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Hypergraphs with No Cycle of a Given Length

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Recently, the authors gave upper bounds for the size of 3-uniform hypergraphs avoiding a given odd cycle using the definition of a cycle due to Berge. In the present paper we extend this bound to m-uniform hypergraphs (for all $m \geqslant 3$), as well as m-uniform hypergraphs avoiding a cycle of length $2k$. Finally we consider non-uniform hypergraphs $\mathcal{H}$ avoiding cycles of length $2k$ or $2k+1$. In both cases we can bound $\sum_{h\in\mathcal{E}(\mathcal{H})} |h|$ by $O(n^{1+1/k})$ under the assumption that all $h \in \mathcal{E}(\mathcal{H})$ satisfy $|h| \geqslant 4k^2$.

1. Introduction

A classical theorem of Bondy and Simonovits is as follows.

Theorem 1.1 (Bondy and Simonovits [2]). *Let $k \geqslant 2$ be fixed and let G be a graph on n vertices containing no cycles of length $2k$. Then $e(G) \leqslant c(k)n^{1+1/k}$.*

We would like to study similar questions in hypergraphs. The cycles in hypergraphs are defined in the Berge sense: a cycle of length k consists of k distinct vertices $v_1, \ldots, v_k$ and k distinct hyperedges $h_1, \ldots, h_k$ such that $v_i, v_{i+1} \in h_i$ for $i = 1, \ldots, k-1$ and $v_1, v_k \in h_k$. Several papers have been written about hypergraph cycles with different definitions: see, *e.g.*, [4] and [7]. For hypergraph cycles in the Berge sense, several results have been proved recently. For triangles, a surprisingly strong and general theorem is proved, which is sharp for 3-uniform or 4-uniform hypergraphs.

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Theorem 1.2 (Győri [5]). *Let $\mathcal{H}$ be a (multi-) hypergraph on n vertices containing no triangles. Then, for large enough n,*

$$\sum_{e\in\mathcal{H}}(|e|-2)\leqslant n^2/8.$$

In [6], we recently proved the following.

Theorem 1.3. *Let $\mathcal{H}$ be a 3-uniform hypergraph on n vertices containing no cycles of length $2k+1$. Then $e(\mathcal{H})<4k^4n^{1+1/k}+15k^4n+10k^2n$.*

In the present paper we extend this result to any m-uniform hypergraph with $m\geqslant 3$. (Note that the theorem is not true for $m=2$.)

Theorem 1.4. *Fix $m\geqslant 3$, and let $\mathcal{H}$ be an m-uniform hypergraph on n vertices containing no cycles of length $2k+1$. Then*

$$e(\mathcal{H})<\left(\sum_{j=m}^{k-1}\binom{j-3}{m-3}+k+1\right)(4k^4n^{1+1/k}+15k^4n+10k^2n)=c(k,m)n^{1+1/k}+O(n).$$

It is interesting that a similar bound holds in the even case.

Theorem 1.5. *Fix $m\geqslant 2$, and let $\mathcal{H}$ be an m-uniform hypergraph on n vertices containing no cycles of length $2k$. Then $e(\mathcal{H})<c(k,m)n^{1+1/k}+O(n)$.*

Both theorems are sharp in the following sense. For $m<k$ we can construct m-uniform hypergraphs avoiding cycles of length $2k$ and $2k+1$ and containing $\Omega(\mathrm{ex}(n;C_4,C_6,\ldots,C_{2k}))$ edges. Here $\mathrm{ex}(n;C_4,C_6,\ldots,C_{2k})$ denotes the largest size of a graph on n vertices containing no even cycles of length $\leqslant 2k$. Thus, even if the orders of magnitude of our bounds are not correct, it will take a major breakthrough in extremal graph theory to improve them.

Remark. To prove Theorems 1.4 and 1.5 by means of Theorem 1.3 and the Bondy–Simonovits theorem, respectively, we use a simple lemma, which says that an m-uniform k-cycle-free hypergraph (or the hyperedges of size m in a hypergraph) can be replaced by l-edges ($l<k$) losing just a constant factor depending only on m and l. (The theorems follow from the lemma easily; we skip the details of the calculations.)

Lemma 1.6 (reduction lemma). *Let $2\leqslant l<m$ and let $\mathcal{H}$ be a m-uniform hypergraph with no cycle of length k. Define the l-uniform hypergraph $\mathcal{H}_1$ in the following way. Order the edges of $\mathcal{H}$ and, going through the edges one by one, pick an l-set from each to be in $\mathcal{H}_1$, but at each point, pick this l-set to have as small multiplicity as possible. Obviously the hypergraph $\mathcal{H}_1$ will also have no k-cycle and each of its hyperedges will have multiplicity no more than $\sum_{j=m}^{k-1}\binom{j-l}{m-l}+k+1$.*

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Remark. Note that if $k \leqslant m$ then the lemma trivially holds: if a hyperedge of $\mathcal{H}_1$ has multiplicity more than $k+1$ then there is an hyperedge h of $\mathcal{H}$ such that every l-subset of h is contained within more than k edges, and we can clearly find a k-cycle within h, a contradiction.

Our main result is a proof that non-uniform hypergraphs without a cycle of length $2k+1$ (or of length $2k$) have $O(n^{1+1/k})$ edges. In fact, for hypergraphs with only large edges we can prove the theorem as follows.

Theorem 1.7. *Fix $k > 1$, and let $\mathcal{H}$ be a (multi-) hypergraph on n vertices with all of its edges of size at least $4k^2$ such that $\mathcal{H}$ contains no cycle of length $2k+p$, where either $p = 0$ or $p = 1$. Then*

$$\sum_{v \in \mathcal{V}(\mathcal{H})} d(v) \leqslant (16k^6 + 8k^2)n^{(k+1)/k} + (16k^7 + 32k^6 + 16k^5)n.$$

Remark. In the case of general hypergraphs, we can apply either Theorem 1.4 or Theorem 1.5 for small hyperedges, and apply Theorem 1.7 for big hyperedges.

2. Proof of Lemma 1.6

Proof of Lemma 1.6. Suppose the lemma is not true. Let $\mathcal{H}$ be an m-uniform hypergraph containing no $\mathcal{C}_k$ and let $\mathcal{H}_1$ be a l-uniform hypergraph defined as in the statement of the lemma, and suppose $\mathcal{H}_1$ has an edge, e_1, of multiplicity greater than $M = \sum_{j=m}^{k-1} \binom{j-l}{m-l} + k + 1$. We will find a sequence of hyperedges $h_1, \ldots, h_k$ in $\mathcal{H}$ forming a k-cycle. To distinguish between the hypergraphs $\mathcal{H}$ and $\mathcal{H}_1$, we will refer to the edges of the former as hyperedges and those of the latter simply as edges. Similarly, we will use the variables h_i for hyperedges in $\mathcal{H}$ and the variables e_i for edges in $\mathcal{H}_1$. For $h \in \mathcal{E}(\mathcal{H})$, we will refer to the l-set chosen in the construction of $\mathcal{H}_1$ to represent h as $e(h)$.

Consider the last hyperedge, which contributed to the multiplicity of e_1 in the construction of $\mathcal{H}_1$. Call this hyperedge h_1. (Thus, for instance, $e(h_1) = e_1$.) By definition, each l-set contained within h_1 must be an edge in $\mathcal{H}_1$ and indeed, must have multiplicity at least M. Otherwise, when we picked $e(h_1)$ from h_1 to be in the graph, we would have chosen one of the other less-represented l-sets in h_1. Given the vertices $x_0, x_1 \in e_1$, pick $e_2 \subset h_1$ with $x_1 \in e_2 \neq e_1$. Let $x_2 \in e_2 \backslash \{x_1\}$.

As e_2 has multiplicity at least M in $\mathcal{H}_1$, we can find $h_2 \in \mathcal{E}(\mathcal{H})$ with $e(h_2) = e_2$ (a copy of e_2) such that every l-set contained in h_2 is an edge in $\mathcal{H}_1$ with multiplicity at least $M-1$. Note that h_1 does not contribute to the multiplicity of e_2 as $e(h_1)$ is a copy of e_1.

Claim 2.1. *Let $i \leqslant k-1$. Suppose we can find distinct vertices $x_1, x_2, \ldots, x_{i-1}$ and distinct edges $e_1, e_2, \ldots, e_{i-1} \in \mathcal{E}(\mathcal{H}_1)$ and $h_1, h_2, \ldots, h_{i-1} \in \mathcal{E}(\mathcal{H})$ such that the following conditions hold:*

- $\forall j \leqslant i-2, x_j \in e_j \cap e_{j+1}$ *and* $x_{i-1} \in e_{i-1}$,
- $\forall j < i-1, e_j \cup e_{j+1} \subseteq h_j$,
- $h_{i-1} \nsubseteq \{x_1, x_2, \ldots, x_{i-1}\}$,

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- *each l-set in h_{i-1} has multiplicity at least $M - \sum_{j=m}^{i-1} \binom{j-l}{m-l} - (i-2)$*

Then we can find an edge of $\mathcal{H}_1$, $e_i \subset h_{i-1}$ and a vertex $x_i \in e_i$ such that x_i is distinct from the vertices $x_1, x_2, \ldots, x_{i-1}$. Moreover, there exists a hyperedge h_i of $\mathcal{H}$, containing e_i, such that h_i is not contained in $\{x_1, x_2, \ldots, x_i\}$ and such that each l-set of h_i occurs with multiplicity at least $M - \sum_{j=m}^{i} \binom{j-l}{m-l} - (i-1)$.

Proof of Claim 2.1. We find x_i, e_i and h_i in the following way. By the third condition, we can pick a new edge $e_i \subset h_{i-1}$ such that there exists a vertex

$$x_i \in e_i \backslash \{x_1, x_2, \ldots, x_{i-1}\}.$$

By the fourth condition, e_i has multiplicity at least $M - \sum_{j=m}^{i-1} \binom{j-l}{m-l} - (i-2)$ in $\mathcal{H}_1$. Let $\mathcal{E}_i^* = \{h \in \mathcal{E}(\mathcal{H}) : e(h) \text{ is a copy of } e_i\}$. We want to pick $h \in \mathcal{E}_i^*$ such that h is not contained in $\{x_1, x_2, \ldots, x_i\}$. There are at most $\binom{i-l}{m-l}$ such forbidden edges in $\mathcal{E}_i^*$. Thus, letting $\mathcal{E}_i = \{h \in \mathcal{E}_i^* : h \nsubseteq \{x_1, x_2, \ldots, x_i\}\}$, we have

$$|\mathcal{E}_i| \geqslant |\mathcal{E}_i^*| - \binom{i-l}{m-l} \geqslant M - \sum_{j=m}^{i} \binom{j-l}{m-l} - (i-2).$$

Then, picking h_i to be the last hyperedge (in our original ordering of the edges) of $\mathcal{E}_i$, we must have that every l-set in h_i has multiplicity in $\mathcal{H}_1$ at least $M - \sum_{j=m}^{i} \binom{j-l}{m-l} - (i-1)$. In this way we have picked x_i, e_i, and h_i such that the above conditions still hold. □

Thus, by induction on i and using Claim 2.1, we can find k vertices $x_1, \ldots, x_k$ such that there are k edges e_i in $\mathcal{H}_1$ and k hyperedges in $\mathcal{H}$ such that for $1 \leqslant j < k$ we have $x_j \in e_j \cap e_{j+1}$. We also have for $2 \leqslant j \leqslant k$ that $e_j \subset h_{j-1} \cap h_j$. Thus we conclude that, for $1 \leqslant j \leqslant k-2$, we have $x_j, x_{j+2} \in h_{j+1}$. Also, we clearly have $x_{k-1}, x_k \in h_k$ and $x_1, x_2 \in h_1$. Thus we have produced the following k-cycle in $\mathcal{H}$: $x_1, x_3, x_5, \ldots, x_{k-1}, x_k, x_{k-2}, x_{k-4}, \ldots, x_2$ if k is even and $x_1, x_3, x_5, \ldots, x_k, x_{k-1}, x_{k-3}, x_{k-5}, \ldots, x_2$ if k is odd, and we have arrived at our contradiction. □

3. Proof of Theorem 1.7

We will use the following lemma.

Lemma 3.1. *Let $\mathcal{H}$ be a multi-hypergraph with all of its edges of size at least $2l$ for some fixed constant l. Suppose also that $\mathcal{H}$ has no Berge path on l vertices. Then the average degree in $\mathcal{H}$ is less than $3l$.*

Proof. We prove this by induction on the number of vertices of $\mathcal{H}$. Clearly, if $\mathcal{H}$ has n vertices $l \leqslant n < 2l$ then there are no edges, so the average degree is less than $3l$. Otherwise, let $n \geqslant 2l$ be the number of vertices of $\mathcal{H}$, and suppose that the average degree in $\mathcal{H}$ is at least $3l$. Let $\mathcal{P}$ be a longest path in $\mathcal{H}$. Let j be the length of this path (so $j < l$.) Let the e_1 be one of the tail edges of the path. Note that the number of vertices in e_1 not in the interior of the path is at least $|e_1| - (j-1) \geqslant 2l - j + 1 \geqslant l$. Each of these vertices is contained only within the edges of $\mathcal{P}$

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and hence each has degree at most j. Pick l of these vertices and delete them and the edges of $\mathcal{P}$ which subsequently have less than $2l$ vertices from $\mathcal{H}$ (forming a new hypergraph $\mathcal{H}^1$). Then $\mathcal{H}^1$ has at most $j(3l-1)$ fewer incidences than does $\mathcal{H}$. And so $\sum_{e\in\mathcal{E}(\mathcal{H}^1)}|e| \geqslant \sum_{e\in\mathcal{E}(\mathcal{H})}|e| - j(3l-1) \geqslant 3ln - j(3l-1) \geqslant 3l(n-l)$. Thus the average degree in $\mathcal{H}^1$ is still at least $3l$, and by induction it must have a path of length l. □

Proof of Theorem 1.7. Suppose the theorem is not true. Let $\mathcal{H}$ be a hypergraph on n vertices containing no $\mathcal{C}_{2k+p}$ where either $p=0$ or $p=1$. Suppose also that all the edges of $\mathcal{H}$ have size at least $4k^2$ and that $\sum_{h\in\mathcal{E}(\mathcal{H})}|h| > (16k^6+8k^4)n^{(k+1)/k} + (16k^7+32k^6+16k^5)n$. Note that we may assume the minimum degree of $\mathcal{H}_0$ to be at least $(4k^4+2k^2)n^{1/k}+4k^5+8k^4+4k^3$. Otherwise we can delete those vertices of smaller degree and any edges that become smaller than $4k^2$, without decreasing the average degree in $\mathcal{H}_0$. The resulting hypergraph will still be a counterexample to Theorem 1.7 and will have minimum degree $\delta \geqslant (4k^4+2k^2)n^{1/k}+4k^5+8k^4+4k^3$ as desired.

To organize the search for special paths, we construct and fix a form of breadth-first search tree (a graph!) rooted at a vertex x. (We follow the idea used in the case of 3-uniform hypergraphs.)

Definition. A *k-neighbourhood skeleton* of the hypergraph $\mathcal{H}$ around the vertex $x\in\mathcal{V}(\mathcal{H})$ consists of $k+1$ pairwise disjoint sets $S_0,\dots,S_k\subset\mathcal{V}(\mathcal{H})$, a subset $\mathcal{E}_S\subset\mathcal{E}(\mathcal{H})$ and maps $\pi:\bigcup_{i=1}^{k}S_i\to\bigcup_{i=0}^{k-1}S_i$ and $\phi:\cup_{i>0}S_i\to\mathcal{E}_S$ such that $\pi(S_i)\subseteq S_{i-1}$, $S_0=\{x\}$, ϕ is a bijection and for each $v\in S_i$, $\{v,\pi(v)\}\subseteq\phi(v)$.

It is clear from the definition that neighbourhood skeletons can be very small (even empty!) so we will always work with neighbourhood skeletons in which each S_i is as large as possible. This is formalized in the following.

Definition. A k-neighbourhood skeleton is *greedily defined* if, for each $i>0$, there is no edge $e\in\mathcal{E}(\mathcal{H})\backslash\phi(\cup_{j=1}^{i}S_j)$ with $e\cap S_0=e\cap S_1=\cdots=e\cap S_{i-2}=\emptyset$, $e\cap S_{i-1}\neq\emptyset$ and $e\nsubseteq S_{i-1}\cup S_i$.

For the sake of convenience, let us fix a greedily defined k-neighbourhood skeleton. (Think of the sets S_i as forming a rooted tree at the vertex x and the map π as a 'parental' map.) We will refer to the composite maps $\pi(\pi(v))$ as $\pi^2(v)$; we will use the same for repeated applications of its inverse: π^{-1}. To help the reader understand the structure of the sets S_i, we mention the following.

Remark. Fix $i,k\geqslant i>0$ and let $u,v\in S_i$. If j is the smallest integer such that $\pi^j(u)=\pi^j(v)$, then there is a path of length $2j$ from u to v in $\mathcal{H}|_{\mathcal{E}_S}$. Note that such a j always exists as $\pi^i(u)=\pi^i(v)=x$

Pick $x\in\mathcal{V}(\mathcal{H})$ and let $(\{S_i\},\mathcal{E}_S,\pi,\phi)$ be a greedily defined k-neighbourhood structure of $\mathcal{H}$ around x. For $1\leqslant i\leqslant k$ let $\mathcal{E}_i=\phi^{-1}(S_i)$. Note that these sets partition $\mathcal{E}_S$.

Proposition 3.2. *$|S_i|\geqslant n^{1/k}|S_{i-1}|$ for $i=1,2,\dots,k$.* □

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Before we prove Proposition 3.2, we note that Theorem 1.7 easily follows from Proposition 3.2. As the sets S_i are pairwise disjoint, Proposition 3.2 implies $|\mathcal{V}(\mathcal{H})| > n$, a contradiction. □

Proof of Proposition 3.2. To show that $|S_i| \geqslant n^{1/k}|S_{i-1}|$ for $1 \leqslant i \leqslant k$, we simply double-count the pairs (v, h), where $v \in h \cap S_{i-1}$ and $h \in \mathcal{E}(\mathcal{H})$. We distinguish between five types of these pairs:

(A) pairs (v, h) with $h \in \mathcal{E}_j$ for $j < i$,
(B) pairs (v, h) with $h \in \mathcal{E}\backslash\mathcal{E}_S$ and $|h \cap S_{i-1}| > 2k^2$,
(C) pairs (v, h) with $h \in \mathcal{E}_i$ and $|h \cap S_{i-1}| > 2k^2$,
(D) pairs (v, h) with $h \in \mathcal{E}\backslash\mathcal{E}_S$ and $|h \cap S_{i-1}| \leqslant 2k^2$,
(E) pairs (v, h) with $h \in \mathcal{E}_i$ and $|h \cap S_{i-1}| \leqslant 2k^2$.

Note that that every pair (v, h) with $v \in h \cap S_{i-1}$ is exactly one of these five types. In particular, there are no edges $h \in \mathcal{E}_j$ for $j > i$ intersecting S_{i-1}.

The heart of Proposition 3.2 lies in the estimate of the number of hyperedges appearing in pairs of type (D) and the following claims.

Claim 3.3. *There are no more than* $(4k^5 + 4k^3)|S_{i-1}|$ *pairs* (v, h) *of type* (A).

Claim 3.4. *There are no more than* $4k^4|S_{i-1}|$ *pairs* (v, h) *of type* (B) *or* (C).

Proof of Claim 3.3. To prove the claim, we need a more regular structure, *i.e.*, a more regular set of edges. For this purpose, we will define a special colouring (a classification) of the edges with a fixed number of colours and then it will be sufficient to study just one (colour) class.

Fix $j < i$. Let $\mathcal{F} = \{h \in \mathcal{E}_j : |h \cap S_{i-1}| > 4k^2\}$. Clearly the number of pairs of type (A) is no more than

$$\sum_{h\in\mathcal{F}} |h \cap S_{i-1}| + 4k^2|S_j|. \tag{3.1}$$

Let $\mathcal{F}_1 = \{h \cap S_{i-1} : h \in \mathcal{F}\}$, and choose $f \in \mathcal{F}_1$, and let $\hat{f}$ represent the edge from which f was derived in $\mathcal{F}$. Let $\mathcal{H}_1$ be the hypergraph consisting of the edges $\mathcal{F}_1$ and vertex set S_{i-1}. Clearly $\mathcal{H}_1$ may have multiple edges. We now attempt to find a linear bound (in terms of $|S_{i-1}|$) for $\sum_{f\in\mathcal{F}_1} |f|$.

Suppose that $\sum_{f\in\mathcal{F}_1} |f| \geqslant 4k^4|S_{i-1}|$. Then we may delete those vertices of $\mathcal{H}_1$ of degree less than $2k^2$ and any edges that become smaller than $2k^2$ without decreasing the average degree of $\mathcal{H}_1$. As we will be bounding the size of $\mathcal{H}_1$ by a linear function of its vertex set, we can thus assume that $\mathcal{H}_1$ has minimum degree $2k^2$ and that all the edges are of size at least $2k^2$. Note that we may have deleted the majority of S_{i-1}, so the sum $\sum_{f\in\mathcal{F}_1} |f|$ might have decreased by just a bit less than $4k^4|S_{i-1}|$. Let $\mathcal{H}_1'$ denote the resulting hypergraph consisting of the set $\mathcal{F}_1'$ of resulting edges and the vertex set $\mathcal{V}(\mathcal{H}_1') \subseteq S_{i-1}$.

We now define possible truncations and an improper colouring of the edges f of $\mathcal{H}_1'$ one by one. Let $f \in \mathcal{F}_1'$. Every vertex $v \in f$ has the ancestors $\pi^{i-1}(v), \pi^{i-2}(v), \ldots, \pi(v), \pi^0(v)$ in $S_0, S_1, \ldots, S_{i-2}$, S_{i-1}, respectively. Take the ancestors of all $v \in f$ in S_1. There are two cases: either they are more or less the same, in which case we can delete at most $2k$ vertices from f such that the remaining vertices have the very same ancestor in S_1, or they are not. If not, then colour f with colour 1.

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If yes, then delete these at most $2k$ vertices from f so the remaining vertices have the very same ancestor in S_1. Then take the ancestors of all the remaining vertices in S_2. Again there are two cases: either they are more or less the same, in which case we can delete (another) at most $2k$ vertices such that the remaining vertices have the very same ancestor in S_2, or they are not. If not, then colour the remaining f with colour 2. If yes, then delete these at most $2k$ vertices from the rest of f so the remaining vertices have the very same ancestor in S_2, and so on. Finally, if we arrive at the point at which only the vertices v are different, then colour the rest of f with colour $i-1$.

In this way we decrease the size of individual edges of $\mathcal{H}_1'$, but not the number of vertices in $\mathcal{H}_1'$. Let $\mathcal{F}_1''$ denote the hypergraph we get from $\mathcal{F}_1'$ after these deletions. Then $\sum_{f\in\mathcal{F}_1''}|f| \geqslant \sum_{f\in\mathcal{F}_1'}|f| - 2k^2|\mathcal{F}_1'|$. Also note that every edge will have at least $2k(k-(i-2)) \geqslant 4k$ vertices remaining after these deletions.

Now we claim that there is no path P on the edges in $\mathcal{F}_1''$ of colour l and of length $2(k-i+l)+p$. Otherwise let e_1 and e_2 be the first and last edges in this path respectively. As e_1 has at least $4k$ vertices, we can pick $v_1 \in e_1$ not in the path P. Now delete v_1 and the $2(k-i+l)+p-1$ inner vertices of P, altogether less then $2k$ vertices from e_2 to get e_2'. The colour of e_2 is l, so the ancestors of the vertices $v \in e_2'$ in S_l are not the same. There are at least two distinct ancestors, so there is a vertex v_2 such that the ancestors of v_1 and v_2 in S_l are distinct. However, the ancestors of v_1 and v_2 in S_{l-1} are the same, as the ancestors of all vertices in the edges of P are the same.

Then it is easy to see that there is a path of length $2(i-l)$ between the vertices v_1 and v_2 in $\mathcal{H}$. The vertices of this path are $v_1, \pi(v_1), \ldots, \pi^{i-l}(v_1) = \pi^{i-l}(v_2), \ldots, \pi(v_2), v_2$. The edges of this path are from the sets $\mathcal{E}_{i-l}, \ldots, \mathcal{E}_{i-1}$. But as the edges of the hypergraph $\mathcal{H}_1''$ are derived from edges of $\mathcal{H}$ within the set $\mathcal{E}_j$, and as the vertices are distinct, we can combine the path P with this second path to get a cycle of length $2k+p$ in $\mathcal{H}$: a contradiction.

Thus, by Lemma 3.1, we conclude that the sum of the sizes of the edges of colour l in $\mathcal{H}_1''$ is strictly less than $3(2(k-i+l)+p)|\mathcal{V}(\mathcal{H}_1')|$. Summing over l, we find

$$\sum_{f\in\mathcal{F}_1''}|f| \leqslant 6k^2|\mathcal{V}(\mathcal{H}_1')|. \tag{3.2}$$

As every edge in $\mathcal{H}_1''$ has at least $4k$ vertices, there can be no more than $1.5k|\mathcal{V}(\mathcal{H}_1')|$ edges in $\mathcal{H}_1'$. It implies that

$$\sum_{h\in\mathcal{F}_1'}|h\cap S_{i-1}| \leqslant 6k^2|\mathcal{V}(\mathcal{H}_1')| + 2k^2|\mathcal{F}_1'| \leqslant (6k^2+3k^3)|\mathcal{V}(\mathcal{H}_1')|. \tag{3.3}$$

We conclude that $\sum_{f\in\mathcal{F}_1}|f| \leqslant 4k^4|S_{i-1} - \mathcal{V}(\mathcal{H}_1')| + (3k^3+6k^2)|\mathcal{V}(\mathcal{H}_1')|$, a contradiction. So the number of pairs of type (A) for this fixed $j<i$ is no more than $4k^4|S_{i-1}| + 4k^2|S_j| < (4k^4 + 4k^2)|S_{i-1}|$. It concludes the proof of Claim 3.3, as summing over $j<i$ we have that the number of pairs (v,h) of type (A) is less than $(4k^5+4k^3)|S_{i-1}|$. □

The proof of Claim 3.4 closely mimics the proof of Claim 3.3, and we just sketch it.

Proof of Claim 3.4. Let

$$\mathcal{F} = \{h \in \mathcal{E}(\mathcal{H}) : \exists v \in \mathcal{V}(\mathcal{H}) \text{ where } (v,h) \text{ is a pair of type (B) or (C)}\}.$$

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Let $\mathcal{F}_2 = \{f \cap S_{i-1} : f \in \mathcal{F}\}$ and let $\mathcal{H}_2$ be the hypergraph on the vertex set S_{i-1} with edges $\mathcal{F}_2$. Again, suppose $\sum_{f\in\mathcal{F}_2} |f| > 4k^4|S_{i-1}|$. Then we may delete those vertices of $\mathcal{H}_2$ of degree less than $2k^2$ and any edges that become smaller than $2k^2$, without decreasing the average degree of $\mathcal{H}_2$. Again we can thus assume that the resulting hypergraph $\mathcal{H}_2'$ with edge set $\mathcal{F}_2'$ has minimum degree $2k^2$ and that all the edges are of size at least $2k^2$. Now, we truncate and colour the edges of $\mathcal{H}_2'$ as in the proof of Claim 3.3.

As before, colouring (and deleting vertices from) each edge, we decrease the total size of the hypergraph $\mathcal{H}_2'$ by less than $2k^2|\mathcal{E}(\mathcal{H}_2')|$.

For fixed l, we claim that there is no path of length $2(k-(i-l))+p$ in $\mathcal{H}_2'$ on edges of colour l. Suppose such a path P exists, and let e_1 and e_2 be the first and last edges respectively of the path. Then, as each edge of $\mathcal{H}_2'$ has size at least $4k$, we can find a vertex $v_1 \in e_1$ that is not contained in the path P. As e_2 has colour l, we can pick a vertex v_2 from e_2 which does not belong to the path P and the ancestors of v_1 and v_2 in S_l are distinct, but the ancestors of v_1 and v_2 in S_{l-1} are the same. Again, it is easy to see that there is a path of length $2(i-l)$ between the vertices v_1 and v_2 in $\mathcal{H}$. The vertices of this path are $v_1, \pi(v_1), \ldots, \pi^{i-l}(v_1) = \pi^{i-l}(v_2), \ldots, \pi(v_2), v_2$. The edges of this path are from the sets $\mathcal{E}_{i-l}, \ldots, \mathcal{E}_{i-1}$. But this, together with the path P, is a $2k+p$ cycle in $\mathcal{H}$, a contradiction. Thus we can apply Lemma 3.1 to find that the sum of the edge sizes of colour l is strictly less than $3[2(k-(i-1-l))+p]|\mathcal{V}(\mathcal{H}_2')|$. Summing over all the colours in $\mathcal{H}_2'$, we get

$$\sum_{f\in\mathcal{E}(\mathcal{H}_2')} |f| \leqslant 6k^2|\mathcal{V}(\mathcal{H}_2')|. \tag{3.4}$$

As every edge in $\mathcal{H}_2'$ has at least $4k$ vertices, there can be no more than $1.5k|\mathcal{V}(\mathcal{H}_2')|$ edges in $\mathcal{H}_2'$. This implies that we have diminished the size of $\mathcal{H}_2'$ by less than $3k^3|\mathcal{V}(\mathcal{H}_2')|$. Therefore, we have

$$\sum_{h\in\mathcal{E}(\mathcal{H}_2')} |h \cap S_{i-1}| \leqslant (3k^3+6k^2)|\mathcal{V}(\mathcal{H}_2')| + 4k^4|S_{i-1} - \mathcal{V}(\mathcal{H}_2')| \leqslant 4k^4|S_{i-1}|, \tag{3.5}$$

a contradiction. Thus the number of pairs (v,h) of types (B) and (C) is at most $4k^4|S_{i-1}|$. □

Now, consider the hyperedges h appearing in pairs of type (D). We recall that $h - S_{i-1}$ is a subset of either S_{i-2} or S_i, and in both cases it has at least $2k^2$ elements. Let $\mathcal{H}_{i-2}$ and $\mathcal{H}_i$ denote the hypergraphs with these hyperedges $h - S_{i-1}$ on the vertex sets S_{i-2} and S_i, respectively. We may assume that $\sum_{f\in\mathcal{E}(\mathcal{H}_{i-2})} |f| \leqslant 4k^4|S_{i-2}|$ and that $\sum_{f\in\mathcal{E}(\mathcal{H}_i)} |f| \leqslant 4k^4|S_i|$, since otherwise we get a forbidden cycle in $\mathcal{H}$ as in the proofs of the Claims 3.3 and 3.4. Furthermore, we may assume that $|S_i| < n^{1/k}|S_{i-1}|$, since we are done otherwise. Then the number of hyperedges h appearing in pairs of type (D) is at most $2k^2(|S_{i-2}| + |S_i|) < 2k^2(1+n^{1/k})|S_{i-1}|$ and since these hyperedges contain at most $2k^2$ vertices in S_{i-1}, the number of pairs of type (D) is less than $4k^4(1+n^{1/k})|S_{i-1}|$.

Letting y represent the number of pairs of type (E), we find

$$(4n^{1/k}k^4 + 4k^5 + 8k^4 + 4k^3)|S_{i-1}| + y \geqslant \delta|S_{i-1}|, \tag{3.6}$$

which implies $y \geqslant (\delta - (4n^{1/k}k^4 + 4k^5 + 8k^4 + 4k^3))|S_{i-1}| \geqslant 2k^2n^{1/k}|S_{i-1}|$. If (v,h) is a pair of type (E) then h intersects S_{i-1} in at most $2k^2$ places, thus there must be at least $n^{1/k}|S_{i-1}|$ edges h such that $\exists v \in h$ with (v,h) a pair of type (E). Then clearly $|S_i| = |\mathcal{E}_i| \geqslant n^{1/k}|S_{i-1}|$. □

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