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Square Banach spaces

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In (2) I described a canonical isometric representation of an arbitrary real Banach space X by vector-valued functions (with the uniform norm) on a compact Hausdorff space Ω with the following properties: (1) the representing function space is invariant under multiplications by continuous real functions on Ω; (2) the norm of each representing function, as a real non-negative function on Ω, is upper semicontinuous; and (3) this decomposition of X is maximally fine. I called attention to the class of spaces X for which at every point of Ω the component space of this representation is one-dimensional or 0, so that the representing functions are in effect real valued. I propose to call such Banach spaces *square*, because of the shape of the unit ball in the two-dimensional case. In (2) I stated without proof, erroneously as it turns out, that the class of square spaces coincides with what Lindenstrauss in (4) called G-spaces. The primary purpose of this paper is to show that the class of square spaces is actually properly contained in that of G-spaces. It is known ((2), p. 620, Example 1) that it contains properly the class of continuous function spaces $C(\Omega)$. Among G-spaces are the M-spaces treated by Kakutani as Banach lattices (3). I shall show further that neither class, square spaces or M-spaces (regarded now purely as Banach spaces), contains the other.

The concrete prototype of a square space is a uniformly closed linear space of real functions on a compact Hausdorff space Ω satisfying

(i) If $x \in X$ and $f \in C(\Omega)$, then fx (pointwise multiplication) belongs to X;

(ii) $|x|$ is upper semicontinuous for all $x \in X$.

For the reasons behind (i) and (ii) see (2). We can assume further for convenience, and without loss of generality, that the set in Ω where all functions in X vanish has void interior. By definition every square space has an isometric representation as a function space of this type. The fact that conversely every such function space is square is not essential to the results of this paper, and will not be proved here.

A Banach space is by definition a G-space if it is isometrically isomorphic to a closed subspace X of a $C(K)$, where K is compact Hausdorff, X being defined by a family of linear relations of the form

$$x(t_\alpha) = \lambda_\alpha x(t'_\alpha) \quad (x \in X). \tag{*}$$

This means that a family of triples $(t_\alpha, t'_\alpha, \lambda_\alpha)$ is given, with $t_\alpha, t'_\alpha \in K$ and λ_α real and X consists of all continuous functions on K satisfying (*).

By way of obtaining a more standard description of a given G-space, consider more generally any Banach space X which is a closed subspace of $C(K)$. Define the *linear dependence decomposition* of K for X to consist of the equivalence classes for the

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following equivalence relation: s, t in K are equivalent if and only if the same functions in X vanish at s and at t. Then there will be one equivalence class L_0 (possibly empty) on which all the functions in X vanish, and for each equivalence class $L \neq L_0$, the restrictions to L of the functions in X will form a one-dimensional set, i.e. all will be constant multiples of any non-zero one. All the relations (*) satisfied by X can easily be written down in terms of the decomposition and a selection of non-vanishing functions λ_L, one for each $L \neq L_0$, each being the restriction to L of a function in X. Now the necessary and sufficient condition that X be a G-space (or rather a representation exhibiting the defining property of a G-space) is that X contains every continuous real function y defined on K which vanishes on L_0 and whose restriction to every $L \neq L_0$ is a constant multiple of λ_L.

THEOREM 1. *Every square space is a G-space.*

In proving the theorem we start with a square space X given in the form of a function space satisfying (i), (ii) as described above. Let $\sigma: \Omega \to X^*$ be defined by evaluation: $\langle x, \sigma(t)\rangle = x(t)$ for $x \in X, t \in \Omega$. It is easy to see that either $\sigma(t) = 0$ or $\|\sigma(t)\| = 1$. We use in X^* the weak-* topology having as a sub-base of neighbourhoods of 0 the sets $\{\mu; |\langle x, \mu\rangle| < \epsilon\}$ for $x \in X$, $\epsilon > 0$. Since the functions in X are not necessarily continuous, σ need not be continuous. For $A \subset X^*$ we shall use the notation star A to mean the set $\{\lambda a; a \in A, |\lambda| \leqslant 1\}$. Let $K = \operatorname{star} \sigma(\Omega)$.

LEMMA 1. *Let $t \in \Omega$, and let U be an open set in X^* containing* $\operatorname{star}\{\sigma(t)\}$. *Then t has a neighbourhood V in Ω such that* $\operatorname{star} \sigma(V) \subset U$.

Proof. $E = \operatorname{star}\{\sigma(t)\}$ being compact, find a finite collection of basic weak* open sets covering E, whose union U' is contained in U. Each of these sets is defined by inequalities involving a finite number of elements of X, so that U' is defined by a logical polynomial of inequalities in some $x_1, \ldots, x_n$ in X. Let $\pi: X^* \to R^n$ be the projection using $x_1, \ldots, x_n$ as coordinates, i.e. $\pi(\mu) = (\langle x_1, \mu\rangle, \ldots, \langle x_n, \mu\rangle)$. Then $\pi(E)$ is a line segment in R^n, symmetric in the origin, and contained in the open set $\pi(U')$. Thus there exists an open convex symmetric polyhedron W such that $\pi(E) \subset W \subset \pi(U')$, and W is defined by inequalities $|y_i| < \alpha_i$ $(i = 1, \ldots, k)$ where $y_1, \ldots, y_k$ are linear functions on R^n and $\alpha_1, \ldots, \alpha_n \in R$. Then each $y_i \circ \pi \circ \sigma$ is a function on Ω which is a linear combination of $x_1, \ldots, x_n$, so belongs to X. For each $i = 1, \ldots, k$ $|y_i \circ \pi \circ \sigma| < \alpha_i$ at t because $\sigma(t) \in E$, and therefore by upper semicontinuity these inequalities persist throughout some neighbourhood V of t. Evidently $\operatorname{star} \sigma(V) \subset U' \subset U$ as required.

LEMMA 2. $K = \operatorname{star} \sigma(\Omega)$ *is weak* compact.*

Proof. Let $\mathscr{F}$ be a given open cover of K. For each $t \in \Omega$, $\operatorname{star}\{\sigma(t)\}$, being compact, is covered by a finite union U_t of sets in $\mathscr{F}$. For each t use Lemma 1 to find a neighbourhood V_t of t such that $\operatorname{star} \sigma(V_t) \subset U_t$. Now, Ω being compact, choose a finite subcover of Ω from $\{V_t\}$. Clearly the collection of sets in $\mathscr{F}$ used in forming the U_t corresponding to the V_t chosen for this subcover form a finite subcover of $\mathscr{F}$.

LEMMA 3. *Let $y: K \to R$ be homogeneous and continuous, and let $t \in \Omega$ be such that $y(\sigma(t)) = 0$. Then $y \circ \sigma$ is continuous at t.*

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Proof. By the homogeneity, $y = 0$ on star $\{\sigma(t)\}$. Hence by continuity and the compactness of star $\{\sigma(t)\}$, given $\epsilon > 0$ there is an open set U containing star $\{\sigma(t)\}$ on which $|y| < \epsilon$. Using Lemma 1, find a neighbourhood V of t with star $\sigma(V) \subset U$. Then $|y \circ \sigma| < \epsilon$ on V.

Proof of Theorem 1. Represent X by functions on K in the natural way; that is, to each $x \in X$ make correspond the function $\tilde{x}$ defined on K by $\tilde{x}(k) = \langle x, k \rangle$ for $k \in K$. An alternative formula for this function is $\tilde{x}(\lambda\sigma(t)) = \lambda x(t)$ for $t \in \Omega$ and $|\lambda| \leqslant 1$, from which it is clear that $x \overset{w}{\mapsto} \tilde{x}$ is an isometry (uniform norm on both sides). The linear dependence decomposition of K for the resulting function space $\tilde{X}$ is also easily discerned: it consists of $\{0\}$ and the one-dimensional sets $E_t = \{\lambda\sigma(t);\ 0 < |\lambda| \leqslant 1\}$ for $t \in \Omega$. Since by Lemma 2 K is compact, to show that X is a G-space it remains to verify that every continuous homogeneous real function y on K belongs to $\tilde{X}$. This is equivalent to showing $z = y \circ \sigma$ belongs to X, since $y = \tilde{z}$. Now for any $t \in \Omega$, if $y(\sigma(t)) \neq 0$, then $\sigma(t) \neq 0$, so there exists $x \in X$ with $x(t) \neq 0$, and we can even have $x(t) = z(t)$. If $z(t) = 0$, take $x = 0$. In either case apply Lemma 3 to $y - \tilde{x}$ to obtain a neighbourhood V_t of t in which $|z - x| < \epsilon$. Since Ω is compact, a finite number of these V_t cover Ω. Call them $V_1, \ldots, V_n$, and let the corresponding x's be $x_1, \ldots, x_n$. Now let $f_1, \ldots, f_n$ be a continuous decomposition of 1 subordinated to the covering $V_1, \ldots, V_n$, i.e. $f_i \in C(\Omega)$, $0 \leqslant f_i \leqslant 1$ and $f_i = 0$ outside V_i $(i = 1, \ldots, n)$, and $\Sigma f_i = 1$. Then $x = \Sigma f_i x_i$ belongs to X, and $|x(t) - z(t)| < \epsilon$ for all $t \in \Omega$. Since X is uniformly closed, and z can be uniformly approximated by functions in X, we have $z \in X$, as was to be proved.

THEOREM 2. *There exist G-spaces which are not square.*

To construct an example, start with any infinite dimensional Banach space Y, and let K be the unit ball in Y^* with the weak* topology, a compact set. Let X be the Banach space of all continuous real functions on K which are homogeneous, i.e. satisfy $x(\lambda k) = \lambda x(k)$ for $k \in K$, $|\lambda| \leqslant 1$. Embedding Y in its second dual in the usual way makes Y into a subspace of X, since linear functions are homogeneous. Thus if k, $k' \in K$ are not in a one-dimensional subspace, there is a function in X vanishing at k but not at k'. This shows that the linear dependence decomposition of K for X is exactly the 'projective' decomposition into $\{0\}$ and the sets

$$L_k = \{\lambda k;\ 0 < |\lambda| \leqslant 1\} \quad \text{for} \quad \|k\| = 1.$$

Clearly X is a G-space.

The proof that X is not square rests on two contrasting facts about K which we give as Lemmas (4 and 5), although it is likely that they are known. The first shows that Y is only a small part of X. The second states the non-existence of weak* continuous functions which are homogeneous of degree 0, that is, that an infinite dimensional weak* projective space carries nonconstant continuous functions.

LEMMA 4. *Given $k \in K$ with $\|k\| = 1$, a weak* neighbourhood U of k, and $\epsilon > 0$, there exists a homogeneous weak* continuous function z on K such that $z(k) = 1$, $|z| < 1 + \epsilon$ on K, and $z = 0$ outside star U.*

Proof. We can suppose U to be a basic neighbourhood:

$$U = \{u \in Y^*;\ |\langle y_i, u \rangle - \langle y_i, k \rangle| < \delta,\ i = 1, \ldots, n\},$$

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where $y_1, \ldots, y_n \in Y$ and $\delta > 0$. Choose $y_0 \in Y$ so that $|\langle y_0, k\rangle| > 1 - \frac{1}{2}\epsilon$. Let π be the projection of Y^* into R^{n+1} defined by the functionals $y_0, y_1, \ldots, y_n$. Then $\pi(K)$ is a symmetric convex compact set, $\pi(k) \in \pi(K)$, $\pi(k) \notin (1 - \frac{1}{2}\epsilon)\pi(K)$, and $\pi(U)$ is a neighbourhood of $\pi(k)$. Let f be a homogeneous continuous function on R^{n+1} such that $f(\pi(k)) = 1$, $\|f\| < 1 + \epsilon$ on $\pi(K)$, and $f = 0$ outside star $\pi(U)$, and set $z = f \circ \pi$. Clearly z has the required properties.

LEMMA 5. *Every weak* continuous function f on $K - \{0\}$ which is constant on every L_k ($\|k\| = 1$) is constant.*

Proof. If f is not constant, then $f(k) < a < b < f(k')$ for some $k, k' \in K$ and real a, b. Then by continuity $f < a$ on a weak* neighbourhood of k, and since f is constant on the rays L_k there exist $y_1, \ldots, y_n$ in Y such that $f < a$ on the symmetric polyhedral cone in $K - \{0\}$ where $y_1, \ldots, y_n$ are all positive or all negative. By continuity, this implies that $f \leqslant a$ on the (weak* closed) subspace which annihilates $y_1, \ldots, y_n$. Similarly $f \geqslant b$ on the subspace annihilating another finite set in Y. The intersection of these subspaces must evidently be $\{0\}$, which is impossible if Y is infinite dimensional.

LEMMA 6. *Let $k \in K$, $\|k\| = 1$. Then the evaluation functional $\tilde{k}$ defined by $\langle x, \tilde{k}\rangle = x(k)$ is an extreme point of the unit ball in X^*.*

Proof. Since $\tilde{k}$ is an evaluation functional, and X uses the uniform norm, clearly $\|\tilde{k}\| \leqslant 1$. Since $\|k\| = 1$ and $Y \subset X$, we have in fact $\|\tilde{k}\| = 1$. Now suppose $\tilde{k}$ is not an extreme point. Then there exists $\mu \neq 0$ in X^* such that $\|\tilde{k} \pm \mu\| \leqslant 1$. To get a contradiction we shall show that $\langle x, \mu\rangle = 0$ for all $x \in X$. In doing this we can suppose $\|x\| = 1$ and let $\langle x, \tilde{k}\rangle = x(k) = \lambda$, where $0 \leqslant \lambda \leqslant 1$. If $\lambda = 1$ we have

$$1 \geqslant |\langle x, \tilde{k} \pm \mu\rangle| = |\lambda \pm \langle x, \mu\rangle| = |1 \pm \langle x, \mu\rangle|$$

whence $\langle x, \mu\rangle = 0$ and we are through. So suppose $\lambda < 1$. Let $\epsilon > 0$ be given, and find a neighbourhood U of k in which $|x - \lambda| < \epsilon$. Use Lemma 4 to find $z \in X$ such that $z(k) = 1$, $\|z\| \leqslant 1 + \epsilon$, and $z = 0$ outside star U, and set $w = x + (1 - \lambda)z \in X$. Then $w(k) = 1$. Outside star U we have $|w| = |x| \leqslant 1$, and inside star U we have

$$|w| < (\lambda + \epsilon) + (1 - \lambda)(1 + \epsilon) < 1 + 2\epsilon,$$

so that $\|w\| \leqslant 1 + 2\epsilon$. Since $\|\tilde{k} \pm \mu\| \leqslant 1$, we have $|1 \pm \langle w, \mu\rangle| = |\langle w, \tilde{k} \pm \mu\rangle| \leqslant 1 + 2\epsilon$ whence $|\langle w, \mu\rangle| \leqslant 2\epsilon$. By the same argument also $|\langle z, \mu\rangle| \leqslant 2\epsilon$. Hence

$$|\langle x, \mu\rangle| \leqslant |\langle w, \mu\rangle| + (1 - \lambda)|\langle z, \mu\rangle| \leqslant 4\epsilon,$$

showing that $\langle x, \mu\rangle = 0$, as promised.

LEMMA 7. *Let X be a square space, let S be the set of extreme points of the unit ball in X^*, and let C be the set of bounded real functions f on S which operate by multiplication on the space of restrictions to S of functions in X (regarded as embedded in X^{**}). Then C separates linearly independent pairs of points in S.*

Proof. Referring back to the construction of the maximal M-structure of X ((2), section 4), we remark that C is a Banach algebra of functions on S, whose spectrum Ω is the domain of the function space representation of X. If s and s' are not separated

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by C, then they correspond to the same point t of Ω. We shall show that then the component space X_t at t in the function space representation is not one-dimensional, so that X is actually not square. Indeed, since s and s' are linearly independent, there exist x and x' in X such that $\langle x, s\rangle = \langle x', s'\rangle = 1$, while $\langle x, s'\rangle = \langle x', s\rangle = 0$. Let $\tilde{x}$ and $\tilde{x}'$ denote the functions on Ω representing x and x', and suppose $\alpha\tilde{x}(t)+\beta\tilde{x}'(t) = 0$. This means $|\alpha x+\beta x'|(t) = 0$, which in turn means ((2), p. 617) that given $\epsilon > 0$ there exists $h \in C$ with $h \geqslant 0$, $h(t) = 1$ and $\|\alpha hx+\beta hx'\| < \epsilon$. Then in particular

$$\epsilon > |\langle \alpha hx+\beta hx', s\rangle| = |h(t)\langle \alpha x+\beta x', s\rangle| = |\alpha|.$$

Since ϵ was arbitrary, we have $\alpha = 0$. Similarly $\beta = 0$. Thus $\tilde{x}(t)$ and $\tilde{x}'(t)$ are linearly independent in X_t, as promised.

Proof of Theorem 2. If X as constructed above were square, then S would be separated (except for antipodal pairs) by functions in C (Lemma 7). In particular (Lemma 6) there would exist a non-constant function f on the set $\{k \in K; \|k\| = 1\}$ which operates by multiplication on the space of restrictions to that set of functions in X. Extend f to $K-\{0\}$ so as to be constant on the sets L_k. Then for $x \in X$ we have also $fx \in X$, because fx agrees on $\{\|k\| = 1\}$ with a function $y \in X$, and then $fx = y$ on $K-\{0\}$, both being homogeneous. Since fx must then be continuous, and since for every $k \in K-\{0\}$ some x in X is not 0 at k, it follows that f is continuous, contradicting Lemma 5.

We turn now to the comparison of square spaces with M-spaces. An M-space is a G-space having a representation such that $\lambda_\alpha \geqslant 0$ in all the relations (*), in other words such that all the functions λ_L are of constant sign. We remark first that a slight modification of the proof of Theorem 2 yields an example of an M-space which is not square. Let the space Y from which the construction starts have a partial order which makes it a Banach lattice, and for K, instead of the unit ball in Y^*, use its intersection with the (closed) positive cone for the induced order in Y^*. Then the continuous homogeneous functions x on K have constant sign on each of the segments

$$\{\lambda k;\ 0 < \lambda \leqslant 1\}, \quad (k \in K, \|k\| = 1)$$

which form the linear dependence decomposition, so that X is an M-space.

THEOREM 3. *There exists a square space which is not an M-space.*

The simplest example is the space X of real functions x on the compact space Ω of real numbers mod 1 (which is topologically a circle), which are continuous except that at $0 \equiv 1$ they are continuous only from the right, the limit of x from the left being equal to $-x(0)$. That X is a square space is clear from the fact that the function space incarnation by which it is given is (except for notation) its maximal M-decomposition as constructed in (2), section 4. Indeed, X^* can be identified (via integration) with the space of bounded Borel measures on $[0, 1)$, and then the classical argument (1) for the case of $C(\Omega)$ carries over to show that the extreme points of the unit ball in X^* are exactly the evaluation functionals at points of Ω and their negatives. Clearly the functions on Ω which operate on X by multiplication are exactly those in $C(\Omega)$. Therefore Ω, being compact, is the spectrum of the maximal decomposition. Furthermore,

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the functions in X have continuous and *a fortiori* upper semicontinuous absolute values.

To show that X is not an M-space, it is convenient to use a different representation. Let Γ be the unit circle in R^2, and let X be the space of all continuous functions on Γ which are antisymmetric, i.e. satisfy $x(-\gamma) = -x(\gamma)$ for all $\gamma \in \Gamma$. Up to a Banach space isomorphism this is the same space as before, as is seen by restricting the functions in X to a semicircle including one end-point. It is obvious that this is a G-space representation of X satisfying relations (*) with $\lambda_\alpha = -1$, but this does not prove that X is not an M-space; we must look at all possible representations. If X is an M-space, then there exists a compact Hausdorff space K and a representation $x \mapsto \tilde{x}$ of X by continuous functions on K satisfying no relations (*) with $\lambda_\alpha < 0$. We shall show that this is impossible.

Define $\sigma: \Gamma \to X^*$ and $\tau: K \to X^*$ by evaluation, that is $\langle x, \sigma(\gamma)\rangle = x(\gamma)$ and $\langle x, \tau(k)\rangle = \tilde{x}(k)$ for $\gamma \in \Gamma$, $k \in K$ and $x \in X$. Because of the continuity of x and $\tilde{x}$ these are continuous maps for the weak* topology of X^*. In fact σ is even a homeomorphism from Γ onto the set of extreme points of the unit ball, embedding Γ in X^*.

LEMMA 8. *If X is any subspace of $C(K)$, and if ξ is an extreme point of the unit ball in X^*, then either ξ or $-\xi$ is evaluation at some point of K.*

Proof. Write $X^\perp$ for the weak* closed subspace of $C(K)^*$ which annihilates X. Extend ξ to a functional μ in $C(K)^*$ of norm 1 by the Hahn–Banach Theorem. Then the intersection of $\mu + X^\perp$ with the unit ball of $C(K)^*$ is a non-empty convex weak* compact set, so by the Krein–Milman Theorem has an extreme point μ'. It is easily shown in two steps that μ' is an extreme point of the unit ball in $C(K)^*$, and therefore by (1), Lemma 3·2, μ' or $-\mu'$ is evaluation at some point of K. Since ξ is the restriction of μ' to X, the same is true of ξ, as was to be proved.

To conclude the proof of Theorem 3, apply Lemma 8 to $\tilde{X}$ to see that the image of K under τ meets every antipodal pair of points in $\sigma(\Gamma)$. Thus $\sigma(\Gamma) \cap \tau(K)$ is, up to a homeomorphism, a closed subset of the circle containing at least one of each pair of antipodal points. But any such set must contain both ends of some diameter, and this provides a relation of the form (*) satisfied by $\tilde{X}$ with $\lambda_\alpha = -1$.

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