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STRUCTURAL STABILITY OF THE GENERALIZED TAYLOR RULE

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This paper analyzes the dynamical properties of monetary models with regime switching. We start with the analysis of the evolution of inflation when policy is guided by a simple monetary rule where coefficients switch with the policy regime. We rule out the possibility of a Hopf bifurcation and demonstrate the possibility of a period-doubling bifurcation. As a result, a small change in the parameters (e.g., a more active policy response) can lead to a drastic change in the path of inflation. We show that the New Keynesian model with a current-looking Taylor rule is not prone to bifurcations. A New Keynesian model with a hybrid rule, however, exhibits the same pattern of period-doubling bifurcations as the analysis with a simple monetary rule.

Keywords: New Keynesian, Taylor Rule, Regime Switching, Bifurcation Analysis, Structural Stability

1. INTRODUCTION

Monetary policy has seen drastic changes over the past decades. The 1970s were plagued by high inflation along with slow economic growth, while the Central bank stayed relatively passive in their actions. With the appointment of Volcker, the Federal Reserve shifted to a more active regime that helped to combat high inflation rates present at the start of the 1980s. The following period of moderate inflation along with stable economic growth has been coined the Great Moderation. With the Great Recession as a result of the financial crises starting in 2007, the Fed had to move aggressively.

This paper studies the dynamical behavior of standard macroeconomic models where the monetary policy regime can switch over time. More specifically, the policy regime follows a Markov chain with a fixed transition matrix. We are interested in the qualitative behavior of the solution not only for one particular calibration of the model but rather the change in the qualitative behavior of the solution in response to changes in the parameterization.

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The standard New Keynesian model is known to display bifurcations, i.e., changes in the dynamical evolution of the variables of interest in response to changes in parameters [see Barnett and Duzhak (2008), Barnett and Duzhak (2010)]. The present paper investigates whether bifurcations are possible when regimes can switch over time. The answer is a-priori not clear. The solution to the model, in the simplest case inflation, evolves differently depending on the state the regime is in. If the standard New Keynesian model displays a bifurcation for a certain region of the parameter space, then the regime switching model would visit this parameter combination only for one of the policy regimes, while the occasional switch to another policy regime can stabilize the solution.

In the first section, we study a basic setup with exogenous real interest rates. The monetary authority sets the nominal interest rate as a function of current inflation. However, the response coefficient varies depending on the policy regime present at the time. The Fisher equation that links the nominal interest rate to future inflation and the real interest rate provides the second relationship. Combining the two, we get an equation that relates future inflation to current inflation and the real interest rate. Taking the latter as given, we get a system of two linear difference equations for inflation in the two regimes.

To perform bifurcation analysis, we study the matrix that governs the evolution of current inflation to future inflation. The relevant properties of this matrix are the sign and magnitude of the eigenvalues. Therefore, we set up and solve the characteristic polynomial.

The solution demonstrates two main findings with respect to bifurcations. First, for the basic setup, there is no possibility of a Hopf bifurcation which would lead to a change from a stable to unstable solution (or vice versa) in response to a (potentially tiny) change in the parameter values. Second, we show the possibility of a period-doubling (PD) bifurcation. In this case, the solution can move to a periodic solution where the periodicity doubles.

Next, we ask whether our analysis of this simple setup carries over to the standard New Keynesian model. There, the monetary policy rule is more involved. The so-called Taylor rule [see Taylor (1999)] has two components, which makes the nominal interest rate a function of both inflation and the output gap. This extra component can lead the solution to become less prone to changes in its characteristics.

Due to the increased dimensionality for the New Keynesian economy as opposed to the basic setup, we move to numerical methods. There, we start with a single bifurcation point and trace out the entire bifurcation boundary through the parameter space, i.e., the parameter combinations at which a bifurcation occurs.

We find that the standard New Keynesian model with regime switching and a standard Taylor rule does not exhibit any bifurcations for the range of feasible parameter combinations. While we do find a bifurcation boundary, it lies outside the relevant range of parameter values since it points to negative coefficients where standard economics tells us that they should be positive.

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Last, we investigate whether a state-of-the-art hybrid Taylor rule exhibits any bifurcations. We solve the same baseline New Keynesian model but use a Taylor rule that allows for forward-looking response to inflation. After going through the same solution steps as in the previous case, we find that this model might exhibit a PD bifurcation. The ideas from the basic setup thus carry over to the most prominent model of monetary policy.

Our paper relates to several strands of the literature. First, we use the New Keynesian model with regime switching and study its properties. The original New Keynesian model has been developed into an important tool for monetary policy [see Gali and Gertler (1999), Bernanke et al. (1999), Leeper and Sims (1994)]. Andrews (1993) and Evans (1985) study monetary policy with parameter instability. Davig and Leeper (2006) and Farmer et al. (2007) study determinacy when the Taylor rule is generalized to allow for regime switching. There is a literature on methods to determine parameter instability in time series [see Hansen (1992), Nyblom (1989)]. Economic models of regime switching had then been investigated previously in different contexts, see, e.g., Hamilton (1989) and Warne (2000). Clarida et al. (1999), Sims and Zha (2006), and Groen and Mumtaz (2008) find empirical support for regime switches in monetary policy.

Second, we relate and make use of a large literature on bifurcation analysis. Seydel (1994) provides an overview on theoretical insights and applications with respect to bifurcations. Benhabib and Nishimura (1979) is an early example of the use of bifurcations in economics. More recently, Barnett and He study various forms of economic settings [see Barnett and He (1999), Barnett and He (2001), Barnett and He (2002), Barnett and He (2004), Barnett and He (2006)]. In related work, we investigate bifurcation properties of New Keynesian models [see Barnett and Duzhak (2008), Barnett and Duzhak (2010)].

The rest of the paper is organized as follows. Section 2 discusses a simple version with exogenous real interest rates where nominal inflation is determined by a policy rule. Section 3 discusses the classic New Keynesian model with regime switching and a current-looking Taylor rule. Section 4 studies the same New Keynesian model with a hybrid Taylor rule where we pay attention to the forward-looking aspect of policy. Section 5 concludes.

2. DYNAMICS WITH A SIMPLE MONETARY POLICY RULE

Before studying a complete New Keynesian macroeconomic model, we study the dynamical properties of the monetary policy rule. A central banker is implementing a policy where he reacts to inflation by changing an interest rate according to

$$i_t = \alpha(s_t)\pi_t, \tag{1}$$

where i_t is the nominal interest rate, $\alpha(s_t)$ a state-dependent coefficient which changes with the policy regime s_t, and π_t denotes the rate of inflation.

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We assume that there are two possible realizations for the policy regime s_t. The policy regime determines the reaction to inflation when setting the nominal interest rate. This linear reaction function to inflation evolves stochastically between two states—$s_t = 1$ and $s_t = 2$, so that

$$\alpha(s_t) = \begin{cases} \alpha_1 & \text{for } s_t = 1 \\ \alpha_2 & \text{for } s_t = 2 \end{cases},$$

where α_i denotes different parameters that govern the aggressiveness of policy to combat inflation. As usual, an active policy regime is the one when policy parameter $\alpha_i > 1$.

The policy regime evolves according to a Markov chain where the transitional probabilities are given by the transition matrix with entries $p_{ij} = P[s_t = j | s_{t-1} = i]$, where $i, j = 1, 2$. Following Davig and Leeper (2006), we study the dynamics of this simple monetary policy rule by using the Fisher equation

$$i_t = E_t \pi_{t+1} + r_t, \tag{2}$$

where r_t is the real interest rate.

The Fisher equation links the nominal interest rate to expected inflation and the real interest rate. We use this relationship to solve for expected inflation that evolves as a function of the nominal and real interest rates. Plugging in for the policy rule where the nominal interest rate is a function of inflation, we end up with the following dynamic system:

$$\begin{bmatrix} E_t[\pi_{1t+1}] \\ E_t[\pi_{2t+1}] \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}^{-1} \begin{bmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{bmatrix} \begin{bmatrix} \pi_{1t} \\ \pi_{2t} \end{bmatrix} - \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}^{-1} \begin{bmatrix} r_t \\ r_t \end{bmatrix}.$$

In this system, we take the real interest rate r_t as exogenously given. A fully specified macroeconomic model, which we study below, endogenizes this rate.

We now analyze this system of linear difference equations. Therefore, we replace the matrix multiplying the vector of inflation by explicitly computing the inverse of the transition matrix

$$\begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}^{-1} \begin{bmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{bmatrix} = \frac{1}{\Delta} \begin{bmatrix} p_{22} & -p_{12} \\ -p_{21} & p_{11} \end{bmatrix} \begin{bmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{bmatrix} = \begin{bmatrix} \frac{p_{22}\alpha}{\Delta} & \frac{-p_{12}\alpha_2}{\Delta} \\ \frac{-p_{21}\alpha_1}{\Delta} & \frac{p_{11}\alpha_2}{\Delta} \end{bmatrix},$$

where Δ denotes the determinant of the transition matrix.

As is standard in the (bifurcation) analysis of difference equations, we study the economy with parameter certainty. Parameter certainty means that agents have no uncertainty about the level of inflation if a certain state occurs. This does not mean that agents know the level of inflation in the following period: The state of the policy regime determines inflation and the state of the policy regime itself switches with given probabilities. Using parameter certainty, we can replace the expected level of inflation conditional on a state occurring by the level of inflation.

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Putting parameter certainty and the adjustments to the matrix and its determinant into our equation, we can restate the system of linear difference equations as

$$\begin{bmatrix} \pi_{1t+1} \\ \pi_{2t+1} \end{bmatrix} = \begin{bmatrix} \frac{p_{22}\alpha}{p_{11}p_{22}-p_{12}p_{21}} & \frac{-p_{12}\alpha_2}{p_{11}p_{22}-p_{12}p_{21}} \\ \frac{-p_{21}\alpha_1}{p_{11}p_{22}-p_{12}p_{21}} & \frac{p_{11}\alpha_2}{p_{11}p_{22}-p_{12}p_{21}} \end{bmatrix} \begin{bmatrix} \pi_{1t} \\ \pi_{2t} \end{bmatrix} - \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}^{-1} \begin{bmatrix} r_t \\ r_t \end{bmatrix}.$$

Since the entries in the transition matrix are probabilities, we know that $p_{11} + p_{21} = 1$ and $p_{22} + p_{12} = 1$. Hence, we can express $\Delta = p_{11}p_{22} - p_{12}p_{21}$ as $\Delta = p_{11} + p_{22} - 1$.

The key component of the dynamical system is the coefficient on current inflation. This Jacobian of the linear difference equation captures the evolution of expected inflation and thus the link between real and nominal interest rates. Given our calculations for the matrix on inflation and the determinant of the transition matrix above, the Jacobian of the linear difference equation is given by

$$J = \begin{bmatrix} \frac{p_{22}\alpha_1}{p_{11}+p_{22}-1} & \frac{-p_{12}\alpha_2}{p_{11}+p_{22}-1} \\ \frac{-p_{21}\alpha_1}{p_{11}+p_{22}-1} & \frac{p_{11}\alpha_2}{p_{11}+p_{22}-1} \end{bmatrix}.$$

To analyze the stability of the evolution of inflation and its dynamic properties, we compute the eigenvalues for the Jacobian matrix. Therefore, we compute the characteristic polynomial $P(\lambda)$, which is quadratic in this case given by

$$P(\lambda) = \lambda^2 - b\lambda + c, \tag{3}$$

where the coefficients are

$$b = \frac{p_{22}\alpha_1 + p_{11}\alpha_2}{p_{11} + p_{22} - 1} \text{ and } c = \frac{\alpha_1\alpha_2}{p_{11} + p_{22} - 1}.$$

The nature of the solution to quadratic equations is mainly determined by the discriminant of the square root that appears in the formula. For the dynamics of inflation, the determinant D is given by

$$D = \left(\frac{p_{22}\alpha_1 + p_{11}\alpha_2}{p_{11} + p_{22} - 1}\right)^2 - \frac{4\alpha_1\alpha_2}{p_{11} + p_{22} - 1}. \tag{4}$$

A negative discriminant D gives rise to complex roots, whereas a positive discriminant leads to real roots.

We are interested in the quality of the dynamics and whether bifurcation exists, i.e., whether the quality of the solution can change drastically despite a negligible change in the parameter values. For a Hopf bifurcation [Hopf (1942)] to exist, the discriminant D must be negative, giving a rise to complex roots of the characteristic polynomial. To check whether this is true, we need to solve for

$$D = \frac{(p_{22}\alpha_1 + p_{11}\alpha_2)^2}{(p_{11} + p_{22} - 1)^2} - \frac{(p_{11} + p_{22} - 1)4\alpha_1\alpha_2}{(p_{11} + p_{22} - 1)^2} < 0.$$

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Given that $(p_{11} + p_{22} - 1)^2$ is always nonnegative, we can simplify the inequality to

$$(p_{22}\alpha_1 + p_{11}\alpha_2)^2 - (p_{11} + p_{22} - 1)4\alpha_1\alpha_2 < 0.$$

The term on the left-hand side stays positive within the feasible set of parameters. Therefore, a Hopf bifurcation that arises only when the roots are complex, is not possible for this economy.

However, we can check the possibility of a PD bifurcation. This type of bifurcation occurs when the root equals negative one and it leads to the doubling of the periodicity of the dynamic solution. The following Lemma provides conditions for the existence of the PD bifurcation [see Kuznetsov (1998, p. 415)].

LEMMA 1 (Period-doubling bifurcation). *Suppose that a one-dimensional system*

$$x \mapsto f(x, \alpha),\ x \in \Re^1,\ \alpha \in \Re^2,$$

with f smooth, so that at $\alpha = 0$ the fixed point $x = 0$, and let the PD bifurcation conditions hold:

$$\mu = f_x(0, 0) = -1, c = \frac{1}{4}[f_{xx}(0, 0)]^2 + \frac{1}{6}(0, 0) = 0.$$

Assume that the following genericity conditions are satisfied:

PD.1 $D(0) = \left(\frac{1}{5}f_{x^5} + \frac{1}{2}f_x f_{x^4} - [f_x]^4\right)(0, 0) = 0$;

PD.2 *the map $\alpha \mapsto (\mu(\alpha) + 1, c(\alpha))^T$ is regular at $\alpha = 0$, where $c(\alpha)$ is given by*

$$c(\alpha) = b(\alpha) + \frac{2a^2(\alpha)}{\mu^2(\alpha) - \mu(\alpha)}.$$

Then there are smooth invertible coordinate and parameter changes transforming the system into

$$\eta \mapsto -(1 + \beta_1)\eta + \beta_2\eta^3 + s\eta^5 + O(\eta^6), where\ s = sign\,[D(0)] = \pm 1.$$

This system without $O(\eta^6)$ terms is called the truncated normal form for the PD bifurcation.

For our model, both conditions for the PD bifurcation hold. To find the combination of parameters that make the variable μ from Lemma 1 equal to negative one, we analyze the eigenvalues of the characteristic polynomial. The characteristic polynomial $P(\lambda)$ has the following roots:

$$\lambda_{1,2} = \frac{1}{2}\left(\frac{\alpha_1 p_{22} + \alpha_2 p_{11}}{p_{11} + p_{22} - 1} \pm \sqrt{D}\right), \tag{5}$$

where D is the discriminant defined above.

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TABLE 1. Standard parameter combination used throughout the paper

Parameter	Value
α_1	1.5
α_2	0.5
γ_1	0.3
γ_2	0.15
p_{11}	0.85
p_{22}	0.9
β	0.98
κ	0.024
σ	0.3

If one of these roots is in the negative part of the unit circle, then there is a possibility of a PD bifurcation, given that the nondegeneracy conditions are satisfied.

From the equation for the roots above, we get $\mu = -1$ whenever one of the roots $\lambda_{1,2} = \frac{1}{2}\left(\frac{\alpha_1 p_{22}+\alpha_2 p_{11}}{p_{11}+p_{22}-1} \pm \sqrt{\frac{(\alpha_1 p_{22}+\alpha_2 p_{11})^2}{(p_{11}+p_{22}-1)^2} - \frac{4\alpha_1\alpha_2}{p_{11}+p_{22}-1}}\right)$ equals -1. As a result, we can rearrange the expression to produce the condition

$$\sqrt{(\alpha_1 p_{22} + \alpha_2 p_{11})^2 - 4\alpha_1\alpha_2(p_{11} + p_{22} - 1)} = 2(p_{11}+p_{22}-1)+(\alpha_1 p_{22}+\alpha_2 p_{11}) \quad (6)$$

that needs to hold for a PD bifurcation to occur.

Simplifying this expression gives us

$$(p_{11} + p_{22} - 1)[p_{11}(1 + \alpha_2) + p_{22}(1 + \alpha_1) + \alpha_1\alpha_2 - 1] = 0. \quad (7)$$

This equation can be described as a bifurcation boundary. It is a function of the parameters of the dynamical model.

The bifurcation boundary is the key object in our analysis of the dynamical system. The quality of the solution on either side of the boundary is very different. Thus, when the parameters of the dynamical system are close or at the bifurcation boundary, small changes to parameters can lead to entirely different behavior of the solution. We chose critical bifurcation parameter to be p_{22}^c.

First look at this equation shows that bifurcation is likely to occur whenever probabilities of staying at each state sum up to one. Thus, a 50% chance of changing a regime might result in a PD bifurcation.

To calibrate the economy, we use the values in Table 1. One of the policy regimes, regime 1, is active with a coefficient greater than 1, whereas regime 2 is a passive regime. The time preference factor β, reaction of inflation to the output gap κ, and the degree of relative risk aversion σ will only become relevant in the latter part of the paper.

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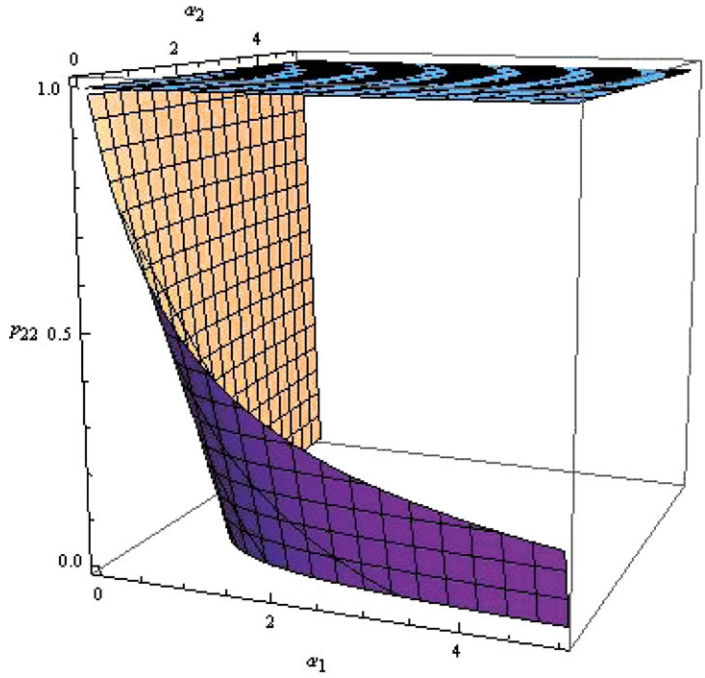


FIGURE 1. Bifurcation boundary for the case of a simple monetary policy rule keeping the probability $p_{11} = 0$.

We furthermore assume that the probability of staying in the active regime conditional on being in the active regime $p_{11} = 0$ is zero. Whenever regime 1 occurs, the economy will be sent to a passive regime with certainty.

Using these assumptions, we determine the critical value for the transitional probability p_{22} being $p_{22}^c = 0.1$. We use this point as a benchmark to trace out the bifurcation boundary. To obtain the entire bifurcation boundary, we vary the other parameters, i.e., policy parameters α_1 and α_2, along with the probability of staying in the passive regime p_{22}.

Consequently, we demonstrate a PD bifurcation boundary as a function of the three control parameters p_{22}, α_1, and α_2, which is shown in Figure 1. This figure illustrates a bifurcation boundary for the feasible set of parameters where transitional probability p_{22} is between 0 and 1 and parameters reflecting the reaction to inflation are greater than zero.

The graphs show the shape of the PD bifurcation boundary. First, if $p_{22} = 1$, then the policy regime would be passive and stay passive indefinitely. In this case, no bifurcation can arise and the bifurcation boundary converges to zero. Second, for the case of $p_{22} = 0$, the two policy regimes are identical in the likelihood with which they occur. The bifurcation boundary is thus symmetric along policy parameters α_1 and α_2.

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Interestingly, however, a bifurcation boundary exists for all probabilities between these two extreme cases. In particular, if the policy reaction coefficient α_2 of the passive regime is small, the policy response coefficient needs be very large for a bifurcation to arise. For a very aggressive policy in the active regime, the rate of inflation will start to evolve in cycles despite the simple nature of its equation of motion. In general, the higher the probability of staying in state one, the lower is the value of policy parameter α_2 on the bifurcation boundary. That is, the model becomes less stable when there is a low probability of switching from an active regime to a regime where policy response is increasingly passive.

3. NEW KEYNESIAN MODEL WITH REGIME SWITCHING

This section describes the well-known equations for the standard New Keynesian setup as laid out in, e.g., Woodford (2003) or Walsh (2003). The standard New Keynesian model traditionally consists of the forward-looking IS equation that describes the demand side of the economy

$$x_t = E_t x_{t+1} - \frac{1}{\sigma}(i_t - E_t \pi_{t+1}) + u_t^D, \tag{8}$$

and the Phillips curve which represents the supply side

$$\pi_t = \beta E_t \pi_{t+1} + \kappa x_t + u_t^S. \tag{9}$$

The IS curve relates the output gap x_t to the nominal interest rate i_t and expectations about the future output gap as well as inflation. The coefficient on the difference between the nominal interest rate and expected inflation, i.e., the coefficient on the real interest rate by the Fisher equation, is given by $1/\sigma$. This coefficient is the inverse of relative risk aversion, which equals the elasticity of intertemporal substitution since preferences with constant relative risk aversion are assumed to derive the equations. The New Keynesian Phillips curve describes how inflation is driven by the output gap and expected inflation. Both equations for demand and supply side allow for a shocks $u_t^{\cdot}$.

The remaining equation to close the economy is a rule for monetary policy that takes the form described in Taylor (1999). According to this Taylor rule, the monetary authority sets the nominal interest rate by targeting both inflation and the output gap

$$i_t = \alpha(s_t)\pi_t + \gamma(s_t)x_t, \tag{10}$$

where α_i governs the Central bank's reaction to inflation and γ_i the reaction to the output gap.

After plugging the Taylor rule into the IS equation, the model can be written in matrix notation.

$$\mathbf{A}\mathbf{Y}_{t+1} = \mathbf{B}\mathbf{Y}_t + \mathbf{u}_t,$$

where the $\mathbf{Y}$ denotes the vector of variables $\mathbf{Y} = [\pi_{1t} \quad \pi_{2t} \quad x_{1t} \quad x_{2t}]^T$ and $\mathbf{u}_t$ the vector of aggregate demand and supply shocks. The matrix multiplying next

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period's endogenous variables, inflation and output in either policy regime, is given by

$$\mathbf{A} = \begin{bmatrix} \beta\, p_{11} & \beta\,(1-p_{22}) & 0 & 0 \\ \beta\,(1-p_{11}) & \beta\, p_{22} & 0 & 0 \\ \frac{p_{11}}{\sigma} & \frac{1-p_{22}}{\sigma} & p_{11} & 1-p_{22} \\ \frac{1-p_{11}}{\sigma} & \frac{p_{22}}{\sigma} & 1-p_{11} & p_{22} \end{bmatrix}.$$

The matrix multiplying current period's variables is given by

$$\mathbf{B} = \begin{bmatrix} 1 & 0 & -\kappa & 0 \\ 0 & 1 & 0 & -\kappa \\ \frac{\alpha_1}{\sigma} & 0 & 1+\frac{\gamma_1}{\sigma} & 0 \\ 0 & \frac{\alpha_2}{\sigma} & 0 & 1+\frac{\gamma_2}{\sigma} \end{bmatrix}.$$

Premultiplying both sides of the equation by the inverse of the matrix $\mathbf{A}$, we obtain the normal form

$$\mathbf{Y}_{t+1} = \mathbf{C}\mathbf{Y}_t + \mathbf{A}^{-1}\mathbf{u}_t$$

which we used in the preceding section where $\mathbf{C} = \mathbf{A}^{-1}\mathbf{B}$.

After going through the steps of the preceding section where we compute the determinant of the matrix $\mathbf{A}$ and then compute the reaction matrix to current period variables, we get

$$\mathbf{C} = \begin{bmatrix} \frac{p_{22}}{(-1+p_{22}+p_{11})\beta} & \frac{-1+p_{22}}{(-1+p_{22}+p_{11})\beta} & -\frac{p_{22}\,k}{(-1+p_{22}+p_{11})\beta} & -\frac{(-1+p_{22})k}{(-1+p_{22}+p_{11})\beta} \\ \frac{-1+p_{11}}{(-1+p_{22}+p_{11})\beta} & \frac{p_{11}}{(-1+p_{22}+p_{11})\beta} & -\frac{(-1+p_{11})k}{(-1+p_{22}+p_{11})\beta} & -\frac{p_{11}\,k}{(-1+p_{22}+p_{11})\beta} \\ \frac{p_{22}\,(-1+\alpha_1\,\beta)}{\sigma\,\beta\,(-1+p_{22}+p_{11})} & \frac{1-p_{22}-\alpha_2\,\beta+\alpha_2\,\beta p_{22}}{\sigma\,\beta\,(-1+p_{22}+p_{11})} & \frac{p_{22}\,(k+\sigma\,\beta+\beta\gamma_1)}{\sigma\,\beta\,(-1+p_{22}+p_{11})} & \frac{(-1+p_{22})(k+\sigma\,\beta+\beta\,\gamma_2)}{\sigma\,\beta\,(-1+p_{22}+p_{11})} \\ \frac{1-p_{11}-\alpha_1\,\beta+\alpha_1\,\beta p_{11}}{\sigma\,\beta\,(-1+p_{22}+p_{11})} & \frac{p_{11}\,(-1+\alpha_2\,\beta)}{\sigma\,\beta\,(-1+p_{22}+p_{11})} & \frac{(-1+p_{11})(k+\sigma\,\beta+\beta\,\gamma_1)}{\sigma\,\beta\,(-1+p_{22}+p_{11})} & \frac{p_{11}\,(k+\sigma\beta+\beta\gamma_2)}{\sigma\,\beta\,(-1+p_{22}+p_{11})} \end{bmatrix}.$$

As before, matrix $\mathbf{C}$ is the key element when performing bifurcation analysis of the model with a generalized Taylor rule.

Given the standard calibration provided in Table 1, we search the parameter space of the model for possible bifurcations. In order to locate bifurcation values, we need to choose our free parameter. Parameters that describe the probabilities of regime changes are held constant, while structural and policy parameters are varied.

At this stage, however, we need to deviate from the path we took for the basic model of the preceding section. While we dealt with a two-by-two matrix in the simple setup, we now have a four-dimensional dynamical system. As a result,

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the computation of the characteristic polynomial and its solution becomes more involved.

Therefore, we move to a numerical method to study our dynamical system. To perform bifurcation analysis and search for PD and Hopf bifurcations, we employ the software continuation package CONTENT. This dynamical system software, developed by Yuri Kuznetsov and V. V. Levitin, traces out bifurcation boundaries for large dynamical systems.

Due to the increased dimensionality of the system, we have four eigenvalues for the Jacobian matrix where we have two pairs of complex-conjugate eigenvalues. Either pair has the potential to display the bifurcation patterns we discussed previously. Therefore, we could potentially see a Hopf bifurcation or a PD bifurcation but also the combination of the two. Then, for example, the output gap could display a qualitatively different pattern in its evolution from inflation.

When using the software, we can show that none of the possible bifurcations can occur for any feasible set of parameters. Hence, there is neither a possibility of a Hopf, nor a possibility of a PD bifurcation, nor the combination thereof for feasible parameters. We do, however, find the bifurcation boundary. A PD-Hopf bifurcation occurs for parameter values $\gamma_2 = 0.179$ and $\kappa = -0.46$. However, κ is determined by time preference and price rigidity which both lie in the unit interval. For any combination of those parameters, the implied κ is positive. Hence, finding a bifurcation at a negative value for κ has no economic significance. After tracing out the entire bifurcation boundary, it never crosses into the subspace of feasible parameter combinations. Hence, we conclude that given the standard parameterization, the general Taylor rule leads to a structurally stable model.

4. THE NEW KEYNESIAN MODEL WITH A HYBRID MONETARY POLICY RULE

This section provides an analysis of the state-of-the-art model for monetary policy. The monetary policy rule consists of a hybrid rule that includes both a current-looking and a forward-looking component. Generally this type of rules can include the features of backward-looking rules such as past values of inflation or output gap, but we will limit our analysis to the following specification:

$$i_t = \alpha(s_t)\pi_{t+1} + \gamma(s_t)x_t. \tag{11}$$

This form of a policy rule was proposed by Clarida et al. (1999) where they provide support for a superior fit of this rule to the policy implemented by the Federal Reserve. According to this rule, a policy maker is forward-looking with respect to inflation and current looking with respect to the output gap.

Using this type of monetary policy rule in a New Keynesian setup with certainty produces the system linear difference equations

$$\mathbf{Y}_{t+1} = \mathbf{D}\mathbf{Y}_t,$$

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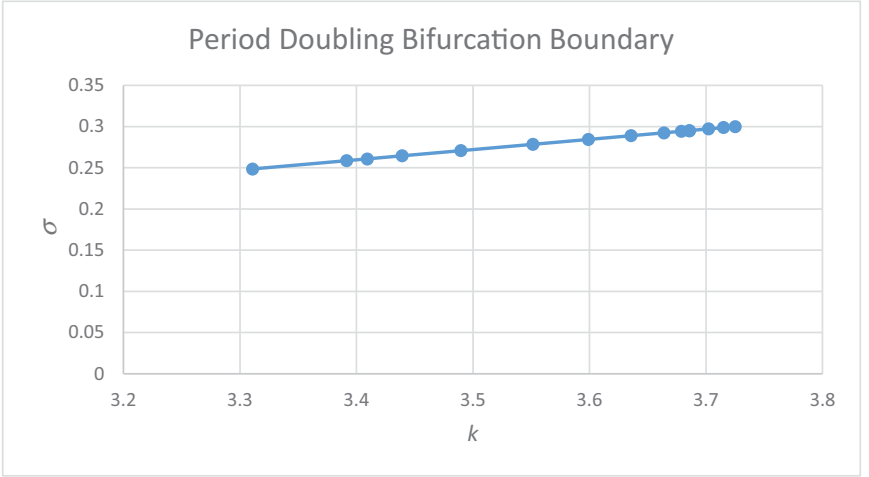


FIGURE 2. Bifurcation for the period-doubling bifurcation in a model with a hybrid policy rule where we vary parameters σ and κ.

where matrix $\mathbf{D}$ is given by

$$\mathbf{D}=\begin{bmatrix} \frac{p_{22}}{\beta(-1+p_{22}+p_{11})} & \frac{-1+p_{22}}{\beta(-1+p_{22}+p_{11})} & -\frac{p_{22}\,k}{\beta\,(-1+p_{22}+p_{11})} & -\frac{(-1+p_{22})k}{\beta\,(-1+p_{22}+p_{11})} \\ \frac{-1+p_{11}}{\beta\,(-1+p_{22}+p_{11})} & \frac{p_{11}}{\beta\,(-1+p_{22}+p_{11})} & -\frac{(-1+p_{11})k}{\beta\,(-1+p_{22}+p_{11})} & -\frac{p_{11}\,k}{\beta\,(-1+p_{22}+p_{11})} \\ \frac{p_{22}\,(-1+\alpha_1)}{\sigma\,\beta\,(-1+p_{22}+p_{11})} & \frac{(-1+p_{22})(-1+\alpha_2)}{\sigma\,\beta\,(-1+p_{22}+p_{11})} & \frac{p_{22}\,(k-k\alpha_1+\sigma\,\beta+\beta\,\gamma_1)}{\sigma\,\beta\,(-1+p_{22}+p_{11})} & \frac{(-1+p_{22})(k-k\alpha_2+\sigma\,\beta+\beta\,\gamma_2)}{\sigma\,\beta\,(-1+p_{22}+p_{11})} \\ \frac{(-1+p_{11})(-1+\alpha_1)}{\sigma\,\beta\,(-1+p_{22}+p_{11})} & \frac{(-1+\alpha_2)p_{11}}{\sigma\,\beta\,(-1+p_{22}+p_{11})} & \frac{(-1+p_{11})(k-k\alpha_1+\sigma\,\beta+\beta\,\gamma_1)}{\sigma\,\beta\,(-1+p_{22}+p_{11})} & \frac{p_{11}\,(k-k\alpha_2+\sigma\,\beta+\beta\,\gamma_2)}{\sigma\,\beta\,(-1+p_{22}+p_{11})} \end{bmatrix}.$$

We analyze the coefficient matrix $\mathbf{D}$ for possibilities of Hopf and PD bifurcations using the same steps as in the preceding section.

Numerical analysis of this dynamic system leads to two findings. First, a Hopf bifurcation cannot occur within feasible set of parameters. Second, however, there is a possibility for a PD bifurcation to emerge. The findings are thus the same as for the simple economy.

Figure 2 plots the bifurcation boundary for the PD bifurcation for the standard parameter combination of Table 1 where we allow risk aversion and the parameter κ to vary.

To find a bifurcation boundary, we need to choose a parameter that will be varied. Assume parameter κ is a free parameter that we use for a numerical bifurcation analysis. We first vary parameter κ while holding all other parameters constant. Assuming the standard calibration in Table 1, the critical value of parameter κ is 3.725. We use this point to trace out the bifurcation boundary. After tracking the first PD bifurcation point, we chose the second parameter that

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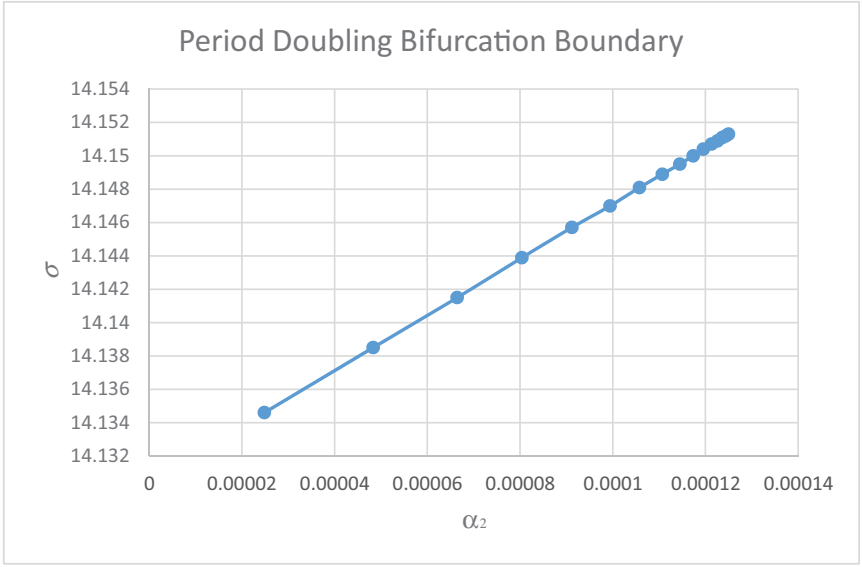


FIGURE 3. Bifurcation for the period-doubling bifurcation in a model with a hybrid policy rule where we vary parameters σ and α_2.

is varied simultaneously with parameter κ. For that case, we can demonstrate a PD bifurcation boundary as a function of two control parameters, κ and the risk aversion parameter σ. Parameter κ is a nonlinear function of the discount factor and the parameter responsible for the degree of price rigidity. It shows that the PD bifurcation will occur when the economy is characterized by a low level of price stickiness. Similarly, we find a PD bifurcation point for parameter $\alpha_2 = 0.00125$. After choosing a second parameter, σ, to be varied, we compute the PD bifurcation boundary demonstrated by Figure 3. It shows the values of parameter σ in the close proximity of zero. Hence, a PD bifurcation will occur for a very narrow set of parameters α_2 and σ corresponding to a risk neutral behavior in combination with very aggressive reaction to inflation in the second regime.

After analyzing further parameter combinations, we find that a PD bifurcation is also possible for lower values of κ accompanied by very high values of the policy parameter α_1 as shown in Figure 4. In other words, aggressive reaction of the Central bank to future inflation will lead to a PD bifurcation in the high-price rigidity environment.

5. CONCLUSION

The New Keynesian model with a Taylor rule generalized to regime switching allows policy parameters to vary over time. This type of policy rule can change the dynamics of inflation and the output gap in a substantial way. We show that

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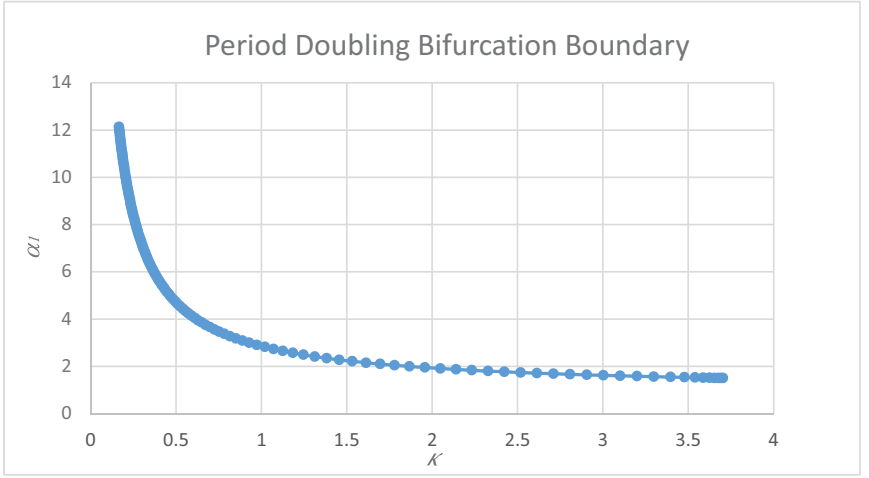


FIGURE 4. Bifurcation for the period-doubling bifurcation in a model with a hybrid policy rule where we vary parameters α_1 and κ.

not only short-term properties change but that regime switching can give rise to changes in the qualitative properties of the solution. To get to this conclusion, we analyze the dynamical properties of these models via bifurcation analysis. We look for Hopf and PD bifurcations within the functional structure considered. The analytical bifurcation analysis detects the possibility of a PD bifurcation for a simple monetary rule using Fisher's equation. This result carries over to the state-of-the-art monetary model with a hybrid Taylor rule which is shown to be subject to a PD bifurcation. Our analysis reveals that PD bifurcations and resulting changes in the dynamics in inflation and output have more tendencies to arise in New Keynesian model with the forward-looking generalized Taylor rule compared to the current-looking counterpart. In order to further study existence and properties of the possible PD bifurcation, the corresponding nonlinear model needs to be analyzed. Higher order terms are necessary to compute the normal form coefficient $c(\alpha)$ of Lemma 1.

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