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A theorem about countable decomposability

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A function $f: X \to \mathbb{R}$ is *countably decomposable* (into continuous functions) if the topological space X can be partitioned into countably many sets A_n with each restriction $f \mid A_n$ continuous. According to L. V. Keldysh (2), the question whether every Baire function is countably decomposable was first raised by N. N. Luzin, and answered by P. S. Novikov. The answer is negative even for Baire-1 functions, as is shown in (2) (see also (1). In this paper we develop a characterization of the countably decomposable functions on a separable metric space X (see Corollary 1). We deduce that when X is complete they include all functions possessing the property P defined by D. E. Peek in (3): *each non-empty σ-perfect set H contains a point at which $f \mid H$ is continuous*. The example given by Peek shows that not every countably decomposable Baire-1 function has property P.

In what follows, it is convenient in the case when Y is empty to regard a function $f \mid (Y$ as being both continuous and totally discontinuous (that is, discontinuous at each point of its domain).

THEOREM. *Given any function $f: X \to \mathbb{R}$, where X is a separable metric space, there exists a partition $X = A \cup B$ for which $f \mid A$ is countably decomposable, while $f \mid (B - Z)$ is totally discontinuous for every set Z such that $f \mid Z$ is countably decomposable.*

Proof. We may assume that X cannot be taken for B. Let $X_0 = X$, and let Y_1 be such that $f \mid (X_0 - Y_1)$ is countably decomposable and Y_1 contains a point at which $f \mid Y_1$ is continuous.

Using the axiom of choice, we first define by transfinite induction a decreasing transfinite sequence of sets X_α, and auxiliary sets Y_α, as follows.

If X_β has been defined for $\beta < \alpha$ and there exists a set $Y \subseteq \cap_{\beta<\alpha} X_\beta$ such that $f \mid ((\cap_{\beta<\alpha} X_\beta) - Y)$ is countably decomposable and $f \mid Y$ has at least one continuity point, then we choose as Y_α some such set, and we define X_α to be the set of discontinuity points of $f \mid Y_\alpha$; if there exists no such set Y then we define X_α to be $\cap_{\beta<\alpha} X_\beta$, and the process terminates.

Next we show that the ordinal α_0 at which the process terminates is countable. For each α with $\alpha + 1 < \alpha_0$ and each $n \in \mathbb{N}$, let $X_\alpha^{(n)}$ denote the set of points of X_α at which $\operatorname{osc} f \mid Y_\alpha \geqslant n^{-1}$. Since $X^{(n)} \uparrow X_\alpha$, and $f \mid Y_{\alpha+1}$ has at least one continuity point, we can choose $n = n(\alpha) \in \mathbb{N}$ such that $X_\alpha^{(n)}$ contains a point at a positive distance from $X_{\alpha+1}^{(n)}$. However, $\beta < \gamma \Rightarrow X_\alpha^{(n)} \supseteq X_\gamma^{(n)}$, so by a classical lemma of point-set topology, for each $m \in \mathbb{N}$ the set $\{\alpha : n(\alpha) = m\}$ is countable; therefore so is α_0.

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Now we can put

$$A = \cup_{\alpha<\alpha_0}[(\cap_{\beta<\alpha} X_\beta) - Y_\alpha] \cup \cup_{\alpha<\alpha_0}[Y_\alpha - X_\alpha],$$

$$B = X_{\alpha_0}.$$

By construction, X is the union of these two sets and they are disjoint. Since f is countably decomposable on each set $(\cap_{\beta<\alpha} X_\beta) - Y_\alpha$, it is also so on their union, while f is continuous on each set $Y_\alpha - X_\alpha$; thus $f \mid A$ is indeed countably decomposable. The property of B is immediate from the definition of X_{α_0}.

Remark. We can take A and B to be Borel sets in the theorem if we require $f \mid (B - Z)$ to be totally discontinuous only for every *Borel* set Z such that $f \mid Z$ is countably decomposable.

COROLLARY 1. *A function $f: X \to \mathbb{R}$, where X is a separable metric space, is countably decomposable if for every uncountable set $B \subseteq X$ there exists a set Z such that $f \mid Z$ is countably decomposable and $f \mid (B - Z)$ has at least one continuity point.*

COROLLARY 2. *If X is a complete separable metric space then every function $f: X \to \mathbb{R}$ possessing property P is countably decomposable.*

Proof. It was shown in (3) that each non-empty set $H \subseteq X$ such that every point of H is a condensation point of H contains a point at which $f \mid H$ is continuous. Since every uncountable set $B \subseteq X$ is the union of such a set H and a countable set D, and $f \mid D$ is automatically countably decomposable, the result follows from Corollary 1.

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