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REVERSE MATHEMATICS, YOUNG DIAGRAMS, AND THE ASCENDING CHAIN CONDITION

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Abstract. Let S be the group of finitely supported permutations of a countably infinite set. Let $K[S]$ be the group algebra of S over a field K of characteristic 0. According to a theorem of Formanek and Lawrence, $K[S]$ satisfies the ascending chain condition for two-sided ideals. We study the reverse mathematics of this theorem, proving its equivalence over RCA_0 (or even over RCA_0^*) to the statement that ω^ω is well ordered. Our equivalence proof proceeds via the statement that the Young diagrams form a well partial ordering.

§1. **Introduction.** *Reverse mathematics* is a research program in the foundations of mathematics. The purpose of reverse mathematics is to discover which axioms are needed to prove specific core-mathematical theorems of analysis, algebra, geometry, combinatorics, etc. Reverse mathematics takes place in the context of subsystems of second-order arithmetic. The standard reference for reverse mathematics is [28, Part A].

This paper is a contribution to the reverse mathematics of algebra. Let $\mathbb{Q}$ be the field of rational numbers. The *Hilbert Basis Theorem* [11] says that, for each positive integer n, there is no infinite ascending sequence of ideals in the polynomial ring $\mathbb{Q}[x_1, \dots, x_n]$. This is one of the most famous theorems of abstract algebra. In a previous paper [25, Theorem 2.7] we have shown that the Hilbert Basis Theorem[1] is reverse-mathematically equivalent to $\mathrm{WO}(\omega^\omega)$. Here $\mathrm{WO}(\omega^\omega)$ is the assertion that (the standard system of Cantor normal form notations for the ordinal numbers less than the ordinal number) ω^ω is well ordered. The place of $\mathrm{WO}(\omega^\omega)$ within the usual hierarchy of subsystems of second-order arithmetic [28] is discussed in our expository paper [32].

Our reverse-mathematical result in [25, Theorem 2.7] implies that the Hilbert Basis Theorem is not finitistically reducible. (See [25, Proposition 2.6].) This foundational outcome is significant with respect to Hilbert's foundational program of finitistic reductionism [12, 26, 35]. In a certain fairly precise philosophical sense

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[1] A related theorem of Hilbert says that for each n there is no infinite ascending sequence of ideals in the power series ring $\mathbb{Q}[[x_1, \dots, x_n]]$. We have shown in [10] that this theorem, like the Hilbert Basis Theorem, is reverse-mathematically equivalent to $\mathrm{WO}(\omega^\omega)$.

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[30, 31], our result in [25, Theorem 2.7] justifies Gordan's famous remark (see [19]) to the effect that the Hilbert Basis Theorem is not mathematics but rather theology.

On the other hand, in analyzing the significance of [25, Theorem 2.7], it is important to note that the Hilbert Basis Theorem refers not to a single ring but to a sequence of rings $\mathbb{Q}[x_1, \ldots, x_n]$ where $n = 1, 2, 3, \ldots$. Moreover, for each specific positive integer n, the special case $\mathbb{Q}[x_1, \ldots, x_n]$ of the Hilbert Basis Theorem is finitistically reducible, in fact provable in RCA_0 [25, Lemmas 3.4 and 3.6]. Therefore, from the foundational viewpoint of [25, Section 1], it would be interesting to find a specific commutative ring which has no infinite ascending sequence of ideals and to prove that the corresponding basis theorem is not finitistically reducible.

At the moment we are unable to provide such a commutative ring, but in this paper we achieve something similar in the noncommutative case. Specifically, we perform a reverse-mathematical analysis of a theorem of Formanek and Lawrence [6]. The *Formanek/Lawrence Theorem* says that there is no infinite ascending sequence of two-sided ideals in the group algebra $\mathbb{Q}[S]$ of the infinite symmetric group S over the field $\mathbb{Q}$. The purpose of this paper is to show that the Formanek/Lawrence Theorem, like the Hilbert Basis Theorem, is reverse-mathematically equivalent to $\mathrm{WO}(\omega^\omega)$. From this result it follows, just as in [25], that the Formanek/Lawrence Theorem is not finitistically reducible.

Our work in this paper involves partitions. A *partition* is a finite sequence of integers $m_1, \ldots, m_k$ such that $m_1 \geq \cdots \geq m_k > 0$ and $k > 0$. It is well known (see for instance [4, Section 28]) that there is a close relationship between two-sided ideals in $\mathbb{Q}[S]$ and partitions. A partition $m_1, \ldots, m_k$ is usually visualized as a *Young diagram*, i.e., a planar array of boxes with k left-justified rows of lengths $m_1, \ldots, m_k$, respectively. For example, the partition $5, 2, 2, 1$ is visualized as the Young diagram in Figure 1 consisting of $10 = 5 + 2 + 2 + 1$ boxes. Note that there is another partition $4, 3, 1, 1, 1$ consisting of the lengths of the columns of this same Young diagram. A standard reference for partition theory is [1].

Our notation and terminology in this paper is as follows. We use $\mathbb{N}$ to denote the set of positive integers. We use i, j, k, l, m, n as variables ranging over $\mathbb{N}$. We use $\mathcal{D}$ to denote the set of *diagrams*, i.e., Young diagrams. We use D, E, F, G, H as variables ranging over $\mathcal{D}$. Given $D \in \mathcal{D}$, the number of boxes in D is denoted $|D|$. We write $r_i(D) =$ the length of the ith row of D, or 0 if $i >$ the number of

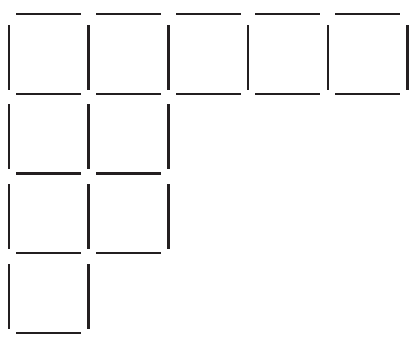


FIGURE 1. The Young diagram corresponding to the partition $5, 2, 2, 1$.

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rows. Similarly, we write $c_i(D)$ = the length of the ith column of D, or 0 if $i >$ the number of columns. Note that $|D| = r_1(D) + \cdots + r_k(D) = c_1(D) + \cdots + c_l(D)$ where $k = c_1(D) = \max(\{i \mid r_i(D) > 0\})$ = the number of rows, and $l = r_1(D) = \max(\{i \mid c_i(D) > 0\})$ = the number of columns. Note also that $|D| \leq r_1(D) \cdot c_1(D)$. For $D, E \in \mathcal{D}$ we write $D \leq E$ to mean that $r_i(D) \leq r_i(E)$ for all i, or equivalently $c_i(D) \leq c_i(E)$ for all i. Thus, $(\mathcal{D}, \leq)$ is a partial ordering. We write $D < E$ to mean that $D \leq E$ and $D \neq E$. We write $D \cup E = \sup(D, E)$ = the unique diagram F such that $r_i(F) = \max(r_i(D), r_i(E))$ for all i, or equivalently $c_i(F) = \max(c_i(D), c_i(E))$ for all i.

As in [25], some of our results in this paper involve well partial ordering theory. A *well partial ordering* is a partial ordering $(\mathcal{P}, \leq)$ such that for all infinite sequences $\langle P_i \rangle_{i \in \mathbb{N}}$ of elements of $\mathcal{P}$ there exist $i, j \in \mathbb{N}$ such that $i < j$ and $P_i \leq P_j$. For example, any well ordering is a well partial ordering, and the product of any two well partial orderings is a well partial ordering. For background on well partial ordering theory, see [5, 17, 21–23, 36]. A key fact for us in this paper is that $(\mathcal{D}, \leq)$ is a well partial ordering. We denote this fact as $\mathrm{WPO}(\mathcal{D})$. In Theorem 2.6 we show that the statement $\mathrm{WPO}(\mathcal{D})$ is robust with respect to equivalence over RCA_0.

The plan of this paper is as follows. In Section 2 we prove that the statements $\mathrm{WO}(\omega^\omega)$ and $\mathrm{WPO}(\mathcal{D})$ are equivalent over RCA_0. In Section 3 we define what it means for a subset of $\mathcal{D}$ to be *closed*, and we prove that $\mathrm{WPO}(\mathcal{D})$ is equivalent over RCA_0 to the statement that there is no infinite ascending sequence of closed subsets of $\mathcal{D}$. In Section 4 we show that the latter statement is equivalent over RCA_0 to the statement that there is no infinite ascending sequence of two-sided ideals in $\mathbb{Q}[S]$. Combining this with the results of Section 2 and Section 3, we obtain our main result. In Section 5 we show how the base theory RCA_0 can be weakened to RCA_0^*. In Section 6 we raise a question for future research.

§2. Well partial ordering of diagrams. In this section we use a result from [25] to prove in RCA_0 that the set $\mathcal{D}$ of all diagrams is well partially ordered if and only if the ordinal number ω^ω is well ordered.

LEMMA 2.1. RCA_0 *proves* $\mathrm{WO}(\omega^\omega) \Rightarrow \mathrm{WPO}(\mathcal{D})$.

PROOF. Reasoning in RCA_0, assume that $\mathrm{WPO}(\mathcal{D})$ fails, i.e., $\mathcal{D}$ is not well partially ordered. Let $\langle D_i \rangle_{i \in \mathbb{N}}$ be a *bad sequence* of diagrams, i.e., $\forall i \, \forall j \, (i < j \Rightarrow D_i \not\leq D_j)$. Let $m = r_1(D_1)$ and $n = c_1(D_1)$. For all $j > 1$ we have $D_1 \not\leq D_j$, hence $m > r_n(D_j)$ and $n > c_m(D_j)$. To each diagram D such that $m > r_n(D)$ and $n > c_m(D)$, we associate a sequence $s(D) \in \mathbb{N}^{m+n-2}$ given by

$$s(D) = \langle r_1(D) + 1, \ldots, r_{n-1}(D) + 1, c_1(D) + 1, \ldots, c_{m-1}(D) + 1 \rangle.$$

The point is that for any two such diagrams D and E, we have $D \leq E$ if and only if $s(D) \leq s(E)$ with respect to the coordinatewise partial ordering of $\mathbb{N}^{m+n-2}$. Since $\langle D_j \rangle_{j>1}$ is a bad sequence of such diagrams, it follows that $\langle s(D_j) \rangle_{j>1}$ is a bad sequence in $\mathbb{N}^{m+n-2}$, hence $\mathbb{N}^{m+n-2}$ is not well partially ordered. It then follows by [25, Lemma 3.6] that ω^ω is not well ordered, i.e., $\mathrm{WO}(\omega^\omega)$ fails. ⊣

LEMMA 2.2. RCA_0 *proves* $\mathrm{WPO}(\mathcal{D}) \Rightarrow \mathrm{WO}(\omega^\omega)$.

PROOF. We reason in RCA_0. Given a nonzero ordinal $\alpha < \omega^\omega$, write α in Cantor normal form as $\alpha = \omega^{n_1} + \cdots + \omega^{n_k}$ where $n_1 \geq \cdots \geq n_k \geq 0$ and $k > 0$. Let $D(\alpha)$

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be the unique diagram D with $r_1(D) = n_1+1, \ldots, r_k(D) = n_k+1$ and $r_{k+1}(D) = 0$. Clearly $D(\alpha) \leq D(\beta)$ implies $\alpha \leq \beta$. Assume now that $\mathrm{WO}(\omega^\omega)$ fails. Let $\langle \alpha_i \rangle_{i\in\mathbb{N}}$ be an infinite descending sequence of ordinals $< \omega^\omega$, i.e., $\omega^\omega > \alpha_i > \alpha_j$ for all $i, j \in \mathbb{N}$ such that $i < j$. Then $D(\alpha_i) \not\leq D(\alpha_j)$ for all $i, j \in \mathbb{N}$ such that $i < j$, hence $\langle D(\alpha_i) \rangle_{i\in\mathbb{N}}$ is a bad sequence in $\mathcal{D}$, so $\mathrm{WPO}(\mathcal{D})$ fails. ⊣

THEOREM 2.3. RCA_0 *proves* $\mathrm{WPO}(\mathcal{D}) \Leftrightarrow \mathrm{WO}(\omega^\omega)$.

PROOF. This is immediate from Lemmas 2.1 and 2.2. ⊣

As a technical supplement to Theorem 2.3, we now prove a theorem to the effect that the statement $\mathrm{WPO}(\mathcal{D})$ is robust up to equivalence over RCA_0.

DEFINITION 2.4. Within RCA_0 we make the following definitions.

1. A set $\mathcal{U}$ of diagrams is said to be *upwardly closed* if $\forall D\, \forall E\, ((D \in \mathcal{U}$ and $D \leq E) \Rightarrow E \in \mathcal{U})$. For each finite set $\mathcal{S}$ of diagrams, there exists a set $\mathrm{ucl}(\mathcal{S}) = \{E \mid \exists D\, (D \in \mathcal{S}$ and $D \leq E)\}$ = the *upward closure* of $\mathcal{S}$, i.e., the smallest upwardly closed set which includes $\mathcal{S}$. An upwardly closed set of diagrams is *finitely generated* if it is of the form $\mathrm{ucl}(\mathcal{S})$ for some finite set $\mathcal{S}$.
2. Consider a sequence of diagrams $\langle D_i \rangle_{i\in\mathbb{N}}$. The sequence is said to be *upwardly closed* if $\forall i\, \forall E\, (D_i \leq E \Rightarrow \exists j\, (D_j = E))$. The sequence is said to be *finitely generated* if there is a finite set $\mathcal{S}$ such that $\forall E\, (E \in \mathrm{ucl}(\mathcal{S}) \Leftrightarrow \exists j\, (D_j = E))$, i.e., there is a finitely generated, upwardly closed set which is the range of the sequence.

LEMMA 2.5. RCA_0 *proves the following. If all upwardly closed sets of diagrams are finitely generated, then all upwardly closed sequences of diagrams are finitely generated.*

PROOF. We reason in RCA_0. Let $\langle D_i \rangle_{i\in\mathbb{N}}$ be an upwardly closed sequence of diagrams which is not finitely generated. Define a subsequence $\langle D_{i_n} \rangle_{i\in\mathbb{N}}$ recursively as follows. Let $i_1 = 1$. Given $i_1 < \cdots < i_n$, let i_{n+1} = the least $i > i_n$ such that $D_i \notin \mathrm{ucl}(\{D_{i_1}, \ldots, D_{i_n}\})$ and $|D_i| > \max(|D_{i_1}|, \ldots, |D_{i_n}|)$. Thus $i_n < i_{n+1}$ and $|D_{i_n}| < |D_{i_{n+1}}|$ for all n. By Δ^0_1 comprehension let $\mathcal{U}$ be the set of diagrams D such that $D \in \mathrm{ucl}(\{D_{i_1}, \ldots, D_{i_n}\})$ where n = the least n such that $|D_{i_n}| > |D|$. Clearly $\mathcal{U} = \bigcup_{n\in\mathbb{N}} \mathrm{ucl}(\{D_{i_1} \ldots, D_{i_n}\})$, hence $\mathcal{U}$ is upwardly closed, and $\mathcal{U} \neq \mathrm{ucl}(\{D_{i_1}, \ldots, D_{i_n}\})$ for all n, hence $\mathcal{U}$ is not finitely generated. ⊣

THEOREM 2.6. *Over* RCA_0 *each of the following statements implies the others.*

1. $\mathrm{WPO}(\mathcal{D})$.
2. *There is no infinite ascending sequence of upwardly closed sets of diagrams.*
3. *There is no infinite ascending sequence of finitely generated, upwardly closed sets of diagrams.*
4. *Every upwardly closed set of diagrams is finitely generated.*
5. *Every upwardly closed sequence of diagrams is finitely generated.*

PROOF. We reason in RCA_0. To prove $1 \Rightarrow 2$, assume that 2 fails. Let $\langle \mathcal{U}_i \rangle_{i\in\mathbb{N}}$ be an infinite ascending sequence of upwardly closed sets of diagrams. For each i let D_i be the least element of $\mathcal{U}_{i+1} \setminus \mathcal{U}_i$. Then $\langle D_i \rangle_{i\in\mathbb{N}}$ is a bad sequence of diagrams, so 1 fails.

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Trivially $2 \Rightarrow 3$. To prove $3 \Rightarrow 4$, assume that 4 fails. Let $\mathcal{U}$ be an upwardly closed set of diagrams which is not finitely generated. Let D_1 be the least element of $\mathcal{U}$. Given $D_1, \ldots, D_n \in \mathcal{U}$, let $D_{n+1} =$ the least element of $\mathcal{U} \setminus \mathcal{U}_n$, where $\mathcal{U}_n = \mathrm{ucl}(\{D_1, \ldots, D_n\})$. Then $\langle \mathcal{U}_n \rangle_{n \in \mathbb{N}}$ is an infinite ascending sequence of upwardly closed sets, so 3 fails.

By Lemma 2.5 we have $4 \Rightarrow 5$. It remains to prove $5 \Rightarrow 1$. Assume that 1 fails. Let $\langle D_i \rangle_{i \in \mathbb{N}}$ be a bad sequence of diagrams. The formula $\Phi(E) \equiv \exists i \, (D_i \leq E)$ is Σ^0_1, so by [28, Lemma II.3.7] there is a sequence of diagrams $\langle E_j \rangle_{j \in \mathbb{N}}$ such that $\forall E \, (\Phi(E) \Leftrightarrow \exists j \, (E = E_j))$. Clearly, $\langle E_j \rangle_{j \in \mathbb{N}}$ is an upwardly closed sequence of diagrams, and by Σ^0_1 bounding we have $\forall m \, \exists n \, (\forall j < m) \, (\exists i < n) \, (D_i \leq E_j)$. Because $\langle D_i \rangle_{i \in \mathbb{N}}$ is bad, it follows that $\langle E_j \rangle_{j \in \mathbb{N}}$ is not finitely generated, so 5 fails. This completes the proof. ⊣

§3. Closed sets of diagrams.

Definition 3.1. A set $\mathcal{U}$ of diagrams is said to be *closed* if

$$\forall D \, (D \in \mathcal{U} \Leftrightarrow \forall E \, (E > D \Rightarrow E \in \mathcal{U})).$$

Remark 3.2. Obviously, all closed sets of diagrams are upwardly closed. However, not all upwardly closed sets of diagrams are closed. For example, letting D_1 and D_2 be the diagrams corresponding to the partitions $4, 2$ and $2, 2, 1, 1$ respectively, the upwardly closed set $\mathrm{ucl}(\{D_1, D_2\})$ is not closed.[2] In this section we clarify the structure of closed sets of diagrams. Definition 3.3 and Lemmas 3.4–3.6 and Theorem 3.8 appear to be new, in the sense that we have not found their counterparts in the partition theory literature such as [1, 4, 6, 7].

Definition 3.3. Given a diagram D, let $(D)_r =$ the diagram obtained by shortening the first row to the length of the second row, or 1 if there is no second row. Thus, $r_1((D)_r) = \max(r_2(D), 1)$ and $r_i((D)_r) = r_i(D)$ for all $i > 1$. Similarly, let $(D)_c =$ the diagram obtained by shortening the first column to the length of the second column, or 1 if there is no second column. Thus $c_1((D)_c) = \max(c_2(D), 1)$ and $c_i((D)_c) = c_i(D)$ for all $i > 1$.

The next three lemmas are proved in RCA_0.

Lemma 3.4. *For all diagrams D we have $D = (D)_r \cup (D)_c$. For all diagrams D and E, if $(D)_r = (E)_r$ or $(D)_c = (E)_c$, then $D \leq E$ or $E \leq D$.*

Proof. These statements are obvious from Definition 3.3. ⊣

Lemma 3.5. *Let D, E, F be diagrams such that $|F| > (r_1(D) - 1)(c_1(E) - 1)$ and $F \geq (D)_r \cup (E)_c$. Then $F \geq D$ or $F \geq E$.*

Proof. Since $|F| > (r_1(D) - 1)(c_1(E) - 1)$, we must have $r_1(F) \geq r_1(D)$ or $c_1(F) \geq c_1(E)$. In the first case, we have $F \geq$ (first row of D) and $F \geq (D)_r$, hence $F \geq D$. In the second case, we have $F \geq$ (first column of E) and $F \geq (E)_c$, hence $F \geq E$. ⊣

[2] To see that $\mathrm{ucl}(\{D_1, D_2\})$ is not closed, let D be the diagram corresponding to the partition 3,3,3. This D does not belong to $\mathrm{ucl}(\{D_1, D_2\})$, but any diagram $> D$ belongs to $\mathrm{ucl}(\{D_1, D_2\})$. And in fact, Theorem 3.8 will tell us that the smallest closed set of diagrams containing D_1 and D_2 is $\mathrm{ucl}(\{E\})$ where E is the diagram corresponding to the partition 2,2.

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LEMMA 3.6. *Let $\mathcal{S}$ be a finite set of diagrams. Suppose F is a diagram such that $\exists n\,\forall G\,((|G| > n \text{ and } G > F) \Rightarrow G \in \mathrm{ucl}(\mathcal{S}))$. Then $F \geq (D)_r \cup (E)_c$ for some $D, E \in \mathcal{S}$.*

PROOF. Let $\mathcal{S}$ and F and n be as in the hypothesis. Since $\mathcal{S}$ is finite, we may safely assume that $n > \max(|F|, \max(\{r_1(D), c_1(E) \mid D, E \in \mathcal{S}\}))$. Let G be the unique diagram with $r_1(G) = n + 1$ and $r_i(G) = r_i(F)$ for all $i > 1$. Then $G > F$ and $|G| > n$, hence $G \in \mathrm{ucl}(\mathcal{S})$, i.e., $G \geq D$ for some $D \in \mathcal{S}$. It then follows that $F \geq (D)_r$. Similarly, let H be the unique diagram with $c_1(H) = n + 1$ and $c_i(H) = c_i(F)$ for all $i > 1$, then $H \geq E$ for some $E \in \mathcal{S}$, hence $F \geq (E)_c$. Thus $F \geq (D)_r \cup (E)_c$. ⊣

DEFINITION 3.7. Given a finite set $\mathcal{S}$ of diagrams, let

$$\widehat{\mathcal{S}} = \{(D)_r \cup (E)_c \mid D, E \in \mathcal{S}\},$$

and let

$$\overline{\mathcal{S}} = \{F \in \mathrm{ucl}(\widehat{\mathcal{S}}) \mid |F| \leq \|\mathcal{S}\|\},$$

where

$$\|\mathcal{S}\| = \max(\{(r_1(D) - 1)(c_1(E) - 1) \mid D, E \in \mathcal{S}\}).$$

Note that $\widehat{\mathcal{S}}$ and $\overline{\mathcal{S}}$ are finite sets of diagrams, and they can be found effectively given the finite set $\mathcal{S}$. Note also that $\mathcal{S} \subseteq \widehat{\mathcal{S}}$, hence $\mathrm{ucl}(\mathcal{S}) \subseteq \mathrm{ucl}(\widehat{\mathcal{S}})$.

THEOREM 3.8. RCA_0 *proves the following. For any finite set $\mathcal{S}$ of diagrams, we have $\mathrm{ucl}(\widehat{\mathcal{S}}) = \overline{\mathcal{S}} \cup \mathrm{ucl}(\mathcal{S})$. Moreover, $\mathrm{ucl}(\widehat{\mathcal{S}})$ is closed and includes $\mathcal{S}$ and is included in all closed sets that include $\mathcal{S}$.*

PROOF. Trivially, $\overline{\mathcal{S}} \cup \mathrm{ucl}(\mathcal{S}) \subseteq \mathrm{ucl}(\widehat{\mathcal{S}})$, and by Lemma 3.5 we have $\mathrm{ucl}(\widehat{\mathcal{S}}) \subseteq \overline{\mathcal{S}} \cup \mathrm{ucl}(\mathcal{S})$, so $\mathrm{ucl}(\widehat{\mathcal{S}}) = \overline{\mathcal{S}} \cup \mathrm{ucl}(\mathcal{S})$. Let $\mathcal{U}$ be a closed set of diagrams which includes $\mathcal{S}$. Then $\mathrm{ucl}(\mathcal{S}) \subseteq \mathcal{U}$, so by Lemma 3.5 we have $F \in \mathcal{U}$ for all $F \in \mathrm{ucl}(\widehat{\mathcal{S}})$ such that $|F| > \|\mathcal{S}\|$. But then, by induction on $\|\mathcal{S}\| - |F|$, it follows that $F \in \mathcal{U}$ for all $F \in \mathrm{ucl}(\widehat{\mathcal{S}})$ such that $|F| \leq \|\mathcal{S}\|$. Thus $\mathrm{ucl}(\widehat{\mathcal{S}}) \subseteq \mathcal{U}$. It remains to prove that $\mathrm{ucl}(\widehat{\mathcal{S}})$ is closed. Let F be a diagram such that $\forall G\,(G > F \Rightarrow G \in \mathrm{ucl}(\widehat{\mathcal{S}}))$, i.e., $\forall G\,(G > F \Rightarrow G \in \overline{\mathcal{S}} \cup \mathrm{ucl}(\mathcal{S}))$. From the definition of $\overline{\mathcal{S}}$ it follows that $\forall G\,((|G| > \|\mathcal{S}\| \text{ and } G > F) \Rightarrow G \in \mathrm{ucl}(\mathcal{S}))$. But then by Lemma 3.6 we have $F \in \mathrm{ucl}(\widehat{\mathcal{S}})$. Thus $\mathrm{ucl}(\widehat{\mathcal{S}})$ is closed. ⊣

We shall now use Theorem 3.8 to prove a result about closed sets of diagrams which is analogous to Theorem 2.6 about upwardly closed sets of diagrams. Our result here will be needed in Section 4.

DEFINITION 3.9. Within RCA_0 we make the following definitions.

1. The *closure* of a finite set $\mathcal{S}$ of diagrams is the set
$$\mathrm{cl}(\mathcal{S}) = \mathrm{ucl}(\widehat{\mathcal{S}}) = \overline{\mathcal{S}} \cup \mathrm{ucl}(\mathcal{S}).$$
This definition is appropriate in view of Theorem 3.8.
2. A sequence of diagrams $\langle D_i \rangle_{i \in \mathbb{N}}$ is said to be *closed* if $\forall D\,((\exists i\,(D = D_i)) \Leftrightarrow \forall E\,(E > D \Rightarrow \exists j\,(E = D_j)))$.

REMARK 3.10. Theorem 3.8 implies that, provably in RCA_0, a closed set of diagrams is finitely generated qua closed set if and only if it is finitely generated qua

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upwardly closed set, i.e., finitely generated in the sense of Definition 2.4. And similarly, a closed sequence of diagrams is finitely generated qua closed sequence if and only if it is finitely generated qua upwardly closed sequence.

LEMMA 3.11. RCA_0 *proves the following. Given a closed sequence of diagrams* $\langle D_i \rangle_{i\in\mathbb{N}}$ *which is not finitely generated, we can find a subsequence* $\langle D_{i_n} \rangle_{n\in\mathbb{N}}$ *such that* $\forall n\,(D_{i_n} \notin \mathrm{cl}(\{D_{i_1}, \ldots, D_{i_{n-1}}\}))$ *and* $\forall m\,\forall n\,(m < n \Rightarrow D_{i_n} \not\leq D_{i_m})$.

PROOF. We reason in RCA_0. Using the results of [28, Section II.3], define i_n recursively as follows. Let $i_1 = 1$. Assume inductively that $i_1 < \cdots < i_n$ have been defined. By bounded Σ^0_1 comprehension [28, Theorem II.3.9], there is a finite set $\mathcal{T}_n = \{D \mid \exists j\,(D = D_j \text{ and } D \leq D_{i_1} \cup \cdots \cup D_{i_n})\}$. Since $\langle D_i \rangle_{i\in\mathbb{N}}$ is not finitely generated, we have $D_i \notin \mathrm{cl}(\mathcal{T}_n)$ for infinitely many i. A fortiori we also have $D_i \notin \mathrm{cl}(\{D_{i_1}, \ldots, D_{i_n}\})$ and $D_i \not\leq D_{i_1} \cup \cdots \cup D_{i_n}$ for infinitely many i, so let $i_{n+1} =$ the least such i which is $> i_n$. Clearly, the subsequence $\langle D_{i_n} \rangle_{n\in\mathbb{N}}$ has the desired properties. ⊣

LEMMA 3.12. RCA_0 *proves the following. If all closed sets of diagrams are finitely generated, then all closed sequences of diagrams are finitely generated.*

PROOF. We reason in RCA_0. Suppose there is a closed sequence of diagrams which is not finitely generated. By Lemma 3.11 let $\langle D_n \rangle_{n\in\mathbb{N}}$ be a sequence of diagrams such that $\forall n\,(D_n \notin \mathrm{cl}(\{D_1, \ldots, D_{n-1}\}))$ and $\forall m\,\forall n\,(m < n \Rightarrow D_n \not\leq D_m)$. In particular we have $\forall m\,\forall n\,(D_m \leq D_n \Rightarrow m = n)$. It then follows by Lemma 3.4 that for all D, there is at most one n such that $(D_n)_r = D$ and there is at most one n such that $(D_n)_c = D$.

Using the results of [28, Section II.3], define a subsequence $\langle D_{n_k} \rangle_{k\in\mathbb{N}}$ recursively as follows. Assume inductively that $n_1 < \cdots < n_{k-1}$ have been defined. Let $\mathcal{S}_k = \{D_{n_1}, \ldots, D_{n_{k-1}}\}$. The set of diagrams D such that $|D| \leq \|\mathcal{S}_k\|$ is finite, so by bounded Σ^0_1 comprehension and Σ^0_1 bounding we have $|(D_n)_r| > \|\mathcal{S}_k\|$ and $|(D_n)_c| > \|\mathcal{S}_k\|$ for all sufficiently large n. Let $n_k =$ the least such n which is also $> n_{k-1}$. This completes the construction. The construction insures that $\forall k\,\forall l\,\forall D$ (if $k < l$ and $D \in \mathrm{cl}(\mathcal{S}_l)$ and $|D| \leq \|\mathcal{S}_k\|$ then $D \in \mathrm{cl}(\mathcal{S}_k)$).

By Δ^0_1 comprehension let $\mathcal{U}$ be the set of diagrams D such that $D \in \mathrm{cl}(\mathcal{S}_k)$ where $k =$ the least k such that $|D| \leq \|\mathcal{S}_k\|$. Then $\mathcal{U} = \bigcup_{k\in\mathbb{N}} \mathrm{cl}(\mathcal{S}_k)$ is a closed set of diagrams which not finitely generated. This proves our lemma. ⊣

THEOREM 3.13. *Over* RCA_0 *each of the following statements implies the others.*

1. $\mathrm{WPO}(\mathcal{D})$.
2. *There is no infinite ascending sequence of closed sets of diagrams.*
3. *There is no infinite ascending sequence of finitely generated closed sets of diagrams.*
4. *Every closed set of diagrams is finitely generated.*
5. *Every closed sequence of diagrams is finitely generated.*

PROOF. We reason in RCA_0. Let D be a diagram, and let $\mathcal{S}$ be a finite set of diagrams. By Theorem 3.8 and Definition 3.9, the relation $D \in \mathrm{cl}(\mathcal{S})$ is Δ^0_1. With this remark in mind, the proofs of $1 \Rightarrow 2$ and $2 \Rightarrow 3$ and $3 \Rightarrow 4$ are similar to the proofs of the corresponding implications in Theorem 2.6.

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By Lemma 3.12 we have 4 $\Rightarrow$ 5. It remains to prove 5 $\Rightarrow$ 1. Assume that 1 fails. Let $\langle D_i \rangle_{i\in\mathbb{N}}$ be a bad sequence of diagrams. The formula $\Phi(E) \equiv \exists i\,(E \in \mathrm{cl}(\{D_1, \ldots, D_i\}))$ is Σ^0_1, so by [28, Lemma II.3.7] there is a sequence of diagrams $\langle E_j \rangle_{j\in\mathbb{N}}$ such that $\forall E\,(\Phi(E) \Leftrightarrow \exists j\,(E = E_j))$. Clearly $\langle E_j \rangle_{j\in\mathbb{N}}$ is closed. We claim that $\langle E_j \rangle_{j\in\mathbb{N}}$ is not finitely generated. Otherwise, by Σ^0_1 bounding there would be an n such that $\{D_1, \ldots, D_n\}$ is a set of generators. Let $\mathcal{S} = \{D_1, \ldots, D_n\}$. By Theorem 3.8 and Definition 3.9, we have $\mathrm{cl}(\mathcal{S}) = \overline{\mathcal{S}} \cup \mathrm{ucl}(\mathcal{S})$. Since $\overline{\mathcal{S}}$ is finite we have $D_i \notin \overline{\mathcal{S}}$ for all sufficiently large i, and since $\langle D_i \rangle_{i\in\mathbb{N}}$ is bad we have $D_i \notin \mathrm{ucl}(\mathcal{S})$ for all $i > n$. It is now clear that $D_i \notin \mathrm{cl}(\mathcal{S})$ for all sufficiently large i, contradicting our assumption that $\mathcal{S}$ is a set of generators. Thus $\langle E_j \rangle_{j\in\mathbb{N}}$ is a counterexample to 5, so 5 fails. $\dashv$

Remark 3.14. For each $n \in \mathbb{N}$, let $\mathcal{D}_n$ be the set of nonempty finite subsets of $\mathbb{N}^n$ which are downwardly closed with respect to the coordinatewise partial ordering of $\mathbb{N}^n$. It is known that $\mathcal{D}_n$ is well partially ordered under inclusion. Recall that Figure 1 is the diagram corresponding to the partition 5, 2, 2, 1. If we rotate Figure 1 counterclockwise by $3\pi/4$ radians, we obtain a picture of an element of $\mathcal{D}_2$, namely, the downwardly closed subset of $\mathbb{N}^2$ consisting of 10 elements:

$$(1,1),\ (1,2),\ (1,3),\ (1,4),\ (1,5),\ (2,1),\ (2,2),\ (3,1),\ (3,2),\ (4,1).$$

In this way we obtain a one-to-one order-preserving correspondence between $\mathcal{D}_2$ and $\mathcal{D}$. Similarly, there is a one-to-one order-preserving correspondence between $\mathcal{D}_{n+1}$ and the so-called n-dimensional partitions which are discussed in [1, Chapter 11]. Our results in this section may be interpreted as results about $\mathcal{D}_2$, and it is straightforward to generalize them to $\mathcal{D}_{n+1}$. It would be interesting to find appropriate similar generalizations of our results in Section 2, from $\mathcal{D}_2$ to $\mathcal{D}_{n+1}$.

§4. Two-sided ideals in $K[S]$. In this section we review the known one-to-one correspondence [6, Theorem 2] between closed sets of diagrams and two-sided ideals of $K[S]$. We observe that this correspondence is provable in RCA_0. We use this observation plus our Theorems 2.3 and 3.13 to obtain a reverse-mathematical classification of the known theorem [6, Theorem 10] that $K[S]$ satisfies the ascending chain condition for two-sided ideals.

Definition 4.1. The following definitions are made in RCA_0.

1. Let S be the *infinite symmetric group*, i.e., the group of finitely supported permutations of $\mathbb{N}$. Let K be a countable field of characteristic 0. For instance, we could take $K = \mathbb{Q}$. The *group algebra* $K[S]$ is the set of formal finite linear combinations $\sum_g a_g g$ where $g \in S$, $a_g \in K$, and $a_g = 0$ for all but finitely many $g \in S$. The algebraic operations on $K[S]$ are given by

$$c \sum_g a_g g = \sum_g (c a_g) g,$$
$$\sum_g a_g g + \sum_g b_g g = \sum_g (a_g + b_g) g,$$
$$\sum_g a_g g \cdot \sum_g b_g g = \sum_g \sum_h (a_g b_h)(gh),$$

where $c \in K$.

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2. A *two-sided ideal of* $K[S]$ is a nonempty set $\mathcal{I} \subseteq K[S]$ such that $(\forall p \in \mathcal{I})\,(\forall q \in \mathcal{I})\,(p+q \in \mathcal{I})$ and $(\forall p \in \mathcal{I})\,(\forall r \in K[S])\,(p \cdot r \in \mathcal{I}$ and $r \cdot p \in \mathcal{I})$.
3. For each $n \in \mathbb{N}$, let S_n be the group of permutations of $\{1, \dots, n\}$. The group algebra $K[S_n]$ and the two-sided ideals of $K[S_n]$ are defined as above, replacing S by S_n. Identifying permutations of $\{1, \dots, n\}$ with permutations of $\mathbb{N}$ having support included in $\{1, \dots, n\}$, we have $S = \bigcup_{n \in \mathbb{N}} S_n$ and $K[S] = \bigcup_{n \in \mathbb{N}} K[S_n]$.

The next lemma summarizes some known facts from the representation theory of S_n.

LEMMA 4.2. RCA_0 *proves the following. There is an explicit one-to-one mapping* $e : \mathcal{D} \to K[S]$ *with the following properties.*

1. *The irreducible central idempotents of* $K[S_n]$ *are* $e(D)$, $|D| = n$.
2. *Consequently, there is a one-to-one correspondence between two-sided ideals of* $K[S_n]$ *and sets of diagrams of size* n. *The two-sided ideal of* $K[S_n]$ *corresponding to* $\mathcal{S} \subseteq \{D \mid |D| = n\}$ *is generated by* $\{e(D) \mid D \in \mathcal{S}\}$.
3. *Let* E *be a diagram of size* $\leq n$. *Then* $e(E)$ *and* $\{e(D) \mid |D| = n, E \leq D\}$ *generate the same two-sided ideal of* $K[S_n]$.

PROOF. As noted in [6, Theorem 1], these facts are proved in [13, 4.27, 4.51, 4.52]. Their formalization within RCA_0 is routine. Some other helpful sources are [4, Sections 23–28] for facts 1 and 2, and [7, Chapter 7] for fact 3. ⊣

LEMMA 4.3. RCA_0 *proves the following. There is a one-to-one correspondence between two-sided ideals of* $K[S]$ *and closed sets of diagrams. The two-sided ideal of* $K[S]$ *corresponding to a closed set* $\mathcal{U}$ *of diagrams is generated by* $\{e(D) \mid D \in \mathcal{U}\}$. *A two-sided ideal of* $K[S]$ *is finitely generated if and only if the corresponding closed set of diagrams is finitely generated.*

PROOF. We reason in RCA_0. Define $\mathcal{T} \subseteq \{E \mid |E| \leq n\}$ to be *n-closed* if

$$\mathcal{T} = \{E \mid |E| \leq n, \forall D\,((|D| = n, E \leq D) \Rightarrow D \in \mathcal{S})\}$$

for some $\mathcal{S} \subseteq \{D \mid |D| = n\}$. Lemma 4.2 implies a one-to-one correspondence between n-closed subsets of $\{E \mid |E| \leq n\}$ and two-sided ideals $\mathcal{I}$ of $K[S_n]$, where the n-closed set corresponding to $\mathcal{I}$ is $\{E \mid |E| \leq n, e(E) \in \mathcal{I}\}$. Our lemma then follows upon noting that (1) a set $\mathcal{U}$ of diagrams is closed if and only if $\forall n\,(\mathcal{U} \cap \{E \mid |E| \leq n\}$ is n-closed), and (2) a set $\mathcal{I} \subseteq K[S]$ is a two-sided ideal of $K[S]$ if and only if $\forall n\,(\mathcal{I} \cap K[S_n]$ is a two-sided ideal of $K[S_n])$. ⊣

We now present the main theorem of this paper.

THEOREM 4.4. *Over* RCA_0 *each of the following statements implies the others.*

1. $\mathrm{WO}(\omega^\omega)$.
2. *There is no infinite ascending sequence of two-sided ideals of* $K[S]$.
3. *There is no infinite ascending sequence of finitely generated, two-sided ideals of* $K[S]$.
4. *Every two-sided ideal of* $K[S]$ *is finitely generated.*

PROOF. We reason in RCA_0. The one-to-one correspondence in Lemma 4.3 is such that each two-sided ideal of $K[S]$ is uniformly Δ^0_1 definable from its corresponding

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closed set of diagrams and vice versa. Therefore, by Δ^0_1 comprehension, the one-to-one correspondence between two-sided ideals $\mathcal{I} \subseteq K[S]$ and closed sets $\mathcal{U} \subseteq \mathcal{D}$ extends to a one-to-one correspondence between sequences $\langle \mathcal{I}_i \rangle_{i \in \mathbb{N}}$ of two-sided ideals of $K[S]$ and sequences $\langle \mathcal{U}_i \rangle_{i \in \mathbb{N}}$ of closed sets of diagrams. We now see that Theorem 4.4 follows from Theorems 2.3 and 3.13. ⊣

§5. **Weakening the base theory.** The usual base theory for reverse mathematics [28, Part A] is RCA_0. However, there is an alternative base theory RCA^*_0 consisting of RCA_0 with Σ^0_1 induction replaced by exponentiation plus Σ^0_0 induction [28, Section IX.4]. Many of the known reversals over RCA_0 can be proved over the weaker base theory RCA^*_0, and in this way one strengthens the reversals [28, Remark X.4.3]. There are also some reversals over RCA^*_0 which have no counterpart over RCA_0, because the theorems in question are provable in RCA_0 and in fact equivalent to RCA_0 over RCA^*_0. The alternative base theory RCA^*_0 was introduced and used in [33] and has been used in several subsequent publications including [8, 9, 14, 29, 34, 37].

The purpose of this section is to strengthen some of our reversals in Theorems 2.6 and 3.13 and 4.4 by weakening the base theory from RCA_0 to RCA^*_0. In order to do so, we exercise care in defining the concept "infinite ascending sequence" within RCA^*_0. The following definition of "infinite ascending sequence" is equivalent over RCA_0 to the usual definition with $I = \mathbb{N}$, but the equivalence does not hold over RCA^*_0.

Definition 5.1. Within RCA^*_0 we define (a code for) an *infinite sequence of ideals of* $K[S]$ to consist of an infinite set $I \subseteq \mathbb{N}$ together with a set $\mathcal{I} \subseteq K[S] \times I$ such that for each $i \in I$ the set $\mathcal{I}_i = \{p \mid (p, i) \in \mathcal{I}\}$ is an ideal of $K[S]$. Such a sequence is denoted $\langle \mathcal{I}_i \rangle_{i \in I}$. The sequence is said to be *ascending* if $(\forall i \in I)\,(\forall j \in I)\,(i < j \Rightarrow \mathcal{I}_i \subsetneqq \mathcal{I}_j)$. Similarly within RCA^*_0 we define (codes for) infinite (ascending) sequences $\langle \mathcal{U}_i \rangle_{i \in I}$ of (closed or upwardly closed) sets of diagrams.

Lemma 5.2. *Over* RCA^*_0 *each of the following statements implies the others.*

1. RCA_0.
2. Σ^0_1 *induction*.
3. *There is no infinite descending sequence in* $\mathbb{N}$. *In other words, there is no function* $f : I \to \mathbb{N}$ *such that* $I \subseteq \mathbb{N}$ *is infinite and* $(\forall i \in I)\,(\forall j \in I)\,(i < j \Rightarrow f(i) > f(j))$.

Proof. We reason in RCA^*_0. For $1 \Leftrightarrow 2$ see [33] or [28, Section X.4]. Clearly we have $2 \Rightarrow 3$, so it remains to prove $3 \Rightarrow 2$. Suppose 2 fails. Let $\Phi(m)$ be a Σ^0_1 formula such that $\Phi(1)$ and $\forall m\,(\Phi(m) \Rightarrow \Phi(m+1))$ and $\exists n \neg \Phi(n)$. Fix an n such that $\Phi(n)$ fails, and write $\Phi(m) \equiv \exists j\, \Theta(m, j)$ where $\Theta(m, j)$ is Σ^0_0. By Σ^0_0 comprehension, there is a function $g : \mathbb{N} \to \{1, \ldots, n\}$ defined by $g(i) =$ the least m such that $\neg\,(\exists j < i)\,\Theta(m, j)$. Clearly we have $\forall i\, \forall j\, (i > j \Rightarrow g(i) \geq g(j))$. By Δ^0_1 comprehension let $I = \{i + 1 \mid g(i+1) > g(i)\}$. If the set I were finite, then letting $m_0 + 1 = g(i_0 + 1)$ where $i_0 + 1 =$ the largest element of I, we would have $\Phi(m_0)$ and $\neg\,\Phi(m_0 + 1)$, a contradiction. Thus I is infinite, so $f : I \to \mathbb{N}$ defined by $f(i) = n - g(i)$ is an infinite descending sequence, so 3 fails. ⊣

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LEMMA 5.3. *Lemmas* 4.2 *and* 4.3 *and Theorem* 3.8 *hold over* RCA_0^*.

PROOF. The proofs in RCA_0 remain valid in RCA_0^*. The proof of Lemma 4.2 involves Gaussian elimination for finite systems of linear equations over K, but this works perfectly well in RCA_0^* despite the use of Σ_1^0 induction in [28, Exercise II.4.11]. The difference here is that K, unlike the real field $\mathbb{R}$, is a countable discrete field. ⊣

The main result of this section is as follows.

THEOREM 5.4. *Over* RCA_0^* *each of the following statements implies the others.*

1. $\mathsf{RCA}_0 + \mathrm{WO}(\omega^\omega)$.
2. *There is no infinite ascending sequence of upwardly closed sets of diagrams.*
3. *There is no infinite ascending sequence of closed sets of diagrams.*
4. *There is no infinite ascending sequence of two-sided ideals of* $K[S]$.

PROOF. We reason in RCA_0^*. By Theorems 2.6 and 3.13 and 4.4, it will suffice to prove that the disjunction $2 \vee 3 \vee 4$ implies RCA_0. Suppose RCA_0 fails. By Lemma 5.2 there is an infinite descending sequence $f : I \to \mathbb{N}$. For each $i \in I$ let F_i be the diagram such that $r_1(F_i) = f(i)$ and $r_2(F_i) = 0$. By Lemma 5.3 let $\mathcal{I}_i$ be the two-sided ideal of $K[S]$ corresponding to the closed set $\mathcal{U}_i = \mathrm{cl}(\{F_i\}) = \mathrm{ucl}(\{F_i\})$. Then $\langle \mathcal{U}_i \rangle_{i \in I}$ is an infinite ascending sequence of (upwardly) closed sets of diagrams, and $\langle \mathcal{I}_i \rangle_{i \in I}$ is an infinite ascending sequence of two-sided ideals in $K[S]$. Thus $2 \vee 3 \vee 4$ fails. ⊣

In a similar fashion, we can strengthen [25, Theorem 2.7] by weakening the base theory from RCA_0 to RCA_0^*. As in [25], let $K[x_1, \ldots, x_n]$ be the ring of polynomials over K in n indeterminates.

THEOREM 5.5. *Over* RCA_0^* *each of the following statements implies the other.*

1. $\mathsf{RCA}_0 + \mathrm{WO}(\omega^\omega)$.
2. *For each* $n \in \mathbb{N}$ *there is no infinite ascending sequence of ideals in* $K[x_1, \ldots, x_n]$.

PROOF. We reason in RCA_0^*. By [25, Theorem 2.7] we have $1 \Rightarrow 2$. Assume that 2 holds and 1 fails. Then by [25, Theorem 2.7] RCA_0 must fail, so by Lemma 5.2 let $f : I \to \mathbb{N}$ be an infinite descending sequence. Let $K[x]$ be the ring of polynomials over K in one indeterminate. For each $i \in I$ let $\mathcal{I}_i$ be the ideal in $K[x]$ generated by the monomial x^{m_i} where $m_i = 2^{f(i)}$. Then $\langle \mathcal{I}_i \rangle_{i \in I}$ is an infinite ascending sequence of ideals in $K[x]$, so 2 fails for $n = 1$. This contradiction completes the proof. ⊣

§6. A question for future research. In this section we raise a question for future research.

Maclagan [15][3] has shown that monomial ideals in the polynomial ring $K[x_1, \ldots, x_n]$ satisfy the *antichain condition*, i.e., there is no infinite sequence of such ideals which are pairwise incomparable under inclusion. Maclagan's argument also gives the same result for the power series ring $K[[x_1, \ldots, x_n]]$. And in a similar vein, we have the following result, which is apparently new.

[3]We thank Florian Pelupessy for calling our attention to [2, 15].

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THEOREM 6.1. *Let $K[S]$ be as in Section 4. Then, the two-sided ideals in $K[S]$ satisfy the antichain condition. In other words, there is no infinite sequence of two-sided ideals in $K[S]$ which are pairwise incomparable under inclusion.*

PROOF. The *better partial orderings* [18][4] are a subclass of the well partial orderings. Many of the well partial orderings which arise in practice are in fact better partial orderings. However, the class of better partial orderings enjoys many infinitary closure properties which are not shared by the class of well partial orderings. A key result of this kind due to Nash-Williams [18] implies the following:

> If $\mathcal{P}$ is a better partial ordering, then the downwardly closed subsets of $\mathcal{P}$ form a better partial ordering under inclusion. (1)

Taking complements, we also have:

> If $\mathcal{P}$ is a better partial ordering, then the upwardly closed subsets of $\mathcal{P}$ form a better partial ordering under reverse inclusion. (2)

Applying (1) to the better partial ordering $\mathbb{N}^n$, we see that $\mathcal{D}_n$ (see Remark 3.14) is a better partial ordering. And then, applying (2) to $\mathcal{D}_n$, we see that the upwardly closed subsets of $\mathcal{D}_n$ are better partially ordered under reverse inclusion. Letting $n = 2$, it follows by Lemma 4.3 that the two-sided ideals of $K[S]$ are better partially ordered under reverse inclusion, so in particular they satisfy the antichain condition. ⊣

REMARK 6.2. The calculations of Aschenbrenner and Pong [2] can be used to show that Maclagan's result is reverse-mathematically equivalent to $\mathrm{WO}(\omega^{\omega^\omega})$. It is also known [25, Theorem 2.9] that the Robson Basis Theorem [20, Theorem 3.15] is reverse-mathematically equivalent to $\mathrm{WO}(\omega^{\omega^\omega})$, and we conjecture that $\forall n\, \mathrm{WPO}(\mathcal{D}_n)$ has the same reverse-mathematical status. In view of Theorem 4.4, it would be interesting to determine the reverse-mathematical status of Theorem 6.1.

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