, C. (2024). Planar reinforced $k$-out percolation. Preprint, arXiv:2407.12484.
[2] BALISTER, P. AND BOLLOBÁS, B. (2013). Percolation in the $k$-nearest neighbor graph. In *Recent Results in Designs and Graphs: A Tribute to Lucia Gionfriddo, Quaderni di Matematica*, eds. M. Buratti, C. Lindner, F. Mazzocca and N. Melone. Vol. 28, pp. 83–100.
[3] BALISTER, P., JOHNSTON, T., SAVERY, M. AND SCOTT, A. (2025). Improved bounds for 1-independent percolation on $\mathbb{Z}^n$. *Electron. J. Prob.* **30**, 1–26.
[4] COUPIER, D., HENRY, B., JAHNEL, B. AND KÖPPL, J. (2024). The planar lattice two-neighbor graph percolates. Preprint, arXiv:2412.20781.
[5] COX, T. AND DURRETT, R. (1983). Oriented percolation in dimensions $d \geq 4$: Bounds and asymptotic formulas. *Math. Proc. Camb. Phil. Soc.* **93**, 151–162.
[6] GRIMMETT, G. (2006). *The Random-Cluster Model*. Springer, New York.
[7] HÄGGSTRÖM, O. AND MEESTER, R. (1996). Nearest neighbor and hard sphere models in continuum percolation. *Random Structures Algorithms* **9**, 295–315.
[8] HARA, T., SLADE, G. AND SOKAL, A. D. (1993). New lower bounds on the self-avoiding-walk connective constant. *J. Statist. Phys.* **72**, 479–517.
[9] HIRSCH, C., HOLMES, M. AND KLEPTSYN, V. (2021). Absence of warm percolation in the very strong reinforcement regime. *Ann. Appl. Prob.* **31**, 199–217.
[10] HOLMES, M. AND SALISBURY, T. S. (2014). Degenerate random environments. *Random Structures Algorithms* **45**, 111–137.
[11] HOLMES, M. AND SALISBURY, T. S. (2016). Forward clusters for degenerate random environments. *Combinatorics Prob. Comput.* **25**, 744–765.
[12] HOLMES, M. AND SALISBURY, T. S. (2021). Phase transitions for degenerate random environments. *Lat. Am. J. Prob. Math. Statist.* **18**, 707–725.
[13] HOLMES, M. AND SALISBURY, T. S. (2021). A shape theorem for the orthant model. *Ann. Prob.* **49**, 1237–1256.
[14] HOLMES, M. AND SALISBURY, T. S. (2025). Percolation of terraces, and enhancements for the orthant model. *Trans. Amer. Math. Soc. Ser. B* **12**, 237–275.
[15] JAHNEL, B. AND TÓBIÁS, A. (2022). Absence of percolation in graphs based on stationary point processes with degrees bounded by two. *Random Structures Algorithms* **62**, 240–255.
[16] LYONS, R. AND PERES, Y. (2017). *Probability on Trees and Networks*. Cambridge University Press.
[17] PÖNITZ, A. AND TITTMANN, P. (2000). Improved upper bounds for self-avoiding walks in $\mathbb{Z}^d$. *Electron. J. Combinatorics* **7**, R21–R21.
[18] SWART, J. (2017). A course in interacting particle systems 2017. Preprint, arXiv:1703.10007.

https://doi.org/10.1017/jpr.2025.15 Published online by Cambridge University Press