Ergod. Th. & Dynam. Sys. (2015), **35**, 2138–2150 © *Cambridge University Press, 2014*
doi:10.1017/etds.2014.17

Directional recurrence for infinite measure preserving $\mathbb{Z}^d$ actions

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(*Received* 18 *June* 2013 *and accepted in revised form* 5 *February* 2014)

Dedicated to the memory of Daniel J. Rudolph

Abstract. We define directional recurrence for infinite measure preserving $\mathbb{Z}^d$ actions both intrinsically and via the unit suspension flow and prove that the two definitions are equivalent. We study the structure of the set of recurrent directions and show it is always a G_δ set. We construct an example of a recurrent action with no recurrent directions, answering a question posed in a 2007 paper of Daniel J. Rudolph. We also show by example that it is possible for a recurrent action to not be recurrent in an irrational direction even if all its sub-actions are recurrent.

1. *Introduction*

Given a dynamical system defined by the action of a group G, it is natural to study the sub-dynamics of the action. In particular, one can ask what dynamical properties of the action of G are inherited by the dynamical systems one obtains by restricting the original action to subgroups of G. In the 1980s Milnor introduced the more general idea of *directional dynamics* [**7**]. He defined the directional entropy of a $\mathbb{Z}^d$ action in all (i.e. including irrational) directions. The study of directional entropy has been a productive line of research (see, for example, [**8–10**, **14**]). In addition, the idea of defining directional dynamical properties more generally has led to other advances in dynamics, most notably expansive sub-dynamics introduced in [**2**]. In this paper we define directional recurrence for infinite measure preserving $\mathbb{Z}^d$ actions and we study the structure of the set of recurrent directions under the assumption that the action is also recurrent.

The motivation for the project originally was a question posed by Rudolph [**12**] in response to Feldman's proof in [**3**] of the ratio ergodic theorem for conservative $\mathbb{Z}^d$ actions. Feldman's proof required that the generators of the action also be recurrent. Rudolph asked

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in [**12**] whether this was an additional assumption or if it is the case that the recurrence properties of a group action are necessarily inherited by its sub-actions. In 2008 the authors answered the question in the negative.

THEOREM 1.1. *There exists an infinite measure preserving, recurrent, and ergodic* $\mathbb{Z}^2$ *action,* $(X, \mu, \{T^{\vec{n}}\}_{\vec{n}\in\mathbb{Z}^2})$*, on a* σ*-finite Lebesgue space, with the property that* $(X, \mu, \{T^{k\vec{n}}\}_{k\in\mathbb{Z}})$ *is not recurrent for all* $\vec{n} \in \mathbb{Z}^2$.

The question became moot from the point of view of the ratio ergodic theorem when Hochman [**5**] gave an alternative proof that does not require the recurrence of any sub-actions. When Rudolph saw the example constructed to prove Theorem 1.1, and others displaying a range of possibilities for the recurrence properties of particular sets of sub-actions, he suggested that we extend the definition of directional recurrence to include irrational directions and that we analyze the structure of possible recurrent directions in this more general setting. The results on directional recurrence presented here were established jointly with Rudolph, but unfortunately the final exposition was completed after his untimely death in 2010. The authors are grateful to have had this chance to collaborate with Rudolph who was both their friend and advisor.

For ease of exposition from here onward we restrict our attention to $d = 2$, but we note that the proofs and examples generalize readily to all $d > 1$.

Denote the directions in $\mathbb{R}^2$ by angles in $[0, \pi)$. There are two ways of defining directional properties for $\mathbb{Z}^2$ actions. Milnor's approach is to define the property intrinsically in the $\mathbb{Z}^2$ action. Namely, we say that an action $\{T^{\vec{n}}\}_{\vec{n}\in\mathbb{Z}^2}$ has a dynamic property in the direction θ if there exist a collection of arbitrarily good rational approximants $\vec{n}_i$ of θ so that the set $\{T^{\vec{n}_i}\}$ has that property. Alternatively, given a direction θ, one can associate to $\{T^{\vec{n}}\}_{\vec{n}\in\mathbb{Z}^2}$ an $\mathbb{R}$ action in that direction by considering the unit suspension flow of $\{T^{\vec{n}}\}_{\vec{n}\in\mathbb{Z}^2}$, restricted to the direction θ. For directional entropy the two approaches yield equivalent definitions [**9**]. Here we define directional recurrence both intrinsically and via the unit suspension, and we prove that the two definitions are equivalent.

THEOREM 1.2. *An ergodic, infinite measure preserving* $\mathbb{Z}^2$ *action on a* σ*-finite Lebesgue space is recurrent in a direction* θ *if and only if its unit suspension restricted to the direction* θ *is recurrent as an* $\mathbb{R}$ *action.*

We show that there are some restrictions on the types of subsets of $[0, \pi)$ that can appear as recurrent directions for a $\mathbb{Z}^2$ action. Deferring the formal definition until later, let $\mathcal{R}_T \subset [0, \pi)$ denote the directions of recurrence for a $\mathbb{Z}^2$ action $\{T^{\vec{n}}\}_{\vec{n}\in\mathbb{Z}^2}$.

THEOREM 1.3. *The set of recurrent directions,* $\mathcal{R}_T$*, of an infinite measure preserving, recurrent, and ergodic* $\mathbb{Z}^2$ *action on a* σ*-finite Lebesgue space,* $(X, \mu, \{T^{\vec{n}}\}_{\vec{n}\in\mathbb{Z}^2})$ *is a* G_δ *subset of* $[0, \pi)$.

There are directional properties with stronger restrictions on the structure of the bad set of directions. For example, an ergodic and probability measure preserving $\mathbb{Z}^2$ action can have at most countably many directions along which it is not ergodic. A similar statement holds for directional weak mixing for a weak mixing $\mathbb{Z}^2$ action (see, for example, [**11**]).

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Here we show that there is a great deal more flexibility in the structure of $\mathcal{R}_T$. In particular, the original example proving Theorem 1.1 has no recurrent directions even under the extended definition of directional recurrence.

THEOREM 1.4. *There exists an infinite measure preserving, recurrent, and ergodic* $\mathbb{Z}^2$ *action on a* σ*-finite Lebesgue space,* $(X, \mu, \{T^{\vec{n}}\}_{\vec{n}\in\mathbb{Z}^2})$*, with the property that* $\mathcal{R}_T = \emptyset$.

The example proving Theorems 1.1 and 1.4 is a rank-one action that is constructed using a cutting and stacking procedure. The construction has enough flexibility built into it so that many specific examples can be obtained by choosing the parameters of the procedure appropriately. Note that by Theorem 1.3 we know that $\mathcal{R}_T$ cannot consist only of the rational directions in $[0, \pi)$. Using our technique we construct an explicit example that shows that it is possible for $\mathcal{R}_T$ to contain all the rational directions, but still not be all of $[0, \pi)$.

THEOREM 1.5. *Let* $\alpha_1, \ldots, \alpha_k \in [0, \pi)$ *be irrational. There exists an infinite measure preserving, recurrent, and ergodic* $\mathbb{Z}^2$ *action on a* σ*-finite Lebesgue space,* $(X, \mu, \{T^{\vec{n}}\}_{\vec{n}\in\mathbb{Z}^2})$*, with the property that* $\mathcal{R}_T$ *contains all rational directions but* $\alpha_i \notin \mathcal{R}_T$ *for* $i = 1, \ldots, k$.

It is, however, an open question as to whether any G_δ subset of $[0, \pi)$ can be achieved as a set of recurrence.

The organization of the paper is as follows. In §2 we define directional recurrence intrinsically and via the unit suspension flow and prove that the two definitions are equivalent. We also show that for rational directions the definition coincides with the usual definition of recurrence for the $\mathbb{Z}$ action obtained by restricting to the corresponding subgroup of $\mathbb{Z}^2$. In §3 we give a third characterization of recurrence in terms of a *sweeping out property* and we prove Theorem 1.3. In §4 we establish the notation we use to construct the cutting and stacking examples, give some basic properties of infinite measure preserving rank-one actions and the proofs of Theorems 1.4 and 1.5.

2. *Directional recurrence*

In what follows we let X denote a σ-finite Lebesgue space and $(X, \mu, \{T^{\vec{n}}\}_{\vec{n}\in\mathbb{Z}^2})$ denote an infinite measure preserving and ergodic $\mathbb{Z}^2$ action on X. Our goal in this section is to extend the notion of recurrence to include directions that do not correspond to a subgroup action of $\{T^{\vec{n}}\}_{\vec{n}\in\mathbb{Z}^2}$. We begin by establishing some notation to describe directions and vectors associated to a direction. We then define directional recurrence intrinsically and via the unit suspension. Finally, we prove that the definitions are equivalent.

2.1. *Notation.*

Recall that a direction is an angle $\theta \in [0, \pi)$. For ease of exposition we introduce some terminology to help associate directions to vectors in the plane.

Notation 2.1. Let $\vec{u} = (u_1, u_2) \in \mathbb{R}^2$. If $u_2 \neq 0$ then the *direction associated to* $\vec{u}$ is the angle it makes with the vector $sign(u_2) \cdot (1, 0)$. If $u_2 = 0$, then the direction associated to $\vec{u}$ is 0.

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Notation 2.2. Given $\theta \in [0, \pi)$, a vector $\vec{v} = \vec{v}(\theta) \in \mathbb{R}^2$ is said to be *associated to the direction* θ if the direction associated to $\vec{v}$ is θ.

Notation 2.3. A direction $\theta \in [0, \pi)$ is rational if it has an associated vector $\vec{n} \in \mathbb{Z}^2$.

The concept of a tunnel is the key geometric tool that we will use to define directional recurrence.

Definition 2.4. Let $\theta \in [0, \pi)$ and $\epsilon > 0$. The ϵ-tunnel of θ is the set of points in $\mathbb{R}^2$ within ϵ of the line through the origin in direction θ, namely the set

$$\bigcup_{t \in \mathbb{R}} B_\epsilon(t\vec{u})$$

where $B_\epsilon(\vec{x})$ denotes the ϵ-ball around the point $\vec{x} \in \mathbb{R}^2$, and $\vec{u} \in \mathbb{R}^2$ is any vector associated to θ.

Finally, we introduce some notation which we will use in working with unit suspension flows.

Notation 2.5. Recall that $\lfloor x \rfloor$ is the greatest integer $n \leq x$. For a vector $\vec{v} = (v_1, v_2) \in \mathbb{R}^2$, we set $\lfloor \vec{v} \rfloor = (\lfloor v_1 \rfloor, \lfloor v_2 \rfloor) \in \mathbb{Z}^2$. For a set $R \subset \mathbb{R}^2$, $\lfloor R \rfloor = \{\lfloor \vec{v} \rfloor : \vec{v} \in R\}$.

Recall also that $\{x\}$ is the fractional part of the real number x, i.e. $\{x\} = x - \lfloor x \rfloor$. We define $\{\vec{v}\} = (\{v_1\}, \{v_2\})$ and $\{R\} = \{\{\vec{v}\} : \vec{v} \in R\}$.

2.2. *Defining directional recurrence by rational approximants.* We begin by recalling the definition of a recurrent group action. For an extensive treatment of the properties of recurrent actions see, for example, [**1**].

Definition 2.6. An ergodic, infinite measure preserving action of a group G on a σ-finite Lebesgue space, $(X, \mu, \{T^\gamma\}_{\gamma \in G})$, is recurrent if for every measurable set A with $\mu(A) > 0$, there exists $\gamma \in G$ such that $\mu(A \cap T^\gamma A) > 0$.

We are now ready to define directional recurrence for actions of the group $\mathbb{Z}^2$.

Definition 2.7. A direction θ is recurrent for an infinite measure preserving, ergodic, and recurrent $\mathbb{Z}^2$ action on a σ-finite Lebesgue space, $(X, \mu, \{T^{\vec{n}}\}_{\vec{n} \in \mathbb{Z}^2})$, if for all $\epsilon > 0$ and all measurable sets A of positive measure there is a vector $\vec{n}$ in the ϵ-tunnel of θ so that $\mu(A \cap T^{\vec{n}} A) > 0$.

Notation 2.8. We denote the set of recurrent directions for $(X, \mu, \{T^{\vec{n}}\}_{\vec{n} \in \mathbb{Z}^2})$ by $\mathcal{R}_T$.

The next result shows that recurrence along a rational direction as given by Definition 2.7 is equivalent to the recurrence of the group elements corresponding to that direction as given by Definition 2.6.

LEMMA 2.9. *Let $(X, \mu, \{T^{\vec{n}}\}_{\vec{n} \in \mathbb{Z}^2})$ be an infinite measure preserving $\mathbb{Z}^2$ action on a σ-finite Lebesgue space. A rational direction $\theta \in [0, \pi) \in \mathcal{R}_T$ if and only if $(X, \mu, \{T^{k\vec{n}}\}_{k \in \mathbb{Z}})$ is a recurrent $\mathbb{Z}$ action for all $\vec{n} \in \mathbb{Z}^2$ whose associated direction is θ.*

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Proof. Fix $(X, \mu, \{T^{\vec{n}}\}_{\vec{n}\in\mathbb{Z}^2})$ as in the statement of the lemma and suppose there is a rational $\theta \in \mathcal{R}_T$. Let $\vec{n} \in \mathbb{Z}^2$ be a vector with associated direction θ and choose $\epsilon > 0$ smaller than the distance between the line $\{t\vec{n}\}_{t\in\mathbb{R}}$ and those elements of $\mathbb{Z}^2$ that do not lie on the line. For any set A of positive measure, the vector $\vec{w} \in \mathbb{Z}^2$ satisfying Definition 2.7 for θ with this ϵ must necessarily be of the form $k\vec{n}$, showing that $T^{\vec{n}}$ satisfies Definition 2.6. The converse is immediate. □

2.3. *Directional recurrence via the unit suspension.* We begin by recalling the definition of the unit suspension of the $\mathbb{Z}^2$ action $(X, \mu, \{T^{\vec{v}}\}_{\vec{v}\in\mathbb{Z}^2})$.

Definition 2.10. The unit suspension of a $\mathbb{Z}^2$ action on a σ-finite measure space, $(X, \mu, \{T^{\vec{n}}\}_{\vec{n}\in\mathbb{Z}^2})$, is the $\mathbb{R}^2$-action $\hat{T}$ defined on $X \times [0, 1)^2$ by

$$\hat{T}^{\vec{v}}(x, \vec{r}) = (T^{\lfloor \vec{v}+\vec{r} \rfloor}x, \{\vec{v} + \vec{r}\}).$$

Note that $\hat{T}$ preserves the product measure $\mu \times \lambda$ where λ is Lebesgue measure on $[0, 1)^2$.

We now prove Theorem 1.2, restated more precisely below.

THEOREM 2.11. *Let $(X, \mu, \{T^{\vec{n}}\}_{\vec{n}\in\mathbb{Z}^2})$ be an infinite measure preserving $\mathbb{Z}^2$ action on a σ-finite Lebesgue space and let $(X \times [0, 1)^2, \mu \times \lambda, \{\hat{T}^{\vec{v}}\}_{\vec{v}\in\mathbb{R}^2})$ denote its unit suspension. The direction $\theta \in \mathcal{R}_T$ if and only if the $\mathbb{R}$ action $(X \times [0, 1)^2, \mu \times \lambda, \{\hat{T}^{t\vec{u}}\}_{t\in\mathbb{R}})$ is recurrent for any $\vec{u}$ associated to the direction θ.*

Proof. Assume first that the $\mathbb{Z}^2$ action is recurrent in direction θ. Because the proofs of Propositions 3.2 and 3.3 given below hold verbatim for $\mathbb{R}^2$ actions, in order to prove that the unit suspension is recurrent in direction θ it suffices to show that it is recurrent for sets of the form $A \times R$ where $A \subset X$ is a subset of positive measure and $R \subset [0, 1)^2$ is a rectangle $[i_1, i_2] \times [j_1, j_2]$. Fix $\epsilon > 0$ to be less than $\frac{1}{4}\min\{i_2 - i_1, j_2 - j_1\}$. By the directional recurrence of the $\mathbb{Z}^2$ action we can find $\vec{n} \in \mathbb{Z}^2$ in the ϵ-tunnel of θ so that $\mu(A \cap T^{\vec{n}}A) > 0$.

Choose $\vec{u} \in \mathbb{R}^2$ to be the vector associated with direction θ that is closest to $\vec{n}$. Note that $\|\vec{u} - \vec{n}\| < \epsilon$. Let $\tilde{R} \subset R$ be the subset of R consisting of those $(i, j) \in R$ that are at least a fraction $\frac{1}{4}\min\{i_2 - i_1, j_2 - j_1\}$ away from the edges of the square. By our choice of ϵ we then know that if $(s, t) \in \tilde{R}$ then $\vec{u} + (s, t) \in \vec{n} + R$ and therefore $\lfloor \vec{u} + \tilde{R} \rfloor = \vec{n}$ and $\{\vec{u} + \tilde{R}\} \subset R$. Thus, if we set $\tilde{A} = A \cap T^{\vec{n}}A$ we have $\hat{T}^{\vec{u}}(\tilde{A} \times \tilde{R}) = (T^{\vec{n}}\tilde{A} \times R')$ for some $R' \subset R$ with area at least half of the area of R and therefore $\mu \times \lambda(\hat{T}^{\vec{u}}(\tilde{A} \times \tilde{R}) \cap (A \times R)) \geq \frac{1}{2}\,\mu \times \lambda(\tilde{A} \times \tilde{R}) > 0$.

Now suppose that the unit suspension is recurrent in a direction θ. Fix $A \subset X$, a set of positive measure and $\epsilon > 0$. Let $R \in [0, 1)^2$ be a square based at the origin of side length $\frac{1}{4}\epsilon$. By the recurrence of the unit suspension there is a vector $\vec{u}$ associated to θ for which the measure of $(A \times R) \cap \hat{T}^{\vec{u}}(A \times R)$ is positive. Consider $(x, \vec{r})$ in this intersection. Since $\hat{T}^{\vec{u}}(x, \vec{r}) = (T^{\lfloor \vec{u}+\vec{r} \rfloor}x, \{\vec{u} + \vec{r}\}) \in A \times R$ we have $\vec{n} = \lfloor \vec{u} + \vec{r} \rfloor$ with $T^{\vec{n}}x \in A$. It remains to prove that $\vec{n}$ must lie in the ϵ-tunnel of θ. To see this, note that both $\vec{r}$ and $\vec{j} = \{\vec{u} + \vec{r}\}$ are in R. On the other hand, $\vec{j} = \vec{u} + \vec{r} - \vec{n}$ so $\|\vec{u} - \vec{n}\| = \|\vec{j} - \vec{r}\|$ and the latter is less than ϵ by the definition of R. □

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3. *The structure of $\mathcal{R}_T$*

3.1. *Uniform sweeping out.* We give an alternative characterization of recurrence that is useful for our arguments in this section. In what follows we fix an infinite measure preserving $\mathbb{Z}^2$ action on a σ-finite Lebesgue space $(X, \mu, \{T^{\vec{n}}\}_{\vec{n}\in\mathbb{Z}^2})$ and we denote the σ-algebra of μ-measurable sets by $\mathcal{B}$.

Definition 3.1. Let $A \in \mathcal{B}$ be such that $0 < \mu(A) < \infty$. A direction θ is said to have the *ϵ sweeping out property* for A if for all α with $0 < \alpha < \frac{1}{2}$, there exist pairwise disjoint subsets $A_1, A_2, \ldots, A_k$ of A of positive measure and vectors $\vec{v}_1, \ldots, \vec{v}_k$ in the ϵ-tunnel of θ so that the sets $T^{\vec{v}_1}A_1, \ldots, T^{\vec{v}_k}A_k$ are a pairwise disjoint collection of subsets of A satisfying

$$\mu\left(\bigcup_{i=1}^{k} A_i\right) > \alpha \cdot \mu(A), \tag{1}$$

$$\mu\left(\bigcup_{i=1}^{k} T^{\vec{v}_i}(A_i)\right) > \alpha \cdot \mu(A). \tag{2}$$

If θ has the ϵ sweeping out property for A for every $\epsilon > 0$ we say it is *uniform sweeping out* for A.

The next two results show that the recurrence of a direction can be characterized in terms of Definition 3.1.

PROPOSITION 3.2. *A recurrent direction is uniform sweeping out for all measurable sets A of finite positive measure.*

Proof. Let θ be a recurrent direction. Choose $\epsilon > 0$, $0 < \alpha < \frac{1}{2}$, and A, a measurable set of finite positive measure. Since θ is recurrent, there is a vector $\vec{v}_1$ in the ϵ-tunnel of θ such that $\mu(A \cap T^{\vec{v}_1}A) > 0$. Define $\hat{A}_1 = A \cap T^{\vec{v}_1}A$ and $A_1 = T^{-\vec{v}_1}\hat{A}_1$.

Assume that we have found $\vec{v}_j$, $\hat{A}_j$, and A_j for $1 \le j \le i$ and set $A'_{i+1} = A - (\bigcup_{j=1}^{i}(\hat{A}_j \cup A_j))$. If $\mu(A'_{i+1}) > 0$, use the recurrence of θ to find $\vec{v}_{i+1}$ in the ϵ-tunnel of θ so that $\mu(A'_{i+1} \cap T^{\vec{v}_{i+1}}A'_{i+1}) > 0$. Define

$$\hat{A}_{i+1} = A'_{i+1} \cap T^{\vec{v}_{i+1}}A'_{i+1} \qquad \text{and} \qquad A_{i+1} = T^{-\vec{v}_{i+1}}\hat{A}_{i+1}.$$

Fix $\vec{v}_1$ and note that the choices of $\vec{v}_i$ with $i > 1$ described above are not unique. Consider all possible extensions of $(A_1, T^{\vec{v}_1}A_1)$ as defined above:

$$(A_1, T^{\vec{v}_1}A_1), \left(\bigcup_{i=1}^{2} A_i, \bigcup_{i=1}^{2} T^{\vec{v}_i}A_i\right), \ldots, \left(\bigcup_{i=1}^{k} A_i, \bigcup_{i=1}^{k} T^{\vec{v}_i}A_i\right), \ldots.$$

Each such sequence forms a chain in the partially ordered set of pairs of measurable subsets of $A \times A$ under the relation $(B, C) \le (B', C')$ if and only if $B \subset B'$ and $C \subset C'$. Since (A, A) is a maximal element of this set, each chain is contained in a maximal chain. Choose such a maximal chain and note that if

$$\mu\left(\bigcup_{i=1}^{\infty}(A_i \cup T^{\vec{v}_i}A_i)\right) < \mu(A)$$

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it would be possible to extend the chain; therefore we must have equality. But then it follows that

$$\mu(A) = \mu\left(\bigcup_{i=1}^{\infty}(A_i \cup T^{\vec{v}_i} A_i)\right) \leq \sum_{i=1}^{\infty} \mu(A_i \cup T^{\vec{v}_i} A_i)$$
$$\leq \sum_{i=1}^{\infty} 2\,\mu(A_i) = 2\sum_{i=1}^{\infty} \mu(A_i).$$

We can thus find k so that

$$\mu\left(\bigcup_{i=1}^{k} A_i\right) = \sum_{i=1}^{k} \mu(A_i) > \alpha\,\mu(A),$$

satisfying (1).

Since the action is measure preserving

$$\sum_{i=1}^{k} \mu(A_i) = \sum_{i=1}^{k} \mu(T^{\vec{v}_i} A_i)$$

and (2) is also satisfied. □

The following is a converse result, showing that if a direction has the uniform sweeping out property for a sufficiently good collection of sets, then it is a recurrent direction.

PROPOSITION 3.3. *Let $\mathcal{A}$ be a countable collection of sets of finite positive measure which are dense in the σ-algebra $\mathcal{B}$. If θ is a direction which is uniform sweeping out for all $A \in \mathcal{A}$, then θ is a recurrent direction.*

Proof. Let C be a subset of X with infinite measure. Since X is σ-finite we can write $C = \bigcup_{i=1}^{\infty} C_i$ with $C_i \subset C_{i+1}$ and $0 < \mu(C_i) < \infty$ for all i. Thus it suffices to prove that θ is a direction of recurrence for sets of finite positive measure.

Fix a set C of finite, positive measure and $\epsilon > 0$. As the collection $\mathcal{A}$ is dense there is a set $A \in \mathcal{A}$ of positive, finite measure, with the property that

$$\mu(A \triangle C) < \epsilon\,\mu(A). \tag{3}$$

Fix $\frac{1}{4} < \alpha < \frac{1}{2}$. Choose sets $A_i \subset A$ and vectors $\{\vec{v}_i\}$ in the ϵ-tunnel of θ, $i = 1, \ldots, k$, satisfying Definition 3.1 for the direction θ. The rest of the argument, while technical, relies on the idea that since the sets $\bigcup A_i$ and $\bigcup T^{\vec{v}_i} A_i$ both cover at least α of the set A, and A is an approximation of the set C, provided ϵ is chosen small enough, there is an index i with the property that both

$$\mu(A_i \cap C) > \tfrac{1}{2}\mu(A_i) \qquad \text{and} \qquad \mu(T^{\vec{v}_i} A_i \cap C) > \tfrac{1}{2}\mu(T^{\vec{v}_i} A_i). \tag{4}$$

We thus have $\vec{v}_i \in \mathbb{Z}^2$ in the ϵ-tunnel of θ for which $\mu(C \cap T^{\vec{v}_i} C) > 0$. Now the details.

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It follows easily from (3) that if we consider $\bigcup A_i \cap C$, rather than $\bigcup A_i$, we lose a set of small measure. More formally,

$$\mu\left(\bigcup_{i=1}^{k} A_i \cap C\right) \geq \mu\left(\bigcup_{i=1}^{k} A_i\right) - \mu(A \triangle C)$$
$$\geq \mu\left(\bigcup_{i=1}^{k} A_i\right) - \epsilon\mu(A).$$

On the other hand, by our choice of α and the statement in (1), $4\mu(\bigcup_{i=1}^{k} A_i) > \mu(A)$ so we have

$$\mu\left(\bigcup_{i=1}^{k} A_i \cap C\right) > (1 - 4\epsilon)\mu\left(\bigcup_{i=1}^{k} A_i\right).$$

We now argue that the collection of sets A_i that are well covered by C make up most of the union, in measure. Let $\mathcal{I} \subset \{1, \ldots, k\}$ be the collection of i's such that $\mu(A_i \cap C) > (1 - 2\sqrt{\epsilon})\mu(A_i)$.

Note that

$$\mu\left(\bigcup_{i=1}^{k} A_i \cap C\right) = \mu\left(\bigcup_{i\in\mathcal{I}} A_i \cap C\right) + \mu\left(\bigcup_{i\notin\mathcal{I}} A_i \cap C\right)$$
$$\leq \mu\left(\bigcup_{i\in\mathcal{I}} A_i\right) + (1 - 2\sqrt{\epsilon})\mu\left(\bigcup_{i\notin\mathcal{I}} A_i\right)$$
$$= \mu\left(\bigcup_{i=1}^{k} A_i\right) - 2\sqrt{\epsilon}\,\mu\left(\bigcup_{i\notin\mathcal{I}} A_i\right).$$

We thus have

$$(1 - 4\epsilon)\mu\left(\bigcup_{i=1}^{k} A_i\right) < \mu\left(\bigcup_{i=1}^{k} A_i \cap C\right) \leq \mu\left(\bigcup_{i=1}^{k} A_i\right) - 2\sqrt{\epsilon}\mu\left(\bigcup_{i\notin\mathcal{I}} A_i\right)$$

which implies that

$$\mu\left(\bigcup_{i\notin\mathcal{I}} A_i\right) \leq 2\sqrt{\epsilon}\mu\left(\bigcup_{i=1}^{k} A_i\right).$$

We now make an analogous argument for the sets $T^{\vec{v}_i} A_i$. We begin by using (2) to argue, similarly to before, that we have

$$\mu\left(\bigcup_{i=1}^{k} T^{\vec{v}_i} A_i \cap C\right) > (1 - 4\epsilon)\mu\left(\bigcup_{i=1}^{k} T^{\vec{v}_i} A_i\right).$$

We can similarly define $\mathcal{J} \subset \{1, \ldots, k\}$ to be the collection of js such that $\mu(T^{\vec{v}_j} A_j \cap C) > (1 - 2\sqrt{\epsilon})\mu(A_j)$ and show that

$$2\sqrt{\epsilon}\mu\left(\bigcup_{j=1}^{k} T^{\vec{v}_j} A_j\right) = 2\sqrt{\epsilon}\mu\left(\bigcup_{j=1}^{k} A_j\right) \geq \mu\left(\bigcup_{j\notin\mathcal{J}} A_j\right) = \mu\left(\bigcup_{j\notin\mathcal{J}} T^{\vec{v}_j} A_j\right).$$

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Finally, if we consider only those indices that lie in both $\mathcal{I}$ and $\mathcal{J}$ we can conclude

$$\mu\Big(\bigcup_{i\in\mathcal{I}\cap\mathcal{J}} A_i\Big) \geq (1-4\sqrt{\epsilon})\mu\Big(\bigcup_{i=1}^{k} A_i\Big) \quad \text{and}$$
$$\mu\Big(\bigcup_{i\in\mathcal{I}\cap\mathcal{J}} T^{\vec{v}_i} A_i\Big) \geq (1-4\sqrt{\epsilon})\mu\Big(\bigcup_{i=1}^{k} T^{\vec{v}_i} A_i\Big).$$

Thus $\mathcal{I}\cap\mathcal{J}\neq\emptyset$ and for any $i\in\mathcal{I}\cap\mathcal{J}$ we will have

$$\mu(A_i\cap C) > (1-2\sqrt{\epsilon})\mu(A_i) \quad \text{and}$$
$$\mu(T^{\vec{v}_i}A_i\cap C) > (1-2\sqrt{\epsilon})\mu(T^{\vec{v}_i}A_i).$$

Clearly if ϵ is chosen small enough then (4) holds. □

3.2. *Proof of Theorem 1.3.* We now have all the ingredients in place to prove that $\mathcal{R}_T$ must be a G_δ set.

Let $\mathcal{A}=\{A_i\}$ be a countable collection of sets of finite positive measure which are dense in $\mathcal{B}$. Then by Propositions 3.2 and 3.3,

$$\mathcal{R}_T = \bigcap_{i=1}^{\infty}\{\theta : \theta \text{ has the uniform sweeping out property under } T \text{ for } A_i\}.$$

By Definition 3.1, we can rewrite this set as:

$$\mathcal{R}_T = \bigcap_{i=1}^{\infty}\bigcap_{n=1}^{\infty}\{\theta : \theta \text{ has the } 1/n \text{ sweeping out property under } T \text{ for } A_i\}.$$

Thus we just need to show that for any set $A\in\mathcal{B}$ of positive finite measure, a set of the form

$$\{\theta : \theta \text{ has the } \epsilon \text{ sweeping out property under } T \text{ for set } A\}$$

is open.

Fix such a set A and $\epsilon>0$, and let θ be in the set defined above. Fix a vector $\vec{v}_i$ in the ϵ-tunnel of θ given by Definition 3.1 and denote its associated direction by ω_i. By the continuity of the sine function there exists an $\eta_i=\eta(\vec{v}_i)$ so that $\vec{v}_i$ lies in the ϵ-tunnel of all directions in the interval $(\omega_i-\eta_i,\omega_i+\eta_i)$. Further, η_i can be chosen in such a way that θ lies in this interval. Choose δ_i so that $(\theta-\delta_i,\theta+\delta_i)\subset(\omega_i-\eta_i,\omega_i+\eta_i)$. Let $\delta=\min_{i=1,\ldots,k}\delta_i$. Then the interval $(\theta-\delta,\theta+\delta)$ consists of directions that contain all the $\vec{v}_i$ in their ϵ-tunnel, and thus all have the ϵ-uniform sweeping out property under T for the set A, as wanted.

4. *Examples of $\mathcal{R}_T$*

4.1. *Infinite measure preserving rank-one actions.* The examples we construct in this paper will be rank-one infinite measure preserving actions defined on measurable subsets of $\mathbb{R}^+$. We will construct them using standard cutting and stacking methods. In this section we establish the notation and facts about such actions which we need for our arguments. For more general discussions of rank-one transformations and cutting and

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stacking we refer the reader to [**4**], for rank-one non-singular transformations to [**13**], and for discussions of rank-one actions in higher dimensions to [**6**, **10**].

The following notation will be of use to us in our constructions.

Notation 4.1. Let $B \subset \mathbb{R}^2$, $\vec{v} \in \mathbb{R}^2$. The $\vec{v}$-tunnel of B is the set $B + \{t\vec{v}\}_{t\in\mathbb{R}}$.

Notation 4.2. For $n \in \mathbb{N}$ we set $B_n = [0, n)^2 \cap \mathbb{Z}^2$ and $\overline{B}_n = (-n, n)^2 \cap \mathbb{Z}^2$. Given a square $B = \vec{v} + B_n$ or $\vec{v} + \overline{B}_n$ we call the point $\vec{v}$ the *base point* of B. A vector $\vec{v} = (x, y) \in \mathbb{Z}^2$ with $x, y > 0$ will be called a *positive vector*.

Fix $n_1 \in \mathbb{N}$ and let $\{I^1_{\vec{v}}\}_{\vec{v}\in B_{n_1}}$ be a pairwise disjoint collection of equal finite length interval subsets of $\mathbb{R}$ indexed by B_{n_1}. Geometrically we think of the intervals as arranged in the shape of B_{n_1} according to their indexing vector. We start defining an action T by mapping levels to each other by translation:

$$T^{\vec{w}}(I^1_{\vec{v}}) = I^1_{\vec{v}+\vec{w}}$$

whenever $\vec{v}, \vec{w} \in B_{n_1}$ are such that $\vec{v} + \vec{w} \in B_{n_1}$. The collection of intervals, denoted by τ_1, is clearly a tower for the $\mathbb{Z}^2$ action. Each interval will be called a *level* of the tower and $I^1_{\vec{0}}$ will be called the *base* of the tower.

Given a tower τ_i, $i \geq 1$, we define the next tower τ_{i+1} by dividing the base $I^i_{\vec{0}}$ into $k(i)$ equal length subintervals, $I^{i,j}_{\vec{0}}$, $j = 1, \ldots, k(i)$, for some choice of $k(i) \in \mathbb{N}$. For $j = 1, \ldots, k(i)$, we call the subsets

$$\tau_i^j = \bigcup_{\vec{v}\in B_{n_i}} T^{\vec{v}} I^{i,j}_{\vec{0}}$$

the *slices* of τ_i. The tower τ_{i+1} is then constructed by first placing these slices of τ_i at $k(i)$ locations in $\mathbb{Z}^2$ so that they do not intersect. These locations are denoted by $\mathcal{P}_i(1)$. In our constructions, $\mathcal{P}_i(1)$ will always include $\vec{0}$. Define $n_{i+1} \in \mathbb{N}$ to be the smallest integer so that $B_{n_{i+1}}$ contains the set

$$\bigcup_{\vec{v}\in\mathcal{P}_i(1)} (B_{n_i} + \vec{v})$$

and set $\mathcal{S}_{i+1} = B_{n_{i+1}} \setminus (\bigcup_{\vec{v}\in\mathcal{P}_i} B_{n_i} + \vec{v})$. For each $\vec{v} \in \mathcal{S}_{i+1}$ we choose an interval $s_{\vec{v}}$ of length equal to that of the intervals constituting the slices of τ_i in such a way that the resulting collection of intervals is pairwise disjoint. These are called the *spacer intervals* of the $i + 1$ stage, or of τ_{i+1}. To complete the construction of τ_{i+1} we assign spacer intervals to the locations in $B_{n_{i+1}}$ not occupied by the slices of τ_i according to their indexing vector.

We rename the intervals $\{I^{i+1}_{\vec{v}}\}_{\vec{v}\in B_{n_{i+1}}}$ with $I^{i+1}_{\vec{0}}$ denoting the base of the tower. Finally, the definition of T is extended to τ_{i+1} by setting

$$T^{\vec{w}}(I^{i+1}_{\vec{v}}) = I^{i+1}_{\vec{v}+\vec{w}}$$

whenever $\vec{v}, \vec{w} \in B_{n_{i+1}}$ are such that $\vec{v} + \vec{w} \in B_{n_{i+1}}$.

Let X denote the subset of $\mathbb{R}$ given by the union of the towers, with the σ-algebra generated by the levels of the towers. In the case where $n_i \to \infty$ and the measure of the levels of the towers goes to 0, the resulting action $(X, \mu, \{T^{\vec{v}}\}_{\vec{v}\in\mathbb{Z}^2})$ is a rank-one

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$\mathbb{Z}^2$ action. We say that $\{\tau_i\}$ is a sequence of towers associated to T. As the levels of τ_i all have the same measure, T is clearly (possibly infinite) measure preserving.

Note that just as we can see slices of τ_i inside τ_{i+1}, we can find slices of those slices inside τ_{i+2}, and so on.

The relative positions of all the slices of τ_i inside later towers are important for our arguments. To this end, for all $j \geq 1$ we define $\mathcal{P}_i(j)$ to be those vectors $\vec{u} \in B_{n_{i+j}}$ with the property that $T^{\vec{u}}$ maps the base of one slice of τ_i in τ_{i+j} to the base of another slice. Let $\mathcal{P}_i = \bigcup_{j\geq 1} \mathcal{P}_i(j)$. We call the elements of $\mathcal{P}_i$ *times of strong recurrence* for the tower τ_i. The next lemma is immediate.

LEMMA 4.3. *Let $I = I_{\vec{v}}^i$ be a level of tower τ_i. Then $T^{\vec{u}} I \cap I \neq \emptyset$ if and only if $\vec{u}$ is a time of strong recurrence for tower τ_i.*

There are several well-known properties of rank-one actions that we will use extensively.

PROPOSITION 4.4. *Let $(X, \mu, \{T^{\vec{v}}\}_{\vec{v}\in\mathbb{Z}^2})$ be a rank-one action, and $\{\tau_i\}$ an associated sequence of towers. Given a measurable set A with $0 < \mu(A) < \infty$ and $\epsilon > 0$, there is an i_0 such that, for all $i > i_0$, there is a level I of τ_i with the property that*

$$\mu(A \cap I) > (1 - \epsilon)\mu(I). \tag{5}$$

THEOREM 4.5. *A rank-one measure preserving $\mathbb{Z}^2$ action is ergodic and recurrent.*

Proof. Let $(X, \mu, \{T^{\vec{v}}\}_{\vec{v}\in\mathbb{Z}^2})$ be a measure preserving, rank-one $\mathbb{Z}^2$ action with $\{\tau_i\}$ an associated sequence of towers. A proof of the ergodicity of T can be found in Theorem 4.2.1 in [**13**]. To see that T is a recurrent $\mathbb{Z}^2$ action, fix $A \subset X$, a measurable set of finite positive measure, and ϵ where $0 < \epsilon < \frac{1}{4}$. Apply Proposition 4.4 to obtain an i_0 and choose $i > i_0$. Let I be the level of τ_i satisfying (5). By construction, τ_{i+1} is created from slices of τ_i: in doing so the level I is divided into $k(i)$ equal measure subintervals, each belonging to a slice τ_i^j of τ_i. By a standard Fubini argument there must be at least two of these subintervals, call them I^{j_1} and I^{j_2}, for which $\mu(A \cap I^{j_k}) > (1 - \sqrt{\epsilon})\mu(I^{j_k}) > \frac{1}{2}\mu(I^{j_k})$ for $k = 1, 2$. Let $\vec{u} \in \mathcal{P}_1(i)$ denote the vector that maps the base of $\tau_i^{j_1}$ to the base of $\tau_i^{j_2}$. In particular, $T^{\vec{u}} I^{j_1} = I^{j_2}$. We are then guaranteed that $\mu(T^{\vec{u}} A \cap A) > 0$. □

Finally, note that to construct a rank-one $\mathbb{Z}^2$ action using the procedure we describe above it suffices to establish a value for n_1 and then for each i to choose the number $k(i)$ of slices of τ_i used to construct τ_{i+1}, along with the set $\mathcal{P}_i(1) \subset \mathbb{Z}^2$ describing the location of these slices. These parameters completely determine the values of the rest of the n_i, $i \geq 2$ and sets $\mathcal{P}_i$.

4.2. *Proof of Theorem 1.4.* To start the construction let $n_1 = 2$, and construct τ_1 from the levels $\{[0, 1), [1, 2), [2, 3), [3, 4)\}$ indexed by B_2 with $I_{\vec{0}}^1 = [0, 1)$. Set $k(i) = 2$ for all i and let $\mathcal{P}_1(1) = \{\vec{0}, (2, 2)\}$.

To define the induction let $\vec{v}_i^*$ denote the non-zero element of all the times of strong recurrence that have thus far been created that has the smallest slope. Call this slope m_i^*.

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Choose a positive vector $\vec{v} \in \mathbb{Z}^2$ and $k \in \mathbb{Z}$ so that if $k\vec{v} = (x, y)$ then

$$y > n_i, \tag{6}$$

$$\frac{y}{x} < \frac{1}{2} m_i^*, \tag{7}$$

$$k\vec{v} \text{ lies below the } \vec{v}_i^*\text{-tunnel of } B_{n_i}, \tag{8}$$

if $\vec{w}$ is a time of strong recurrence that has been created up to this point then

$$k\vec{v} + B_{n_i} \text{ does not intersect the } \vec{w}\text{-tunnels of } B_{n_i}. \tag{9}$$

Let $\mathcal{P}_i(1) = \{\vec{0}, k\vec{v}\}$.

Let $(X, \mu, \{T^{\vec{v}}\}_{\vec{v}\in\mathbb{Z}^2})$ denote the ergodic, infinite measure preserving, conservative $\mathbb{Z}^2$ action constructed in this way. To see that $\mathcal{R}_T = \emptyset$ first consider the case where there exists a rational direction θ with an associated vector $\vec{v} \in \mathbb{Z}^2$ so that $\vec{v} \in \mathcal{P}_i$ for some i. Choose the first stage i for which this is true and let s denote a spacer level from this stage. If there exists $n \in \mathbb{Z}$ such that $\mu(T^{n\vec{v}}(s) \cap s) > 0$ then by Lemma 4.3 one slice of τ_i must intersect the $\vec{v}$-tunnel of another slice of τ_i at some stage $k > i$. By condition (9) this is impossible since it implies that one of the two slices of τ_{k-1} must then intersect the $\vec{v}$-tunnel of the other.

Now suppose that θ does not have an associated vector in $\mathbb{Z}^2$ which is a time of strong recurrence. If $\theta = 0$, then if $\epsilon = 1$, condition (6) implies that all slices of τ_1 save one have to lie above the 1-tunnel of the horizontal axis. If $\theta \neq 0$, then by (7) we can find i such that $\vec{v}_k^*$ has slope smaller than $\tan\theta$ for all $k \geq i$. Let s be a spacer level in τ_{i+1} and suppose there exists $\vec{w} \in \mathbb{Z}^2$ in the ϵ-tunnel of θ for some $\epsilon > 0$ such that $\mu(T^{\vec{w}}s \cap s) > 0$. Arguing as above, this implies there exists a $k > i$ such that one slice of τ_k intersects the $\vec{u}(\theta)$-tunnel of the other at stage $k + 1$. But by (8) and (9) this is impossible.

4.3. *Proof of Theorem 1.5.* Let $\{\theta_j\}$ denote the set of rational directions. We first describe the construction that will guarantee $\theta_j \in \mathcal{R}_T$ for all j. Let $\mathcal{U} = \{u_i\}$ denote an enumeration of the elements of $\mathbb{Z}^2$. The towers will be of shape $\overline{B}_n$ centered at the origin, and the construction can start with an arbitrary choice for n_1. For $i > 1$ the remaining parameters are chosen to be $k(i) = 2$ and $\mathcal{P}_i(1) = \{\vec{0}, \vec{w}_i\}$ with $\vec{w}_i = t_i\vec{u}_i$, where $t_i \in \mathbb{Z}$ is chosen so that $\vec{w}_i + \overline{B}_{n_i}$ is disjoint from $\overline{B}_{n_i}$.

Let $(X, \mu, \{T^{\vec{v}}\}_{\vec{v}\in\mathbb{Z}^2})$ denote the resulting $\mathbb{Z}^2$ action. To see that $\mathcal{R}_T$ contains all rational directions fix a rational θ_j, a measurable set A of finite positive measure, and $0 < \epsilon < \frac{1}{4}$ and apply Proposition 4.4 to obtain i_0. Note that there are infinitely many vectors $\vec{u}_k \in \mathbb{Z}^2$ associated with θ_j and therefore there are infinitely many $i > i_0$ so that $\mathcal{P}_i(1)$ consists of $\vec{0}$ and a vector associated with θ_j. For such an i let I be the level in τ^i satisfying (5). This implies that both slices of I in τ^{i+1} must be strictly more than $\frac{1}{2}$ covered by A and therefore $\mu(T^{\vec{u}}A \cap A) > 0$ for the non-zero vector $\vec{u} \in P_i(1)$ associated with θ_j.

It follows from Theorem 1.3 that $\mathcal{R}_T$ must contain other directions besides the θ_j. We now describe how to modify the construction so that we can guarantee the given collection of irrational directions $\alpha_1, \ldots, \alpha_k$ are not in $\mathcal{R}_T$. First note that for any $\epsilon > 0$ every time a vector $t\vec{w}$ associated to θ_k appears as a time of strong recurrence, it is in the ϵ-tunnel for

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only the interval of directions $(\theta_k - \delta(\epsilon, t), \theta_k + \delta(\epsilon, t))$ where $\delta(\epsilon, t) = \arcsin(\epsilon/\|t\vec{w}\|)$. There is some flexibility in the construction to choose t as large as we like, thereby shrinking this interval as much as we like. We next describe how to use this flexibility to guarantee that none of the α_i are recurrent directions for T.

Fix a sequence ϵ_i decreasing to zero and at each stage of the construction choose t_i so that the interval $(\theta_{k(i)} - \delta(\epsilon_i, t), \theta_{k(i)} + \delta(\epsilon_i, t))$ does not contain any of the directions α_j. For any $\epsilon > 0$ there exists i_o such that, for all $i \geq i_0$, $\epsilon_i < \epsilon$. Let I be a level of τ_{i_0}. By Lemma 4.3 we have that for all j, α_j cannot be ϵ recurrent for I.

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