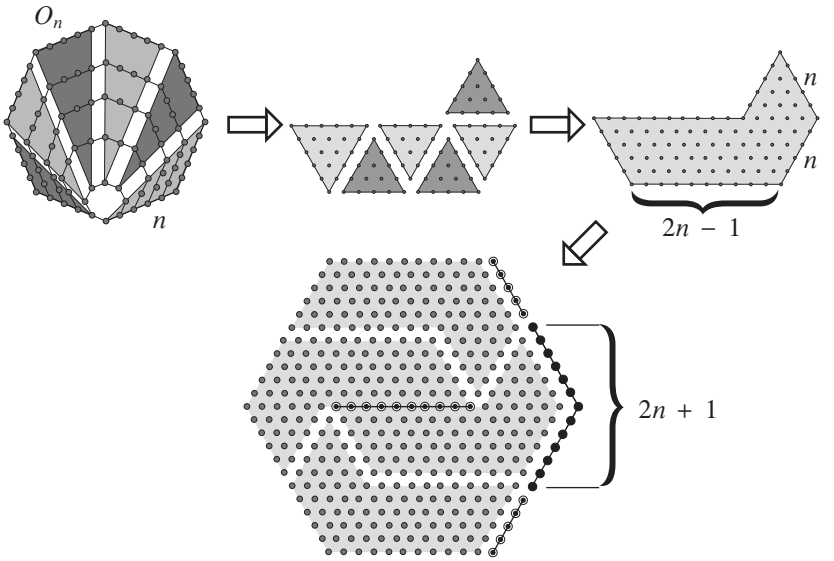
Proof: The proof is demonstrated for $n = 6$.



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109.18 Transitivity, transpositions and permutation groups

Let $n \geqslant 2$ be fixed. A permutation group G on the set $\{1, 2, \ldots, n\}$, or a subgroup G of the symmetric group S_n, is called transitive when, for all $i \neq j$ in $\{1, 2, \ldots, n\}$, there is a permutation in G sending i to j.

Of course, S_n itself is transitive, since there is a permutation sending 1 to every other number. Moreover, S_n is generated by transpositions, e.g., by transpositions of adjacent elements: (1, 2), (2, 3), (3, 4), …, $(n - 1, n)$. As the next result shows, within the class of permutation groups generated by transpositions, S_n is the only transitive one.

Proposition: Let G be a permutation group on $\{1, 2, \ldots, n\}$ generated by transpositions. If G is transitive, then $G = S_n$.

Proof: Let $\tau = (i, j)$ be a transposition contained in G. For $k \in \{1, 2, \ldots, n\}$, and since G is generated by transpositions and is transitive, there exist transpositions

$$(j, i_1), (i_1, i_2), \ldots, (i_{r-1}, r) \text{ and } (i_r, k)$$

connecting j and k, all contained in the set of generators of G. Without loss

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of generality, we can assume that i_s are mutually different $(s = 1, 2, \dots, r)$. First, assume that i_s are different from i, too. Since G contains the element

$$\sigma = (j, i_1), (i_1, i_2), \dots, (i_r, k),$$

G also contains $\sigma\tau\sigma^{-1} = (i, k)$. Next, assume that $i_t = i$ for some t. Since G contains the element

$$\pi = (i_t, i_{t+1}), (i_{t+1}, i_{t+2}), \dots, (i_r, k),$$

G contains $\pi\tau\pi^{-1} = (j, k)$ and also contains $\tau(\pi\tau\pi^{-1})\tau^{-1} = (i, k)$. So G is the symmetric group S_n.

Examples

(i) The Klein group K_4 is a group with four elements, in which each element is self-inverse (composing it with itself produces the identity) and in which composing any two of the three non-identity elements produces the third one. It can be described as the symmetry group of a non-square rectangle (with the three non-identity elements being horizontal and vertical reflection and 180-degree rotation). K_4 is generated by the transpositions (1 2) and (3 4), but is not transitive - otherwise, by the above Proposition, K_4 would be isomorphic to S_4, which is not true.

(ii) The dihedral group D_4, i.e, the group of symmetries of a square, which includes rotations and reflection is a transitive subgroup of S_4 (cf. [1]), but is not generated by transpositions - otherwise, by the above Proposition, D_4 would be isomorphic to S_4, which is not true.

(iii) The alternating group A_3, i.e., the group of even permutations on $\{1, 2, \dots, n\}$, is not generated by transpositions (instead, it is generated by all 3-cycles); but is transitive, as the 3-cycles (1 2 3) and (1 3 2) send 1 to every other number.

(iv) The kth cyclic group C_k in S_n, i.e., the group generated by a single element, is neither transitive, nor generated by transpositions for any $3 \leqslant k \leqslant n$.

Reference

1. B. L. Van der Waerden, *Algebra Vol. 1,* Springer Verlag, New York, MSC: 20B10 (2003).

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