Appendix

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The notations and the references are the same as in the article by Alvis and Lusztig (0). For a finite group H let $H^\wedge$ be the set of isomorphism classes of irreducible complex representations of H. Let $|\theta|$ denote the degree of $\theta \in H^\wedge$.

Let $u \in G$ be unipotent. The group $Z_G(u)$ acts on $\mathscr{B}_u^G$ and therefore the finite group $A^G(u) = Z_G(u)/Z_G^0(u)$ acts on the cohomology of $\mathscr{B}_u^G$. Let $A^G(u)_0^\wedge$ be the set of all $\chi \in A^G(u)^\wedge$ which occur in $H^{2\beta(u)}(\mathscr{B}_u^G)$. Springer (11), (12) associates to each $\chi \in A^G(u)_0^\wedge$ an irreducible complex representation of W. In accordance with (0) we denote $\rho_{u,\chi}^G$ (or $\rho_{u,\chi}$) the tensor product of this representation by the sign representation of W. Springer shows that $(u, \chi) \mapsto \rho_{u,\chi}$ defines a bijection from the set of conjugacy classes of pairs (u, χ) (u unipotent, $\chi \in A^G(u)_0^\wedge$) to $W^\wedge$. We always have $1_{A^G(u)} \in A^G(u)_0^\wedge$ and by definition $\rho_u = \rho_{u,1}$. So Alvis and Lusztig describe the restriction of this map to the set of pairs of the form $(u, 1)$, for groups of type E_n ($n = 6, 7, 8$). The tables below give the Springer map for the remaining elements.

We may assume that G is adjoint. The groups $A^G(u)$ have been computed by A. V. Alekseevsky (13). They are all of the form $\mathfrak{S}_r$ with $1 \leqslant r \leqslant 5$. In the tables the elements of $A^G(u)^\wedge$ are labelled by the partition of r associated to the corresponding representation of $\mathfrak{S}_r$ (with (r) for the trivial representation and $(11 \ldots 1)$ for the sign representation of $\mathfrak{S}_r$). The representations $\rho \in W^\wedge$ are denoted d_a where $d = |\rho|$ and $a = a_\rho$ is defined as in (0). We write also $d_a(W)$ when we want to specify which Weyl group we consider.

For E_6 and E_7 we always have $A^G(u)_0^\wedge = A^G(u)^\wedge$. In E_8 there is a unique unipotent class with $A^G(u) \cong \mathfrak{S}_5$. For this class the sign representation of $\mathfrak{S}_5$ is missing, as predicted by Lusztig. For the other classes $A^G(u)_0^\wedge = A^G(u)^\wedge$.

From the tables we see that the classes with Dynkin diagrams $^{2010101}_{0}$ and $^{1010101}_{0}$ in E_8 (resp. $^{010101}_{0}$ in E_7) give the representations of W of degree 4096 (resp. 512). These representations are known to have peculiar properties (see e.g. (3)). For these unipotent classes $A^G(u) \cong \mathfrak{S}_2$ and the fact that for each of them the two representations $\rho_{u,\chi}$ ($\chi \in A^G(u)^\wedge$) have the same degree means that no component of $\mathscr{B}_u^G$ is stable under the action of $A^G(u)$. It can be checked that these are the only cases where this happens.

We give some indications on the way the computations can be carried out. We assume that G is of type E_8 and that the results are known for groups of smaller rank. Throughout L is a Levi factor of a parabolic subgroup of G and W' is the Weyl group of L, identified with a subgroup of W.

We use the following properties.

(C) The representations $\rho \in W^\wedge$ which occur in the W-module $H^{2\beta(u)}(\mathscr{B}_u^G)$ are exactly the $\rho_{u,\chi}$'s ($\chi \in A^G(u)^\wedge$) and they don't occur in $H^i(\mathscr{B}_u^G)$, for $i \neq 2\beta(u)$. In particular they have non-zero multiplicities in $\hat{\rho}_u^G = \Sigma_{i \geqslant 0}(-1)^i H^i(\mathscr{B}_u^G)$.

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(D) If $\rho \in W^\wedge$ occurs in $H^i(\mathscr{B}_u^G)$ for some $i \neq 2\beta(u)$, then ρ is of the form $\rho_{u',\chi'}$ for some u' such that $u \in \overline{G.u'} - G.u'$, where $G.u'$ is the G-orbit of u'.

(E) Same as (E) of (0).

The demonstrations are the same as in (0).

Let $u \in G$, $v \in L$ be unipotent. Springer (12) has defined a finite set $X_{u,v}$ on which $A^G(u) \times A^L(v)$ acts, with the following property. Decompose the character $\epsilon_{u,v}$ of the permutation representation afforded by $X_{u,v}$ as

$$\epsilon_{u,v} = \Sigma_{\chi,\psi} n_{u,v,\chi,\psi} \chi \otimes \psi \ (\chi \in A^G(u)^\wedge, \psi \in A^L(v)^\wedge).$$

Then [loc. cit. 4·4].

(F) $\mathrm{Res}^W_{W'}(\rho^G_{u,\chi}) = \Sigma_{(v)} \Sigma_\psi n_{u,v,\chi,\psi} \rho^L_{v,\psi}$, where the first sum is over a set of representatives of unipotent classes in L.

We shall use in particular the following special cases.

(F′) If $n_{u,v,1,1} = 0$, then $X_{u,v} = \emptyset$ and all $n_{u,v,\chi,\psi} = 0$ $(\chi \in A^G(u)^\wedge_0, \psi \in A^L(v)^\wedge_0)$.

(F″) If the class of u is obtained by induction from that of v, then $X_{u,v}$ is of the form $A^G(u)/N$, where N is a subgroup of $A^G(u)$, and $A^G(u)$ acts by left translation; moreover there is a subgroup H of $A^G(u)$ containing N as a normal subgroup with $H/N \cong A^L(v)$, and the action of $A^L(v)$ corresponds to the action of H by right translation. If $\chi \in A^G(u)^\wedge$ and $\psi \in A^L(v)^\wedge$, then $n_{u,v,\chi,\psi}$ is equal to the multiplicity of the lift $\tilde{\psi}$ of ψ to H in $\mathrm{Res}^{A^G(u)}_H(\chi)$.

(F′) is obvious and (F″) can be deduced from ((8), 1·3, 1·5, proof of 3·5).

The following property is weaker than (F′) but easier to use.

(G) $|\rho_{u,\chi}| \leqslant |\chi|\,|\rho_u|$.

This follows from the fact that the multiplicity of any irreducible character θ in a transitive permutation representation cannot exceed $|\theta|$.

We compute first the $\rho_{u,\chi}$'s is the case where $\beta(u) \leqslant 15$. There are 18 representations $\rho \in W^\wedge - \bar{S}_W$ which are of type X or Z in Frame's tables (14). Using (1) we can check that they all occur in a representation of the form $\mathrm{Ind}^W_{W'}(\rho^L_{v,\psi})$ with L of type E_7 and v in one of the classes $^{0020002}_{0}$, $^{2101001}_{0}$, $^{2100010}_{1}$, $^{2010102}_{0}$. We deduce then from (C), (D), (E) that these 18 representations must be of the form $\rho_{u,\chi}$ for some u with $\beta(u) \leqslant 15$. It should be noted that we don't need to care about possible cancellations in $\mathrm{Ind}^W_{W'}(\hat{\rho}^L_v)$ since (C) and (D) hold also in L. It follows that $A^G(u)^\wedge_0 = A^G(u)^\wedge$ if $\beta(u) \leqslant 15$, and moreover the $\rho_{u,\chi}$'s are of type X or Z in this case. Let $W^\wedge_u = \{\rho_{u,\chi} | \chi \in A^G(u)^\wedge_0\}$. We give two examples of computations.

Example 1. Take u in the class $^{2220202}_{0}$. Here $A^G(u) \cong \mathfrak{S}_2$. By (G) $|\rho_{u,11}| \leqslant 112$. Take L of type E_7. Then $^{2220202}_{0}$ is induced from $^{220202}_{0}$ and using (F), (F″) we find that $21_6(W')$ occurs in $\mathrm{Res}^W_{W'}(\rho_{u,11})$. This leaves two possibilities, 28_8 and 56_{19}. However $35_{13}(W')$ occurs in $\mathrm{Res}^W_{W'}(56_{19})$. By (F′) this rules 56_{19} out. Hence $\rho_{u,11} = 28_8$.

Example 2. Take u in the class $^{2000200}_{0}$. We have $A^G(u) \cong \mathfrak{S}_3$. We assume that the $\rho_{u',\chi'}$'s with $\beta(u') \leqslant 9$ or with u' in the class $^{2010102}_{0}$ are already known. Take u_1 in $^{2010101}_{0}$, u_2 in $^{0110101}_{0}$ and let L be of type E_7. We may assume that $u_1, u_2 \in L$. Using (E) we find that 840_{13} and 4096_{12} occur in $\hat{\rho}^G_{u_1}$ and 175_{12} in $\hat{\rho}^G_{u_2}$. Using (C), (D) we find then that $u_1, u_2 \in \overline{G.u}$, $W^\wedge_u \subset \{2240_{10}, 840_{13}, 175_{12}, 4096_{12}\}$ and $W^\wedge_{u_1} \subset \{4096_{11}, 4096_{12}, 840_{13}\}$. However $512_{12}(W')$ occurs in $\mathrm{Res}^W_{W'}(4096_{12})$ and by (F′) this implies $4096_{12} \notin W^\wedge_u$. Hence $\rho_{u_1,11} = 4096_{12}$ and $W^\wedge_u = \{2240_{10}, 175_{12}, 840_{13}\}$.

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We take now L of type D_5+A_2. We can use (F″). The Dynkin diagram of u gives an $A^G(u)$-stable component of $\mathscr{B}_u^G$. Using this component we find that $H=A^G(u)$, $|N|=3$. It is then easily checked that $\rho_{u,21}=175_{12}, \rho_{u,111}=840_{13}$.

We discuss now in details the case where $A^G(u)\cong\mathfrak{S}_5$. It is the most delicate one.

Example 3. Take u in the class $^{0002000}_{0}$. We have $A^G(u)\cong\mathfrak{S}_5$ and $\beta(u)=16$. Let u' be in the class $^{0010100}_{0}$. We have $A^G(u')\cong\mathfrak{S}_3$. We fix L of type E_7.

(i) Using (F′) systematically, we get $W_u^\wedge\subset\{4480_{16}, 5670_{18}, 4536_{18}, 1400_{20}, 1680_{22}, 1134_{20}, 70_{32}, 448_{25}\}$.

(ii) Let $v\in L$ belong to the class $^{200200}_{0}$ or $^{000020}_{0}$. In both cases we get the following intertwinning numbers:

	4480_{16} $=\rho_{u,5}$	7168_{17} $=\rho_{u',3}$	4536_{18}	5670_{18}	1400_{20}	1134_{20}	1680_{22}	70_{32}	5600_{19}	448_{25}
$\mathrm{Ind}_{W'}^{W}(\rho_{v,3}^{L})$	1	1	1	1	0	0	0	0	0	0
$\mathrm{Ind}_{W'}^{W}(\rho_{v,21}^{L})$	0	0	1	1	1	0	1	0	1	0
$\mathrm{Ind}_{W'}^{W}(\rho_{v,111}^{L})$	0	0	0	0	0	0	1	1	0	1

These are the only significant intertwinning numbers in the situation we consider.

Take first v in $^{200200}_{0}$. By (C), (D), (E) we get $\{7168_{17}, 5600_{19}\}\subset W_{u'}^\wedge\subset\{7168_{17}, 5600_{19}, 5670_{18}, 4536_{18}, 1400_{20}, 1680_{22}, 70_{32}, 448_{25}\}$ and $\{7168_{19}, 4480_{16}, 5670_{18}, 4536_{18}, 1400_{20}, 1680_{22}, 70_{32}, 5600_{19}, 448_{24}\}\subset W_u^\wedge\cup W_{u'}^\wedge$.

If $A^G(u')_0^\wedge\neq A^G(u')^\wedge$, then $W_u=\{4480_{16}, 5670_{18}, 4536_{18}, 1400_{20}, 1680_{22}, 70_{32}, 448_{25}\}$. Then taking v in $^{000020}_{0}$ and using (F″), we must have $H\subset\mathfrak{S}_5$, $H/N=\mathfrak{S}_3$ and $\mathrm{Ind}_H^{\mathfrak{S}_5}(1_H)$ must be the sum of three distinct irreducible representations. This forces

$$H\cong\mathfrak{S}_3\times\mathfrak{S}_2\subset\mathfrak{S}_5,$$

N being the factor $\mathfrak{S}_2$. However this gives multiplicities which do not match the table.

Therefore $A^G(u')_0^\wedge=A^G(u')^\wedge$. Taking again v in $^{000020}_{0}$ and looking for the various possibilities for the set $X_{u',v}$, we find that $\rho_{u',21}=5600_{19}$ and $\rho_{u',111}\in\{70_{32}, 448_{25}\}$.

Using (F″) with v in $^{000020}_{0}$, we find that H and N must be the same as in the discussion above, and we get the following informations:

$\{\rho_{u,41}, \rho_{u,32}\}=\{5670_{18}, 4536_{18}\}$, $\rho_{u,311}=1680_{22}$, $\rho_{u,221}=1400_{20}$ and $\rho_{u,2111}\in\{70_{32}, 448_{25}\}$.

Moreover if the sign representation of $\mathfrak{S}_5$ is contained in $A^G(u)_0^\wedge$, then $\rho_{u,11111}=1134_{20}$.

(iii) We take v in $^{020020}_{0}$. Now $A^L(v)\cong\mathfrak{S}_2$. We have the following intertwinning numbers:

	4480_{16}	4536_{18}	5670_{18}	1400_{20}	1680_{22}	70_{32}	448_{25}	1134_{20}
$\mathrm{Ind}_{W'}^{W}(\rho_{v,2}^{L})$	1	1	1	0	0	0	0	0
$\mathrm{Ind}_{W'}^{W}(\rho_{v,11}^{L})$	0	0	1	0	1	0	1	1

The multiplicity of the trivial character $(2)\in A^L(v)^\wedge$ in $\epsilon_{u,v}$ is 10. As $\epsilon_{u,v}$ arises from a permutation the multiplicity of $(11)\in A^L(v)^\wedge$ is at most 10. This implies that 448_{25} and 1134_{20} are not in $W_u^\wedge$ and that $5670_{18}=\rho_{u,\chi}$ with $|\chi|=4$. The sign character is therefore missing in $A^G(u)_0^\wedge$, $\rho_{u,41}=5670_{18}$, $\rho_{u,32}=4536_{18}$, $\rho_{u,2111}=70_{32}$ and $\rho_{u',111}=448_{25}$.

For the remaining classes it is enough to use (C), (D), (E), except in two cases where $A^G(u)\cong\mathfrak{S}_3$, but there it is easy to use (F) to get the complete result.

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E_6

Class of u	Diagram	$\beta(u)$	r	χ	$\rho_{u,\chi}$
A_5+A_1	20202 0	3	2	11	15_5
$D_4(a_1)$	00200 0	7	3	21 111	90_8 20_{10}
A_2	00000 2	15	2	11	15_{17}

E_7

Class of u	Diagram	$\beta(u)$	r	χ	$\rho_{u,\chi}$
D_6+A_1	220202 0	3	2	11	21_6
$E_6(a_1)$	020202 0	4	2	11	105_5
$D_6(a_1)+A_1$	200202 0	5	2	11	15_7
$D_6(a_2)+A_1$	200200 0	7	3	21 111	280_9 35_{13}
$(A_5+A_1)'$	020020 0	8	2	11	189_{10}
$D_5(a_1)$	010102 0	10	2	11	336_{11}
A_4+A_1	010101 0	11	2	11	512_{12}
A_4	020002 0	13	2	11	336_{14}
A_3+A_2	010100 0	14	2	11	84_{15}
$D_4(a_1)+A_1$	100010 1	15	2	11	189_{17}
$D_4(a_1)$	000020 0	16	3	21 111	280_{18} 35_{22}
A_2+A_1	010001 0	25	2	11	105_{26}
A_2	000002 0	30	2	11	21_{33}

E_8

Class of u	Diagram	$\beta(u)$	r	χ	$\rho_{u,\chi}$
E_7+A_1	2220202 0	3	2	11	28_8
D_8	2020202 0	4	2	11	160_7
$E_7(a_1)+A_1$	2200202 0	5	2	11	50_8
$D_8(a_1)$	0200202 0	6	2	11	300_8
$E_7(a_2)+A_1$	2200200 0	7	3	21 111	1008_9 56_{19}
A_8	0200200 0	8	3	21 111	1575_{10} 350_{14}

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E_8 continued

Class of u	Diagram	$\beta(u)$	r	χ	$\rho_{u,\chi}$
$D_7(a_1)$	2002002 0	9	2	11	1050_{10}
D_6+A_1	2010102 0	10	2	11	1296_{13}
$D_8(a_3)$	2000200 0	10	3	21 111	175_{12} 840_{13}
$E_6(a_1)+A_1$	2010101 0	11	2	11	4096_{12}
$D_7(a_2)$	1010101 0	12	2	11	3360_{13}
$E_6(a_1)$	2020002 0	13	2	11	2100_{16}
D_5+A_2	2002000 0	13	2	11	840_{14}
$D_6(a_1)+A_1$	2010100 0	14	2	11	700_{16}
$D_6(a_1)$	2100010 1	15	2	11	2400_{17}
$2A_4$	0002000 0	16	5	41 32 311 221 2111 11111	5670_{18} 4536_{18} 1680_{22} 1400_{20} 70_{32} —
A_5+A_2	0010100 0	17	3	21 111	5600_{19} 448_{25}
A_5+2A_1	0101001 0	18	2	11	1134_{20}
$D_6(a_2)$	0100010 1	18	2	11	2688_{20}
$(A_5+A_1)''$	0200002 0	21	2	11	2400_{23}
D_4+A_2	2000000 2	21	2	11	168_{24}
A_4+2A_1	1000100 0	24	2	11	3360_{25}
$D_5(a_1)$	2010001 0	25	2	11	2100_{28}
A_4+A_1	1010001 0	26	2	11	4096_{27}
$D_4(a_1)+A_2$	0000000 2	28	2	11	840_{31}
A_4	2000002 0	30	2	11	1296_{33}
A_3+A_2	0010001 0	31	2	11	972_{32}
$D_4(a_1)+A_1$	0100000 1	32	3	21 111	1575_{34} 350_{38}
$D_4(a_1)$	0200000 0	37	3	21 111	1008_{39} 56_{49}
$2A_2$	0000002 0	42	2	11	300_{44}
A_2+A_1	1000001 0	52	2	11	160_{55}
A_2	2000000 0	63	2	11	28_{68}

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