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A new description of orthogonal bases

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We show that an orthogonal basis for a finite-dimensional Hilbert space can be equivalently characterised as a commutative †-Frobenius monoid in the category **FdHilb**, which has finite-dimensional Hilbert spaces as objects and continuous linear maps as morphisms, and tensor product for the monoidal structure. The basis is normalised exactly when the corresponding commutative †-Frobenius monoid is special. Hence, both orthogonal and orthonormal bases are characterised without mentioning vectors, but just in terms of the categorical structure: composition of operations, tensor product and the †-functor. Moreover, this characterisation can be interpreted operationally, since the †-Frobenius structure allows the cloning and deletion of basis vectors. That is, we capture the basis vectors by relying on their ability to be cloned and deleted. Since this ability distinguishes classical data from quantum data, our result has important implications for categorical quantum mechanics.

1. Introduction

Given any orthonormal basis $\{|\phi_i\rangle\}_i$ in a finite dimensional Hilbert space H, we can always define the linear maps

$$\delta : H \to H \otimes H :: |\phi_i\rangle \mapsto |\phi_i\rangle \otimes |\phi_i\rangle \tag{1}$$

and

$$\epsilon : H \to \mathbb{C} :: |\phi_i\rangle \mapsto 1\,. \tag{2}$$

It was observed in Coecke and Pavlovic (2007) that the triple (H, δ, ϵ) is a *commutative special †-Frobenius comonoid* in the category **FdHilb** of finite dimensional Hilbert spaces and linear maps, with tensor product as the monoidal structure. Meanwhile, the fact that orthonormal bases can be encoded as commutative special †-Frobenius comonoids has resulted in many important applications in the area of categorical quantum mechanics, such as in describing the flow of classical information in quantum informatic protocols (Coecke and Pavlovic 2007; Coecke *et al.* 2008) and in defining complementarity and special quantum logic gates in quantum computational schemes (Coecke and Duncan 2008). In this paper, we establish that every commutative †-Frobenius monoid arises from an orthogonal basis, and that adding the specialty condition gives an orthonormal basis.

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The plan of the paper is as follows:

— Section 2 provides some category-theoretic preliminaries.

— Section 3 spells out in more detail how a commutative †-Frobenius monoid arises from an orthogonal basis.

— Section 4 describes how to extract an orthogonal basis from any commutative †-Frobenius monoid in **FdHilb**.

— Section 5 states our main theorem.

— Section 6 spells out that within this established bijective correspondence, normalisation of basis vectors means speciality of the corresponding commutative †-Frobenius monoid. We also compare this result to an already-known result that classifies *arbitrary* bases on a finite-dimensional complex vector space as special Frobenius algebras.

— Section 7 describes some categorical statements arising from these results; in particular, we describe how to obtain the category of finite sets as a category of commutative †-Frobenius monoids in **FdHilb**.

2. Preliminaries

The research area of categorical quantum mechanics emerged from the observation that the subtle details of important, experimentally established quantum-informatic protocols can already be specified at an abstract category-theoretical level (Abramsky and Coecke 2004). The background structure is that of a symmetric monoidal †-category (Abramsky and Coecke 2004; Selinger 2007), which is a symmetric monoidal category together with an identity-on-objects involutive endofunctor that coherently preserves the symmetric monoidal structure. Within this context, the aim is then to maximise the ratio of results to assumptions: how much additional structure is required to describe and prove the correctness of a variety of physical phenomena. One such piece of additional structure is that of commutative †-*Frobenius monoids* (Coecke and Pavlovic 2007), which are a specialisation of earlier concepts introduced by Lawvere and Carboni and Walters (Lawvere 1969; Carboni and Walters 1987). Relative to the *quantum universe* modelled by the symmetric monoidal †-category, these commutative †-Frobenius monoids model the *classical interfaces*, and enable us to specify projector spectra, measurements and classical data flows (Coecke and Pavlovic 2007; Coecke *et al.* 2008).

Definition 2.1. A *Frobenius monoid* in a symmetric monoidal category is a quintuple $(X, m, u, \delta, \epsilon)$ consisting of a monoid

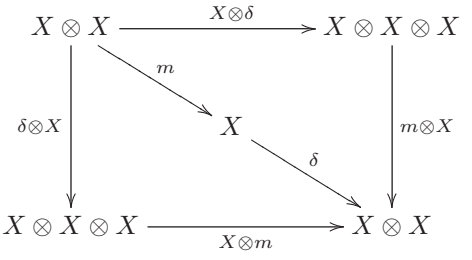
$$I \xrightarrow{u} X \xleftarrow{m} X \otimes X$$

and a comonoid

$$I \xleftarrow{\epsilon} X \xrightarrow{\delta} X \otimes X$$

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that together satisfy the *Frobenius condition*:



(3)

A Frobenius monoid is said to be *special* if $m \circ \delta = \mathrm{id}_X$. It is *commutative* if the multiplication m satisfies $m \circ \sigma_{X,X} = m$, and *cocommutative* if the comultiplication δ satisfies $\sigma_{X,X} \circ \delta = \delta$, where $\sigma_{X,X} : X \otimes X \to X \otimes X$ is the swap map that forms part of the structure of a symmetric monoidal category.

In a monoidal †-category, a †-*Frobenius monoid* is a Frobenius monoid of the form $(X, m, u, m^\dagger, u^\dagger)$. We will sometimes refer to such a structure as a †-Frobenius *co*monoid if we prefer to emphasise the comonoid part rather than the monoid part.

Example 2.2. The monoidal unit object I can be equipped with a canonical special commutative †-Frobenius monoid structure, namely $(I, \lambda_I^{-1} : I \otimes I \to I, \mathrm{id}_I)$.

Example 2.3. The most important example of a special commutative †-Frobenius monoid, which is the central example of relevance to this paper, is given by an orthonormal basis for a finite-dimensional Hilbert spaces. This gives a †-Frobenius monoid in **FdHilb**. Details of the construction of this are given in the next section.

Recall that in a monoidal category, a morphism $f : X \to Y$ is a *monoid homomorphism* for monoids (X, m, u) and (Y, m', u') if

$$\begin{aligned} f \circ m &= m' \circ (f \otimes f) \\ f \circ u &= u' \end{aligned} \tag{4}$$

and that it is a comonoid homomorphism for comonoids (X, δ, ϵ) and (Y, δ', ϵ') if

$$\begin{aligned} \delta \circ f &= (f \otimes f) \circ \delta' \\ \epsilon \circ f &= \epsilon' . \end{aligned} \tag{5}$$

Definition 2.4. A *copyable element* of a †-Frobenius monoid (X, m_X, u_X) is a comonoid homomorphism $\alpha : I \to X$.

3. Turning an orthogonal basis into a commutative †-Frobenius monoid

Given an orthogonal basis $\{|\phi_i\rangle\}_i$, the maps defined in (1) and (2) are the linear extensions of *copying* and *uniformly deleting* these basis vectors.

It is easy to see that we can recover the basis from δ alone by solving

$$\delta(|\psi\rangle) = |\psi\rangle \otimes |\psi\rangle . \tag{6}$$

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No other vectors besides those in $\{|\phi_i\rangle\}_i$ will satisfy this equation since for

$$|\psi\rangle \equiv \sum_{i=1}^{\dim(H)} \frac{\langle\phi_i|\psi\rangle}{\langle\phi_i|\phi_i\rangle}\,|\phi_i\rangle \tag{7}$$

with at least two non-zero scalars in $\{\langle\phi_i|\psi\rangle\}_i$, the quantity

$$\delta(|\psi\rangle) = \sum_{i=1}^{\dim(H)} \frac{\langle\phi_i|\psi\rangle}{\langle\phi_i|\phi_i\rangle}\,(|\phi_i\rangle \otimes |\phi_i\rangle) \tag{8}$$

will always be *entangled*, that is, it cannot be written in the form $|\psi_a\rangle \otimes |\psi_b\rangle$ for some $|\psi_a\rangle$ and $|\psi_b\rangle$, and is thus not equal to $|\psi\rangle \otimes |\psi\rangle$. So δ, and thus also the triple (H, δ, ϵ), faithfully encodes $\{|\phi_i\rangle\}_i$.

To see that $\delta^\dagger$ and δ obey the Frobenius condition, it suffices to note that

$$\delta^\dagger : H \otimes H \to H :: |\phi_i\rangle \otimes |\phi_j\rangle \mapsto \begin{cases} |\phi_i\rangle & i = j \\ 0 & i \neq j \end{cases} \tag{9}$$

so

$$|\phi_i\rangle \otimes |\phi_j\rangle \overset{\delta^\dagger}{\mapsto} \begin{cases} |\phi_i\rangle \\ 0 \end{cases} \overset{\delta}{\mapsto} \begin{cases} |\phi_i\rangle \otimes |\phi_i\rangle & i = j \\ 0 & i \neq j \end{cases}$$

and

$$|\phi_i\rangle \otimes |\phi_j\rangle \overset{\mathrm{id}_H \otimes \delta}{\mapsto} |\phi_i\rangle \otimes |\phi_j\rangle \otimes |\phi_j\rangle \overset{\delta^\dagger \otimes \mathrm{id}_H}{\mapsto} \begin{cases} |\phi_i\rangle \otimes |\phi_i\rangle & i = j \\ 0 & i \neq j. \end{cases}$$

As a consequence, by linearity, $\delta \circ \delta^\dagger = (\delta^\dagger \otimes \mathrm{id}_H) \circ (\mathrm{id}_H \otimes \delta)$. It is easy to verify the fact that (H, δ, ϵ) is a comonoid. The unit of the corresponding monoid is

$$\epsilon^\dagger : \mathbb{C} \to H :: 1 \mapsto \sum_{i=1}^{\dim(H)} |\phi_i\rangle. \tag{10}$$

4. Turning a commutative †-Frobenius monoid into an orthogonal basis

We will freely switch between denoting elements of H as linear maps

$$\alpha : \mathbb{C} \to H :: 1 \mapsto |\alpha\rangle$$

and as kets $|\alpha\rangle = \alpha(1) \in H$. Taking the adjoint of α gives us

$$\alpha^\dagger : H \to \mathbb{C} :: |\psi\rangle \mapsto \langle\alpha|\psi\rangle$$

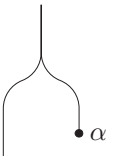
and hence $\langle\alpha| = \alpha^\dagger \in H^*$.

Let (H, m, u) be a commutative †-Frobenius monoid. Any element $\alpha \in H$ induces a linear map

$$R_\alpha := m \circ (\mathrm{id}_A \otimes \alpha) : H \to H, \tag{11}$$

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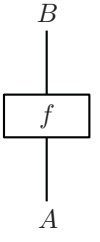
called the *right action* of α. We draw this right action as follows:



The diagram is read from bottom to top. This is a direct representation of our definition of R_α as right-multiplication by the element α: vertical lines represent the vector space H, the dot represents the element α and the merging of the two lines represents the multiplication operation m.

This graphical calculus is an important and well-studied tool for working with monoidal categories. While related to Penrose's intuitive diagrammatic notations for tensor calculus (Penrose 2005, Section 12.8), it in fact provides a completely formal graphical syntax for monoidal categories and their extensions, together with the free constructions and coherence theorems. Formal graphical calculi for various types of monoidal category go back to the early days of category theory (Kelly 1974), but modern versions are largely based on Joyal and Street (1991) – an excellent survey is given in Selinger (2010).

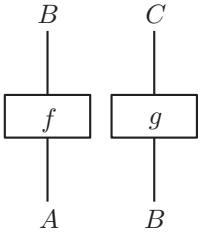
In general, an object in our monoidal category is represented by a vertical line, and a morphism $f : A \to B$ is represented as a 'junction' between wires, with an input wire of type A and an output wire of type B:



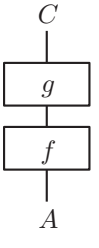
(12)

For the purposes of this paper, our 'junctions' will usually be drawn as intersections of curving wires, rather than as boxes.

Here we have labelled each wire with its type, but we will often omit this label when the type is obvious. Given morphisms $g : A \to B$ and $h : B \to C$, we represent tensor products and composites as follows:



$f \otimes g : A \otimes B \to B \otimes C$

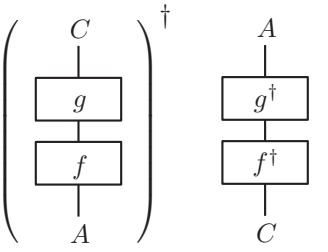


$g \circ f : A \to C$

(13)

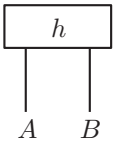
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The adjoint of a diagram is constructed by flipping the diagram about a horizontal axis, as demonstrated by the following example:



(14)

The tensor unit object I has the property that $I \otimes X \simeq X$ for all objects X, so its wire is 'invisible' in the graphical language. For example, the graphical representation of a morphism $h : A \otimes B \to I$ is

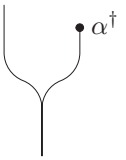


(15)

The elegance of the graphical notation lies in the fact that any two such diagrams represent equal morphisms in the category when one can be topologically deformed into the other, in the following precise sense.

Theorem 4.1 (Joyal and Street 1991; Selinger 2007). An equation expressed in the language of dagger symmetric monoidal categories follows from the axioms of dagger symmetric monoidal categories if and only if it holds up to 'isomorphism of diagrams' in the graphical language, where 'isomorphism of diagrams' mean that there is a bijective correspondence between junction boxes that is also respected by the wires connecting them.

Since H is a Hilbert space, the right action R_α defined in equation (11) has an adjoint $R_\alpha{}^\dagger : H \to H$. We draw the adjoint $R_\alpha{}^\dagger$ by flipping the diagram on a horizonal axis, in the manner described above:



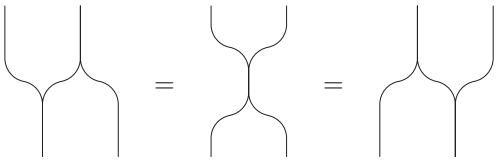
(16)

The splitting of the line into two represents the comultiplication, and the dot represents the linear map $\alpha^\dagger : H \to \mathbb{C}$. We can show that this adjoint to a right action is also the right action of some element of H.

Lemma 4.2. If (H, m, u) is a commutative †-Frobenius monoid in a symmetric monoidal †-category, then $R_\alpha{}^\dagger = R_{\alpha'}$ for $\alpha' = (\mathrm{id}_H \otimes \alpha^\dagger) \circ m^\dagger \circ u$.

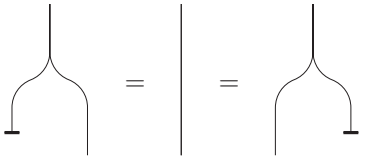
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Proof. We draw the Frobenius law of Definition 2.1 as follows:



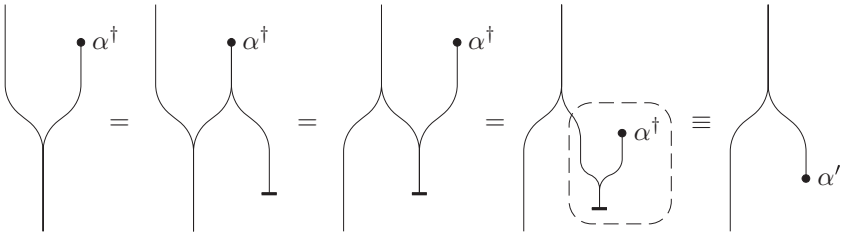
(17)

Representing the unit $u : I \to X$ as a small horizontal bar, we draw the unit law as follows:



(18)

We can now use the unit law and the Frobenius law to redraw the graphical representation of $R_\alpha{}^\dagger$ as follows:



(19)

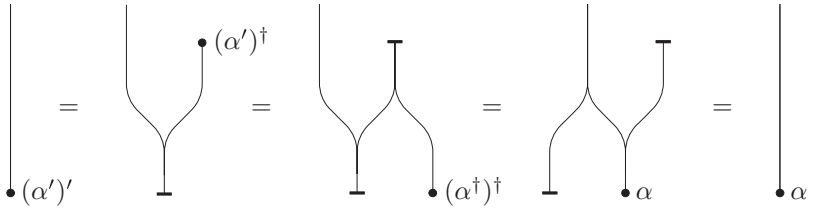
So the adjoint of R_α is indeed itself a right action of another element, which we call α'. Following the rules of the graphical calculus described earlier, it can be seen that $\alpha' = (\mathrm{id}_H \otimes \alpha^\dagger) \circ m^\dagger \circ u$. □

Lemma 4.3. Let $\mathbf{C}$ be a symmetric monoidal †-category and (H, m, u) be a commutative †-Frobenius monoid in $\mathbf{C}$. The right action mapping

$$\mathbf{C}(\mathrm{I}, H) \to \mathbf{C}(H, H) :: \alpha \mapsto R_\alpha$$

is an involution-preserving monoid embedding when the set $\mathbf{C}(\mathrm{I}, H)$ is equipped with the involutive monoid structure induced by (H, m, u). When $\mathbf{C} = \mathbf{FdHilb}$, this mapping also preserves the vector space structure.

Proof. Using the Frobenius and unit identities and the fact that the †-functor is an involution, we first show that $(-)'$ as in Lemma 4.2 is involutive:



(20)

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Here, we have applied the definition of the involution twice, and then simplified using the Frobenius and unit laws, to conclude that $(\alpha')' = \alpha$.

Taking adjoints gives an involution on $\mathbf{C}(X, X)$, and since $R_{\alpha'} = R_\alpha{}^\dagger$, we can see that the embedding $\alpha \mapsto R_\alpha$ does indeed preserve the involution. Moreover, the mapping is injective since, by the unit equation, we have $R_\alpha \circ u = \alpha$. It is straightforward to show that multiplication and unit are preserved. Preservation of the vector space structure when $\mathbf{C} = \mathbf{FdHilb}$ follows by the linearity of m. □

Corollary 4.4. Any †-Frobenius monoid in **FdHilb** is a C*-algebra.

Proof. Let (H, m, u) be a †-Frobenius monoid in **FdHilb**. The endomorphism monoid $\mathbf{FdHilb}(H, H)$ is a C*-algebra. By the above Lemma, we know that (H, m, u) has an involution-preserving embedding into this endomorphism monoid. However, any finite-dimensional involution-closed subalgebra of a C*-algebra is also a C*-algebra, so it follows that any †-Frobenius monoid in **FdHilb** is a C*-algebra, and, in particular, that it can be given a C*-algebra norm. □

Remark 4.5. Note that in the above we did not assume the †-Frobenius monoid to be commutative – see Vicary (2008) for further details.

Corollary 4.6. The copyable elements for any commutative †-Frobenius monoid on H in **FdHilb** form a basis for H.

Proof. For $\phi : \mathbb{C} \to H$ to be a copyable element for the commutative †-Frobenius monoid (H, m, u) means that $m^\dagger \circ \phi = \phi \otimes \phi$. Taking adjoints, this is equivalent to $\phi^\dagger \circ m = \phi^\dagger \otimes \phi^\dagger$. By the structure theorem for commutative C*-algebras (Murphy 1990), the monoid (H, m, u) is a direct sum of copies of the ground field $\mathbb{C}$, with pointwise multiplication, so $H = \mathbb{C}^N$ where $N = \dim H$. Writing $1_i \in \mathbb{C}^N$ for the unit element of the ith copy of $\mathbb{C}$ in the direct sum, it is clear that $m(1_i \otimes 1_j) = \delta_{i,j} 1_i$, and that any map $\psi : \mathbb{C}^N \to \mathbb{C}$ is defined by the values $\psi(1_i)$ it takes at each unit element. We now suppose that such a ψ satisfies $\psi \circ m = \psi \otimes \psi$. Then, for each $1_i \otimes 1_j \in \mathbb{C}^N \otimes \mathbb{C}^N$, we have

$$(\psi \circ m)(1_i \otimes 1_j) = (\psi \otimes \psi)(1_i \otimes 1_j).$$

Simplifying, we obtain

$$\delta_{i,j} \psi(1_i) = \psi(1_i)\psi(1_j),$$

and it is then clear that

$$\psi(1_i) = \delta_{i,n},$$

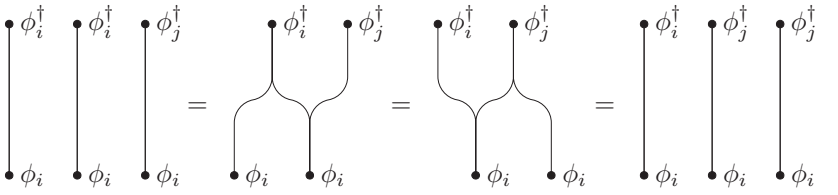
where n labels some particular element of the direct sum. The finite set of all such ψ evidently form a basis for H^*, so their adjoints, which as established above are the copyable elements of the †-Frobenius algebra, form a basis for H, as required. □

We still need to show that this basis is orthogonal.

Lemma 4.7. If $\phi_i, \phi_j : I \to X$ are comonoid homomorphisms for a cocommutative †-Frobenius comonoid, and if $\langle \phi_i | \phi_j \rangle := \phi_i^\dagger \circ \phi_j$, $\langle \phi_i | \phi_i \rangle$ and $\langle \phi_j | \phi_j \rangle$ are all cancellable scalars, then they must all be real and equal.

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Proof. Given that ϕ_i and ϕ_j are comonoid homomorphisms, and making use of one of the Frobenius identities, we can derive the following equation:



(21)

The graphical representation of $\langle \phi_i | \phi_j \rangle$ as a line with dots at each end follows from its definition as the composite $\phi_i^\dagger \circ \phi_j : \mathbb{C} \to \mathbb{C}$, together with the rules of the graphical language. Switching the roles of ϕ_i and ϕ_j, we can obtain another similar equation, and writing both equations in bra-ket notation we obtain

$$\langle \phi_i | \phi_i \rangle \langle \phi_i | \phi_i \rangle \langle \phi_i | \phi_j \rangle = \langle \phi_i | \phi_i \rangle \langle \phi_i | \phi_j \rangle \langle \phi_i | \phi_j \rangle,$$

$$\langle \phi_j | \phi_j \rangle \langle \phi_j | \phi_j \rangle \langle \phi_j | \phi_i \rangle = \langle \phi_j | \phi_j \rangle \langle \phi_j | \phi_i \rangle \langle \phi_j | \phi_i \rangle.$$

If we cancel $\langle \phi_i | \phi_i \rangle$, $\langle \phi_j | \phi_j \rangle$ and $\langle \phi_i | \phi_j \rangle$, we obtain

$$\langle \phi_i | \phi_i \rangle = \langle \phi_i | \phi_j \rangle$$
$$\langle \phi_j | \phi_j \rangle = \langle \phi_j | \phi_i \rangle.$$

It follows that these inner products are real, and that they are all equal:

$$\langle \phi_i | \phi_i \rangle = \langle \phi_i | \phi_j \rangle = \langle \phi_j | \phi_i \rangle = \langle \phi_j | \phi_j \rangle.$$ □

Corollary 4.8. The copyable elements for any commutative †-Frobenius monoid on H in **FdHilb** form an orthogonal basis for H.

Proof. Let $\phi_i, \phi_j : \mathbb{C} \to H$ be two different vectors in this basis. This will only be impossible to satisfy when H is one-dimensional, but in that case the basis is trivially orthogonal. Hence, we assume that ϕ_i and ϕ_j are *not* orthogonal, that is, $\langle \phi_i | \phi_j \rangle$ is cancellable, so Lemma 4.7 applies. This allows us to conclude that

$$\langle \phi_i - \phi_j | \phi_i - \phi_j \rangle = \langle \phi_i | \phi_i \rangle - \langle \phi_i | \phi_j \rangle - \langle \phi_j | \phi_i \rangle + \langle \phi_j | \phi_j \rangle = 0, \tag{22}$$

and hence $|\phi_i\rangle = |\phi_j\rangle$, which contradicts our assumption that ϕ_i and ϕ_j were different, so the basis $|\phi_i\rangle$ is orthogonal. □

5. Statement of the main result

Theorem 5.1. Every commutative †-Frobenius monoid in **FdHilb** determines an orthogonal basis, consisting of its copyable elements, and every orthogonal basis determines a commutative †-Frobenius monoid in **FdHilb** through prescriptions (1) and (2). These constructions are inverse to each other.

We now only need to explain why the construction described in Section 4, which produces an orthogonal basis from a commutative †-Frobenius algebra, is inverse to the

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construction described in Section 3, which produces a commutative †-Frobenius algebra from an orthogonal basis.

We assume we begin with an orthogonal basis. We construct the †-Frobenius monoid as the unique one with a comultiplication that perfectly copies our original basis, and we construct a new orthogonal basis consisting of those elements that are perfectly copied by the comultiplication. This new basis must be at least as large as our original basis. However, since any two orthogonal bases for a finite-dimensional Hilbert space must have the same number of elements, the new basis is actually the same as the original basis.

We now assume we begin with a commutative †-Frobenius monoid. We construct our orthogonal basis as those elements that are copied by the comultiplication, and construct our new monoid as the unique †-Frobenius monoid that perfectly copies this basis. However, since the original monoid also perfectly copies this basis, by uniqueness, the two monoids are the same.

Remark 5.2. In other papers on this subject, abstract basis vectors are required to be *self-conjugate* comonoid homomorphisms, where the conjugate of a morphism $f : X \to Y$ relative to †-Frobenius monoids (X, m_X, u_X) and (Y, m_Y, u_Y) is defined to be

$$f_* := (\mathrm{id}_Y \otimes {\eta_X}^\dagger) \circ (\mathrm{id}_Y \otimes f^\dagger \otimes \mathrm{id}_X) \circ (\eta_Y \otimes \mathrm{id}_X) \tag{23}$$

with $\eta_Z = {m_Z}^\dagger \circ u_Z : \mathrm{I} \to Z \otimes Z$. In **FdHilb**, such a morphism is self-conjugate if all entries of its matrix representation in the corresponding bases are self-conjugate, which follows automatically in **FdHilb** from it being a comonoid homomorphism. In other categories the additional constraint of the basis vectors being self-conjugate guarantees that they are involution preserving – *cf.* the last part of the proof of Corollary 4.6.

6. Other types of basis

The elements of the orthogonal basis are normalised exactly when the corresponding commutative †-Frobenius algebra is *special*, meaning $m \circ \delta = \mathrm{id}_H$. This is most straightforwardly seen from the explicit form of the comultiplication in terms of the orthogonal basis as given in (1). From this we see that

$$m \circ \delta = \delta^\dagger \circ \delta = \sum_{i=1}^{\dim(H)} |\phi_i\rangle\langle\phi_i|, \tag{24}$$

and it is clear that this is the identity if and only if the vectors $|\phi_i\rangle$ are normalised.

Interestingly, it is already known that for a finite-dimensional complex *vector* space, a basis exactly corresponds to a choice of special commutative Frobenius algebra[†]. This result follows from the fact that a special commutative Frobenius algebra is necessarily strongly separable, and since the ground field is of characteristic 0, it is therefore necessarily finite-dimensional and semisimple (Aguiar 2000). By the Artin–Wedderburn theorem of ring theory, such an algebra is canonically isomorphic to a finite cartesian product of the complex numbers, up to permutation, and the corresponding basis elements are given by

[†] We are grateful to John Baez for pointing this out to us.

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the number 1 in each of the complex factors. Of course, it does not make sense to ask whether such a Frobenius algebra is †-Frobenius, or whether the elements of such a basis are normalised or orthogonal.

In summary, on a finite-dimensional complex Hilbert space, we can describe different types of basis very precisely with the following structures:

Type of basis	**Algebraic structure**
Arbitrary	Commutative special Frobenius algebra
Orthogonal	Commutative †-Frobenius algebra
Orthonormal	Commutative special †-Frobenius algebra

The inner product on the Hilbert space does not play a role for the case of the arbitrary basis. In every case, the basis is recovered from the Frobenius algebra as those vectors that are perfectly copied by the comultiplication, and the Frobenius algebra is recovered from the basis as the unique Frobenius algebra of the correct type with a comultiplication that perfectly copies the basis.

It is interesting to consider whether an arbitrary commutative Frobenius algebra on a complex vector space might also correspond to some type of basis structure. In fact, it does not, and such Frobenius algebras can be very wild indeed; see Kock (2004) for plenty of examples. It is perhaps surprising that both the specialness axiom and the †-Frobenius axiom can serve independently to tame this wildness.

7. Categorical statements

We have shown that a commutative †-Frobenius monoid in **FdHilb** corresponds to a basis of a finite-dimensional Hilbert space. Any such basis is determined up to unitary isomorphism by the norms of the basis elements, which constitute a list of positive real numbers.

If a homomorphism between two commutative †-Frobenius monoids preserves all of the structure – the multiplication, unit, comultiplication and counit – then it is necessarily an isomorphism, and in **FdHilb**, it is necessarily unitary. Such a homomorphism will map one basis onto another, taking basis elements onto basis elements of the same norm. This leads to the following result.

Corollary 7.1. The category of commutative †-Frobenius monoids in **FdHilb**, with morphisms preserving all of the Frobenius structure, is equivalent to the groupoid of 'finite lists of real numbers and isomorphisms that preserve the numbers', which has objects given by finite sets equipped with functions into the positive real numbers, and morphisms given by isomorphisms of sets that preserve the functions into the real numbers.

This is interesting from the perspective of unitary 2-dimensional topological quantum field theory (Kock 2004), since such things are given by commutative †-Frobenius monoids in **FdHilb**, and the natural notion of homomorphism is one that preserves all of the Frobenius structure.

If we only require that our homomorphisms preserve the comultiplication and counit, this gives arbitrary functions between bases, without the requirement of

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preserving the length of the basis vector. This follows from the spectral theorem for commutative C*-algebras: a comonoid homomorphism gives rise to an oppositely-directed monoid homomorphism by taking the adjoint, and any involution-preserving monoid homomorphism is equivalent to an oppositely-directed continuous function between the spectra of the C*-algebras.

Corollary 7.2. The category of commutative †-Frobenius monoids in **FdHilb**, with morphisms preserving comultiplication and counit, is equivalent to the category **FinSet** of finite sets and functions.

This gives an interesting new perspective on the relationship between **FdHilb** and **FinSet**. There is an obvious functor F : **FinSet** $\rightarrow$ **FdHilb** that takes a set with n elements to a Hilbert space of dimension n and chosen basis, and takes a function between sets to the induced linear map. This can be thought of as a 'free' functor since it generates the free Hilbert space on a finite set. But using the equivalence between **FinSet** and the category of commutative †-Frobenius monoids described in Corollary 7.2, we see that this functor F is equivalent to the *forgetful* functor that regards each finite set as a Hilbert space equipped with a commutative †-Frobenius monoid structure, and forgets the monoid structure. So a finite set can be considered as a finite-dimensional Hilbert space with the extra structure of a commutative †-Frobenius monoid, or a finite-dimensional Hilbert space can be considered as a finite set with the extra structure of a vector space and inner product.

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