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Numerical range of Toeplitz and weighted composition operators on weighted Bergman spaces

Anirban Sen, Subhadip Halder, Riddhick Birbonshi, and Kallol Paul

Abstract. In this article, we completely describe the numerical range of Toeplitz operators on weighted Bergman spaces with harmonic symbol. We also characterize the numerical range of weighted composition operators on weighted Bergman spaces and classify some sets which are the numerical range of composition operators. We investigate the inclusion of zero in the numerical range, and compute the radius of circular and elliptical disk contained in the numerical range of weighted composition operators on weighted Bergman spaces.

1 Introduction

Let $\mathbb{B}(\mathbb{H})$ be the C^*-algebra of all bounded linear operators on a complex Hilbert space $\mathbb{H}$. For $T \in \mathbb{B}(\mathbb{H})$, the numerical range of T, denoted by $W(T)$, is the subset of the complex plane $\mathbb{C}$ defined by

$$W(T) = \{\langle Tf, f\rangle : f \in \mathbb{H}, \|f\| = 1\}.$$

It is well known that $W(T)$ is a bounded and convex subset of $\mathbb{C}$. The spectrum of T, denoted by $\sigma(T)$, is contained in the closure of $W(T)$. We refer to [8, 19] for the detailed proofs of these results and other properties of the numerical range.

The numerical ranges of Toeplitz and composition operators have been studied over the years. In 1972, Klein [12] completely described the numerical range of Toeplitz operators on the Hardy–Hilbert space of the unit disk. Later, the numerical ranges of Toeplitz operators on Bergman space and polydisk were studied in [6, 17, 18]. In [13],

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the numerical ranges of composition operators on the Hardy–Hilbert space induced by monomials were characterized. Bourdon and Shapiro [2, 3] studied the numerical range of composition operators and the containment of the origin. Recently, the numerical ranges of weighted composition operators on Hardy–Hilbert space and weighted Bergman spaces were explored in [7, 21].

In this article, we study the numerical ranges of Toeplitz operators and weighted composition operators on weighted Bergman spaces. The article is structured as follows. In Section 2, we introduce some notation, recall some definitions, and present some preliminary results. In Section 3, we completely describe the numerical range of Toeplitz operators on the weighted Bergman spaces with harmonic symbol. We provide an example to justify that the harmonic condition is necessary for the characterization given in Theorem 3.5. Then, we obtain the numerical range of some particular classes of weighted composition operators on the weighted Bergman spaces. Further, we characterize some bounded and convex sets which are the numerical range of weighted composition operators. Our main aim of Section 4 is to study when the origin is contained in the numerical range of weighted composition operators acting on weighted Bergman spaces. Then, we derive some sufficient conditions on the closedness of the numerical range of weighted composition operators. In Section 5, we identify several classes of weighted composition operators whose numerical range includes a circular disk or an elliptical disk. Furthermore, we determine the radius of the circular disk as well as the lengths of the minor and major axes of the elliptical disk.

2 Preliminaries

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk. Let $H(\mathbb{D})$ be the space of all analytic functions on $\mathbb{D}$ and H^∞ be the space of all bounded analytic functions on $\mathbb{D}$. We will use the notation $\overline{X}, \partial X, \operatorname{int} X, \operatorname{Rel\,int} X$, and $X^\wedge$ for the closure, boundary, interior, relative interior, and convex hull, respectively, of the set $X \subset \mathbb{C}$.

For $\alpha > -1$, the weighted Bergman space $L^2_a(dA_\alpha)$ on the unit disk is defined as

$$L^2_a(dA_\alpha) = \left\{ f \in H(\mathbb{D}) : \int_{\mathbb{D}} |f(z)|^2 dA_\alpha(z) < \infty \right\},$$

where dA denotes the normalized Lebesgue area measure on $\mathbb{D}$ and

$$dA_\alpha(z) = (\alpha + 1)(1 - |z|^2)^\alpha dA(z).$$

Clearly, $L^2_a(dA_\alpha) = H(\mathbb{D}) \cap L^2(\mathbb{D}, dA_\alpha)$ and when the weight parameter $\alpha = 0$, the weighted Bergman space becomes the classical Bergman space $L^2_a(dA)$. Here, we note that $L^2_a(dA_\alpha)$ is a closed subspace of $L^2(\mathbb{D}, dA_\alpha)$ and has the orthonormal basis $\{e_n\}_{n=0}^\infty$, where

$$e_n(z) = \sqrt{\frac{\Gamma(n + \alpha + 2)}{n!\Gamma(\alpha + 2)}} z^n \quad \text{for all } n \geq 0.$$

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For $f, g \in L^2_a(dA_\alpha)$, the inner product on $L^2_a(dA_\alpha)$ can also be expressed as

$$\langle f, g\rangle = \sum_{n=0}^{\infty} \frac{n!\Gamma(\alpha+2)}{\Gamma(n+\alpha+2)} \hat{f}_n \bar{\hat{g}}_n,$$

where $f(z) = \sum_{n=0}^{\infty} \hat{f}_n z^n$ and $g(z) = \sum_{k=0}^{\infty} \hat{g}_n z^n$. It is well known that the weighted Bergman spaces are reproducing kernel Hilbert space and the reproducing kernel of $L^2_a(dA_\alpha)$ at the point $w \in \mathbb{D}$ is given by

$$k^\alpha_w(z) = \frac{1}{(1-\bar{w}z)^{\alpha+2}}.$$

The normalized reproducing kernel at $w \in \mathbb{D}$ is given by

$$\hat{k}^\alpha_w(z) = \frac{(1-|w|^2)^{\frac{\alpha}{2}+1}}{(1-\bar{w}z)^{\alpha+2}}.$$

For more details about the weighted Bergman spaces, we refer to the book [20].

Let P_α denote the orthogonal projection of $L^2(\mathbb{D}, dA_\alpha)$ onto $L^2_a(dA_\alpha)$. Let $L^\infty(\mathbb{D}, dA_\alpha)$ be the space of all complex measurable functions ϕ on $\mathbb{D}$ such that

$$\|\phi\|_{\infty,\alpha} = \sup\{c \geq 0 : A_\alpha(\{z \in \mathbb{D} : |\phi(z)| > c\}) > 0\} < \infty.$$

For $\phi \in L^\infty(\mathbb{D}, dA_\alpha)$, the operator T_ϕ on $L^2_a(dA_\alpha)$ defined by

$$T_\phi f = P_\alpha(\phi f), \ f \in L^2_a(dA_\alpha)$$

is called the Toeplitz operator on $L^2_a(dA_\alpha)$ with symbol ϕ. It is easy to observe that T_ϕ is a bounded linear operator on $L^2_a(dA_\alpha)$ with $\|T_\phi\| \leq \|\phi\|_{\infty,\alpha}$. Furthermore, if $\phi \in H^\infty$, then $T^*_\phi k^\alpha_w = \overline{\phi(w)} k^\alpha_w$ for all $w \in \mathbb{D}$ (see [5]).

Let $\phi : \mathbb{D} \to \mathbb{D}$ be an analytic self-map on $\mathbb{D}$ and $\psi \in H(\mathbb{D})$. The weighted composition operator $C_{\psi,\phi} : H(\mathbb{D}) \to H(\mathbb{D})$ is defined by

$$C_{\psi,\phi} f = \psi(f \circ \phi) \quad \text{for all } f \in H(\mathbb{D}).$$

In particular, when ϕ is the identity mapping on $\mathbb{D}$ then $C_{\psi,\phi}$ becomes the multiplication operator M_ψ and for $\psi = 1$, $C_{\psi,\phi}$ becomes the unweighted composition operator C_ϕ. In this article, we limit our analysis to weighted composition operators on $L^2_a(dA_\alpha)$.

In [4, Theorem 1], Čučković et al. proved the following boundedness condition of weighted composition operators on $L^2_a(dA_\alpha)$:

$$C_{\psi,\phi} \in \mathbb{B}(L^2_a(dA_\alpha)) \text{ if and only if } \sup_{a\in\mathbb{D}} I_{\phi,\alpha}(\psi)(a) < \infty, \tag{2.1}$$

where

$$I_{\phi,\alpha}(\psi)(a) = \int_{\mathbb{D}} \left(\frac{1-|a|^2}{|1-\bar{a}\phi(w)|^2}\right)^{\alpha+2} |\psi(w)|^2 dA_\alpha(w).$$

Clearly, it follows from (2.1) that if $\psi \in H^\infty$, then $C_{\psi,\phi} \in \mathbb{B}(L^2_a(dA_\alpha))$. Next relation follows from the reproducing property of $L^2_a(dA_\alpha)$ that if $C_{\psi,\phi} \in \mathbb{B}(L^2_a(dA_\alpha))$, then $C^*_{\psi,\phi} k^\alpha_w = \overline{\psi(w)} k^\alpha_{\phi(w)}$ for all $w \in \mathbb{D}$.

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3 Shape of the numerical range

We begin with the definition of α-essential range.

Definition 3.1 For $\phi \in L^\infty(\mathbb{D}, dA_\alpha)$, the α-essential range of ϕ is denoted by $R_{\phi,\alpha}$ and defined as

$$R_{\phi,\alpha} = \{w : A_\alpha(\{z \in \mathbb{D} : |\phi(z) - w| < \varepsilon\}) > 0 \text{ for any } \varepsilon > 0\}.$$

It is easy to observe that $R_{\phi,\alpha}$ is a compact subset of $\mathbb{C}$ and

$$\|\phi\|_{\infty,\alpha} = \max\{|w| : w \in R_{\phi,\alpha}\}.$$

Now, we prove the following lemma.

Lemma 3.1 *Let $\phi \in L^\infty(\mathbb{D}, dA_\alpha)$ be such that ϕ is continuous on $\mathbb{D}$, then $\overline{\phi(\mathbb{D})} = R_{\phi,\alpha}$.*

Proof If $u \in \phi(\mathbb{D})$, then there exists $z_0 \in \mathbb{D}$ such that $u = \phi(z_0)$. Since ϕ is continuous so for any $\varepsilon > 0$, there exists $\delta > 0$ such that $|\phi(z) - u| < \varepsilon$ whenever $z \in S = \{z \in \mathbb{D} : |z - z_0| < \delta\}$. Clearly, $A_\alpha(\{z \in \mathbb{D} : |\phi(z) - u| < \varepsilon\}) \geq A_\alpha(S)$. If possible let $A_\alpha(S) = 0$. Then, we have $\int_S (\alpha+1)(1-|z|^2)^\alpha dA = 0$. This implies that $(\alpha+1)(1-|z|^2)^\alpha = 0$ almost all on $z \in S$, which is not possible. Thus, $A_\alpha(\{z \in \mathbb{D} : |\phi(z) - u| < \varepsilon\}) \geq A_\alpha(S) > 0$. Therefore, $u \in R_{\phi,\alpha}$ and since $R_{\phi,\alpha}$ is closed so $\overline{\phi(\mathbb{D})} \subseteq R_{\phi,\alpha}$.

Now, if $u \in R_{\phi,\alpha}$, then from the definition, it follows that for any $\varepsilon > 0$, there exists $z \in \mathbb{D}$ such that $|\phi(z) - u| < \varepsilon$. Hence, $u \in \overline{\phi(\mathbb{D})}$ and this completes the proof. ∎

Remark 3.2 Here, we note that if $\phi \in L^\infty(\mathbb{D}, dA_\alpha)$ and ϕ is continuous on $\mathbb{D}$, then ϕ is bounded on $\mathbb{D}$.

To prove our next result, we need the following lemma which was proved in [1].

Lemma 3.3 *If ϕ is harmonic and integrable over $\mathbb{D}$ then so is $\phi \circ \xi$ for every Möbius transformation ξ of $\mathbb{D}$, and $\int_{\mathbb{D}} (\phi \circ \xi) dA = \phi(\xi(0))$.*

In the following proposition, we obtain the spectrum of Toeplitz operator on $L^2_a(dA_\alpha)$ with real and harmonic symbol, which generalizes the existing result on the Bergman space given in [14, Proposition 12].

Proposition 3.4 *If $\phi \in L^\infty(\mathbb{D}, dA_\alpha)$ is real and harmonic, then $\sigma(T_\phi) = [\inf \phi, \sup \phi]$.*

Proof First, we have to prove $\sigma(T_\phi) \subseteq [\inf \phi, \sup \phi]$ and so we have to show that $T_{\phi-\lambda}$ is invertible whenever $\lambda \notin [\inf \phi, \sup \phi]$. Since $\lambda \notin [\inf \phi, \sup \phi]$ so either $\phi(z) - \lambda > 0$ or $\phi(z) - \lambda < 0$ for all $z \in \mathbb{D}$. For the former case, we choose $\varepsilon > 0$ such that

$$\|\varepsilon(\phi - \lambda) - 1\|_{\infty,\alpha} \leq \sup_{z \in \mathbb{D}} |\varepsilon(\phi(z) - \lambda) - 1| < 1$$

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and so we have

$$\|T_{\varepsilon(\phi-\lambda)} - I\| = \|T_{\varepsilon(\phi-\lambda)-1}\| \leq \|\varepsilon(\phi-\lambda)-1\|_{\infty,\alpha} < 1.$$

Thus, $T_{\varepsilon(\phi-\lambda)}$ is invertible and so $T_{\phi-\lambda}$ is invertible. For the latter case, $\phi(z)-\lambda<0$ for all $z\in\mathbb{D}$ implies that $-(\phi(z)-\lambda)>0$ for all $z\in\mathbb{D}$. Proceeding similarly as the former case and using the relation $T_{\phi-\lambda} = -T_{-\phi+\lambda}$, we get the desired result.

Next, we prove the opposite inclusion. Since $\sigma(T_\phi)$ is a closed subset of $\mathbb{C}$ so it suffices to show that $\sigma(T_\phi)\supseteq(\inf\phi,\sup\phi)$. As $\sigma(T_{\phi-\lambda})\supseteq(\inf\phi-\lambda,\sup\phi-\lambda)$ for any $\lambda\in\mathbb{R}$, we only have to prove that T_ϕ is not invertible whenever ϕ takes both positive and negative values on $\mathbb{D}$. Since ϕ is continuous, there exists $w\in\mathbb{D}$ such that $\phi(w)=0$. Now, to reach our main goal, we only show that $k_w^\alpha\notin\text{Range}(T_\phi)$. We suppose that $k_w^\alpha\in\text{Range}(T_\phi)$, then there exists $f\in L_a^2(dA_\alpha)$ such that $T_\phi f = k_w^\alpha$, i.e., $P_\alpha(\phi f) = k_w^\alpha$. Since $\phi\in L^\infty(\mathbb{D},dA_\alpha)$ so $\phi f\in L^2(\mathbb{D},dA_\alpha) = L_a^2(dA_\alpha)\oplus L_a^2(dA_\alpha)^\perp$ and we have $\psi\in L_a^2(dA_\alpha)^\perp$ such that

$$\phi f = k_w^\alpha + \psi. \tag{3.1}$$

For any $g\in H^\infty$, from (3.1), we have

$$\int_{\mathbb{D}}\phi|f|^2\bar{g}dA_\alpha = \int_{\mathbb{D}}k_w^\alpha\bar{f}\bar{g}dA_\alpha + \int_{\mathbb{D}}\psi\bar{f}\bar{g}dA_\alpha. \tag{3.2}$$

Since $\psi\in L_a^2(dA_\alpha)^\perp$ and $fg\in L_a^2(dA_\alpha)$ so

$$\int_{\mathbb{D}}\psi\bar{f}\bar{g}dA_\alpha = 0.$$

Now, we get

$$\int_{\mathbb{D}}k_w^\alpha\bar{f}\bar{g}dA_\alpha = \langle k_w^\alpha, fg\rangle = \langle k_w^\alpha, T_g f\rangle = \langle T_g^* k_w^\alpha, f\rangle = \overline{g(w)}\langle k_w^\alpha, f\rangle = \overline{g(w)f(w)}.$$

Therefore, from (3.2), we have

$$\int_{\mathbb{D}}\phi|f|^2\bar{g}dA_\alpha = \overline{g(w)f(w)}. \tag{3.3}$$

In particular considering $g=1$, the equality (3.3) implies that $f(w)\in\mathbb{R}$. Thus, for any $g\in H^\infty$, from (3.3), we get

$$\int_{\mathbb{D}}\phi|f|^2 Re g dA_\alpha = f(w)Re(g(w)). \tag{3.4}$$

Since $\text{Re}H^\infty$ is weak*-dense in the bounded real harmonic functions, then there exists a sequence $\{g_n\}$ in $\text{Re}H^\infty$ such that

$$\lim_{n\to\infty}\int_{\mathbb{D}}\psi\text{Re}g_n dA = \int_{\mathbb{D}}\psi\phi dA \text{ for all } \psi\in L^1(\mathbb{D},dA). \tag{3.5}$$

As ϕ is bounded and $f\in L_a^2(dA_\alpha)$ so we have $(\alpha+1)\phi|f|^2(1-|z|^2)^\alpha\in L^1(\mathbb{D},dA)$. Thus, from (3.5), we get

$$\lim_{n\to\infty}\int_{\mathbb{D}}\phi|f|^2\text{Re}g_n dA_\alpha = \int_{\mathbb{D}}\phi^2|f|^2dA_\alpha.$$

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By applying (3.4), we obtain that

$$\lim_{n\to\infty} f(w)\mathrm{Re}g_n(w) = \int_{\mathbb{D}} \phi^2|f|^2 dA_\alpha. \tag{3.6}$$

Let $\hat{k}_w$ be the normalized reproducing kernel of $L^2_a(dA)$ at the point w. Then, the function $|\hat{k}_w(z)|^2 \in L^1(\mathbb{D}, dA)$ and from (3.5), we get

$$\lim_{n\to\infty} \int_{\mathbb{D}} \mathrm{Re}g_n|\hat{k}_w(z)|^2 dA = \int_{\mathbb{D}} \phi|\hat{k}_w(z)|^2 dA. \tag{3.7}$$

Let ϕ_w be the Möbius map on $\mathbb{D}$, given by $\phi_w(z) = \frac{w-z}{1-\bar{w}z}$ for all $z \in \mathbb{D}$. Since the real Jacobian of ϕ_w is given by $|\hat{k}_w(z)|^2$, we have

$$\lim_{n\to\infty} \int_{\mathbb{D}} \mathrm{Re}g_n \circ \phi_w dA = \int_{\mathbb{D}} \phi \circ \phi_w dA. \tag{3.8}$$

Now, by Lemma 3.3 and (3.8), we obtain

$$\lim_{n\to\infty} \mathrm{Re}g_n(w) = \phi(w). \tag{3.9}$$

Therefore, combining (3.6) and (3.9), we get

$$\int_{\mathbb{D}} \phi^2|f|^2 dA_\alpha = f(w)\phi(w) = 0.$$

This implies that $\phi^2|f|^2 \equiv 0$ on $\mathbb{D}$. As ϕ takes positive value on $\mathbb{D}$ and f is analytic on $\mathbb{D}$ so we have $f = 0$. This implies that $T_\phi f = k^\alpha_w = 0$, which is a contradiction as $k^\alpha_w \neq 0$. Thus, $k^\alpha_w \notin \mathrm{Range}(T_\phi)$ and this completes the proof. ∎

In the next result, we completely determine the numerical range of Toeplitz operators with harmonic symbol acting on $L^2_a(dA_\alpha)$.

Theorem 3.5 *If $\phi \in L^\infty(\mathbb{D}, dA_\alpha)$ is a nonconstant harmonic function on $\mathbb{D}$, then $W(T_\phi) = \mathrm{Rel\ int}\overline{\phi(\mathbb{D})}^\wedge$.*

Proof Let $\phi \in L^\infty(\mathbb{D}, dA_\alpha)$ be a nonconstant harmonic function on $\mathbb{D}$. We prove this theorem by considering the following two cases.

Case 1. ϕ is real-valued.

Clearly, T_ϕ is self-adjoint. By Proposition 3.4, we get that $\overline{W(T_\phi)} = \sigma(T_\phi)^\wedge = [\inf\phi, \sup\phi]$. As $W(T_\phi)$ is convex so $(\inf\phi, \sup\phi) \subseteq W(T_\phi)$. Now, we will show that T_ϕ has no eigenvectors. As $T_{\phi-\lambda} = T_\phi - \lambda I$ for all $\lambda \in \mathbb{C}$, we have to show that for any $f \in L^2_a(dA_\alpha)$, $T_\phi f = 0$ implies that $f = 0$. If $T_\phi f = 0$, then $\phi f \in L^2_a(dA_\alpha)^\perp$. For any $g \in H^\infty$, we have $fg \in L^2_a(dA_\alpha)$. Thus, we get

$$\int_{\mathbb{D}} \phi|f|^2 \bar{g} dA_\alpha = \langle \phi f, fg\rangle = 0.$$

Hence, we obtain

$$\int_{\mathbb{D}} \phi|f|^2 Re g dA_\alpha = \int_{\mathbb{D}} \phi|f|^2 Re\bar{g} dA_\alpha = 0.$$

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Now, proceeding similarly as in Proposition 3.4, we get

$$\int_{\mathbb{D}} \phi^2 |f|^2 dA_\alpha = 0.$$

This implies that $f = 0$. Thus, $\inf \phi, \sup \phi \notin W(T_\phi)$ because if either $\inf \phi$ or $\sup \phi$ are in $W(T_\phi)$ then they are extreme points of T_ϕ and hence they are eigenvalues of T_ϕ. This completes the proof for the first case.

Case 2. ϕ is complex-valued.

We first prove the inclusion $W(T_\phi) \subseteq \text{Rel int}\overline{\phi(\mathbb{D})}^\wedge$. Let M_ϕ be the multiplication operator on $L^2(\mathbb{D}, dA_\alpha)$. Since M_ϕ is a normal operator and $\sigma(M_\phi) = R_{\phi,\alpha}$ (see [9, Problem 67]), so $\overline{W(M_\phi)} = R^\wedge_{\phi,\alpha}$. Therefore, from Lemma 3.1, we get

$$\overline{W(M_\phi)} = \overline{\phi(\mathbb{D})}^\wedge. \tag{3.10}$$

As T_ϕ dilates to M_ϕ,

$$W(T_\phi) \subseteq \overline{W(M_\phi)}. \tag{3.11}$$

Now, combining (3.10) and (3.11), we obtain

$$W(T_\phi) \subseteq \overline{\phi(\mathbb{D})}^\wedge. \tag{3.12}$$

Suppose that $W(T_\phi)$ is not contained in $\text{Rel int}\overline{\phi(\mathbb{D})}^\wedge$. Then, there exists $\theta \in \mathbb{R}$, $\gamma \in \mathbb{C}$, and $f \in L^2_a(dA_\alpha)$ with $\|f\| = 1$ such that

$$\langle T_{Re(e^{i\theta}(\phi+\gamma))} f, f\rangle = \max \overline{\text{Re}(e^{i\theta}(\phi+\gamma))(\mathbb{D})}^\wedge = \delta.$$

This implies that

$$\langle M_{Re(e^{i\theta}(\phi+\gamma))} f, f\rangle = \langle \text{Re}(e^{i\theta}(\phi+\gamma)) f, f\rangle = \delta.$$

Since $M_{Re(e^{i\theta}(\phi+\gamma))} \leq \delta I$ so we get $\text{Re}(e^{i\theta}(\phi+\gamma)) f = cf$. The analyticity of non-zero f implies that $\text{Re}(e^{i\theta}(\phi+\gamma)(z)) = \delta$ for all $z \in \mathbb{D}$. Therefore, $\overline{(\phi+\gamma)(\mathbb{D})}^\wedge$ is contained in a line. Repeating the above process with $\text{Im}(e^{i\theta}(\phi+\gamma))$ yields that ϕ is constant, which contradicts our assumption. Therefore, $W(T_\phi)$ is contained in the relative interior of $\overline{\phi(\mathbb{D})}^\wedge$.

Suppose they are not equal. Then, there exist $\theta \in \mathbb{R}$ and $c \in \mathbb{C}$ such that $W(T_\psi) \subsetneq \text{Rel int}\overline{\psi(\mathbb{D})}^\wedge$, where $\psi = \text{Re}(e^{i\theta}(\phi + c))$, which is real and harmonic and this contradicts Case 1. Thus, we obtain the desired relation. ∎

The following corollary follows from Theorem 3.5.

Corollary 3.6 *If $\phi \in H^\infty$ then $W(M_\phi) = \phi(\mathbb{D})^\wedge$.*

The next example demonstrates that the condition ϕ is harmonic in $\mathbb{D}$ is essential in Theorem 3.5.

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Example 3.7 If we consider the function $\phi(z) = |z|^2$ on $\mathbb{D}$, then ϕ is continuous but not harmonic in $\mathbb{D}$. For any $n, m \geq 0$, we have

$$\begin{aligned}
&\langle T_\phi e_n, e_m \rangle \\
&= \frac{\sqrt{\Gamma(n+\alpha+2)\Gamma(m+\alpha+2)}}{\sqrt{n!m!}\Gamma(\alpha+2)}(\alpha+1)\int_{\mathbb{D}} \phi(z) z^n \bar{z}^m dA_\alpha(z) \\
&= \frac{\sqrt{\Gamma(n+\alpha+2)\Gamma(m+\alpha+2)}}{\pi\sqrt{n!m!}\Gamma(\alpha+1)}\left(\int_{r=0}^{1} r^{n+m+3}(1-r^2)^\alpha dr\right)\left(\int_{\theta=0}^{2\pi} e^{i(n-m)\theta} d\theta\right) \\
&= \begin{cases} \lambda_n & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases},
\end{aligned}$$

where

$$\lambda_n = \frac{\Gamma(n+\alpha+2)}{n!\Gamma(\alpha+1)}\int_{r=0}^{1} r^{n+1}(1-r)^\alpha dr = \frac{n+1}{n+\alpha+2}.$$

Clearly, $\{\lambda_n\}_{n=0}^{\infty}$ is an increasing sequence with $\lambda_n \to 1$. Thus, the matrix representation of T_ϕ is a diagonal matrix with diagonal elements λ_n, relative to the standard ordered basis $\{e_n\}_{n=0}^{\infty}$ of $L_a^2(dA_\alpha)$. Hence, $W(T_\phi) = \left[\frac{1}{\alpha+2}, 1\right)$. Again Rel $\text{int}\overline{\phi(\mathbb{D})}^{\wedge} = (0,1)$. Hence, in this example, $W(T_\phi)$ is not equal to Rel $\text{int}\overline{\phi(\mathbb{D})}^{\wedge}$ furthermore $W(T_\phi)$ is not a relatively open subset of $\mathbb{C}$.

We now introduce the following definition.

Definition 3.2 For $n \geq 2$ and $n > j \geq 0$, we define the subset L_j of $L_a^2(dA_\alpha)$ as

$$L_j = \left\{ f \in L_a^2(dA_\alpha) : f(z) = z^j g(z^n), g \in L_a^2(dA_\alpha) \right\}.$$

Now, we prove the following lemma which will be useful when proving the next result.

Lemma 3.8 *If $m \in \mathbb{N}$ and $c > 1$, then the sequence $\left\{ x_n = \frac{n!\Gamma(nm+c)}{(nm)!\Gamma(n+c)} \right\}$ is bounded.*

Proof If c is an integer, then it easily follows that $\{x_n\}$ is bounded. Now, the function $f(x) = \frac{\Gamma(mn+x)}{\Gamma(n+x)}$ is increasing on the interval $[1, \infty)$. So, for an arbitrary $c > 1$, by choosing an integer greater than c and using the boundedness of the sequence for the integer case, the desired result follows. ∎

Next, we prove the following decomposition of $L_a^2(dA_\alpha)$, which will be an essential tool to compute the numerical range of weighted composition operators on $L_a^2(dA_\alpha)$.

Proposition 3.9 *For each $n \geq 2$, $L_a^2(dA_\alpha)$ can be decomposed as*

$$L_a^2(dA_\alpha) = L_0 \oplus L_1 \oplus \cdots \oplus L_{n-1}.$$

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Proof By applying Lemma 3.8, it follows that each element of $L_a^2(dA_\alpha)$ of the form $\sum_{k=0}^{\infty} a_k z^{kn+j}$ lies in L_j and conversely. Then, it follows easily that each L_j is a closed subspace and for all $n \geq 2$,

$$L_a^2(dA_\alpha) = L_0 \oplus L_1 \oplus \cdots \oplus L_{n-1}.$$

■

Our next result reads as follows.

Lemma 3.10 *If $\psi \in H^\infty$ and $M_\psi(L_j) \subseteq L_j$, then $W(M_\psi|_{L_j}) = \phi(\mathbb{D})^\wedge$.*

Proof Let p_j be the orthogonal projection from $L_a^2(dA_\alpha)$ onto L_j. Now, for $w \in \mathbb{D}\backslash\{0\}$, we denote $k_{w,j}^\alpha = p_j k_w^\alpha$. Then, we have

$$\langle M_\psi \hat{k}_{w,j}^\alpha, \hat{k}_{w,j}^\alpha \rangle = \frac{1}{\|k_{w,j}^\alpha\|^2} \langle \psi k_{w,j}^\alpha, k_{w,j}^\alpha \rangle = \frac{1}{\|k_{w,j}^\alpha\|^2} \psi(w) k_{w,j}^\alpha(w) = \psi(w).$$

Thus, for any $w \in \mathbb{D}\backslash\{0\}$, $\psi(w) \in W(M_\psi|_{L_j})$ and $\psi(0) \in W(M_\psi|_{L_j})$ by open mapping theorem. Thus, we get $W(M_\psi|_{L_j}) \supseteq \phi(\mathbb{D})^\wedge$. Now, the desired result follows from Corollary 3.6 and $W(M_\psi|_{L_j}) \subseteq W(M_\psi)$. ■

Now, we are in a position to prove the following result.

Theorem 3.11 *Let $\phi(z) = \lambda z$ with $\lambda = e^{2\pi i/n}$ and $\psi(z) = g(z^n)$ for some $g \in H^\infty$. Then,*

$$W(C_{\psi,\phi}) = (\psi(\mathbb{D}) \cup \lambda\psi(\mathbb{D}) \cup \cdots \cup \lambda^{n-1}\psi(\mathbb{D}))^\wedge.$$

Proof If $f \in L_j$, then $f(\phi(z)) = f(\lambda z) = \lambda^j f(z)$ and we have $C_\phi(L_j) \subseteq L_j$. Since $\psi(z) = g(z^n)$ and ψ is bounded on $\mathbb{D}$ so $M_\psi(L_j) \subseteq L_j$ and so $C_{\psi,\phi}(L_j) \subseteq L_j$. This implies that

$$C_{\psi,\phi}(L_j) = C_0 \oplus C_1 \oplus \cdots \oplus C_{n-1},$$

where $C_j = C_{\psi,\phi}|_{L_j}$. For any $h \in L_j$ with $\|h\| = 1$, we have $\langle C_j h, h\rangle = \lambda^j \langle \psi h, h\rangle$ and this implies that $W(C_j) = \lambda^j W(M_\psi|_{L_j})$. Hence, from Lemma 3.10, we have $W(C_j) = \lambda^j \phi(\mathbb{D})^\wedge$. Thus,

$$\begin{aligned} W(C_{\psi,\phi}) &= (W(C_0) \cup W(C_1) \cup \cdots \cup W(C_{n-1}))^\wedge \\ &= (\psi(\mathbb{D}) \cup \lambda\psi(\mathbb{D}) \cup \cdots \cup \lambda^{n-1}\psi(\mathbb{D}))^\wedge, \end{aligned}$$

as desired. ■

The following corollary follows immediately from Theorem 3.11.

Corollary 3.12 *If $\phi(z) = -z$, then $W(C_\phi) = [-1, 1]$ and if $\phi(z) = e^{2\pi i/n} z$ with $n > 2$, then $W(C_\phi)$ is the closed, regular polygonal region with n sides and inscribed in the unit circle.*

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In the following, we classify some subsets of $\mathbb{C}$ which are the numerical ranges of weighted composition operators acting on $L^2_a(dA_\alpha)$. To do this, we start with the following definition.

A subset S of $\mathbb{C}$ is said to have n-fold symmetry about the origin if it satisfies $e^{2\pi i/n}S = S$.

Theorem 3.13 *Let S be a non-empty, open, bounded, and convex subset of $\mathbb{C}$. If S has n-fold symmetry about the origin then for any $\alpha > -1$, there exists $C_{\psi,\phi} \in \mathbb{B}\left(L^2_a(dA_\alpha)\right)$ such that $W(C_{\psi,\phi}) = S$.*

Proof Let f be a Riemann map from $\mathbb{D}$ onto S. Consider the function $\psi : \mathbb{D} \to S$ such that $\psi(z) = f(z^n)$ for all $z \in \mathbb{D}$. It is easy to verify that $\psi(\mathbb{D}) = f(\mathbb{D})$ and so we obtain $\psi(\mathbb{D}) = S$. Let $\phi(z) = \lambda z$, where $\lambda = e^{2\pi i/n}$. By applying Theorem 3.11, we get

$$W(C_{\psi,\phi}) = (\psi(\mathbb{D}) \cup \lambda\psi(\mathbb{D}) \cup \cdots \cup \lambda^{n-1}\psi(\mathbb{D}))^\wedge. \tag{3.13}$$

Since S has an n-fold symmetry about the origin so $\lambda^k\psi(\mathbb{D}) = S$ for all $0 \le k \le n-1$. As S is convex so from (3.13), we obtain the desired result. ∎

Corollary 3.14 *Let $f \in H^\infty$ be nonconstant. If $n > 1$, then for any $\alpha > -1$, there exists $C_{\psi,\phi} \in \mathbb{B}\left(L^2_a(dA_\alpha)\right)$ such that $W(C_{\psi,\phi})$ is the smallest convex set with n-fold symmetry about the origin with $W(C_{\psi,\phi}) \supseteq f(\mathbb{D})$.*

Proof Let us define $\psi : \mathbb{D} \to \mathbb{C}$ such that $\psi(z) = f(z^n)$ for all $z \in \mathbb{D}$. Then, $\psi(\mathbb{D}) = f(\mathbb{D})$. Let $\phi(z) = \lambda z$, where $\lambda = e^{2\pi i/n}$. From Theorem 3.11, we get

$$W(C_{\psi,\phi}) = (f(\mathbb{D}) \cup \lambda f(\mathbb{D}) \cup \cdots \cup \lambda^{n-1} f(\mathbb{D}))^\wedge. \tag{3.14}$$

As $\lambda W(C_{\psi,\phi}) = W(C_{\psi,\phi})$ so $W(C_{\psi,\phi})$ has an n-fold symmetry about the origin and it is a convex set. If M is a convex set and has an n-fold symmetry about the origin with $M \supseteq f(\mathbb{D})$ then $M \supseteq \lambda^k f(\mathbb{D})$ for all $0 \le k \le n-1$. Thus, from (3.14), it follows that $M \supseteq W(C_{\psi,\phi})$. Therefore, $W(C_{\psi,\phi})$ is the smallest convex set with n-fold symmetry about the origin with $W(C_{\psi,\phi}) \supseteq f(\mathbb{D})$. ∎

We conclude this section by asking which non-empty, bounded, and convex subsets of $\mathbb{C}$ can be the numerical range of weighted composition operators acting on $L^2_a(dA_\alpha)$.

4 Containment of zero in the numerical range

Our main focus in this section is to investigate the containment of the origin in the interior of the numerical range of weighted composition operators on $L^2_a(dA_\alpha)$. We study the case when the origin is contained in the numerical range as well as in its closure for the sum of two weighted composition operators. To do so, we recall the definition of radial limit.

A function $f \in H(\mathbb{D})$ is said to have a radial limit if $\lim_{r\to 1} f(re^{i\theta})$ exists almost everywhere in $\partial\mathbb{D}$. It is proved in [16, Theorem 11.32] that for every $f \in H^\infty$, there

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corresponds a function $f^* \in L^\infty(\partial\mathbb{D})$, defined almost everywhere by

$$f^*(e^{i\theta}) = \lim_{r\to 1} f(re^{i\theta}).$$

Moreover, if $f^*(e^{i\theta}) = 0$ for almost all $e^{i\theta}$ on some arc $I \subseteq \partial\mathbb{D}$, then $f(z) = 0$ for all $z \in \mathbb{D}$.

Now, we are in a position to prove the first result of this section.

Theorem 4.1 *Let ϕ_1 and ϕ_2 be two holomorphic self-maps on $\mathbb{D}$ and $\psi_1, \psi_2 \in H(\mathbb{D})$ be such that $C_{\psi_1,\phi_1}, C_{\psi_2,\phi_2} \in \mathbb{B}\left(L_a^2(dA_\alpha)\right)$.*

(i) *If ϕ_1 and ϕ_2 are identity maps on $\mathbb{D}$, and ϕ_1, ϕ_2 have a common zero in $\mathbb{D}$, then $0 \in W(C_{\psi_1,\phi_1} + C_{\psi_2,\phi_2})$.*

(ii) *If ϕ_1, ϕ_2 are not identity maps on $\mathbb{D}$ and $\psi_1, \psi_2 \in H^\infty$, then $0 \in \overline{W(C_{\psi_1,\phi_1} + C_{\psi_2,\phi_2})}$.*

Proof For any $w \in \mathbb{D}$, we have

$$\begin{aligned}
&\langle (C_{\psi_1,\phi_1} + C_{\psi_2,\phi_2})\hat{k}_w^\alpha, \hat{k}_w^\alpha\rangle \\
&= \frac{1}{\|k_w^\alpha\|^2}\langle k_w^\alpha, (C_{\psi_1,\phi_1} + C_{\psi_2,\phi_2})^* k_w^\alpha\rangle \\
&= \frac{1}{\|k_w^\alpha\|^2}\left(\langle k_w^\alpha, \overline{\psi_1(w)}k_{\phi_1(w)}^\alpha\rangle + \langle k_w^\alpha, \overline{\psi_2(w)}k_{\phi_2(w)}^\alpha\rangle\right) \\
&= \frac{\psi_1(w)}{\|k_w^\alpha\|^2}k_w^\alpha(\phi_1(w)) + \frac{\psi_2(w)}{\|k_w^\alpha\|^2}k_w^\alpha(\phi_2(w)) \\
&= \frac{\psi_1(w)(1-|w|^2)^{\alpha+2}}{(1-\bar{w}\phi_1(w))^{\alpha+2}} + \frac{\psi_2(w)(1-|w|^2)^{\alpha+2}}{(1-\bar{w}\phi_2(w))^{\alpha+2}}. \qquad (4.1)
\end{aligned}$$

(i) Since ϕ_1 and ϕ_2 are identity maps on $\mathbb{D}$ then from (4.1), we have

$$\langle (C_{\psi_1,\phi_1} + C_{\psi_2,\phi_2})\hat{k}_w^\alpha, \hat{k}_w^\alpha\rangle = \psi_1(w) + \psi_2(w).$$

If $\psi_1(w_0) = \psi_2(w_0) = 0$ for $w_0 \in \mathbb{D}$, then we get

$$\langle (C_{\psi_1,\phi_1} + C_{\psi_2,\phi_2})\hat{k}_{w_0}^\alpha, \hat{k}_{w_0}^\alpha\rangle = 0,$$

as desired.

(ii) As ϕ_1, ϕ_2 are not identity maps then the sets $\{e^{i\theta} : \phi_1^*(e^{i\theta}) = e^{i\theta}\}$ and $\{e^{i\theta} : \phi_2^*(e^{i\theta}) = e^{i\theta}\}$ have measure zero on $\partial\mathbb{D}$. So there exists a $w_0 \in \partial\mathbb{D}$ such that $\psi_i^*(w_0)$ exists and $\phi_i^*(w_0) \neq w_0$ for $i = 1, 2$. Hence, from (4.1), we have

$$\lim_{w\to w_0}\langle (C_{\psi_1,\phi_1} + C_{\psi_2,\phi_2})\hat{k}_w^\alpha, \hat{k}_w^\alpha\rangle = 0.$$

Therefore, $0 \in \overline{W(C_{\psi_1,\phi_1} + C_{\psi_2,\phi_2})}$ and this completes the proof. ∎

Next, we completely characterize the numerical range of the bounded weighted composition operators induced by constant composition maps.

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Proposition 4.2 *Let $C_{\psi,\phi} \in \mathbb{B}\left(L_a^2(dA_\alpha)\right)$ be such that $\phi \equiv w$ with $|w| < 1$.*

(i) *If $k_w^\alpha = \mu\psi$ for some $\mu \neq 0$, then $W(C_{\psi,\phi}) = [0, \bar{\mu}\|\psi\|^2]$.*
(ii) *If $k_w^\alpha \perp \psi$, then $W(C_{\psi,\phi})$ is the closed disk centered at the origin and radius $\frac{\|\psi\|}{2(1-|w|^2)^{\frac{\alpha}{2}+1}}$.*
(iii) *Otherwise, $W(C_{\psi,\phi})$ is a closed elliptical disk with foci at 0 and $\psi(w)$.*

Proof Since $C_{\psi,\phi} \in \mathbb{B}\left(L_a^2(dA_\alpha)\right)$ and ϕ is a constant function so for any $f \in L_a^2(dA_\alpha)$,

$$C_{\psi,\phi} f = \psi f(w) = \langle f, k_w^\alpha \rangle \psi.$$

Thus, $C_{\psi,\phi}$ is a rank one operator and the desired result follows from [2, Proposition 2.5]. ∎

Now, we state the following lemma which follows from [11, Theorem 2.6].

Lemma 4.3 *Let $C_{\psi,\phi} \in \mathbb{B}\left(L_a^2(dA_\alpha)\right)$, ϕ be a nonconstant analytic self-map on $\mathbb{D}$ and ψ be non-zero. If either ψ has a zero on $\mathbb{D}$ or ϕ is not one-to-one, then $0 \in int\ W(C_{\psi,\phi})$.*

In the following results, we investigate the inclusion of the origin in the interior of numerical range of weighted composition operators on $L_a^2(dA_\alpha)$.

Theorem 4.4 *Let $C_{\psi,\phi} \in \mathbb{B}\left(L_a^2(dA_\alpha)\right)$ and $\phi(0) = 0$. If ϕ is not of the form $\phi(z) = tz$, where $t \in \overline{\mathbb{D}}$, then $0 \in int\ W(C_{\psi,\phi})$.*

Proof If $\phi'(0) = 0$, then ϕ is not one-to-one and by Lemma 4.3, we have $0 \in int\ W(C_{\psi,\phi})$. We now consider the case $\phi'(0) = \lambda \neq 0$. As ϕ is not of the form $\phi(z) = tz$, where $t \in \overline{\mathbb{D}}$, ϕ can be written as

$$\phi(z) = \lambda z\left(1 + bz^m(1 + h(z))\right),$$

where m is a positive integer, $b \neq 0$, and $h \in H(\mathbb{D})$ with $h(0) = 0$. Therefore, for any $n \geq 1$,

$$\phi^n(z) = \lambda^n z^n + nb\lambda^n z^{n+m} + \text{higher-order terms in } z. \tag{4.2}$$

The matrix of $C_{\psi,\phi}$ with respect to the orthonormal basis $\{e_n\}_{n=0}^\infty$ has its n-th column given by the sequence of coefficients of the power series expansion of $\sqrt{\frac{\Gamma(n+\alpha+2)}{n!\Gamma(\alpha+2)}}\psi\phi^n$. Let M_n be the subspace of $L_a^2(dA_\alpha)$ spanned by e_n and e_{n+m}. Clearly, M_n is a two-dimensional subspace of $L_a^2(dA_\alpha)$. Let $\sum_{k=0}^\infty \hat{\psi}_k z^k$ be the power series expansion of ψ. Then, the matrix representation of $C_{\psi,\phi}$ on M_n with respect to the basis $\{e_n, e_{n+m}\}$ is given by

$$\begin{pmatrix} \hat{\psi}_0\lambda^n & 0 \\ \sqrt{\frac{(m+n)!\Gamma(n+\alpha+2)}{n!\Gamma(m+n+\alpha+2)}}\lambda^n(nb\hat{\psi}_0 + \hat{\psi}_m) & \hat{\psi}_0\lambda^{m+n} \end{pmatrix} = \lambda^n C_n,$$

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where

$$C_n = \begin{pmatrix} \hat{\psi}_0 & 0 \\ \sqrt{\frac{(m+n)!\Gamma(n+\alpha+2)}{n!\Gamma(m+n+\alpha+2)}}(nb\hat{\psi}_0 + \hat{\psi}_m) & \hat{\psi}_0\lambda^m \end{pmatrix}.$$

Since the numerical range of compression is contained in the numerical range of the operator (see [19, Proposition 1.4]), it is sufficient to show that $0 \in \text{int } W(C_n)$ for some n. If $\hat{\psi}_0 = 0$, then it follows from Lemma 4.3 that $0 \in \text{int } W(C_n)$. If $\hat{\psi}_0 \neq 0$, then $W(C_n)$ is an elliptical disk with foci $\hat{\psi}_0$ and $\hat{\psi}_0\lambda^m$ with the length of the minor axis $\sqrt{\frac{(m+n)!\Gamma(n+\alpha+2)}{n!\Gamma(m+n+\alpha+2)}}|nb\hat{\psi}_0 + \hat{\psi}_m|$ (see [8, Example 3]). A simple computation shows that

$$\lim_{n\to\infty} \frac{(m+n)!\Gamma(n+\alpha+2)}{n!\Gamma(m+n+\alpha+2)} = 1.$$

Thus, if we choose n large enough, then the length of the minor axis of $W(C_n)$ will be larger than the modulus of its center. Hence, there exits n for which $0 \in int\ W(C_n)$, as desired. ∎

Theorem 4.5 *Let $C_{\psi,\phi} \in \mathbb{B}\left(L_a^2(dA_\alpha)\right)$ and ψ be nonconstant. If $\phi(z) = tz$, where $-1 \leq t \leq 0$, then $0 \in int\ W(C_{\psi,\phi})$.*

Proof First, we prove this result for $\psi(0) = 0$. For $-1 \leq t < 0$, the result follows from Lemma 4.3 and for $t = 0$, Proposition 4.2 implies that $0 \in \text{int } W(C_{\psi,\phi})$.

To prove the result for $\psi(0) \neq 0$, it is enough to show this for $\psi(0) = 1$. So, $\psi(z) = 1 + \eta(z)$, where η is a nonconstant analytic function with $\eta(0) = 0$. Now, for any $f \in L_a^2(dA_\alpha)$ with $\|f\| = 1$, we get

$$\langle C_{\psi,\phi}f, f\rangle = \langle C_\phi f, f\rangle + \langle C_{\eta,\phi}f, f\rangle.$$

For $t = 0$, the result follows from Proposition 4.2. Since $\eta(0) = 0$, for $-1 \leq t < 0$, it follows from Lemma 4.3 that $W(C_{\eta,\phi})$ contains a disk of positive radius and centered at the origin. So there exists $f_1 \in L_a^2(dA_\alpha)$ with $\|f_1\| = 1$ such that $\text{Im}\langle C_{\eta,\phi}f_1, f_1\rangle > 0$. As $\langle C_\phi f_1, f_1\rangle$ is real, $p_1 = \langle C_{\psi,\phi}f_1, f_1\rangle$ is in the upper half plane. Similarly, we get another point p_2 in the lower half plane. Again we have $\langle C_{\psi,\phi}e_1, e_1\rangle = t$ and $\langle C_{\psi,\phi}e_0, e_0\rangle = 1$. Thus, $0 \in \text{int}\{p_1, p_2, t, 1\}^\wedge \subseteq \text{int}W(C_{\psi,\phi})$, as desired. ∎

Remark 4.6

(i) If ψ is constant and $\phi(z) = tz$, where $-1 \leq t < 1$, then $W(C_{\psi,\phi})$ is a line segment of $\mathbb{C}$ and thus $0 \notin \text{int}W(C_{\psi,\phi})$.

(ii) If ψ is nonconstant with $\psi(0) = 0$ and $\phi(z) = tz$, where $0 < t < 1$, then by Lemma 4.3, we have $0 \in \text{int } W(C_{\psi,\phi})$.

(iii) If ψ is nonconstant with $\psi(0) \neq 0$ and $\phi(z) = tz$, where $0 < t < 1$, then the following two cases are possible:

(a) Let ψ be such that $\psi(0) \neq 0$ but $\psi(w_0) = 0$ for some $w_0 \in \mathbb{D}$. Then, by applying Lemma 4.3, we conclude that $0 \in \text{int } W(C_{\psi,\phi})$.

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(b) Now, we consider $\phi(z) = z/2$ and $\psi(z) = 1 + z/4$. Let $f \in L_a^2(dA_\alpha)$ with $\|f\| = 1$ and $f(z) = \sum_{k=0}^{\infty} \hat{f}_k z^k$ be the power series expansion. Then, we have

$$\begin{aligned}\langle C_{\psi,\phi} f, f\rangle &= \sum_{k=0}^{\infty} \frac{k!\Gamma(\alpha+2)}{2^k\Gamma(k+\alpha+2)}|\hat{f}_k|^2 + \frac{1}{4}\sum_{k=0}^{\infty} \frac{(k+1)!\Gamma(\alpha+2)}{2^k\Gamma(k+\alpha+3)}\hat{f}_k\bar{\hat{f}}_{k+1} \\ &= \gamma + \frac{1}{4}\delta,\end{aligned}$$

where $\gamma = \sum_{k=0}^{\infty} \frac{k!\Gamma(\alpha+2)}{2^k\Gamma(k+\alpha+2)}|\hat{f}_k|^2$ and $\delta = \sum_{k=0}^{\infty} \frac{(k+1)!\Gamma(\alpha+2)}{2^k\Gamma(k+\alpha+3)}\hat{f}_k\bar{\hat{f}}_{k+1}$. Since $|\hat{f}_k\bar{\hat{f}}_{k+1}| \le \frac{1}{2}\left(|\hat{f}_k|^2 + |\hat{f}_{k+1}|^2\right)$ and by a simple computation, we have

$$|\delta| \le \frac{3}{2}\gamma \quad \text{as } \gamma > 0.$$

Therefore, $\gamma + \frac{1}{4}\delta \neq 0$ and this implies that $0 \notin W(C_{\psi,\phi})$.

Finally, we introduce some sufficient condition for the closedness of numerical range of compact weighted composition operators. The results follow from the previous results of this section and the well-known result that if the numerical range of a compact operator contains 0 then it is closed (see [9, p. 115]).

Corollary 4.7 *Let $C_{\psi,\phi}$ be a compact operator on $L_a^2(dA_\alpha)$. Then, $W(C_{\psi,\phi})$ is closed if one of the following condition holds:*

(i) *ϕ is the identity map on $\mathbb{D}$ and ψ has a zero in $\mathbb{D}$.*
(ii) *$\psi \neq 0$ and either ψ has a zero in $\mathbb{D}$ or ϕ is not one-to-one.*
(iii) *$\phi(0) = 0$ and ϕ is not of the form $\phi(z) = tz$ for $t \in \overline{\mathbb{D}}$.*
(iv) *ψ is nonconstant and $\phi(z) = tz$, $-1 \le t \le 0$.*

Example 4.8 If we consider the example $\psi(z) = z^2$ and $\phi(z) = az^2$ with $|a| < 1$ then ϕ is not of the form $\phi(z) = tz$ for $t \in \overline{\mathbb{D}}$, and $\phi(0) = 0$. Again from [15, Corollary 1], it follows that $C_{\psi,\phi}$ is compact on $L_a^2(dA_\alpha)$. Thus, for this example, $W(C_{\psi,\phi})$ is closed.

5 Containment of circular disk or elliptical disk in the numerical range

In the previous section, we studied the containment of the origin in the interior of numerical range. Our next focus is to investigate the weighted composition operators for which the numerical range contains a circular disk or elliptical disk and accordingly we find the radius of the disk or lengths of the major and minor axes.

Theorem 5.1 *Let $C_{\psi,\phi} \in \mathbb{B}\left(L_a^2(dA_\alpha)\right)$ be such that $\phi(0) = 0$ and ψ has a zero of order $m > 0$ at the origin. If $\hat{\psi}_m$ denotes the m-th Taylor coefficient of ψ, then $W(C_{\psi,\phi})$ contains the disk of radius $\frac{m!\Gamma(\alpha+2)}{\Gamma(m+\alpha+2)+m!\Gamma(\alpha+2)}|\hat{\psi}_m|$, centered at the origin.*

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Proof Let us consider $f(z) = \sqrt{\frac{\Gamma(m+\alpha+2)}{\Gamma(m+\alpha+2)+m!\Gamma(\alpha+2)}}(\lambda + z^m)$ for all $z \in \mathbb{D}$ with $|\lambda| = 1$. Then, $f \in L_a^2(dA_\alpha)$ with $\|f\| = 1$. Let $\sum_{k=1}^{\infty} \hat{\phi}_k z^k$ and $\sum_{k=m}^{\infty} \hat{\psi}_k z^k$ be the Taylor series of ϕ and ψ, respectively. Then,

$$f(\phi(z)) = \sqrt{\frac{\Gamma(m+\alpha+2)}{\Gamma(m+\alpha+2)+m!\Gamma(\alpha+2)}}\left(\lambda + \left(\sum_{k=1}^{\infty} \hat{\phi}_k z^k\right)^m\right)$$
$$= \sqrt{\frac{\Gamma(m+\alpha+2)}{\Gamma(m+\alpha+2)+m!\Gamma(\alpha+2)}}\left(\lambda + \hat{\phi}_1^m z^m + \text{higher-order terms in } z\right).$$

Thus, we have

$$\begin{aligned}
&\langle C_{\psi,\phi} f, f\rangle \\
&= \langle \psi(z) f(\phi(z)), f(z)\rangle \\
&= \frac{\Gamma(m+\alpha+2)}{\Gamma(m+\alpha+2)+m!\Gamma(\alpha+2)}\left\langle \left(\sum_{k=m}^{\infty} \hat{\psi}_k z^k\right)\left(\lambda + \hat{\phi}_1^m z^m + \text{higher-order terms in } z\right), \lambda + z^m \right\rangle \\
&= \frac{m!\Gamma(\alpha+2)}{\Gamma(m+\alpha+2)+m!\Gamma(\alpha+2)}\lambda\hat{\psi}_m.
\end{aligned}$$

Since λ is an arbitrary complex number with $|\lambda| = 1$ so $W(C_{\psi,\phi})$ contains the disk of radius $\frac{m!\Gamma(\alpha+2)}{\Gamma(m+\alpha+2)+m!\Gamma(\alpha+2)}|\hat{\psi}_m|$ and center at the origin. ∎

Theorem 5.2 *Let $C_{\psi,\phi} \in \mathbb{B}\left(L_a^2(dA_\alpha)\right)$ be such that $\phi(z) = \lambda z$ with $\lambda \neq 0$ and $\psi(z) = \sum_{k=1}^{\infty} \hat{\psi}_k z^k$. Then, for all $m \geq 2$, $W(C_{\psi,\phi})$ contains the disk of radius $\frac{1}{2}\sqrt{\frac{m!\Gamma(\alpha+3)}{\Gamma(m+\alpha+2)}}|\lambda\hat{\psi}_{m-1}|$ and center at the origin.*

Proof For $m \geq 2$, let M_m be the subspace of $L_a^2(dA_\alpha)$ spanned by e_1 and e_m. Now, we have

$$C_{\psi,\phi} e_1(z) = \lambda\sqrt{\alpha+2}\left(\sum_{k=1}^{\infty} \hat{\psi}_k z^{k+1}\right)$$

and

$$C_{\psi,\phi} e_m(z) = \lambda^m \sqrt{\frac{\Gamma(m+\alpha+2)}{m!\Gamma(\alpha+2)}}\left(\sum_{k=1}^{\infty} \hat{\psi}_k z^{k+m}\right).$$

Thus, the compression of $C_{\psi,\phi}$ to M_m has the matrix representation

$$\begin{pmatrix} 0 & 0 \\ \lambda\sqrt{\frac{m!\Gamma(\alpha+3)}{\Gamma(m+\alpha+2)}}\hat{\psi}_{m-1} & 0 \end{pmatrix}.$$

Therefore, the numerical range of the compression of $C_{\psi,\phi}$ to M_m is a closed disk centered at the origin and radius $\frac{1}{2}\sqrt{\frac{m!\Gamma(\alpha+3)}{\Gamma(m+\alpha+2)}}|\lambda\hat{\psi}_{m-1}|$ (see [8, Example 3]).

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Thus, $W(C_{\psi,\phi})$ contains the disk of radius $\frac{1}{2}\sqrt{\frac{m!\Gamma(\alpha+3)}{\Gamma(m+\alpha+2)}}|\lambda\hat{\psi}_{m-1}|$ and center at the origin. ∎

Remark 5.3 Here, we remark that the case $m = 0$ cannot be included in Theorems 5.1 and 5.2. If we consider $\psi(z) = 1 + \varepsilon z$ and $\phi(z) = -z/2$ with $\varepsilon > 0$ then it is easy to observe that the radius of the largest circular disk at the origin and contained in $W(C_{\psi,\phi})$ is at most ε.

Theorem 5.4 *Let $C_{\psi,\phi} \in \mathbb{B}\left(L_a^2(dA_\alpha)\right)$ be such that $\phi(z) = e^{2\pi i/n}z$ and $\psi(z) = \sum_{k=0}^{\infty}\hat{\psi}_k z^k$. If m_1, m_2 are two positive integers with $m_2 > m_1$ and $\hat{\psi}_{nm_1}\hat{\psi}_{nm_2}\hat{\psi}_{n(m_1-m_2)} = 0$ but all of the three terms $\hat{\psi}_{nm_1}, \hat{\psi}_{nm_2}$, and $\hat{\psi}_{n(m_1-m_2)}$ are not equal to zero. Then, $W(C_{\psi,\phi})$ contains the circular disk centered at $\hat{\psi}_0$ with radius*

$$\frac{1}{2}\sqrt{\frac{(nm_2)!\Gamma(\alpha+2)}{\Gamma(nm_2+\alpha+2)}\left(c|\hat{\psi}_{nm_1}|^2 + |\hat{\psi}_{n(m_1-m_2)}|^2 + \frac{1}{c}|\hat{\psi}_{nm_2}|^2\right)},$$

where $c = \frac{(nm_1)!\Gamma(nm_2+\alpha+2)}{(nm_2)!\Gamma(nm_1+\alpha+2)}$.

Proof Let M be the subspace of $L_a^2(dA_\alpha)$ spanned by e_0, e_{nm_1}, and e_{nm_2}. Then, we have

$$C_{\psi,\phi}e_0(z) = \sum_{k=0}^{\infty}\hat{\psi}_k z^k,$$

$$C_{\psi,\phi}e_{nm_1}(z) = \sqrt{\frac{\Gamma(nm_1+\alpha+2)}{(nm_1)!\Gamma(\alpha+2)}}\sum_{k=0}^{\infty}\hat{\psi}_k z^{nm_1+k}$$

and

$$C_{\psi,\phi}e_{nm_2}(z) = \sqrt{\frac{\Gamma(nm_2+\alpha+2)}{(nm_2)!\Gamma(\alpha+2)}}\sum_{k=0}^{\infty}\hat{\psi}_k z^{nm_2+k}.$$

Thus, the compression of $C_{\psi,\phi}$ to M has the matrix representation

$$\begin{pmatrix} \hat{\psi}_0 & 0 & 0 \\ \sqrt{\frac{(nm_1)!\Gamma(\alpha+2)}{\Gamma(nm_1+\alpha+2)}}\hat{\psi}_{nm_1} & \hat{\psi}_0 & 0 \\ \sqrt{\frac{(nm_2)!\Gamma(\alpha+2)}{\Gamma(nm_2+\alpha+2)}}\hat{\psi}_{nm_2} & \frac{\sqrt{\Gamma(nm_1+\alpha+2)\Gamma(\alpha+2)}(nm_2)!}{\sqrt{(nm_1)!}\Gamma(nm_2+\alpha+2)}\hat{\psi}_{n(m_1-m_2)} & \hat{\psi}_0 \end{pmatrix}.$$

It follows from [10, Theorem 4.1] that the numerical range of the compression of $C_{\psi,\phi}$ to M is the circular disk centered at $\hat{\psi}_0$ with radius

$$\frac{1}{2}\sqrt{\frac{(nm_2)!\Gamma(\alpha+2)}{\Gamma(nm_2+\alpha+2)}\left(c|\hat{\psi}_{nm_1}|^2 + |\hat{\psi}_{n(m_1-m_2)}|^2 + \frac{1}{c}|\hat{\psi}_{nm_2}|^2\right)},$$

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where $c = \frac{(nm_1)!\Gamma(nm_2+\alpha+2)}{(nm_2)!\Gamma(nm_1+\alpha+2)}$. As the numerical range of compression is contained in the numerical range of the operator, this proves the desired result. ∎

Theorem 5.5 *Let $C_{\psi,\phi} \in \mathbb{B}\left(L_a^2(dA_\alpha)\right)$ be such that $\phi(z) = \lambda z$ with $\lambda = e^{2\pi i/n}$ and $\psi(z) = \sum_{k=0}^{\infty} \hat{\psi}_k z^k$ with $\hat{\psi}_{np+j} \neq 0$ for some $0 < j < n$. Then, $W(C_{\psi,\phi})$ contains the elliptical disk foci $\hat{\psi}_0$ and $\lambda^{np+j}\hat{\psi}_0$, with major axis*

$$\sqrt{|\hat{\psi}_0|^2|1 - e^{2\pi ij/n}|^2 + \frac{(np+j)!\Gamma(\alpha+2)}{\Gamma(np+j+\alpha+2)}|\hat{\psi}_{np+j}|^2}$$

and minor axis

$$\sqrt{\frac{(np+j)!\Gamma(\alpha+2)}{\Gamma(np+j+\alpha+2)}}|\hat{\psi}_{np+j}|.$$

Proof Let M be the subspace of $L_a^2(dA_\alpha)$ spanned by e_0 and e_{np+j}. We have

$$C_{\psi,\phi}e_0(z) = \sum_{k=0}^{\infty} \hat{\psi}_k z^k$$

and

$$C_{\psi,\phi}e_{np+j}(z) = \lambda^{np+j}\sqrt{\frac{\Gamma(np+j+\alpha+2)}{(np+j)!\Gamma(\alpha+2)}}\left(\sum_{k=0}^{\infty} \hat{\psi}_k z^{np+j+k}\right).$$

Thus, the compression of $C_{\psi,\phi}$ to M has the matrix representation

$$\begin{pmatrix} \hat{\psi}_0 & 0 \\ \sqrt{\frac{(np+j)!\Gamma(\alpha+2)}{\Gamma(np+j+\alpha+2)}}\hat{\psi}_{np+j} & \lambda^{np+j}\hat{\psi}_0 \end{pmatrix}.$$

Since $0 < j < n$ so $\lambda^{np+j} \neq 1$ and hence the numerical range of the compression of $C_{\psi,\phi}$ to M is the elliptical disk foci $\hat{\psi}_0$ and $\lambda^{np+j}\hat{\psi}_0$, and with major axis $\sqrt{|\hat{\psi}_0|^2|1 - e^{2\pi ij/n}|^2 + \frac{(np+j)!\Gamma(\alpha+2)}{\Gamma(np+j+\alpha+2)}|\hat{\psi}_{np+j}|^2}$ and minor axis $\sqrt{\frac{(np+j)!\Gamma(\alpha+2)}{\Gamma(np+j+\alpha+2)}}|\hat{\psi}_{np+j}|$ (see [8, Example 3]). The desired result follows from the fact that the numerical range of compression is contained in the numerical range of the operator. ∎

Theorem 5.6 *Let $C_{\psi,\phi} \in \mathbb{B}\left(L_a^2(dA_\alpha)\right)$ be such that $\phi(z) = e^{2\pi i\theta}z$ and $\psi(z) = \sum_{k=0}^{\infty} \hat{\psi}_k z^k$, where θ is irrational. If $n \geq 0$ and $m > 0$, then $W(C_{\psi,\phi})$ contains the elliptical disk with foci at $e^{2\pi in\theta}$ and $e^{2\pi i(n+m)\theta}$, and with major axis*

$$\sqrt{|e^{2\pi in\theta} - e^{2\pi i(n+m)\theta}|^2 + \frac{(n+m)!\Gamma(n+\alpha+2)}{n!\Gamma(n+m+\alpha+2)}|\hat{\psi}_m|^2}$$

and minor axis $\sqrt{\frac{(n+m)!\Gamma(n+\alpha+2)}{n!\Gamma(n+m+\alpha+2)}}|\hat{\psi}_m|$.

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Proof Let M be the subspace of $L_a^2(dA_\alpha)$ spanned by e_n and e_{n+m}. Then, we have

$$C_{\psi,\phi}e_n(z) = e^{2\pi in\theta}\sqrt{\frac{\Gamma(n+\alpha+2)}{n!\Gamma(\alpha+2)}}\sum_{k=0}^{\infty}\hat{\psi}_k z^{k+n}$$

and

$$C_{\psi,\phi}e_{n+m}(z) = e^{2\pi i(n+m)\theta}\sqrt{\frac{\Gamma(n+m+\alpha+2)}{(n+m)!\Gamma(\alpha+2)}}\sum_{k=0}^{\infty}\hat{\psi}_k z^{k+n+m}.$$

Thus, the compression of $C_{\psi,\phi}$ to M has the matrix representation

$$\begin{pmatrix} e^{2\pi in\theta} & 0 \\ e^{2\pi i(n+m)\theta}\sqrt{\frac{(n+m)!\Gamma(n+\alpha+2)}{n!\Gamma(n+m+\alpha+2)}}\hat{\psi}_m & e^{2\pi i(n+m)\theta} \end{pmatrix}.$$

Therefore, the numerical range of the compression of $C_{\psi,\phi}$ to M is the elliptical disk with foci $e^{2\pi in\theta}$ and $e^{2\pi i(n+m)\theta}$, and with major axis

$$\sqrt{|e^{2\pi in\theta} - e^{2\pi i(n+m)\theta}|^2 + \frac{(n+m)!\Gamma(n+\alpha+2)}{n!\Gamma(n+m+\alpha+2)}|\hat{\psi}_m|^2}$$

and minor axis $\sqrt{\frac{(n+m)!\Gamma(n+\alpha+2)}{n!\Gamma(n+m+\alpha+2)}}|\hat{\psi}_m|$ (see [8, Example 3]). The desired result follows as the numerical range of compression is contained in the numerical range of the operator. ∎

We end with the conclusion that the results of this section may be useful to estimate the lower bounds of numerical radius of some classes of weighted composition operators acting on $L_a^2(dA_\alpha)$.

Conflict of interest The authors declare that data sharing is not applicable to this article as no datasets were generated or analyzed during the current study. The authors declare no conflict of interest.

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Mathematical Institute, Silesian University in Opava, Na Rybnicku 1, Opava 74601, Czech Republic
e-mail: anirbansenfulia@gmail.com

Department of Mathematics, Jadavpur University, Kolkata 700032, West Bengal, India
e-mail: subhadiphalderju@gmail.com riddhick.math@gmail.com

Vice-Chancellor, Kalyani University, West Bengal 741235, and Professor (on lien), Department of Mathematics, Jadavpur University, Kolkata 700032, West Bengal, India
e-mail: kalloldada@gmail.com

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