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A splicing formula for the LMO invariant

Gwénaël Massuyeau and Delphine Moussard

Abstract. We prove a "splicing formula" for the LMO invariant, which is the universal finite-type invariant of rational homology three-spheres. Specifically, if a rational homology three-sphere M is obtained by gluing the exteriors of two framed knots $K_1 \subset M_1$ and $K_2 \subset M_2$ in rational homology three-spheres, our formula expresses the LMO invariant of M in terms of the Kontsevich–LMO invariants of (M_1, K_1) and (M_2, K_2). The proof uses the techniques that Bar-Natan and Lawrence developed to obtain a rational surgery formula for the LMO invariant. In low degrees, we recover Fujita's formula for the Casson–Walker invariant, and we observe that the second term of the Ohtsuki series is not additive under "standard" splicing. The splicing formula also works when each M_i comes with a link L_i in addition to the knot K_i, hence we get a "satellite formula" for the Kontsevich–LMO invariant.

1 Introduction

The LMO invariant of closed oriented three-manifolds was constructed by Le, Murakami, and Ohtsuki [LMO98]: from the Kontsevich integral of a surgery link, they derived a quantity which is invariant under Kirby moves. According to Le [Le97], the LMO invariant of integral homology three-spheres is universal among all $\mathbb{Q}$-valued finite-type invariants in the sense of [O96a, GGP01, H00]. A more general result is true for rational homology three-spheres whose first homology groups have a fixed cardinality [M12b, M14]. In the case of a rational homology three-sphere M, the LMO invariant of M coincides with its Aarhus integral $Z(M)$ [BGRT02, BGRT04]. The invariant $Z(M)$ takes values in a graded vector space of trivalent diagrams, which is denoted by $\mathcal{A}(\emptyset)$. Its first nontrivial term, the coefficient of the θ-shaped diagram, is given by the Casson–Walker invariant $\lambda_W(M)$ as normalized in [W92]:

$$Z(M) = \emptyset + \frac{\lambda_W(M)}{4} \ominus + \begin{pmatrix} \text{trivalent diagrams} \\ \text{with} \geq 4 \text{ vertices} \end{pmatrix} \in \mathcal{A}(\emptyset).$$

The *splicing operation* is the general procedure by which, given two framed knots in oriented three-manifolds, one creates a new three-manifold by gluing the exteriors of these knots. The term "splicing" is sometimes used in the literature for what is called below the *standard* splicing, namely the case when the gluing homeomorphism

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identifies the meridian of each of the two knots with the parallel of the other one. Standard splicing has the property to preserve the class of integral homology three-spheres (a knot in such a manifold being framed with its "preferred" parallel). Another special case of splicing is given by the rational surgery on a knot.

The main result of this paper expresses the LMO invariant of a rational homology three-sphere that is defined as the splice of two framed knots in rational homology three-spheres, in terms of the Kontsevich–LMO invariants of the two knots. For simplicity, we will first consider the case of null-homologous knots (which are framed with the "preferred" parallel).

Theorem *Let $K_1 \subset M_1$ and $K_2 \subset M_2$ be null-homologous knots in rational homology three-spheres. Consider a splice M of (M_1, K_1) and (M_2, K_2) that is also a rational homology three-sphere. Then, we have*

$$Z(M) = \omega \exp\left(\frac{1}{48}\left(-S\left(\frac{p}{r}\right) + \frac{p+s}{r}\right)\ominus\right)\left\langle \partial_{p,r}\left(Z^{\ddot{\ }}(K_1)\right),\ \partial_{s,r}\left(Z^{\ddot{\ }}(K_2)\right)\right\rangle_r \tag{1.1}$$

where $Z^{\ddot{\ }}(K_i)$ is the "wheeled" version of the Kontsevich–LMO invariant of $K_i \subset M_i$, the integers p, r, and s depend on the gluing homeomorphism, the term $\omega \in \mathcal{A}(\emptyset)$ is a constant, $S(p/r)$ is a Dedekind symbol , and $\partial_{p,r}, \partial_{s,r}, \langle -, - \rangle_r$ are diagrammatic operations depending only on the indicated integers p, r, and s.

The reader is referred to Theorem 3.1 for a precise statement of this "splicing formula." Although the LMO invariant does not separate rational homology three-spheres [BL04], it is still unknown whether it separates integral homology three-spheres; nevertheless, Bar-Natan and Lawrence proved that it does separate Seifert fibered spaces that are integral homology three-spheres. They also computed the LMO invariant of all lens spaces. We expect that the splicing formula will be useful to compute (to some extent) the LMO invariant of new families of rational homology three-spheres. We particularly think of graph manifolds (within this homology type) since they are obtained from Seifert fibered spaces by repeated splicings along their fibers. A possible outcome of this study would be to decide whether the LMO invariant also distinguishes graph manifolds that are integral homology three-spheres.

The paper is organized as follows. In Section 2, we describe the splice of two framed knots in closed oriented three-manifolds—say $K_1 \subset M_1$ and $K_2 \subset M_2$—as the result of the surgery along a framed link $K_1 \sqcup H \sqcup K_2$ in the connected sum $M_1 \sharp M_2$, where H is a "chain" of Hopf links "clasping" K_1 and K_2. This surgery description of the splicing operation is well-known in, at least, two special cases: for standard splicings and for rational surgeries. Even in its most general form, the reader could deduce it from [G75, Section 5.2] by determining the handle decomposition of the plumbed four-manifold that is constructed there. Here, the surgery description of a splice is proved by purely three-dimensional arguments.

In Section 3, we prove the above theorem. Starting with the surgery description of a splice, we use the techniques that Bar-Natan and Lawrence developed to produce a rational surgery formula for the LMO invariant [BL04]; their work relies on the "Wheels and Wheeling" conjectures proved in [BLT03]. The rational surgery formula is reproved here as a consequence of the splicing formula (Corollary 3.3).

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In Section 4, the splicing formula is made explicit in low degrees. In degree 2, we recover Fujita's splicing formula [F93] for the Casson–Walker invariant λ_W, which involves the second derivatives of the Alexander polynomial of knots (Proposition 4.1). In degree 4, we obtain a splicing formula for the second term λ_2 of the Ohtsuki series which, in addition to the second and fourth derivatives of the Alexander polynomial, needs the third coefficient of the expansion of the Jones polynomial (Proposition 4.4). It turns out that, in contrast with λ_W, the invariant λ_2 is not additive with respect to standard splicing.

The last two sections provide more general versions of the splicing formula, which will be needed for the above-mentioned project of studying the LMO invariant of graph manifolds. In Section 5, we consider the situation where each M_i in the above theorem comes with an additional link L_i (disjoint from K_i). Then, the Kontsevich–LMO invariant of $L_1 \sqcup L_2 \subset M$ is expressed by the same formula (1.1) in terms of the Kontsevich–LMO invariants of $K_1 \sqcup L_1 \subset M_1$ and $K_2 \sqcup L_2 \subset M_2$ (Theorem 5.1). As explained in the monograph [EN85], splicing is a fundamental operation in knot theory: indeed, splicing subsumes all satellite operations (such as connected sum, cabling, Whitehead doubling, etc). Thus, we give a "satellite formula" for the Kontsevich–LMO invariant (Corollary 5.3) and we derive from this a result of Suetsugu [S96]. Finally, we prove in Section 6 a generalization of the above theorem where each knot K_i is allowed to be nontrivial in homology. Then formula (1.1) extends by also taking into account the self-linking numbers of K_1 and K_2 (Theorem 6.2).

The paper ends with an appendix which collects useful facts about tridiagonal matrices and their signatures in terms of Dedekind sums.

Conventions. Throughout the paper, all manifolds are assumed to be compact, connected, and oriented. The boundary of a manifold (if any) is oriented with the "outward normal first" convention.

Unless otherwise stated, all knots are oriented. Given a framed knot K in a three-manifold M, the parallel $\rho(K)$ of K (defining the framing) inherits from K an orientation, and the meridian $\mu(K)$ of K is oriented so that $\mathrm{lk}(K, \mu(K)) = +1$. Given a framed link L in a three-manifold M, the manifold obtained from M by surgery on L is denoted M_L.

2 Splicing and surgery

Let M_1 and M_2 be closed three-manifolds, and let $K_1 \subset M_1$ and $K_2 \subset M_2$ be knots. For $i = 1, 2$, let $\mathrm{N}(K_i)$ be a tubular neighborhood of K_i and set $X_i := M_i \backslash \mathrm{Int}(\mathrm{N}(K_i))$. Given an orientation-reversing homeomorphism

$$f : \partial \mathrm{N}(K_1) \xrightarrow{\cong} \partial \mathrm{N}(K_2),$$

define the *splice* of (M_1, K_1) and (M_2, K_2) along f as the closed three-manifold

$$M := X_1 \bigcup_f X_2. \tag{2.1}$$

The goal of this section is to describe M as a surgery along a framed link in the connected sum $M_1 \sharp M_2$.

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Remark 2.1 There is a more general "self-splicing" operation, which is defined as follows: given a closed (possibly disconnected) three-manifold M_0, two disjoint knots $K_1, K_2 \subset M_0$, and an orientation-reversing homeomorphism $f : \partial \mathrm{N}(K_1) \to \partial \mathrm{N}(K_2)$, the *self-splice* M of (M_0, K_1, K_2) is the result of self-gluing $M_0 \backslash \mathrm{Int}(\mathrm{N}(K_1) \cup \mathrm{N}(K_2))$ using f. Let M_i be the connected component of M_0 containing K_i for $i = 1, 2$, and assume for simplicity that $M_0 = M_1 \cup M_2$. If $M_1 \neq M_2$, then M is the splice of (M_1, K_1) and (M_2, K_2). If $M_1 = M_2$, then M is not a rational homology three-sphere even if M_0 is assumed to be so. Since we are only interested in rational homology three-spheres in this paper, we will not further consider self-splices.

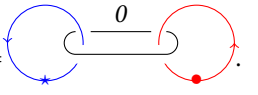
Fix a "model" $\mathbb{T}$ of the two-dimensional torus, together with a basis (α, β) of its fundamental group. The surface $\mathbb{T}$ is oriented so that the intersection number $\alpha \cdot \beta$ is +1. If some identifications $\mathbb{T} \cong -\partial \mathrm{N}(K_1)$ and $\mathbb{T} \cong \partial \mathrm{N}(K_2)$ are fixed, then the splice (2.1) can be achieved by gluing the mapping cylinder of an orientation-preserving self-homeomorphism of $\mathbb{T}$ to $X_1 \sqcup X_2$. We will use this point of view to get a surgery description of the splicing operation.

To this aim, we study *toroidal cobordisms*, namely three-manifolds C whose boundary $\partial C = (-\partial_- C) \sqcup \partial_+ C$ is decomposed into two parts, $\partial_+ C$ and $\partial_- C$, each being identified to $\mathbb{T}$. The *composition* $D \circ C$ of two toroidal cobordisms, C and D, is defined by gluing C to D using the given identifications $\partial_+ C \cong \mathbb{T} \cong \partial_- D$. There is a one-to-one correspondence between pairs (M, J), where J is a two-component framed link in a closed three-manifold M, and toroidal cobordisms C. Specifically, the cobordism corresponding to a two-component framed link $J = J' \sqcup J''$ in a closed three-manifold M is its exterior

$$C := M \backslash \big(\mathrm{Int}(\mathrm{N}(J')) \sqcup \mathrm{Int}(\mathrm{N}(J'')) \big)$$

where $\partial_- C := \partial \mathrm{N}(J')$ is identified to $\mathbb{T}$ by $(\mu(J'), \rho(J')) \mapsto (\alpha, \beta)$ and $\partial_+ C := -\partial \mathrm{N}(J'')$ is identified to $\mathbb{T}$ by $(\mu(J''), \rho(J'')) \mapsto (\alpha, -\beta)$. Here, we use the conventions that have been set at the very end of the introduction.

In this way, toroidal cobordisms can be presented by a surgery link in the complement of the trivial two-component framed link in S^3. For instance, Figure 1 represents M as the three-manifold obtained by surgery on the black framed unoriented link, while J' and J'' are the copies in M of the blue (blue $\star$–marked) knot and the red (red $\bullet$–marked) knot, respectively; each $k \in \mathbb{Z}$ decorating a black (unmarked) component is here to mean that we are performing a surgery with a framing number that differs by k from the "blackboard framing." In the sequel, we shall use these diagrammatic conventions for surgery presentations.

Lemma 2.2 *The trivial toroidal cobordism is* $I =$ 

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Proof The manifold obtained by surgery is $S^2 \times S^1$; the knots J' and J'' that correspond to the blue (blue $\star$–marked) component and to the red (red $\bullet$–marked) component, respectively, are of the form $\{*\} \times S^1$. Hence, the cobordism is an annulus times S^1, namely a thickened torus. Moreover, the meridian $\mu(J')$ can be slid over the disk glued by surgery to get $\mu(J'')$. Finally, $\rho(J')$ is clearly homotopic to $-\rho(J'')$. ■

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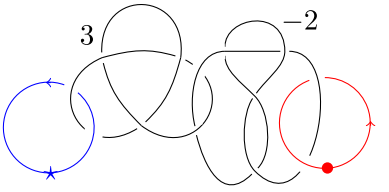


Figure 1: Surgery presentation of a toroidal cobordism.

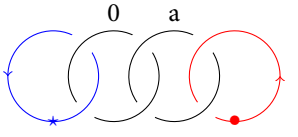
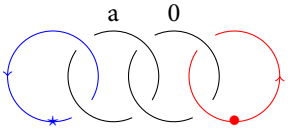
Lemma 2.3 *Let C and D be toroidal cobordisms, which are respectively associated to pairs (M, J) and (N, L) of framed links in 3–manifolds. Then the composed cobordism $D \circ C$ is associated to the pair $\big((M \sharp N)_{J'' \sharp L'}, J' \sqcup L''\big)$.*

Proof Perform the connected-sum inside the tubular neighborhoods $\mathrm{N}(J'')$ and $\mathrm{N}(L')$. Then the cobordism associated to $\big((M \sharp N)_{J'' \sharp L'}, J' \sqcup L''\big)$ is

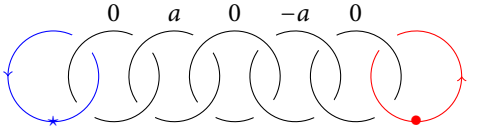
$$C' \circ \big(\mathrm{N}(J'') \sharp \mathrm{N}(L')\big)_{J'' \sharp L'} \circ C.$$

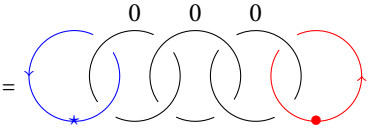
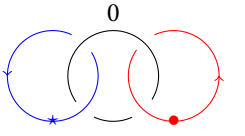
It is easily checked that $\big(\mathrm{N}(J'') \sharp \mathrm{N}(L')\big)_{J'' \sharp L'}$ is the trivial cobordism (e.g., using Lemma 2.2). ∎

For any $a \in \mathbb{Z}$, we introduce the following toroidal cobordisms:

(2.2) $\mathcal{R}_a =$  $\mathcal{L}_a =$ 

Let us compute the composition $\mathcal{L}_{-a} \circ \mathcal{R}_a$ using Lemma 2.3:

$\mathcal{L}_{-a} \circ \mathcal{R}_a =$ 

$=$  $=$  $= I$

The second and third equalities are given by slam-dunk moves as represented in the righthand side of Figure 2 (see [CG88] for the lefthand side and deduce the righthand side by sliding first the rightmost component on the leftmost one). We deduce that the cobordisms $\mathcal{R}_a$ and $\mathcal{L}_a$ are invertible, which implies that they are mapping cylinders. We focus on the $\mathcal{L}_a$ and compute their action in homology.

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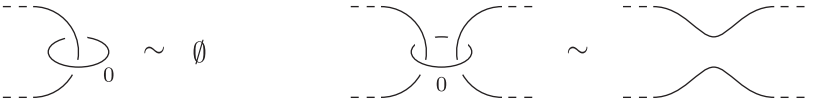


Figure 2: A slam-dunk move and a corollary.

Lemma 2.4 *The action of the cobordism $\mathcal{L}_a$ on the homology of $\mathbb{T}$ is given by the matrix $\begin{pmatrix} a & -1 \\ 1 & 0 \end{pmatrix}$ in the basis (α, β) of $H_1(\mathbb{T};\mathbb{Z})$.*

Proof Denote by L_a and L_0 the surgery components of $\mathcal{L}_a$ in (2.2) framed by a and 0, respectively, and orient each of them in the counter-clockwise direction. We have the following identities in $H_1(\mathcal{L}_a;\mathbb{Z})$:

$$\begin{array}{lll} \rho(J') = \mu(L_a) & \rho(L_a) = \mu(J') + \mu(L_0) & \rho(L_a) + a\,\mu(L_a) = 0 \\ \rho(J'') = \mu(L_0) & \rho(L_0) = \mu(L_a) + \mu(J'') & \rho(L_0) = 0 \end{array}$$

This gives $\mu(J') = a\,\mu(J'') - \rho(J'')$ and $\rho(J') = -\mu(J'')$, and we can conclude. ∎

Lemma 2.5 *For any integers $a_1, \dots, a_n$, we have*

$$\mathcal{L}_0 \circ \mathcal{L}_{a_n} \circ \cdots \circ \mathcal{L}_{a_1} = \begin{cases} a_1 \quad \cdots \quad a_{n-1}\ a_n & \text{if } n \text{ is even,} \\ a_1 \quad \cdots \quad a_{n-1}\ a_n & \text{if } n \text{ is odd.} \end{cases}$$

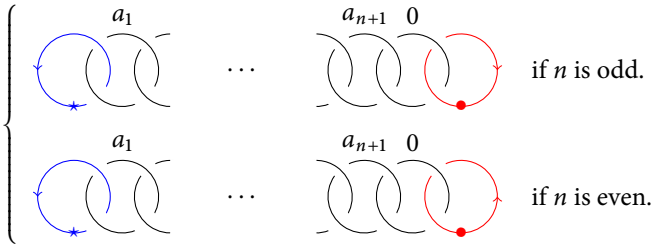
Proof For any integers a and b, we have

$$\mathcal{L}_a \circ \mathcal{L}_b = b\ 0\ 0\ a\ 0 = b\ 0\ 0\ a\ 0 = b\ a\ 0$$

where the first equality is given by Lemma 2.3, the second one by an isotopy, and the third one by slam-dunk. For any integers $a_1, \dots, a_n, a_{n+1}$, a similar computation gives

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by induction:

$\mathcal{L}_{a_{n+1}} \circ \mathcal{L}_{a_n} \circ \cdots \circ \mathcal{L}_{a_1} =$ 

Setting $a_{n+1} := 0$ and applying a slam-dunk move leads to the result. ∎

We now come back to the splice M of (M_1, K_1) and (M_2, K_2) defined by an orientation-reversing homeomorphism $f : \partial \mathrm{N}(K_1) \to \partial \mathrm{N}(K_2)$, and we assume that K_1, K_2 are framed. Using the framing, we make the following identifications:

$$-\partial X_2 = \partial \mathrm{N}(K_2) \xrightarrow{\cong} \mathbb{T}, \quad (\mu(K_2), \rho(K_2)) \longmapsto (\alpha, \beta) \tag{2.3}$$
$$\partial X_1 = -\partial \mathrm{N}(K_1) \xrightarrow{\cong} \mathbb{T}, \quad (\mu(K_1), \rho(K_1)) \longmapsto (\alpha, -\beta)$$

Thanks to these identifications, we view f as an orientation-preserving self-homeomorphism of $\mathbb{T}$, and we consider the matrix giving the action of f on $H_1(\mathbb{T}; \mathbb{Z})$:

$$\begin{pmatrix} p & r \\ q & s \end{pmatrix} \quad \text{where} \quad \begin{cases} f_*(\alpha) = p\,\alpha + q\,\beta \\ f_*(\beta) = r\,\alpha + s\,\beta \end{cases} \tag{2.4}$$

The following lemma gives a decomposition of this matrix. See [G75, Lemma 4] for a proof.

Lemma 2.6 *Any matrix in* $\mathrm{SL}_2(\mathbb{Z})$ *is a product of matrices* $\begin{pmatrix} a & -1 \\ 1 & 0 \end{pmatrix}$ *with* $a \in \mathbb{Z}$.

Applying Lemma 2.6 to $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} p & r \\ q & s \end{pmatrix}$, we get some integers $a_1, \ldots, a_n$ such that

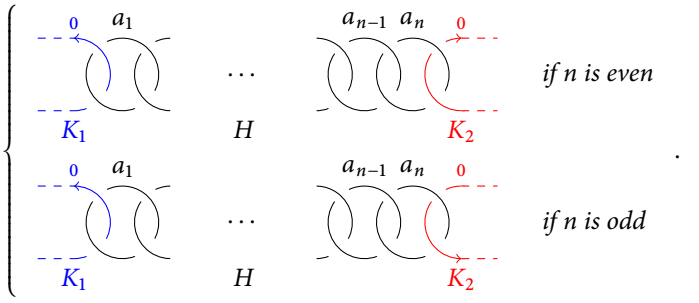
$$\begin{pmatrix} p & r \\ q & s \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_n & -1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_1 & -1 \\ 1 & 0 \end{pmatrix}. \tag{2.5}$$

Proposition 2.7 *The splice M of (M_1, K_1) and (M_2, K_2) defined by a homeomorphism f, whose matrix (2.4) in homology is decomposed as in (2.5), is the surgered manifold*

$$(M_1 \sharp M_2)_{K_1 \sqcup H \sqcup K_2}$$

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where $H := H(a_1, \dots, a_n)$ is the "chain" of Hopf links "clasping" K_1 and K_2 as shown below:



Proof Let C_f be the mapping cylinder of f, which we view as a toroidal cobordism. Since C_f is determined by its action on homology, Lemma 2.4 implies

$$C_f = \mathcal{L}_0 \circ \mathcal{L}_{a_n} \circ \cdots \circ \mathcal{L}_{a_1}.$$

Viewing M as a cobordism $\emptyset \to \emptyset$, X_1 as a cobordism $\emptyset \to \mathbb{T}$ and X_2 as a cobordism $\mathbb{T} \to \emptyset$, we have $M = X_2 \circ C_f \circ X_1$. Hence, we conclude with Lemma 2.3 and Lemma 2.5. ∎

3 Splicing formula

In the sequel, we mainly follow the notations of [BL04] which we recall in part. Let N be a rational homology three-sphere (in short, a $\mathbb{Q}HS$) and let $L \subset N$ be a framed knot. We denote by

$$Z(N, L) \in \mathcal{A}(\circlearrowleft_l)$$

the Kontsevich–LMO invariant of the pair (N, L). Here the "abstract" oriented one-manifold $\circlearrowleft$ is labeled with the letter l in order to refer to the unique connected component of L: this convention (lower-to-upper case letters) will be used throughout the text to label connected components. In the absence of knot, $Z(N) := Z(N, \emptyset)$ is the LMO invariant of N which is denoted by $\hat{Z}^{\mathrm{LMO}}(N) \in \mathcal{A}(\emptyset)$ in [BL04][1] . In the case of the standard three-sphere, $Z(L) := Z(S^3, L)$ is the Kontsevich integral of L in the version that is also denoted by $Z(L) \in \mathcal{A}(\circlearrowleft_l)$ in [BL04]. Even if $N \neq S^3$, $Z(N, L)$ will often be abbreviated to $Z(L)$ when N is clear from the context.

Let $\chi : \mathcal{A}(*_X) \to \mathcal{A}(\uparrow_X)$ be the diagrammatic analogue of the PBW isomorphism, which is defined for any finite set X. Denote by

$$\Omega := \chi^{-1} Z(\text{unknot}) \in \mathcal{A}(*)$$

the "symmetrized" value of the unknot. The latter has been found in [BLT03] to be equal to

$$\Omega = \exp\Big(\sum_{m \geq 1} b_{2m}\, \omega_{2m} \Big) \tag{3.1}$$

[1] This corresponds to the notation $\hat{\Omega}(M)$ in [LMO98] and to $\mathring{A}(M)$ in [BGRT02].

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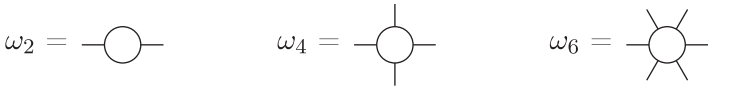


Figure 3: The diagrams ω_{2m}.

where ω_{2m} denotes the Jacobi diagram consisting of one "wheel" with $2m$ "spokes" (as depicted in Figure 3), and the *modified Bernoulli numbers*

$$b_2 = \frac{1}{48}, \quad b_4 = -\frac{1}{5760}, \quad \text{etc}, \tag{3.2}$$

are defined by the formal power series

$$\sum_{m\geq 1} b_{2m} X^{2m} := \frac{1}{2}\log\left(\frac{\sinh(X/2)}{X/2}\right) \in \mathbb{Q}[[X]].$$

Given a finite set X and $D \in \mathcal{A}(*_X)$, we denote by ∂_D (resp. $\langle D, - \rangle$) the operator that maps any Jacobi diagram E (having univalent vertices colored by X) to the sum of all ways of gluing all the x-colored vertices of D to some of (resp . to all) the x-colored vertices of E, for each $x \in X$. In particular, we will need the constant

$$\omega := \langle \Omega, \Omega \rangle = \emptyset + \frac{1}{24}\ \bigcirc\!\!\!\!-\!\!\!\!\bigcirc + \left(\begin{array}{c}\text{trivalent diagrams}\\ \text{with} \geq 6 \text{vertices}\end{array}\right) \in \mathcal{A}(\emptyset).$$

It has been proved in [BLT03] that the composition of linear isomorphisms

$$\mathcal{A}(*) \xrightarrow[\cong]{\partial_\Omega} \mathcal{A}(*) \xrightarrow[\cong]{\chi} \mathcal{A}(\uparrow) \cong \mathcal{A}(\circlearrowleft)$$

preserves the algebra structures. Thus, as in [BL04], we will work with the *wheeled* version of the Kontsevich–LMO invariant, namely

$$Z^{\urcorner}(N, L) := \partial_{\Omega_l}^{-1} \chi^{-1}(Z(N, L)) \in \mathcal{A}(*_l).$$

Here, $\Omega_l \in \mathcal{A}(*_l)$ is the same as $\Omega \in \mathcal{A}(*)$ but with each univalent vertex now labeled with l. For instance, [BL04, Lemma 3.10] tells us that

$$Z^{\urcorner}(\text{unknot}) = \omega^{-1}\Omega \in \mathcal{A}(*). \tag{3.3}$$

Let now M_1 and M_2 be $\mathbb{Q}$HS. Let $K_1 \subset M_1$ and $K_2 \subset M_2$ be null-homologous knots. We give K_i the *preferred* parallel $\rho(K_i)$, i.e ., the one that bounds a surface in the knot exterior $X_i = M_i \backslash \text{Int}(\text{N}(K_i))$. We identify ∂X_i with $\mathbb{T}$ as in (2.3), and we consider the splice

$$M := X_1 \bigcup_f X_2 \tag{3.4}$$

defined by the homeomorphism $f : \mathbb{T} \to \mathbb{T}$ that is encoded by four integers p, q, r, s as in (2.4). A Mayer–Vietoris argument shows that M is a $\mathbb{Q}$HS if and only if $r \neq 0$ (see the proof of Lemma 6.1 below).

Our main result computes the LMO invariant $Z(M)$ in terms of the wheeled Kontsevich–LMO invariants $Z^{\urcorner}(K_i) = Z^{\urcorner}(M_i, K_i)$. The formula involves Dedekind

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symbols whose definition is recalled in Appendix A. We also denote by θ the θ-shaped diagram $\ominus \in \mathcal{A}(\emptyset)$.

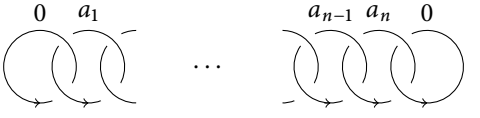
Theorem 3.1 *Let $K_1 \subset M_1$ and $K_2 \subset M_2$ be null-homologous knots in $\mathbb{Q}HS$, and let M be the splice of (M_1, K_1) and (M_2, K_2) as described by (3.4) in terms of four integers p, q, r, s. If M is also a $\mathbb{Q}HS$, then we have*

$$Z(M) = \omega \exp\left(\frac{\theta}{48}\left(-S\left(\frac{p}{r}\right) + \frac{p+s}{r}\right)\right)\left\langle \partial_{D_1}\big(Z^{\sqcup}(K_1)\big)\big|_{k_1 \to -k_2/r}, \partial_{D_2}\big(Z^{\sqcup}(K_2)\big)\right\rangle$$

where

$$D_1 := \exp\left(-\frac{p}{2r}\ \overset{k_1}{\smile}\ \overset{k_1}{}\right) \quad \text{and} \quad D_2 := \exp\left(-\frac{s}{2r}\ \overset{k_2}{\smile}\ \overset{k_2}{}\right). \tag{3.5}$$

Proof Choose a matrix decomposition as in (2.5), which leads to $n \geq 1$ integers $a_1, \ldots, a_n$, and consider the Hopf chain $\widetilde{H} := H(0, a_1, \ldots, a_n, 0) \subset S^3$ shown below:



Reading from left to right, the components of $\widetilde{H}$ are denoted $H', H_1, \ldots, H_n, H''$.

Let $\varepsilon_n K_2$ be the knot K_2 if n is odd, and the knot K_2 *with reversed orientation* if n is even. According to Proposition 2.7, M is the result of doing surgery in $M_1 \sharp M_2$ along the link $K_1 \sqcup H \sqcup \varepsilon_n K_2$ where H is the Hopf chain $H(a_1, \ldots, a_n)$ "clasping" positively K_1 on one side and $\varepsilon_n K_2$ on the other side. This link in $M_1 \sharp M_2 = (M_1 \sharp S^3) \sharp M_2$ is obtained from $(K_1 \sqcup \widetilde{H}) \sqcup K_2$ by doing the connected sum of $K_1 \subset M_1$ with $H' \subset S^3$, in a first time, and by doing the connected sum of $H'' \subset M_1 \sharp S^3$ with $\varepsilon_n K_2 \subset M_2$, in a second time. The way the wheeled Kontsevich integral behaves under connected sums of links is described in [BL04, Lemma 3.9]; the proof given there works as well for the wheeled Kontsevich–LMO invariant. Applying this result, we get

$$Z^{\sqcup}(K_1 \sqcup H \sqcup \varepsilon_n K_2) = \omega^2\, \Omega_{k_1}^{-1}\, \Omega_{k_2}^{-1}\, Z^{\sqcup}(K_1)\, Z^{\sqcup}(\widetilde{H})|_{h' \to k_1}|_{h'' \to k_2}\, Z^{\sqcup}(\varepsilon_n K_2) \tag{3.6}$$

where $Z^{\sqcup}(K_1 \sqcup H \sqcup \varepsilon_n K_2) := Z^{\sqcup}(M_1 \sharp M_2, K_1 \sqcup H \sqcup \varepsilon_n K_2)$. Besides, according to [BL04, Proposition 4.3] which computes the value of $Z^{\sqcup}$ on any Hopf chain, we have

$$Z^{\sqcup}(\widetilde{H}) = \omega^{-1} \exp\left(-\frac{\theta \sum_{i=1}^n a_i}{48}\right) \prod_{i=1}^n \Omega_{h_i}^{-1} \cdot \tag{3.7}$$

$$\cdot \exp\Big(\overset{h'}{\frown}\ \overset{h_1}{} + \sum_{i=1}^{n-1} \overset{h_i}{\frown}\ \overset{h_{i+1}}{} + \overset{h_n}{\frown}\ \overset{h''}{} + \frac{1}{2}\sum_{i=1}^n a_i \overset{h_i}{\frown}\ \overset{h_i}{} \Big).$$

Next, the way the LMO invariant of a $\mathbb{Q}HS$ can be computed from the wheeled Kontsevich integral of a surgery presentation in S^3 is described in [BL04, equation (22)]; the proof given there works as well for a surgery presentation in any $\mathbb{Q}HS$ other than S^3—see [BL04, Remark 1.7] in this connection. Applying this result to

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$M = (M_1 \sharp M_2)_{K_1 \sqcup H \sqcup \varepsilon_n K_2}$, we get the diagrammatic Gaussian integral

$$Z(M) = \exp\left(\frac{\theta\zeta(\Lambda)}{16}\right) \int Z^{\sqcup}(K_1 \sqcup H \sqcup \varepsilon_n K_2)\, \Omega_{k_1}\, \Omega_{k_2} \prod_{i=1}^{n} \Omega_{h_i}\, \mathrm{d}k_1\, \mathrm{d}h_1 \cdots \mathrm{d}h_n\, \mathrm{d}k_2,$$

where $\zeta(\Lambda)$ denotes the signature of the linking matrix Λ of $K_1 \sqcup H \sqcup \varepsilon_n K_2$ in $M_1 \sharp M_2$. Note that Λ is the tridiagonal matrix

$$\Lambda = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 1 & a_1 & 1 & \cdots & 0 & 0 \\ 0 & 1 & a_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_n & 1 \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}.$$

Combining this with (3.6) and (3.7), we deduce that

$$Z(M) = \omega\, \exp\left(\frac{\theta(3\zeta(\Lambda) - \mathrm{tr}(\Lambda))}{48}\right) \int Z^{\sqcup}(K_1)\, Z^{\sqcup}(\varepsilon_n K_2) \cdot$$
$$\cdot \exp\Big(\overset{\frown}{{}_{k_1}\ {}_{h_1}} + \sum_{i=1}^{n-1} \overset{\frown}{{}_{h_i}\ {}_{h_{i+1}}} + \overset{\frown}{{}_{h_n}\ {}_{k_2}} + \frac{1}{2} \sum_{i=1}^{n} a_i \overset{\frown}{{}_{h_i}\ {}_{h_i}} \Big) \mathrm{d}k_1\, \mathrm{d}h_1 \cdots \mathrm{d}h_n\, \mathrm{d}k_2.$$

Since K_1 is endowed with its preferred parallel, $Z^{\sqcup}(K_1)$ has no strut; similarly, $Z^{\sqcup}(\varepsilon_n K_2)$ has no strut. Thus, the "strut" part of the above integrand is fully given by the " exp" part that is apparent there. To compute this Gaussian integral, we need to invert the tridiagonal matrix Λ but, knowing that $Z^{\sqcup}(K_1)\, Z^{\sqcup}(\varepsilon_n K_2)$ has no univalent vertex colored by $h_1, \ldots, h_n$, we only need to determine the four corners of Λ^{-1}. To this purpose, we compute the 2×2 matrix associated to the tridiagonal matrix Λ (see Appendix A):

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 & -1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_n & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} a_1 & 1 \\ -1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_n & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$
$$\overset{(2.5)}{=} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} p & r \\ q & s \end{pmatrix}^{T} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} -p & q \\ r & -s \end{pmatrix}$$

and we deduce from Proposition A.1 that

$$\Lambda^{-1} = \begin{pmatrix} p/r & ? & \cdots & ? & (-1)^{n+1}/r \\ ? & ? & \cdots & ? & ? \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ ? & ? & \cdots & ? & ? \\ (-1)^{n+1}/r & ? & \cdots & ? & s/r \end{pmatrix}.$$

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Therefore, performing Gaussian integration, we get

$$Z(M) = \omega \exp\left(\frac{\theta(3\zeta(\Lambda) - \mathrm{tr}(\Lambda))}{48}\right) \cdot \cdot \left\langle \exp\left(-\frac{p}{2r} \overset{k_1}{\smile}\overset{k_1}{} + \frac{(-1)^n}{r} \overset{k_1}{\smile}\overset{k_2}{} - \frac{s}{2r} \overset{k_2}{\smile}\overset{k_2}{} \right), Z^{\ddot{\cup}}(K_1)\, Z^{\ddot{\cup}}(\varepsilon_n K_2) \right\rangle$$

and, since $Z^{\ddot{\cup}}(-K_2)$ is obtained from $Z^{\ddot{\cup}}(K_2)$ by the change $k_2 \to -k_2$, this is equivalent to

$$Z(M) = \omega \exp\left(\frac{\theta(3\zeta(\Lambda) - \mathrm{tr}(\Lambda))}{48}\right) \cdot \cdot \left\langle \exp\left(-\frac{p}{2r} \overset{k_1}{\smile}\overset{k_1}{} - \frac{1}{r} \overset{k_1}{\smile}\overset{k_2}{} - \frac{s}{2r} \overset{k_2}{\smile}\overset{k_2}{} \right), Z^{\ddot{\cup}}(K_1)\, Z^{\ddot{\cup}}(K_2) \right\rangle$$

whatever the parity of n is. Using now Theorem A.3, we obtain

$$Z(M) = \omega \exp\left(\frac{\theta}{48}\left(-S\left(\frac{p}{r}\right) + \frac{p+s}{r}\right)\right) \cdot \cdot \left\langle \exp\left(-\frac{p}{2r} \overset{k_1}{\smile}\overset{k_1}{} - \frac{1}{r} \overset{k_1}{\smile}\overset{k_2}{} - \frac{s}{2r} \overset{k_2}{\smile}\overset{k_2}{} \right), Z^{\ddot{\cup}}(K_1)\, Z^{\ddot{\cup}}(K_2) \right\rangle.$$

Using the notation (3.5), this is equivalent to

$$Z(M) = \omega \exp\left(\frac{\theta}{48}\left(-S\left(\frac{p}{r}\right) + \frac{p+s}{r}\right)\right) \cdot \cdot \left\langle \exp\left(-\frac{1}{r} \overset{k_1}{\smile}\overset{k_2}{} \right), \partial_{D_1}\left(Z^{\ddot{\cup}}(K_1)\right) \partial_{D_2}\left(Z^{\ddot{\cup}}(K_2)\right) \right\rangle$$

and the conclusion easily follows. ∎

Example 3.2 When the splice of (M_1, K_1) and (M_2, K_2) is standard, Theorem 3.1 simplifies as follows. Then $\mu(K_1)$ and $\rho(K_1)$ are glued along $\rho(K_2)$ and $\mu(K_2)$, respectively: thus $p = s = 0$ and $q = -r = 1$. Hence, by identifying the symbols k_1 and k_2, we get

$$Z(M) = \omega \left\langle Z^{\ddot{\cup}}(K_1)\,,\, Z^{\ddot{\cup}}(K_2) \right\rangle. \tag{3.8}$$

Bar-Natan and Lawrence developed their techniques (which we have intensively used in the proof of Theorem 3.1) to prove a *rational* surgery formula for the LMO invariant [BL04, equation (23)]. We now explain how their formula—in the case of a rational surgery along a single knot—is recovered from Theorem 3.1.

Corollary 3.3 (Bar-Natan and Lawrence) *Let $L \subset N$ be a null-homologous knot in a $\mathbb{Q}$HS, with framing given by the preferred parallel. Let r, s be nonzero coprime integers,*

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and let M be obtained from N by an (r/s)–surgery along L. Then

$$Z(M) = \exp\left(\frac{\theta}{48}\left(-S\left(\frac{s}{r}\right)+\frac{s}{r}\right)\right)\left\langle \Omega_{l/r}\,,\, \partial_D\big(Z^{\curlywedge}(L)\big)\right\rangle \quad \textit{with } D := \exp\Big(-\frac{s}{2r}\ {}^{l}\!\smile^{l}\Big)$$

and the latter identity is equivalent to [BL04, equation (23)].

Proof We consider $(M_1, K_1) := (S^3, \text{unknot})$, $(M_2, K_2) := (N, L)$ and the integers p, q are chosen so that $ps - qr = 1$. Then, the corresponding splice is the three-manifold M. Recall from (3.3) that $Z^{\curlywedge}(K_1) = \omega^{-1}\Omega_{k_1}$. Hence, in this particular situation, Theorem 3.1 gives

$$Z(M) = \exp\left(\frac{\theta}{48}\left(-S\left(\frac{p}{r}\right)+\frac{p+s}{r}\right)\right)\left\langle \partial_{D_1}\big(\Omega_{k_1}\big)\big|_{k_1\to -k_2/r}\,,\, \partial_{D_2}\big(Z^{\curlywedge}(K_2)\big)\right\rangle$$

where

$$D_1 := \exp\left(-\frac{p}{2r}\ {}^{k_1}\!\smile^{k_1}\right) \quad \text{and} \quad D_2 := \exp\left(-\frac{s}{2r}\ {}^{k_2}\!\smile^{k_2}\right).$$

Besides, recall from [BL04, Corollary 3.4] that, for any $\alpha \in \mathbb{Q}$, we have

$$\partial_E(\Omega) = \exp\left(\frac{\alpha\theta}{48}\right)\Omega \ \in \mathcal{A}(*) \quad \text{where } E := \exp\left(\frac{\alpha}{2}\ \smile\right). \tag{3.9}$$

In particular, $\partial_{D_1}\Omega_{k_1} = \exp\big(-\frac{p}{r}\frac{\theta}{48}\big)\Omega_{k_1}$. Hence, using that Ω only shows diagrams with an even number of univalent vertices (see (3.1)) and that $S(p/r) = S(s/r)$ (since $p \equiv s^{-1} \mod r$), we get

$$Z(M) = \exp\left(\frac{\theta}{48}\left(-S\left(\frac{s}{r}\right)+\frac{s}{r}\right)\right)\left\langle \Omega_{l/r}\,,\, \partial_D\big(Z^{\curlywedge}(L)\big)\right\rangle. \tag{3.10}$$

We now check the equivalence between this formula and [BL04, equation (23)], which states (in the case of $N = S^3$) that

$$Z(M) = \exp\left(\frac{\theta}{48}\left(3\,\mathrm{sign}\left(\frac{r}{s}\right)+S\left(\frac{r}{s}\right)-\frac{r}{s}\right)\right)\left\langle \exp\Big(\frac{-s}{2r}\ {}^{l}\!\smile^{l}\Big), Z^{\curlywedge}(L)\,\Omega_{l/s}\right\rangle. \tag{3.11}$$

Note that

$$\left\langle \exp\Big(\frac{-s}{2r}\ {}^{l}\!\smile^{l}\Big), Z^{\curlywedge}(L)\,\Omega_{l/s}\right\rangle = \left\langle \partial_D(\Omega_{l/s})\big|_{l\to -sl/r}\,,\, \partial_D\big(Z^{\curlywedge}(L)\big)\right\rangle;$$

besides, using (3.9) one more time, we obtain

$$\partial_D(\Omega_{l/s}) = \exp\left(-\frac{\theta}{48}\frac{1}{rs}\right)\Omega_{l/s};$$

hence, we deduce that

$$\left\langle \exp\Big(\frac{-s}{2r}\ {}^{l}\!\smile^{l}\Big), Z^{\curlywedge}(L)\,\Omega_{l/s}\right\rangle = \exp\left(-\frac{\theta}{48}\frac{1}{rs}\right)\left\langle \Omega_{l/r}\,,\, \partial_D(Z^{\curlywedge}(L))\right\rangle.$$

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Then the equivalence between (3.10) and (3.11) follows from the reciprocity law (A.6) for Dedekind symbols. ■

Example 3.4 Think of the lens space $L(r,s)$ as S^3 surgered along the (r/s)–framed unknot. Then, a direct application of Corollary 3.3 using (3.3) and (3.9), again, gives

$$Z\big(L(r,s)\big) = \omega^{-1} \exp\left(-\frac{\theta}{48} S\left(\frac{s}{r}\right)\right)\big\langle \Omega_{l/r}, \Omega_l \big\rangle.$$

This formula is easily seen to be equivalent to [BL04, Proposition 5.1].

4 Low degree formulas

In this section, we apply Theorem 3.1 to get splicing formulas for the lowest degree terms of the LMO invariant. In the sequel, the *degree* of a Jacobi diagram refers to its *internal* degree, *ie* its number of trivalent vertices, and the symbol " $\equiv_n$ " stands for an equality "up to terms of degree at least n."

According to [LMO98, Proposition 5.3], the LMO invariant of any $\mathbb{Q}$HS M can be written as follows up to degree 5:

$$Z(M) \equiv_6 \emptyset + \frac{1}{4}\lambda_{\mathrm{W}}(M)\, \theta + \lambda_2(M)\, \Theta + \frac{1}{32}\lambda_{\mathrm{W}}(M)^2\, \theta\theta. \tag{4.1}$$

Here, $\lambda_{\mathrm{W}}(M) \in \mathbb{Q}$ is the Casson–Walker invariant of M as normalized in [W92], and $\lambda_2(M) \in \mathbb{Q}$ is a finite-type invariant of degree 4 (which, up to a multiplicative constant, is the second term of the Ohtsuki series of M [O96b]).

In [F93, Corollary 1.2], Fujita gives a formula for the Casson–Walker invariant of a $\mathbb{Q}$HS obtained as the splice of two knots in integral homology three-spheres. The next proposition generalizes his result to the splice of null-homologous knots in $\mathbb{Q}$HS. Given a null-homologous knot K in a $\mathbb{Q}$HS M, denote by

$$\Delta_K(t) \in \mathbb{Q}[t^{\pm 1}]$$

its Alexander polynomial normalized so that $\Delta_K(t^{-1}) = \Delta_K(t)$ and $\Delta_K(1) = 1$ (see [M12a, Theorem 1.5]).

Proposition 4.1 (Fujita) *Let $K_1 \subset M_1$ and $K_2 \subset M_2$ be null-homologous knots in $\mathbb{Q}$HS and let M be the splice of (M_1,K_1) and (M_2,K_2) described by (3.4) in terms of four integers p,q,r,s. If M is also a $\mathbb{Q}$HS, then*

$$\lambda_{\mathrm{W}}(M) = \lambda_{\mathrm{W}}(M_1) + \lambda_{\mathrm{W}}(M_2) - \frac{1}{12} S\left(\frac{p}{r}\right) + \frac{p}{r}\Delta''_{K_1}(1) + \frac{s}{r}\Delta''_{K_2}(1).$$

Example 4.2 For a standard splicing, we have $\lambda_{\mathrm{W}}(M) = \lambda_{\mathrm{W}}(M_1) + \lambda_{\mathrm{W}}(M_2)$. This additivity formula for λ_{W} was obtained in [BN90, FN88].

Remark 4.3 In the case $(M_1,K_1) := (S^3, \text{unknot})$, Proposition 4.1 specializes to Walker's rational surgery formula for λ_{W}— see [W92, Proposition 6.2].

To express the splicing formula for λ_2, we need the following invariant of a null-homologous knot K in a $\mathbb{Q}$HS M: let $\nu(K)$ be the coefficient of Θ in $\chi^{-1}Z(K)$

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where $Z(K) = Z(M, K)$ is the Kontsevich–LMO invariant. When $M = S^3$, we have $v(K) = -\frac{1}{24} j_3(K)$ where $j_3(K)$ is the coefficient of h^3 in the Jones polynomial $V_K(t)$ evaluated at $t := e^h$ (see [I17, Lemma 2.1]).

Proposition 4.4 *Let $K_1 \subset M_1$ and $K_2 \subset M_2$ be null-homologous knots in $\mathbb{Q}HS$ and let M be the splice of (M_1, K_1) and (M_2, K_2) described by (3.4) in terms of four integers p, q, r, s. If M is also a $\mathbb{Q}HS$, then*

$$\begin{aligned}\lambda_2(M) = \lambda_2(M_1) + \lambda_2(M_2) + \frac{1}{1152}\left(\frac{1}{r^2} - 1\right)\\ + \frac{1}{96}\left(1 - \frac{1}{r^2}\right)\left(\Delta''_{K_1}(1) + \Delta''_{K_2}(1)\right) + \frac{9}{16}\frac{p^2}{r^2}\Delta''_{K_1}(1) + \frac{9}{16}\frac{s^2}{r^2}\Delta''_{K_2}(1)\\ + \frac{7}{32}\frac{p^2}{r^2}\left(\Delta''_{K_1}(1)\right)^2 + \frac{7}{32}\frac{s^2}{r^2}\left(\Delta''_{K_2}(1)\right)^2 + \frac{1}{8r^2}\Delta''_{K_1}(1)\Delta''_{K_2}(1)\\ - \frac{5}{96}\frac{p^2}{r^2}\Delta^{(4)}_{K_1}(1) - \frac{5}{96}\frac{s^2}{r^2}\Delta^{(4)}_{K_2}(1) - \frac{p}{r}v(K_1) - \frac{s}{r}v(K_2).\end{aligned}$$

Example 4.5 For a standard splicing, we have $\lambda_2(M) = \lambda_2(M_1) + \lambda_2(M_2) + \frac{1}{8}\Delta''_{K_1}(1)\,\Delta''_{K_2}(1)$. We note that the same nonadditivity formula is satisfied for the "SU(3) Casson invariant" in the case when K_1 and K_2 are torus knots in S^3 [BH09], but the relation between this gauge-theoretical invariant and finite-type invariants does not seem to be known yet.

Remark 4.6 In the case $(M_1, K_1) := (S^3, \text{unknot})$, Proposition 4.4 specializes to Ito's rational surgery formula for λ_2—see [I17, Theorem 1.2].

To deduce from Theorem 3.1, the above splicing formulas for λ_W and λ_2, we need first to determine the low-degree terms of the wheeled Kontsevich–LMO invariant $Z^{\curlywedge}$. The following result of Kricker (consequence of [K00, Theorem 1.0.8]) gives the "one-loop" part of $Z^{\curlywedge}$. Kricker indeed works with a knot in an integral homology three-sphere, but the whole article adapts in the case of a null-homologous knot in a $\mathbb{Q}HS$, with the above normalization of the Alexander polynomial.

Theorem 4.7 (Kricker) *Let K be a null-homologous knot in a $\mathbb{Q}HS$, with framing given by the preferred parallel. Then the Kontsevich–LMO invariant of K writes $\chi^{-1}Z(K) = \mathrm{Wh}(K)\,\exp(R)$ where R is a $\mathbb{Q}$-linear combination of connected Jacobi diagrams with at least two loops and*

$$\mathrm{Wh}(K) := \Omega \exp\left(-\frac{1}{2}\log\left(\Delta_K(e^h)\right)\Big|_{h^{2m}\to\omega_{2m}}\right).$$

In the computations to come, it will be more convenient to use the Conway normalization

$$\nabla_K(z) = 1 + \sum_{m>0} a_{2m}(K) z^{2m}$$

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of the Alexander polynomial of a knot K, given by $\nabla_K(t-t^{-1}) = \Delta_K(t^2)$. In particular, derivating four times this equality gives

$$\Delta_K''(1) = 2a_2(K) \quad \text{and} \quad \Delta_K^{(4)}(1) = 24(a_2(K)+a_4(K)). \tag{4.2}$$

Lemma 4.8 *Let K be a null-homologous knot in a $\mathbb{Q}$HS M, and let $a_{2i} := a_{2i}(K)$ with $i \geq 1$ be the coefficients of the Alexander–Conway polynomial of K. Then*

$$\chi^{-1}Z(K) \equiv_6 \emptyset + \frac{\lambda_W(M)}{4}\,\ominus + \lambda_2(M)\,\boxminus + \frac{\lambda_W(M)^2}{32}\,\ominus\ominus + \Big(b_2 - \frac{1}{2}a_2\Big)\,\overset{k}{\bigcirc}\overset{k}{}$$
$$+ \frac{\lambda_W(M)}{4}\Big(b_2 - \frac{1}{2}a_2\Big)\,\ominus\,\overset{k}{\bigcirc}\overset{k}{} + \frac{1}{2}\Big(b_2 - \frac{1}{2}a_2\Big)^2\,\overset{k}{\bigcirc}\overset{k}{}\;\overset{k}{\bigcirc}\overset{k}{}$$
$$+ \nu(K)\,\overset{k}{\bigcirc}\overset{k}{} + \Big(b_4 - \frac{1}{24}a_2 + \frac{1}{4}a_2^2 - \frac{1}{2}a_4\Big)\,\overset{k}{\underset{k}{\times}}\overset{k}{}$$

where b_2, b_4 are the first modified Bernoulli numbers (3.2).

Proof It is well-known (and easily verified from the AS and IHX relations) that the following diagrams constitute a basis of the "strut-less" *and* "connected" part of $\mathcal{A}(*_k)$ up to degree 5:

$$\emptyset,\quad \ominus,\quad \overset{k}{\bigcirc}\overset{k}{},\quad \boxminus,\quad \overset{k}{\ominus}\overset{k}{},\quad \overset{k}{\underset{k}{\times}}\overset{k}{}$$

Hence, by the group-like property of $\chi^{-1}Z(K)$ and the fact that its purely trivalent part reduces to $Z(M)$, it suffices to show that its connected one-loop part equals

$$\Big(b_2 - \frac{1}{2}a_2\Big)\,\overset{k}{\bigcirc}\overset{k}{} + \Big(b_4 - \frac{1}{24}a_2 + \frac{1}{4}a_2^2 - \frac{1}{2}a_4\Big)\,\overset{k}{\underset{k}{\times}}\overset{k}{}$$

up to degree 5. We have

$$\Delta_K\big(e^h\big) = \nabla_K\Big(e^{\frac{h}{2}} - e^{-\frac{h}{2}}\Big) = 1 + a_2h^2 + \Big(\frac{a_2}{12} + a_4\Big)h^4 + (\deg \geq 6)$$

thus

$$\log\big(\Delta_K(e^h)\big) = a_2h^2 + \Big(\frac{a_2}{12} - \frac{a_2^2}{2} + a_4\Big)h^4 + (\deg \geq 6)$$

and we deduce that

$$\log \mathrm{Wh}(K) \equiv_6 \Big(b_2 - \frac{1}{2}a_2\Big)\omega_2 + \Big(b_4 - \frac{1}{24}a_2 + \frac{1}{4}a_2^2 - \frac{1}{2}a_4\Big)\omega_4.$$

We conclude thanks to Theorem 4.7. ∎

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Lemma 4.9 *Let K be a null-homologous knot in a $\mathbb{Q}$HS M, and let $a_{2i} := a_{2i}(K)$ with $i \geq 1$ be the coefficients of the Alexander–Conwayy polynomial of K. Then*

$$Z^{\Box}(K) \equiv_6 \emptyset + \frac{\lambda_W(M)}{4}\ \ominus + \left(a_2b_2 - 2b_2^2 + \lambda_2(M)\right)\ \oslash + \frac{\lambda_W(M)^2}{32}\ \ominus\ominus$$
$$+ \frac{\lambda_W(M)}{4}\left(b_2 - \frac{1}{2}a_2\right)\ \ominus\ \theta^{k\ k} + \frac{1}{2}\left(b_2 - \frac{1}{2}a_2\right)^2\ \theta^{k\ k}\ \theta^{k\ k}$$
$$+ \left(b_2 - \frac{1}{2}a_2\right)\ \theta^{k\ k} + v(K)\ \oslash^{k\ k} + \left(b_4 - \frac{1}{24}a_2 + \frac{1}{4}a_2^2 - \frac{1}{2}a_4\right)\ \theta^{k\ k}_{k\ k}.$$

Proof We have $\Omega^{-1} \equiv_3 \emptyset - b_2\omega_2$. Since $\chi^{-1}Z(K)$ is strut-less, we have

$$Z^{\Box}(K) = \partial_{\Omega^{-1}}\left(\chi^{-1}Z(K)\right) \equiv_6 \chi^{-1}Z(K) - 2b_2\big(\text{coef. of } \omega_2 \text{ in } \chi^{-1}Z(K)\big)\ \oslash$$

and we conclude with Lemma 4.8. ∎

We have now gathered all we need to prove the splicing formulas for λ_W and λ_2.

Proof of Proposition 4.1 In the equality of Theorem 3.1, the coefficient of θ in the left-hand side is $\lambda_W(M)/4$, while the coefficient of θ in the right-hand side is

$$\frac{1}{48}\left(\frac{s+p}{r} - S\left(\frac{p}{r}\right)\right) + \frac{1}{4}\lambda_W(M_1) + \frac{1}{4}\lambda_W(M_2)$$
$$- \frac{p}{r}\left(b_2 - \frac{1}{2}a_2(K_1)\right) - \frac{s}{r}\left(b_2 - \frac{1}{2}a_2(K_2)\right).$$

We conclude by substituting the values (3.2) and (4.2). ∎

Proof of Proposition 4.4 We compute the coefficient of $\oslash$ in the righthand side of the equality in Theorem 3.1. First of all, the contribution of ω is $2b_2^2$. Next, $\partial_{D_i}\left(Z^{\Box}(K_i)\right)$ contributes in two ways to this coefficient: there is an "individual" contribution through its $\oslash$ part, and there is a "mutual" contribution through its θ part. On the one hand, the individual contribution of $\partial_{D_i}\left(Z^{\Box}(K_i)\right)$ is

$$a_2b_2 - 2b_2^2 + \lambda_2(M_i) - \frac{t}{r}v(K_i) + \frac{t^2}{r^2}\left(b_2 - \frac{1}{2}a_2\right)^2 + \frac{5}{2}\frac{t^2}{r^2}\left(b_4 - \frac{1}{24}a_2 + \frac{1}{4}a_2^2 - \frac{1}{2}a_4\right)$$

where $a_\ell := a_\rho(K_i)$, $t := p$ if $i = 1$ and $t := s$ if $i = 2$ (to compute the subcontribution resulting from the Θ term of $Z^{\Box}(K_i)$, we use an IHX relation that gives $\oslash = 2\ \triangle$). On the other hand, the "mutual" contribution of $\partial_{D_1}\left(Z^{\Box}(K_1)\right)$ and $\partial_{D_2}\left(Z^{\Box}(K_2)\right)$ is

$$\frac{2}{r^2}\left(b_2 - \frac{1}{2}a_2(K_1)\right)\left(b_2 - \frac{1}{2}a_2(K_2)\right).$$

Summing all these contributions and substituting the values (3.2) and (4.2), we obtain the desired formula for $\lambda_2(M)$. ∎

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5 Splicing and satellite operations

The following generalization of Theorem 3.1 is proved exactly in the same way.

Theorem 5.1 *Let $K_1 \subset M_1$ and $K_2 \subset M_2$ be null-homologous knots in $\mathbb{Q}HS$ and let M be the splice of (M_1, K_1) and (M_2, K_2) as described by (3.4) in terms of four integers p, q, r, s, assuming that $r \neq 0$. For each $i \in \{1, 2\}$, let $L_i \subset M_i$ be a framed link disjoint from K_i and let*

$$Z^{\curlywedge_{k_i}}(M_i, K_i \sqcup L_i) := \partial_{\Omega_{k_i}}^{-1} \chi_{k_i}^{-1} Z(M_i, K_i \sqcup L_i)$$

be the Kontsevich–LMO invariant of $(M_i, K_i \sqcup L_i)$ "wheeled" at K_i. Then

$$Z(M, L_1 \sqcup L_2) = \omega \exp\left(\frac{\theta}{48}\left(-S\left(\frac{p}{r}\right) + \frac{p+s}{r}\right)\right) \cdot \\ \cdot \left\langle \partial_{D_1}\left(Z^{\curlywedge_{k_1}}(M_1, K_1 \sqcup L_1)\right)\Big|_{k_1 \to -k_2/r}, \partial_{D_2}\left(Z^{\curlywedge_{k_2}}(M_2, K_2 \sqcup L_2)\right)\right\rangle$$

where

$$D_1 := \exp\left(-\frac{p}{2r} \overset{k_1}{\smile} \overset{k_1}{}\right) \quad \text{and} \quad D_2 := \exp\left(-\frac{s}{2r} \overset{k_2}{\smile} \overset{k_2}{}\right).$$

Example 5.2 For a standard splice, identifying the symbols k_1 and k_2, we get

$$Z(M, L_1 \sqcup L_2) = \omega \left\langle Z^{\curlywedge_{k_1}}(M_1, K_1 \sqcup L_1), Z^{\curlywedge_{k_2}}(M_2, K_2 \sqcup L_2)\right\rangle. \tag{5.1}$$

Let P be a framed link in the solid torus $S^1 \times D^2$ and let L be a framed knot in an oriented three-manifold N. We identify $S^1 \times D^2$ with the tubular neighborhood $\mathrm{N}(L)$ so that $S^1 \times \{0\}$ and $S^1 \times \{1\}$ correspond to the knot L and its parallel $\rho(L)$, respectively. The *P-satellite* of L is the image of P in N by this identification $S^1 \times D^2 \cong \mathrm{N}(L)$; this framed link is denoted by

$$L_P \subset N.$$

It is well-known that L_P can be constructed by splicing: identify $S^1 \times D^2$ with the exterior of the unknot $U \subset S^3$, so that both $S^1 \times \{0\}$ and $S^1 \times \{1\}$ are oriented meridians of U; then the standard splice of (N, L) and (S^3, U) is a copy of N where $P \subset S^3 \setminus \mathrm{N}(U)$ corresponds to L_P. Hence, the following is an immediate consequence of Theorem 5.1 in the simplified version of (5.1).

Corollary 5.3 *Let $L \subset N$ be a null-homologous knot in a $\mathbb{Q}HS$ (with framing given by the preferred parallel), and let $P \subset S^1 \times D^2$ be a framed link. Then, by identifying the symbols l and u, we have*

$$Z(N, L_P) = \omega \left\langle Z^{\curlywedge}(N, L), Z^{\curlywedge_u}(S^3, U \sqcup P)\right\rangle. \tag{5.2}$$

We now derive from the "satellite formula" (5.2) a generalization of a formula of Suetsugu [S96]. The latter involves the extension of the Kontsevich integral to framed links in a solid torus [S96] or, equivalently, in a thickened annulus [AMR98]. For an

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n-component framed link $P \subset S^1 \times D^2$, this invariant

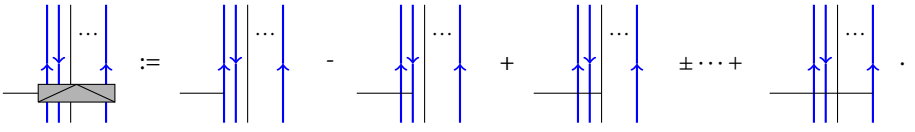
$$Z^{\circ}(P) \in \mathcal{A}^{\circ}(\circlearrowleft^1 \cdots \circlearrowleft^n)$$

is valued in the space generated by homotopy classes of immersions from chord diagrams on $\circlearrowleft^1 \cdots \circlearrowleft^n$ to the annulus, modulo the 4T relation.

Let D be a Jacobi diagram on $\circlearrowleft$ (possibly with some univalent vertices colored by a finite set X) and let E be a chord diagram on $\circlearrowleft^1 \cdots \circlearrowleft^n$ in the annulus. One can produce from D and E a linear combination of Jacobi diagrams

$$D|_{\circlearrowleft \to E}$$

on $\circlearrowleft^1 \cdots \circlearrowleft^n$ (possibly with some univalent vertices colored by X) by the following procedure: (1) thicken the one-manifold $\circlearrowleft$ in D to an annulus, replacing every univalent vertex of D by a box directed in the same direction as $\circlearrowleft$; (2) put E inside this annulus; and (3) apply the "box notation" below:



Here, some pieces of chords of E and some arcs of the one-manifold $\circlearrowleft^1 \cdots \circlearrowleft^n$ (which are shown in bold blue) go through the box, and each of them contributes to one summand in the box notation; an arc contributes with a + or a −, depending on the compatibility of its orientation with the direction of the box; a piece of chord always contributes with a +, the orientation of the new trivalent vertex being determined by the direction of the box. Of course, for step (2) to make sense, it is necessary to assume that (after a small homotopy) no univalent vertex of E (i.e., no extremity of any chord) is contained in one of the boxes. That the resulting operation

$$\mathcal{A}(\circlearrowleft, *_X) \times \mathcal{A}^{\circ}(\circlearrowleft^1 \cdots \circlearrowleft^n) \longrightarrow \mathcal{A}(\circlearrowleft^1 \cdots \circlearrowleft^n, *_X), \quad (D, E) \longmapsto D|_{\circlearrowleft \to E}$$

is well-defined follows from the STU, AS, and IHX relations in the target. Besides, it is obvious that this operation is compatible with additional "link relations" on X. This corresponds to the operation $\diamond$ in [S96].

Corollary 5.4 (Suetsugu) *Let $L \subset N$ be a null-homologous knot in a $\mathbb{Q}$HS (with framing given by the preferred parallel), and let $P \subset S^1 \times D^2$ be a framed link. Then, we have*

$$Z(N, L_P) = \left(Z(N, L) \sharp v^{-1}\right)\Big|_{\circlearrowleft \to Z^{\circ}(P)} \tag{5.3}$$

where $\sharp$ denotes the usual multiplication in $\mathcal{A}(\circlearrowleft) \cong \mathcal{A}(\uparrow)$ and $v = Z(\mathit{unknot})$.

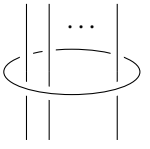
Proof There is a tangle Q in $[-1, +1] \times D^2$ whose "closure" in $S^1 \times D^2$ is P. Thus, we obtain a decomposition of the following form for $U \sqcup P \subset S^3$ in the strict monoidal

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category of tangles:

$$U \sqcup P = \begin{array}{|c|}\hline \text{“caps”} \\ \hline H \mid Q \\ \hline \text{“cups”} \\ \hline \end{array}\,. \tag{5.4}$$

Here, “caps” (resp. “cups”) denote the appropriate cabling (with possible orientation changes) of the elementary tangle $\cap$ (resp. $\cup$), and H is the “open” Hopf link whose “opened” component has been cabled accordingly:

$H :=$ 

In the decomposition (5.4), U is the closed component of H. By choosing a parenthesization of the top boundary points of Q, we can upgrade (5.4) to a decomposition in the nonstrict monoidal category of q-tangles. The value of $Z(H)$ is deduced from [BL04, Theorem 4] by cabling the opened component of the “open” Hopf link. Then, using the definition of the invariant $Z^{\bullet}$ as it is done in [AMR98], we obtain that

$$\chi_u^{-1} Z(S^3, U \sqcup P) = \Omega_u \cdot \exp_{\sharp}\Big({}^{u}\!\!\circlearrowleft \Big)\Big|_{\circlearrowleft \to Z^{\bullet}(P)}.$$

Then, we deduce from formula (5.2) that

$$\begin{aligned} Z(N, L_P) &= \omega\left\langle Z^{\ddagger}(N,L)\cdot\Omega_l^{-1},\ \Omega_l\cdot\exp_{\sharp}\Big({}^{l}\!\!\circlearrowleft\Big)\Big|_{\circlearrowleft\to Z^{\bullet}(P)}\right\rangle \\ &= \omega\left\langle \partial_{\Omega_l}\big(Z^{\ddagger}(N,L)\cdot\Omega_l^{-1}\big),\ \exp_{\sharp}\Big({}^{l}\!\!\circlearrowleft\Big)\Big|_{\circlearrowleft\to Z^{\bullet}(P)}\right\rangle \end{aligned}$$

and, by “wheeling” [BL04, Theorem 3], we get

$$\begin{aligned} Z(N, L_P) &= \omega\left\langle \chi_l^{-1}\big(Z(N,L)\ \sharp\ \chi_l\partial_{\Omega_l}(\Omega_l^{-1})\big),\ \exp_{\sharp}\Big({}^{l}\!\!\circlearrowleft\Big)\Big|_{\circlearrowleft\to Z^{\bullet}(P)}\right\rangle \\ &= \omega\left\langle \chi_l^{-1}\big(Z(N,L)\ \sharp\ \big(\chi_l\partial_{\Omega_l}(\Omega_l)\big)^{-1}\big),\ \exp_{\sharp}\Big({}^{l}\!\!\circlearrowleft\Big)\Big|_{\circlearrowleft\to Z^{\bullet}(P)}\right\rangle. \end{aligned}$$

By [BL04, Proposition 3.3], we have $\partial_{\Omega}(\Omega) = \omega\Omega$. Hence $\chi\partial_{\Omega}(\Omega) = \omega\nu$ and we conclude that

$$\begin{aligned} Z(N, L_P) &= \left\langle \chi_l^{-1}\big(Z(N,L)\ \sharp\ \nu^{-1}\big),\ \exp_{\sharp}\Big({}^{l}\!\!\circlearrowleft\Big)\Big|_{\circlearrowleft\to Z^{\bullet}(P)}\right\rangle \\ &= \big(Z(N,L)\ \sharp\ \nu^{-1}\big)\big|_{\circlearrowleft\to Z^{\bullet}(P)}. \end{aligned}$$

■

Remark 5.5 Formula (5.3) is stated in [S96, Theorem 2.5] for the case $N := S^3$. An indication of proof is given, but the factor “ $\sharp\, \nu^{-1}$ ” seems to be missing there.

Remark 5.6 If $N := S^3$ and P is connected in Corollary 5.4, then we can view L and P as one-component “bottom tangles” in handlebodies of genus 0 and 1, respectively. In such a case, formula (5.3) follows immediately from the functoriality of the invariant of “bottom tangles in handlebodies” that is constructed in [HM17] as an extension of the Kontsevich integral.

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6 The case of knots that are not trivial in homology

Let now M_1 and M_2 be $\mathbb{Q}$HS and let $K_1 \subset M_1$ and $K_2 \subset M_2$ be framed knots. We identify the boundary of $X_i = M_i \backslash \mathrm{Int}(\mathrm{N}(K_i))$ with $\mathbb{T}$ as in (2.3) and we consider the splice

$$M := X_1 \bigcup_f X_2 \tag{6.1}$$

defined by the homeomorphism $f : \mathbb{T} \to \mathbb{T}$ that is encoded by four integers p, q, r, s as in (2.4). The self-linking number of K_i is denoted by

$$\frac{u_i}{v_i} := \mathrm{lk}\big(K_i, \rho(K_i)\big) \in \mathbb{Q}, \tag{6.2}$$

where u_i, v_i are coprime integers such that $v_i > 0$. Observe that, when K_i is null-homologous and $\rho(K_i)$ is the preferred parallel, we have $u_i = 0$ and $v_i = 1$.

Lemma 6.1 *With the above notations, M is a $\mathbb{Q}$HS if and only if we have*

$$q\,u_1u_2 + r\,v_1v_2 + s\,u_2v_1 + p\,v_2u_1 \neq 0.$$

Proof Recall that the *longitude* $\ell(K_i)$ of K_i is the oriented simple closed curve in $\partial\,\mathrm{N}(K_i)$ that is homologous to a positive multiple of K_i in $\mathrm{N}(K_i)$ and is rationally null-homologous in the exterior of $\mathrm{N}(K_i)$. Thus we have $\mathrm{lk}(K_i, \ell(K_i)) = 0$ and we deduce from (6.2) that

$$\ell(K_i) = -u_i\,\mu(K_i) + v_i\,\rho(K_i) \ \in H_1(\partial\,\mathrm{N}(K_i); \mathbb{Z}).$$

A Mayer–Vietoris argument shows that $H_1(M; \mathbb{Q})$ is generated by $\mu(K_2)$ and $\rho(K_2)$, subject to the relations

$$-u_2\,\mu(K_2) + v_2\,\rho(K_2) = 0 \quad \text{and} \quad (-pu_1 - rv_1)\,\mu(K_2) + (-qu_1 - sv_1)\,\rho(K_2) = 0$$

which express the triviality in homology of $\ell(K_2)$ and $\ell(K_1)$, respectively. Hence $H_1(M; \mathbb{Q}) = 0$ if and only if

$$\begin{vmatrix} -u_2 & v_2 \\ pu_1 + rv_1 & qu_1 + sv_1 \end{vmatrix} \neq 0 \qquad \blacksquare$$

For any framed knot L in a $\mathbb{Q}$HS N, with self-linking number $\varphi \in \mathbb{Q}$, we denote by

$$\underline{Z}(N, L) := Z(N, L) \sharp \exp_\sharp\Big(-\frac{\varphi}{2} \ \Big) \in \mathcal{A}(\circlearrowleft_l) \cong \mathcal{A}(\uparrow^l)$$



the *unframed* version of the Kontsevich–LMO invariant $Z(N, L)$ and we denote by

$$\underline{Z}^{\circlearrowleft}(N, L) := \partial_{\Omega_l}^{-1}\chi^{-1}(\underline{Z}(N, L)) \ \in \mathcal{A}(*_l)$$

the wheeled version of the latter. Note that $\underline{Z}^{\circlearrowleft}(N, L)$ does not show any strut and, by "wheeling" [BL04, Theorem 3], we have

$$\underline{Z}^{\circlearrowleft}(N, L) = Z^{\circlearrowleft}(N, L) \cdot \exp\Big(-\frac{\varphi}{2} \ \Big) + \frac{\varphi}{48} \Big). \tag{6.3}$$





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Theorem 6.2 *For $i \in \{1,2\}$, let $K_i \subset M_i$ be a framed knot in a $\mathbb{Q}$HS, with self-linking number $u_i/v_i \in \mathbb{Q}$, and let M be the splice of (M_1,K_1) and (M_2,K_2) as described by (6.1) in terms of four integers p,q,r,s. If M is also a $\mathbb{Q}$HS, then we have*

$$Z(M) = \omega \exp\left(\frac{\theta\kappa}{48}\right) \langle \partial_{D_1}(\underline{Z}^{\curlywedge}(K_1))|_{k_1 \to -v_1v_2k_2/\lambda} \,,\, \partial_{D_2}(\underline{Z}^{\curlywedge}(K_2))\rangle,$$

where we have set $\lambda := qu_1u_2 + rv_1v_2 + su_2v_1 + pv_2u_1$, $\tau_1 := qu_1 + sv_1$, $\tau_2 := qu_2 + pv_2$,

$$\kappa := \begin{cases} S(s/q) - (s+p)/q + 3\operatorname{sgn}(q\tau_1) + 3\operatorname{sgn}(\lambda\tau_1) - u_1/v_1 - u_2/v_2 & \text{if } q \neq 0, \\ s\big(3\operatorname{sgn}(\lambda) - r\big) - u_1/v_1 - u_2/v_2 & \text{if } q = 0, \end{cases}$$

and

$$D_1 := \exp\left(-\frac{v_1\tau_2}{2\lambda} \overset{k_1}{\smile} \overset{k_1}{}\right), \qquad D_2 := \exp\left(-\frac{v_2\tau_1}{2\lambda} \overset{k_2}{\smile} \overset{k_2}{}\right). \tag{6.4}$$

Proof The first part of the proof of Theorem 3.1 works verbatim in the more general situation that we are now considering. Thus we choose a matrix decomposition as in (2.5), which leads to $n \geq 1$ integers $a_1,\dots,a_n$, and we consider the Hopf chain $H := H(a_1,\dots,a_n) \subset M_1 \sharp M_2$ that "clasps" positively K_1 and $\varepsilon_n K_2$. We obtain that

$$Z(M) = \omega \exp\left(\frac{\theta\big(3\zeta(\Lambda) - \sum_{i=1}^n a_i\big)}{48}\right) \int Z^{\curlywedge}(K_1)\, Z^{\curlywedge}(\varepsilon_n K_2) \cdot$$
$$\cdot \exp\Big(\overset{k_1}{\frown} \overset{h_1}{} + \sum_{i=1}^{n-1} \overset{h_i}{\frown} \overset{h_{i+1}}{} + \overset{h_n}{\frown} \overset{k_2}{} + \frac{1}{2}\sum_{i=1}^n a_i \overset{h_i}{\frown} \overset{h_i}{} \Big)\, \mathrm{d}k_1\, \mathrm{d}h_1 \cdots \mathrm{d}h_n\, \mathrm{d}k_2.$$

Here Λ denotes the linking matrix of $K_1 \sqcup H \sqcup \varepsilon_n K_2$ in $M_1 \sharp M_2$, which is the tridiagonal matrix

$$\Lambda := \begin{pmatrix} u_1/v_1 & 1 & 0 & \cdots & 0 & 0 \\ 1 & a_1 & 1 & \cdots & 0 & 0 \\ 0 & 1 & a_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_n & 1 \\ 0 & 0 & 0 & \cdots & 1 & u_2/v_2 \end{pmatrix}.$$

Using (6.3), we deduce that

$$Z(M) = \omega \exp\left(\frac{\theta\big(3\zeta(\Lambda) - \operatorname{tr}(\Lambda)\big)}{48}\right) \int \underline{Z}^{\curlywedge}(K_1)\, \underline{Z}^{\curlywedge}(\varepsilon_n K_2) \cdot$$
$$\cdot \exp\Big(\overset{k_1}{\frown} \overset{h_1}{} + \sum_{i=1}^{n-1} \overset{h_i}{\frown} \overset{h_{i+1}}{} + \overset{h_n}{\frown} \overset{k_2}{} + \frac{1}{2}\sum_{i=1}^n a_i \overset{h_i}{\frown} \overset{h_i}{} + \frac{u_1}{2v_1} \overset{k_1}{\frown} \overset{k_1}{} + \frac{u_2}{2v_2} \overset{k_2}{\frown} \overset{k_2}{} \Big) \cdot$$
$$\cdot\, \mathrm{d}k_1\, \mathrm{d}h_1 \cdots \mathrm{d}h_n\, \mathrm{d}k_2$$

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The 2×2 matrix associated to the tridiagonal matrix Λ is

$$\begin{aligned}
&\begin{pmatrix}0&-1\\1&0\end{pmatrix}\cdot\begin{pmatrix}u_1/v_1&-1\\1&0\end{pmatrix}\begin{pmatrix}a_1&-1\\1&0\end{pmatrix}\cdots\begin{pmatrix}a_n&-1\\1&0\end{pmatrix}\begin{pmatrix}u_2/v_2&-1\\1&0\end{pmatrix}\\
&=\begin{pmatrix}0&-1\\1&0\end{pmatrix}\begin{pmatrix}u_1/v_1&-1\\1&0\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}a_1&1\\-1&0\end{pmatrix}\cdots\begin{pmatrix}a_n&1\\-1&0\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}u_2/v_2&-1\\1&0\end{pmatrix}\\
&=\begin{pmatrix}0&-1\\1&0\end{pmatrix}\begin{pmatrix}u_1/v_1&1\\1&0\end{pmatrix}\left(\begin{pmatrix}a_n&-1\\1&0\end{pmatrix}\cdots\begin{pmatrix}a_1&-1\\1&0\end{pmatrix}\right)^T\begin{pmatrix}u_2/v_2&-1\\-1&0\end{pmatrix}\\
&\overset{(2.5)}{=}\begin{pmatrix}0&-1\\1&0\end{pmatrix}\begin{pmatrix}u_1/v_1&1\\1&0\end{pmatrix}\begin{pmatrix}q&s\\-p&-r\end{pmatrix}^T\begin{pmatrix}u_2/v_2&-1\\-1&0\end{pmatrix}\\
&=\begin{pmatrix}-(qu_2+pv_2)/v_2&q\\(qu_1u_2+su_2v_1+pu_1v_2+rv_1v_2)/(v_1v_2)&-(qu_1+sv_1)/v_1\end{pmatrix}
\end{aligned}$$

so that, by Proposition A.1, we have

$$\Lambda^{-1}=\frac{1}{\lambda}\begin{pmatrix}v_1\tau_2&?&\cdots&?&(-1)^{n+1}v_1v_2\\?&?&\cdots&?&?\\\vdots&\vdots&\ddots&\vdots&\vdots\\?&?&\cdots&?&?\\(-1)^{n+1}v_1v_2&?&\cdots&?&v_2\tau_1\end{pmatrix}$$

where $\lambda:=qu_1u_2+su_2v_1+pu_1v_2+rv_1v_2$, $\tau_1:=qu_1+sv_1$ and $\tau_2:=qu_2+pv_2$. Therefore, performing Gaussian integration and using that $\underline{Z}^{\sqcup}(-K_2)$ is obtained from $\underline{Z}^{\sqcup}(K_2)$ by the change $k_2\to -k_2$, we get

$$\begin{aligned}
Z(M)=\omega\,\exp\left(\frac{\theta(3\zeta(\Lambda)-\mathrm{tr}(\Lambda))}{48}\right)\cdot\\
\cdot\left\langle\exp\left(-\frac{v_1\tau_2}{2\lambda}\overset{k_1}{\smile}\overset{k_1}{}-\frac{v_1v_2}{\lambda}\overset{k_1}{\smile}\overset{k_2}{}-\frac{v_2\tau_1}{2\lambda}\overset{k_2}{\smile}\overset{k_2}{}\right),\underline{Z}^{\sqcup}(K_1)\,\underline{Z}^{\sqcup}(K_2)\right\rangle.
\end{aligned}$$

Using the notation (6.4), this is equivalent to

$$Z(M)=\omega\,\exp\left(\frac{\theta(3\zeta(\Lambda)-\mathrm{tr}(\Lambda))}{48}\right)\cdot\left\langle\partial_{D_1}\left(\underline{Z}^{\sqcup}(K_1)\right)\big|_{k_1\to -v_1v_2k_2/\lambda},\,\partial_{D_2}\left(\underline{Z}^{\sqcup}(K_2)\right)\right\rangle.$$

It remains to compute $\kappa:=3\zeta(\Lambda)-\mathrm{tr}(\Lambda)\in\mathbb{Q}$. By the above computation of the 2×2 matrix $A(u_1/v_1,a_1,\dots,a_n,u_2/v_2)$ associated to $\Lambda=\Lambda(u_1/v_1,a_1,\dots,a_n,u_2/v_2)$, we deduce from (A.1) that

$$\zeta(\Lambda)=\zeta(\Lambda(u_1/v_1,a_1,\dots,a_n))+\mathrm{sgn}(\lambda\tau_1).$$

Besides, since the 2×2 matrix associated to $\Lambda(u_1/v_1,a_1,\dots,a_n)$ is

$$\begin{aligned}
&A(u_1/v_1,a_1,\dots,a_n,u_2/v_2)\begin{pmatrix}0&1\\-1&u_2/v_2\end{pmatrix}\\
&=\begin{pmatrix}-\tau_2/v_2&q\\\lambda/(v_1v_2)&-\tau_1/v_1\end{pmatrix}\begin{pmatrix}0&1\\-1&u_2/v_2\end{pmatrix}=\begin{pmatrix}-q&-p\\\tau_1/v_1&(pu_1+rv_1)/v_1\end{pmatrix},
\end{aligned}$$

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it follows from (A.2) that

$$\zeta(\Lambda(u_1/v_1, a_1, \dots, a_n)) = \zeta(\Lambda(a_1, \dots, a_n)) + \operatorname{sgn}(q\tau_1).$$

Finally, since the 2×2 matrix associated to $\Lambda(a_1, \dots, a_n)$ is

$$\begin{aligned}
&\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} a_1 & -1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_n & -1 \\ 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a_1 & 1 \\ -1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_n & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\
&\overset{(2.5)}{=} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} q & s \\ -p & -r \end{pmatrix}^T \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} s & r \\ q & p \end{pmatrix},
\end{aligned}$$

we deduce from Theorem A.3 that

$$3\zeta\big(\Lambda(a_1, \dots, a_n)\big) - \sum_{i=1}^{n} a_i = \begin{cases} S(s/q) - (s+p)/q & \text{if } q \neq 0, \\ -r/s & \text{if } q = 0. \end{cases}$$

We conclude that

$$\kappa = \begin{cases} S(s/q) - (s+p)/q + 3\operatorname{sgn}(q\tau_1) + 3\operatorname{sgn}(\lambda\tau_1) - u_1/v_1 - u_2/v_2 & \text{if } q \neq 0, \\ -r/s + 3\operatorname{sgn}(\lambda s) - u_1/v_1 - u_2/v_2 & \text{if } q = 0. \end{cases}$$

■

Remark 6.3 At the end of the proof of Theorem 6.2, we have applied (A.1) before (A.2) to compute $\kappa \in \mathbb{Q}$. If we went the other way, we would have obtained an equivalent formula:

$$\kappa = \begin{cases} S(s/q) - (s+p)/q + 3\operatorname{sgn}(q\tau_2) + 3\operatorname{sgn}(\lambda\tau_2) - u_1/v_1 - u_2/v_2 & \text{if } q \neq 0, \\ p\big(3\operatorname{sgn}(\lambda) - r\big) - u_1/v_1 - u_2/v_2 & \text{if } q = 0. \end{cases}$$

Remark 6.4 In the same way as Theorem 3.1 extends to Theorem 5.1, there is a generalization of Theorem 6.2 which includes additional links L_i in M_i (disjoint from K_i).

Appendix A Signatures of tridiagonal matrices

A *tridiagonal matrix* is a square matrix of the form

$$\Lambda(c_1, \dots, c_\ell) := \begin{pmatrix} c_1 & 1 & 0 & \dots & \dots & 0 \\ 1 & c_2 & 1 & \ddots & & \vdots \\ 0 & 1 & c_3 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \ddots & 1 \\ 0 & \dots & \dots & 0 & 1 & c_\ell \end{pmatrix}$$

where $c_1, \dots, c_\ell \in \mathbb{R}$. The *associated* 2×2 matrix is

$$\mathrm{A}(c_1, \dots, c_\ell) := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} c_1 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c_2 & -1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} c_\ell & -1 \\ 1 & 0 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}).$$

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Proposition A.1 (Bar-Natan and Lawrence) *Let $c_1, \dots, c_\ell \in \mathbb{R}$ and write*

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} := \mathrm{A}(c_1, \dots, c_\ell).$$

The matrix $\Lambda(c_1, \dots, c_\ell)$ is invertible if and only if $\gamma \neq 0$. In such a case, the four "corners" of the inverse of $\Lambda(c_1, \dots, c_\ell)$ are given by

$$\Lambda(c_1, \dots, c_\ell)^{-1} = \begin{pmatrix} -\alpha/\gamma & ? & \cdots & ? & (-1)^{\ell+1}/\gamma \\ ? & ? & \cdots & ? & ? \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ ? & ? & \cdots & ? & ? \\ (-1)^{\ell+1}/\gamma & ? & \cdots & ? & -\delta/\gamma \end{pmatrix}.$$

Proof The arguments given in [BL04, Proposition 2.4] under the assumption $c_1, \dots, c_\ell \in \mathbb{Z}$ work in the same way for arbitrary $c_1, \dots, c_\ell \in \mathbb{R}$. But it seems that the $(-1)^\ell$ in the statement of [BL04, Proposition 2.4] should be replaced by $(-1)^{\ell+1}$. ∎

We explain how to compute inductively the signature $\zeta(\Lambda)$ of a tridiagonal matrix Λ.

Proposition A.2 *Let $c_1, \dots, c_\ell \in \mathbb{R}$ and write*

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} := \mathrm{A}(c_1, \dots, c_\ell).$$

Then, we have the following inductive formulas:

$$\zeta\big(\Lambda(c_1, c_2, \dots, c_\ell)\big) = \zeta\big(\Lambda(c_1, \dots, c_{\ell-1})\big) - \operatorname{sgn}(\gamma)\,\operatorname{sgn}(\delta), \tag{A.1}$$

$$\zeta\big(\Lambda(c_1, c_2, \dots, c_\ell)\big) = \zeta\big(\Lambda(c_2, \dots, c_\ell)\big) - \operatorname{sgn}(\gamma)\,\operatorname{sgn}(\alpha). \tag{A.2}$$

Proof The proof of (A.1) being very similar to the proof of (A.2), we give the latter and omit the former. A straightforward computation gives

$$A(c_2, \dots, c_\ell) = \begin{pmatrix} \gamma + \alpha c_1 & \delta + \beta c_1 \\ -\alpha & -\beta \end{pmatrix}. \tag{A.3}$$

Thus, by Proposition A.1, the matrix $\underline{\Lambda} := \Lambda(c_2, \dots, c_\ell)$ is invertible if and only if $\alpha \neq 0$.

Assume first that $\alpha \neq 0$. We shall use the fact that $\Lambda := \Lambda(c_1, c_2, \dots, c_\ell)$ is obtained from $\underline{\Lambda}$ by a kind of "plumbing" operation—compare with the proof of [GR16, Proposition 2.5.3]. Specifically, we have the following congruence where $v := (1, 0, \dots, 0) \in \mathbb{R}^{\ell-1}$:

$$\underbrace{\begin{pmatrix} c_1 & v \\ v^T & \underline{\Lambda} \end{pmatrix}}_{\Lambda} = \begin{pmatrix} 1 & v\underline{\Lambda}^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} c_1 - v\underline{\Lambda}^{-1}v^T & 0 \\ 0 & \underline{\Lambda} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \underline{\Lambda}^{-1}v^T & I \end{pmatrix}.$$

Therefore,

$$\zeta(\Lambda) = \operatorname{sgn}(c_1 - v\underline{\Lambda}^{-1}v^T) + \zeta(\underline{\Lambda}).$$

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By Proposition A.1, the upper left "corner" of $\underline{\Lambda}^{-1}$ can be read from (A.3). Hence, we get $v\underline{\Lambda}^{-1}v^T = \gamma/\alpha + c_1$ and we deduce that $\zeta(\Lambda) = -\operatorname{sgn}(\gamma/\alpha) + \zeta(\underline{\Lambda})$.

Assume now that $\alpha = 0$. Let $x = (x_2, \dots, x_\ell) \in \mathbb{R}^{\ell-1}$ be such that $x \neq 0$ and $\underline{\Lambda}x^T = 0$. By writing the linear system $\underline{\Lambda}x^T = 0$, we see that $x_2 = 0$ would successively imply $x_3 = 0, \dots, x_\ell = 0$. Hence we must have $x_2 \neq 0$ and we get a new basis of $\mathbb{R}^\ell$ by substituting $x \in \{0\} \oplus \mathbb{R}^{\ell-1} \subset \mathbb{R}^\ell$ to the second vector of the canonical basis: we shall express in this new basis the symmetric bilinear form in $\mathbb{R}^\ell$ that is given by Λ in the canonical basis. Specifically, we have the following congruence where $\underline{x} := (x_3, \dots, x_\ell)$, $w := (1, 0, \dots, 0) \in \mathbb{R}^{\ell-2}$ and $\underline{\underline{\Lambda}} := \Lambda(c_3, \dots, c_\ell)$:

$$\begin{pmatrix} 0 & 0 \\ 0 & \underline{\underline{\Lambda}} \end{pmatrix} = \underbrace{\begin{pmatrix} x_2 & \underline{x} \\ 0 & I \end{pmatrix}}_{X^T} \underbrace{\begin{pmatrix} c_2 & w \\ w^T & \underline{\underline{\Lambda}} \end{pmatrix}}_{\underline{\Lambda}} \underbrace{\begin{pmatrix} x_2 & 0 \\ \underline{x}^T & I \end{pmatrix}}_{X} \tag{A.4}$$

Therefore, we also have the following congruence:

$$\begin{pmatrix} c_1 & x_2 & 0 \\ x_2 & 0 & 0 \\ 0 & 0 & \underline{\underline{\Lambda}} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & X^T \end{pmatrix} \underbrace{\begin{pmatrix} c_1 & v \\ v^T & \underline{\Lambda} \end{pmatrix}}_{\Lambda} \begin{pmatrix} 1 & 0 \\ 0 & X \end{pmatrix} \tag{A.5}$$

We deduce from (A.4) that $\zeta(\underline{\underline{\Lambda}}) = \zeta(\underline{\Lambda})$ and we deduce from (A.5) that $\zeta(\underline{\underline{\Lambda}}) = \zeta(\Lambda)$. ∎

We now recall how signatures of tridiagonal matrices with *integral* coefficients can be computed from Dedekind sums. For any pair of coprime integers p, q with $q \neq 0$, define the *Dedekind sum* $\mathfrak{s}(p, q)$ by

$$\mathfrak{s}(p, q) := \sum_{k=1}^{|q|-1} \left(\!\!\left(\frac{k}{q} \right)\!\!\right) \cdot \left(\!\!\left(\frac{kp}{q} \right)\!\!\right)$$

where $((-))$ denotes the *sawtooth function* defined by $((x)) := x - \lfloor x \rfloor - 1/2$ for $x \in \mathbb{R}\backslash\mathbb{Z}$ and $((x)) := 0$ for $x \in \mathbb{Z}$. Then the *Dedekind symbol*

$$S(p/q) := 12 \operatorname{sgn}(q)\, \mathfrak{s}(p, q)$$

is easily seen to satisfy $S(-p/q) = -S(p/q)$ and $S(p/q + 1) = S(p/q)$. Furthermore, we have

$$S(p/q) + S(q/p) = p/q + q/p + 1/pq - 3\operatorname{sgn}(pq); \tag{A.6}$$

see [KM94] and references therein.

The following theorem is proved in [KM94, (1.12) and (2.2)]—see also [BG92]. For the sake of completeness, we provide a proof based on Proposition A.2 and assuming the reciprocity law (A.6).

Theorem A.3 (Kirby and Melvin) *Let $c_1, \dots, c_\ell \in \mathbb{Z}$ and write*

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} := \mathrm{A}(c_1, \dots, c_\ell).$$

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The signature $\zeta(\Lambda)$ *and the trace* $\mathrm{tr}(\Lambda)$ *of* $\Lambda := \Lambda(c_1, \dots, c_\ell)$ *are related as follows:*

$$3\zeta(\Lambda) - \mathrm{tr}(\Lambda) = \begin{cases} S(\alpha/\gamma) - (\alpha+\delta)/\gamma & \text{if } \gamma \neq 0, \\ -\beta/\alpha & \text{if } \gamma = 0. \end{cases} \tag{A.7}$$

Proof The proof is by induction on $\ell \geq 1$. For $\ell = 1$, (A.7) asserts that

$$3\,\mathrm{sgn}(c_1) - c_1 = \begin{cases} -S(1/c_1) + 2/c_1 & \text{if } c_1 \neq 0, \\ 0 & \text{if } c_1 = 0, \end{cases}$$

which is trivial for $c_1 = 0$ and follows from (A.6) for $c_1 \neq 0$. Hence we suppose that (A.7) holds true for $\underline{\Lambda} := \Lambda(c_2, \dots, c_\ell)$; the 2×2 matrix associated to $\underline{\Lambda}$ has been computed at (A.3).

Assume that $\gamma = 0$. It follows from (A.2) that

$$3\zeta(\Lambda) - \mathrm{tr}(\Lambda) = 3\zeta(\underline{\Lambda}) - \mathrm{tr}(\underline{\Lambda}) - c_1.$$

Since $\alpha\delta - \beta\gamma = 1$, we must have $\alpha \neq 0$. Then the induction hypothesis gives

$$3\zeta(\Lambda) - \mathrm{tr}(\Lambda) = S(-\gamma/\alpha - c_1) - (\gamma + \alpha c_1 - \beta)/(-\alpha) - c_1 \;=\; -\beta/\alpha.$$

Assume now that $\gamma \neq 0$. If $\alpha = 0$, we have $\beta = -\gamma \in \{-1, +1\}$, and it follows from (A.2) and the induction hypothesis that

$$\begin{aligned} 3\zeta(\Lambda) - \mathrm{tr}(\Lambda) &= 3\zeta(\underline{\Lambda}) - \mathrm{tr}(\underline{\Lambda}) - c_1 \\ &= -(\delta + \beta c_1)/(\gamma + \alpha c_1) - c_1 \;=\; -\delta/\gamma \;=\; S(\alpha/\gamma) - (\alpha+\delta)/\gamma. \end{aligned}$$

If $\alpha \neq 0$, it follows from (A.2) and the induction hypothesis that

$$\begin{aligned} 3\zeta(\Lambda) - \mathrm{tr}(\Lambda) &= 3\zeta(\underline{\Lambda}) - \mathrm{tr}(\underline{\Lambda}) - 3\,\mathrm{sgn}(\gamma\alpha) - c_1 \\ &= S((\gamma + c_1\alpha)/(-\alpha)) - (\gamma + c_1\alpha - \beta)/(-\alpha) - 3\,\mathrm{sgn}(\gamma\alpha) - c_1 \\ &= -S(\gamma/\alpha) + \gamma/\alpha - \beta/\alpha - 3\,\mathrm{sgn}(\gamma\alpha) \\ &\overset{(A.6)}{=} S(\alpha/\gamma) - \alpha/\gamma - 1/(\gamma\alpha) - \beta/\alpha \;=\; S(\alpha/\gamma) - \alpha/\gamma - \delta/\gamma. \end{aligned}$$

■

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