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On exterior powers of reflection representations, II

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Abstract. Let W be a group endowed with a finite set S of generators. A representation (V, ρ) of W is called a reflection representation of (W, S) if $\rho(s)$ is a (generalized) reflection on V for each generator $s \in S$. In this article, we prove that for any irreducible reflection representation V, all the exterior powers $\bigwedge^d V$, $d = 0, 1, \ldots, \dim V$, are irreducible W-modules, and they are non-isomorphic to each other. This extends a theorem of R. Steinberg which is stated for Euclidean reflection groups. Moreover, we prove that the exterior powers (except for the 0th and the highest power) of two non-isomorphic reflection representations always give non-isomorphic W-modules. This allows us to construct numerous pairwise non-isomorphic irreducible representations for such groups, especially for Coxeter groups.

1 Introduction

1.1 Overview

In [9, Section 14], R. Steinberg proved a theorem stating that the exterior powers of the irreducible reflection representation of a Euclidean reflection group are again irreducible and pairwise non-isomorphic (see also [1, Chapter V, Section 2, Exercise 3]). For Weyl groups, the exterior powers of the standard reflection representation are well studied (see, for example, [2, 4, 5, 8]).

The proof of Steinberg's theorem relies on the existence of an inner product which stays invariant under the group action. In a previous paper [6], the author extended Steinberg's result to a more general context where the inner product may not exist. Let W be a group and $S = \{s_1, \ldots, s_k\}$ be a set of generators of W. We say a representation $\rho : W \to \mathrm{GL}(V)$ is a reflection representation of (W, S) if each of the generators s_i acts by a generalized reflection, and denote by α_i the chosen reflection vector (see Section 2.1 for related notions). The main theorem in [6] reads:

Theorem 1.1 [6, Theorem 1.2] *Let (V, ρ) be an n-dimensional irreducible reflection representation of (W, S) over a field $\mathbb{F}$ of characteristic 0, with reflection vectors $\alpha_1, \ldots, \alpha_k$. Suppose*

$$\text{for any two indices } i, j, s_i \cdot \alpha_j \neq \alpha_j \text{ if and only if } s_j \cdot \alpha_i \neq \alpha_i. \tag{1.1}$$

Then the W-modules $\{\bigwedge^d V \mid 0 \leq d \leq n\}$ are irreducible and pairwise non-isomorphic.

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As pointed out in [6], usually there is no W-invariant bilinear form on the reflection representation, so that our result is not a trivial generalization.

The first aim of this article is to show that the assumption (1.1) can be removed.

Theorem 1.2 *Let (V,ρ) be an n-dimensional irreducible reflection representation of (W,S) over a field $\mathbb{F}$ of characteristic 0. Then the W-modules $\{\bigwedge^d V \mid 0 \le d \le n\}$ are irreducible and pairwise non-isomorphic.*

The readers may find that the proof of Theorem 1.2 is similar to that of Theorem 1.1 [6, Theorem 1.2]. The proof here simplifies the proof in [6] a little bit. See Section 4 for more details.

The major contribution of this article is the second main result, stating that the exterior powers of two different reflection representations are also different. To be precise, we have

Theorem 1.3 *Let (V_ι,ρ_ι), $\iota = 1,2$, be two irreducible reflection representations of (W,S) over a field $\mathbb{F}$ of characteristic 0, with dimensions n_1 and n_2, respectively. Suppose $\bigwedge^{d_1} V_1 \simeq \bigwedge^{d_2} V_2$ as W-modules for some integers d_1, d_2 with $1 \le d_\iota \le n_\iota - 1$ $(\iota = 1,2)$. Then $d_1 = d_2$, $n_1 = n_2$, and $V_1 \simeq V_2$ as W-modules.*

Remark 1.4 Note that $\bigwedge^0 V$ is the one-dimensional W-module with trivial W-action. While $\bigwedge^n V$ carries the one-dimensional representation $\det\circ\rho$ for any n-dimensional representation (V,ρ) of W, and different ρ's might share the same determinant $\det\circ\rho$ (for example, if each generator s_i is of order two, then $\det\circ\rho(s_i) = -1$ for any reflection representation (V,ρ) and any i). Thus, in Theorem 1.3 the range $1 \le d_\iota \le n_\iota - 1$ is the best we can expect.

By combining the results in Theorems 1.2 and 1.3, immediately we have the following corollary, which allows us to construct numerous pairwise non-isomorphic irreducible representations for the group W.

Corollary 1.5 *Suppose we have a family of irreducible reflection representations $\{V_i \mid i \in I\}$ of (W,S). Then $\{\bigwedge^d V_i \mid i \in I, 1 \le d \le \dim V_i - 1\}$ is a family of simple W-modules, and they are pairwise non-isomorphic.*

1.2 Motivation and application

The motivation of this work (as well as the previous [6]) comes as follows. Suppose W is a Coxeter group with the finite set S of defining generators. In another paper [7] by the author, all the reflection representations (over $\mathbb{C}$) of (W,S) are determined. The most essential thing in this process is the classification of isomorphism classes of the so-called *generalized geometric representations* (that is, those reflection representations admitting a basis formed by the reflection vectors). In [7], such representations are classified using the characters of the first integral homology group of simple graphs which are closely related to the Coxeter graph. Moreover, "most" of them are irreducible. While if a generalized geometric representation is reducible, then it has a semisimple quotient, each of whose direct summand is an irreducible reflection representation of some parabolic subgroup. Therefore, the results in this article are applicable, and then we obtain a large class of irreducible representations which are non-isomorphic to each other.

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For example, if (W, S) is the affine Weyl group of type $\widetilde{A}_n$, that is,

$$W = \langle s_0, s_1, \ldots, s_n \mid s_i^2 = (s_i s_{i+1})^2 = e, \forall\, i = 0, 1, \ldots, n \rangle$$

(regard $n+1$ as 0), then the Coxeter graph is a cycle. The corresponding first homology group with integral coefficients is isomorphic to $\mathbb{Z}$, and its characters are parameterized by $\mathbb{C}^\times$, and so are the generalized geometric representations. For $x \in \mathbb{C}^\times$, the corresponding generalized geometric representation, denoted by V_x, is an $(n+1)$-dimensional $\mathbb{C}$-vector space with basis $\{\alpha_0, \alpha_1, \ldots, \alpha_n\}$, and the W-action can be defined by

$$\begin{aligned} s_i \alpha_i &= -\alpha_i, \quad \forall\, i = 0, 1, \ldots, n; \\ s_0 \alpha_n &= \alpha_n + x\alpha_0; \\ s_n \alpha_0 &= \alpha_0 + \frac{1}{x}\alpha_n; \\ s_i \alpha_{i+1} = s_{i+1}\alpha_i &= \alpha_i + \alpha_{i+1}, \quad \forall\, i = 0, 1, \ldots, n-1; \\ s_i \alpha_j &= \alpha_j, \quad \text{if } i \neq j \text{ and } i, j \text{ are not adjacent.} \end{aligned}$$

All of these representations are irreducible except when $x = 1$. If $x = 1$, the representation V_1 is nothing but the geometric representation in the sense of [1, Chapter V, Section 4], and it admits an n-dimensional simple quotient V_1/U where $U := \langle \alpha_0 + \alpha_1 + \cdots + \alpha_n \rangle$. The quotient V_1/U is also a reflection representation. Applying Theorems 1.2 and 1.3 yields uncountably many simple modules for the affine Weyl group W:

$$\left\{ \bigwedge^d V_x \;\middle|\; 1 \leq d \leq n, x \in \mathbb{C}^\times \backslash \{1\} \right\} \cup \left\{ \bigwedge^d (V_1/U) \;\middle|\; 1 \leq d \leq n-1 \right\}.$$

1.3 Outline of this article

The paper is organized as follows. In Section 2, we recollect the basic definitions and some preliminary results. In Section 3, we recollect some basic results on exterior powers of reflection representations. In Sections 4 and 5, we prove Theorems 1.2 and 1.3, respectively.

2 Preliminaries

Throughout this article, we work over a field $\mathbb{F}$ of characteristic 0. We require $\operatorname{char} \mathbb{F} = 0$ only to ensure the exterior powers of an irreducible representation are semisimple (see Remark 3.7). In fact, the notions of reflections and reflection representations can be defined over fields of arbitrary characteristic.

For any positive integer k, we denote $[k] := \{1, 2, \ldots, k\}$. For a fixed representation $\rho : W \to \mathrm{GL}(V)$ and an element $s \in W$, we also denote simply by s the linear map $\rho(s) \in \mathrm{GL}(V)$ if there is no ambiguity.

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2.1 Reflections and reflection representations

Definition 2.1 [6, Definition 2.1] Let V be a finite-dimensional vector space over $\mathbb{F}$.

(1) A linear map $s : V \to V$ is called a *generalized reflection* (and *reflection* for short) if s is diagonalizable and $\operatorname{rank}(s - \mathrm{Id}_V) = 1$.
(2) Suppose s is a reflection on V. The hyperplane $H_s := \ker(s - \mathrm{Id}_V)$, which is fixed pointwise by s, is called the *reflection hyperplane* of s. Let α_s be a nonzero vector in $\operatorname{Im}(s - \mathrm{Id}_V)$. Then, $s \cdot \alpha_s = \lambda_s \alpha_s$ for some $\lambda_s \in \mathbb{F}\backslash\{1\}$, and α_s is called a *reflection vector* of s.

Note that if s is an invertible map, then $\lambda_s \neq 0$.

The following lemma is immediate.

Lemma 2.2 [6, Lemma 2.2] *Let s be a reflection on V and α_s be a reflection vector. Then there exists a nonzero linear function $f : V \to \mathbb{F}$ such that $s \cdot v = v + f(v)\alpha_s$ for any $v \in V$.*

The main object of our study, reflection representation, is defined as follows.

Definition 2.3 Let W be a group endowed with a finite set of generators $S = \{s_1, \ldots, s_k\}$. A representation (V, ρ) of W over $\mathbb{F}$ is called a *reflection representation of* (W, S) if the linear map $\rho(s_i) \in \mathrm{GL}(V)$ is a reflection on V for any $i \in [k]$.

2.2 Digraphs

Digraphs will be helpful to investigate the structure of reflection representations. In what follows we recall some relevant basic definitions.

By definition, a *directed graph* (or *digraph* for short) $G = (I, A)$ consists of a set I of vertices and a set A of arrows, where each arrow in A is an ordered binary subset (i, j) of I. We also denote by $i \to j$ the arrow (i, j). For our purpose, we only consider finite digraphs without loops and multiple arrows, that is, (1) I is a finite set, (2) there is no arrow of the form $i \to i$ and (3) each arrow $i \to j$ occurs at most once in A.

Suppose $i, j \in I$ are two vertices of a digraph G. A *walk* in G from i to j is a sequence of vertices

$$i = i_0, \quad i_1, \quad \ldots, \quad i_{l-1}, \quad i_l = j$$

such that $i_{m-1} \to i_m$ is an arrow in A for each $m \in [l]$.

An *undirected walk* in G from i to j is an alternating sequence

$$i = i_0, \quad a_1, \quad i_1, \quad a_2, \quad i_2, \quad \ldots, \quad i_{l-1}, \quad a_l, \quad i_l = j$$

of vertices $i_0, i_1, \ldots, i_l \in I$ and arrows $a_1, \ldots, a_l \in A$ such that either $a_m = i_{m-1} \to i_m$ or $a_m = i_m \to i_{m-1}$ for each $m \in [l]$.

A digraph G is called *weakly connected* if for any two vertices i, j there exists an undirected walk from i to j. In other words, G is weakly connected if the undirected graph obtained by forgetting the directions of all arrows in A is connected. Moreover, G is called *strongly connected* if for any two vertices i, j there exists two walks, one from i to j and the other from j to i.

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Suppose $J \subset I$ is a subset of the vertices of G. We define a digraph $G(J)$, called the *sub-digraph spanned by* J, to be the digraph $(J, A(J))$ with the set J of vertices, and the set $A(J) := \{i \to j \mid i, j \in J, \text{ and } i \to j \text{ is an arrow in } A\}$ of arrows.

Definition 2.4 Let $G = (I, A)$ be a digraph and $J, J' \subseteq I$ be subsets of vertices. Suppose there exist vertices $i \in J$ and $j \in J'$ such that $i \to j$ is an arrow in A and $J \backslash \{i\} = J' \backslash \{j\}$. Then we say J' is obtained from J by a *move-forward*, and J is obtained from J' by a *move-back*. We also say uniformly that J or J' is obtained from the other by a *move*.

Intuitively, we obtain J' from J by moving the vertex i to the vertex j along the arrow $i \to j$.

The following lemma is essentially [6, Lemma 4.3].

Lemma 2.5 *Let $G = (I, A)$ be a weakly connected digraph. Let $J, J' \subseteq I$ be two subsets with the same cardinality. Then J' can be obtained from J by finite steps of moves.*

Proof Forgetting the directions of arrows in A, this lemma follows from [6, Lemma 4.3]. ∎

Digraphs and reflection representations are related via the following definition.

Definition 2.6 Let W be a group endowed with a finite set of generators $S = \{s_1, \ldots, s_k\}$, and (V, ρ) be a reflection representation of (W, S). For each $i \in [k]$, let α_i be an arbitrarily chosen reflection vector of s_i. For any subset $I \subseteq [k]$, we define the *associated digraph* G_I to be a digraph (I, A) where I is the set of vertices and

$$A := \{i \to j \mid i, j \in I, s_j \cdot \alpha_i \neq \alpha_i\}$$

is the set of arrows. We also denote simply by G the associated digraph $G_{[k]}$.

Clearly, for subsets $J \subseteq I \subseteq [k]$, the digraph G_J is the sub-digraph $G_I(J)$ of G_I spanned by J.

Immediately we have the following fact about the associated digraph.

Lemma 2.7 *If $i \to j$ is an arrow in the digraph G_I, then α_j belongs to the subrepresentation generated by α_i.*

Proof By definition, we have $s_j \cdot \alpha_i \neq \alpha_i$. In view of Lemma 2.2, the vector $\alpha_i - s_j \cdot \alpha_i$ is a nonzero multiple of α_j. But this vector lies in the subrepresentation generated by α_i. ∎

2.3 Some numerical lemmas

We will need the following lemmas.

Lemma 2.8 *Let n_1, n_2, d_1, d_2 be positive integers and $1 \leq d_\iota \leq n_\iota - 1$ for $\iota = 1, 2$. Suppose $\frac{d_1}{n_1} = \frac{d_2}{n_2}$ and $\binom{n_1}{d_1} = \binom{n_2}{d_2}$. Then $n_1 = n_2$ and $d_1 = d_2$.*

Proof Without loss of generality, we may assume $n_1 \leq n_2$ and $d_\iota \leq \frac{n_\iota}{2}$ for $\iota = 1, 2$. Suppose $n_1 < n_2$. Then $d_1 < d_2$. Then we have $\binom{n_2}{d_2} > \binom{n_2}{d_1} > \binom{n_1}{d_1}$ which is a contradiction. Therefore, $n_1 = n_2$ and hence $d_1 = d_2$. ∎

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Lemma 2.9 *Let n_1, n_2, d_1, d_2 be positive integers and $1 \le d_\iota \le n_\iota - 1$ for $\iota = 1, 2$. Suppose*

$$\binom{n_1 - 1}{d_1} = \binom{n_2 - 1}{d_2} \tag{2.1}$$

and

$$\binom{n_1 - 1}{d_1 - 1} = \binom{n_2 - 1}{d_2 - 1}. \tag{2.2}$$

Then $n_1 = n_2$ and $d_1 = d_2$.

Proof By direct computations, for $\iota = 1, 2$ we have

$$\begin{aligned}\binom{n_\iota - 1}{d_\iota} - \binom{n_\iota - 1}{d_\iota - 1} &= \frac{(n_\iota - 1)!}{d_\iota!(n_\iota - d_\iota - 1)!} - \frac{(n_\iota - 1)!}{(d_\iota - 1)!(n_\iota - d_\iota)!} \\ &= \frac{(n_\iota - 1)!}{d_\iota!(n_\iota - d_\iota)!}(n_\iota - 2d_\iota) \\ &= \binom{n_\iota}{d_\iota}\Big(1 - \frac{2d_\iota}{n_\iota}\Big).\end{aligned}$$

By Equations (2.1) and (2.2), we then have

$$\binom{n_1}{d_1}\Big(1 - \frac{2d_1}{n_1}\Big) = \binom{n_2}{d_2}\Big(1 - \frac{2d_2}{n_2}\Big). \tag{2.3}$$

Adding the Equations (2.1) and (2.2) together yields

$$\binom{n_1}{d_1} = \binom{n_2}{d_2}. \tag{2.4}$$

We combine Equations (2.3) and (2.4), then we obtain

$$\frac{d_1}{n_1} = \frac{d_2}{n_2}.$$

By Lemma 2.8 we have $n_1 = n_2$ and $d_1 = d_2$. ∎

3 Exterior powers of reflection representations

In this section we recollect some first results about exterior powers. Let W be a group endowed with a set of generators $S = \{s_1, \dots, s_k\}$ as before. Suppose (V, ρ) is an n-dimensional representation of W. The action $\bigwedge^d \rho$ of W on the dth exterior power $\bigwedge^d V$ $(0 \le d \le n)$ is given by

$$w \cdot (v_1 \wedge \cdots \wedge v_d) = (w \cdot v_1) \wedge \cdots \wedge (w \cdot v_d), \quad \forall w \in W, v_1, \dots, v_d \in V.$$

In particular, $\bigwedge^0 V$ is the one-dimensional W-module with trivial action, and $\bigwedge^n V$ carries the one-dimensional representation $\det \circ \rho$. If $\{v_1, \dots, v_n\}$ is a basis of V, then

$$\{v_{i_1} \wedge \cdots \wedge v_{i_d} \mid 1 \le i_1 < \cdots < i_d \le n\}$$

is a basis of $\bigwedge^d V$. In particular, $\dim \bigwedge^d V = \binom{n}{d}$. For more details about exterior powers, see, for example, [3].

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Suppose further that (V, ρ) is a reflection representation and α_i is a chosen reflection vector of s_i with eigenvalue λ_i ($\neq 1$) for each $i \in [k]$ (see Definitions 2.1 and 2.3). We also denote by H_i the reflection hyperplane of s_i. For each $i \in [k]$ and $0 \leq d \leq n$, we define

$$V_{d,i}^{+} := \left\{ v \in \bigwedge^{d} V \;\middle|\; s_i \cdot v = v \right\}, \quad V_{d,i}^{-} := \left\{ v \in \bigwedge^{d} V \;\middle|\; s_i \cdot v = \lambda_i v \right\}$$

to be the eigen-subspaces of s_i in $\bigwedge^d V$, for the eigenvalues 1 and λ_i, respectively.

Retain the notations W, V, s_i, α_i, etc.

Lemma 3.1 [6, Lemma 3.2 and Corollary 3.3] *Let $i \in [k]$ and $0 \leq d \leq n$.*

(1) *We have $V_{d,i}^{+} = \bigwedge^d H_i$ and $\dim V_{d,i}^{+} = \binom{n-1}{d}$. Here we regard $\binom{n-1}{n} = 0$ if $d = n$.*
(2) *Extend the reflection vector α_i arbitrarily to a basis of V, say, $\alpha_i, v_2, \ldots, v_n$. Then, $V_{d,i}^{-}$ has a basis*

$$\{\alpha_i \wedge v_{i_1} \wedge \cdots \wedge v_{i_{d-1}} \mid 2 \leq i_1 < \cdots < i_{d-1} \leq n\}.$$

In particular, $\dim V_{d,i}^{-} = \binom{n-1}{d-1}$. Here we regard $\binom{n-1}{-1} = 0$ if $d = 0$.
(3) *As a vector space, $\bigwedge^d V = V_{d,i}^{+} \oplus V_{d,i}^{-}$. In particular, the only possible eigenvalues of s_i on $\bigwedge^d V$ are 1 and λ_i.*

Lemma 3.2 [6, Proposition 3.5] *Suppose the reflection vectors $\alpha_1, \ldots, \alpha_m$ ($m \leq k$) are linearly independent. We extend these vectors to a basis of V, say,*

$$\{\alpha_1, \ldots, \alpha_m, v_{m+1}, \ldots, v_n\}.$$

(1) *If $0 \leq d < m$, then $\bigcap_{1 \leq i \leq m} V_{d,i}^{-} = 0$.*
(2) *If $m \leq d \leq n$, then $\bigcap_{1 \leq i \leq m} V_{d,i}^{-}$ has a basis*

$$\{\alpha_1 \wedge \cdots \wedge \alpha_m \wedge v_{i_{m+1}} \wedge \cdots \wedge v_{i_d} \mid m+1 \leq i_{m+1} < \cdots < i_d \leq n\}.$$

In particular, if $d = m$, then $\bigcap_{1 \leq i \leq m} V_{d,i}^{-}$ is one-dimensional with a basis vector $\alpha_1 \wedge \cdots \wedge \alpha_m$.

Lemma 3.3 *Suppose $m \leq d$, $m \leq k-1$, and the reflection vectors $\alpha_1, \ldots, \alpha_m$ are linearly independent. Suppose α_{m+1} is a linear combination of $\alpha_1, \ldots, \alpha_m$. Then $\bigcap_{1 \leq i \leq m+1} V_{d,i}^{-} = \bigcap_{1 \leq i \leq m} V_{d,i}^{-} \neq 0$ (that is, $s_{m+1} \cdot v = \lambda_{m+1} v$ for any $v \in \bigcap_{1 \leq i \leq m} V_{d,i}^{-}$).*

Proof The fact that $\bigcap_{1 \leq i \leq m} V_{d,i}^{-} \neq 0$ follows from Lemma 3.2. Moreover, the subspace $\bigcap_{1 \leq i \leq m} V_{d,i}^{-}$ admits a basis of the form

$$\{\alpha_1 \wedge \cdots \wedge \alpha_m \wedge v_{i_{m+1}} \wedge \cdots \wedge v_{i_d} \mid m+1 \leq i_{m+1} < \cdots < i_d \leq n\} \tag{3.1}$$

where $\{\alpha_1, \ldots, \alpha_m, v_{m+1}, \ldots, v_n\}$ is a basis of V.

Suppose $\alpha_{m+1} = c_1\alpha_1 + \cdots + c_m\alpha_m$, $c_i \in \mathbb{F}$. Without loss of generality, we may assume further $c_1 \neq 0$. Then for any basis vector in (3.1) we have

$$\begin{aligned} &\alpha_1 \wedge \cdots \wedge \alpha_m \wedge v_{i_{m+1}} \wedge \cdots \wedge v_{i_d} \\ &\quad = c_1^{-1}(c_1\alpha_1) \wedge \alpha_2 \wedge \cdots \wedge \alpha_m \wedge v_{i_{m+1}} \wedge \cdots \wedge v_{i_d} \end{aligned}$$

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$$= c_1^{-1}(c_1\alpha_1 + \cdots + c_m\alpha_m) \wedge \alpha_2 \wedge \cdots \wedge \alpha_m \wedge v_{i_{m+1}} \wedge \cdots \wedge v_{i_d}$$
$$= c_1^{-1}\alpha_{m+1} \wedge \alpha_2 \wedge \cdots \wedge \alpha_m \wedge v_{i_{m+1}} \wedge \cdots \wedge v_{i_d}.$$

Note that this is a nonzero vector, and that $\{\alpha_{m+1}, \alpha_2, \ldots, \alpha_m, v_{m+1}, \ldots, v_n\}$ is also a basis of V. By Lemma 3.2 again, we have

$$\alpha_{m+1} \wedge \alpha_2 \wedge \cdots \wedge \alpha_m \wedge v_{i_{m+1}} \wedge \cdots \wedge v_{i_d} \in V^-_{d,m+1}.$$

Therefore, $\bigcap_{1\le i\le m} V^-_{d,i} \subseteq V^-_{d,m+1}$, and thus $\bigcap_{1\le i\le m+1} V^-_{d,i} = \bigcap_{1\le i\le m} V^-_{d,i}$. ∎

Lemma 3.4 [6, Proposition 3.6] *If $0 \le d, d' \le n$ are integers and $\bigwedge^d V \simeq \bigwedge^{d'} V$ as W-modules, then $d = d'$.*

Remark 3.5 Lemma 3.4 holds for any representation on which some element $s \in W$ acts by a reflection, not necessary a reflection representation. See [6, Proposition 3.6] for details.

Lemma 3.6 [6, Corollary 3.8] *If the representation (V, ρ) is irreducible, then the W-module $\bigwedge^d V$ is semisimple for any $d = 0, 1, \ldots, n$.*

Remark 3.7 Recall that $\operatorname{char}\mathbb{F}$ is assumed to be 0. This is used in the proof of Lemma 3.6. See [6, Lemma 3.7 and Corollary 3.8] for details.

4 Proof of Theorem 1.2

In this section we give the proof of Theorem 1.2.

Recall that W is a group endowed with a set of generators $S = \{s_1, \ldots, s_k\}$, and (V, ρ) is an n-dimensional irreducible reflection representation of (W, S) over a field $\mathbb{F}$ of characteristic 0. We denote by α_i the chosen reflection vector of s_i as before, and by λ_i ($\neq 1$) the corresponding eigenvalue, for each $i \in [k]$.

By Lemma 3.4, the W-modules $\{\bigwedge^d V \mid 0 \le d \le n\}$ are pairwise non-isomorphic. Therefore, to prove Theorem 1.2, it suffices to show that $\bigwedge^d V$ is a simple W-module for each d. But we have seen in Lemma 3.6 that $\bigwedge^d V$ is semisimple, so the problem reduces to proving

(4.1) any endomorphism of $\bigwedge^d V$ is a scalar multiplication.

Recall in Definition 2.6 that a digraph G_I is associated with the reflection representation (V, ρ) and an arbitrary subset $I \subseteq [k]$. We have the following lemma.

Lemma 4.1 *There exists a subset $I \subseteq [k]$ such that*

(1) *the digraph G_I is weakly connected, and*
(2) *$\{\alpha_i \mid i \in I\}$ is a basis of V.*

Proof Suppose we have found a subset $J \subseteq [k]$ such that

(a) the digraph G_J is weakly connected, and
(b) $\{\alpha_i \mid i \in J\}$ is linearly independent.

For example, any singleton $\{j\} \subseteq [k]$ is such a subset.

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If $|J| = n$ (the dimension of V), then we are done. Otherwise, suppose $|J| < n$. Let $V_J := \bigoplus_{i\in J} \mathbb{F}\alpha_i$, which is a proper subspace of V. Since V is a simple W-module, there exists $j \in J$ and $i_0 \in [k]$ such that $s_{i_0} \cdot \alpha_j \notin V_J$. By Lemma 2.2, $s_{i_0} \cdot \alpha_j$ is of the form

$$s_{i_0} \cdot \alpha_j = \alpha_j + x\alpha_{i_0}$$

for some $x \in \mathbb{F}$. Then we must have $x \neq 0$ and $i_0 \notin J$, otherwise $s_{i_0} \cdot \alpha_j$ would belong to V_J. Now let $J' = J \sqcup \{i_0\}$. Then the associated digraph $G_{J'}$ is also weakly connected since we have an arrow $j \to i_0$. Moreover, the set of vectors $\{\alpha_i \mid i \in J'\}$ is linearly independent since $\alpha_{i_0} \notin \bigoplus_{i\in J} \mathbb{F}\alpha_i$. Therefore, the subset J' satisfies the conditions (a) and (b). Moreover, we have $|J'| = |J| + 1$. By induction on cardinality, there exists a subset $I \subseteq [k]$ satisfying (1) and (2). ∎

The following corollary of Lemma 4.1 will be used in Section 5.

Corollary 4.2 (See also [6, Claim 5.2]) *Suppose (V, ρ) is an n-dimensional irreducible reflection representation of (W, S) with reflection vectors $\{\alpha_i \mid i \in [k]\}$. Then the space V is spanned by $\{\alpha_i \mid i \in [k]\}$, that is, $V = \sum_{i\in[k]} \mathbb{F}\alpha_i$. In particular, $n \leq k$.*

Let I be obtained as in Lemma 4.1. Without loss of generality, we may assume $I = [n]$, the first n indices of $[k]$ (note that we have $n \leq k$ by Lemma 4.1). The vectors $\{\alpha_i \mid i \in I\}$ form a basis of V. For each fixed d with $0 \leq d \leq n$, the set of vectors

$$\{\alpha_{i_1} \wedge \cdots \wedge \alpha_{i_d} \mid 1 \leq i_1 < \cdots < i_d \leq n\}$$

is a basis of $\bigwedge^d V$.

For any set of distinct indices $1 \leq i_1, \ldots, i_d \leq n$, by Lemma 3.2, the intersection $\bigcap_{1\leq j\leq d} V^-_{d,i_j}$ of the d eigen-subspaces is one-dimensional,

$$\bigcap_{1\leq j\leq d} V^-_{d,i_j} = \mathbb{F}\alpha_{i_1} \wedge \cdots \wedge \alpha_{i_d}.$$

Suppose now $\varphi \in \mathrm{End}_W(\bigwedge^d V)$ is an endomorphism. Then φ preserves the subspace $\bigcap_{1\leq j\leq d} V^-_{d,i_j}$. Therefore,

$$\varphi(\alpha_{i_1} \wedge \cdots \wedge \alpha_{i_d}) = \gamma_{i_1,\ldots,i_d}\alpha_{i_1} \wedge \cdots \wedge \alpha_{i_d} \text{ for some } \gamma_{i_1,\ldots,i_d} \in \mathbb{F}.$$

Notice that $\alpha_{i_{\sigma(1)}} \wedge \cdots \wedge \alpha_{i_{\sigma(d)}} = \mathrm{sign}(\sigma)\alpha_{i_1} \wedge \cdots \wedge \alpha_{i_d}$ for any permutation $\sigma \in \mathfrak{S}_d$, and hence that $\gamma_{i_1,\ldots,i_d}$ depends only on the set $\{i_1, \ldots, i_d\}$, not on the order of the indices. To prove the statement (4.1), it suffices to show that the coefficients $\gamma_{i_1,\ldots,i_d}$ are independent of the choice of the indices $\{i_1, \ldots, i_d\}$. The following result is essentially the same as [6, Claim 5.5].

Lemma 4.3 *Let $J = \{i_1, \ldots, i_d\}$, $J' = \{j_1, \ldots, j_d\}$ be two subsets of I, both consisting of d elements. Suppose J' can be obtained from J by a move (see Definition 2.4) in the digraph G_I. Then $\gamma_{i_1,\ldots,i_d} = \gamma_{j_1,\ldots,j_d}$.*

Proof Without loss of generality, we may assume that $d \leq n-1$, $J = \{1, \ldots, d\}$, $J' = \{1, 2, \ldots, d-1, d+1\}$, and $d \to d+1$ is an arrow in G_I. Then $s_{d+1} \cdot \alpha_d \neq \alpha_d$.

For $i = 1, \ldots, d$, by Lemma 2.2 we assume that

$$s_{d+1} \cdot \alpha_i = \alpha_i + c_i\alpha_{d+1}, \quad c_i \in \mathbb{F}.$$

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Then $c_d \neq 0$. We have

$$\begin{aligned} & s_{d+1} \cdot (\alpha_1 \wedge \cdots \wedge \alpha_d) \\ & \quad = (\alpha_1 + c_1\alpha_{d+1}) \wedge \cdots \wedge (\alpha_d + c_d\alpha_{d+1}) \\ & \quad = \alpha_1 \wedge \cdots \wedge \alpha_d + \sum_{1 \le i \le n} (-1)^{d-i} c_i \cdot \alpha_1 \wedge \cdots \wedge \widehat{\alpha_i} \wedge \cdots \wedge \alpha_d \wedge \alpha_{d+1}. \end{aligned}$$

Hence,

$$\begin{aligned} & \varphi\big(s_{d+1} \cdot (\alpha_1 \wedge \cdots \wedge \alpha_d)\big) \\ & \quad = \varphi\big(\alpha_1 \wedge \cdots \wedge \alpha_d + \sum_{i=1}^{d} (-1)^{d-i} c_i \cdot \alpha_1 \wedge \cdots \wedge \widehat{\alpha_i} \wedge \cdots \wedge \alpha_{d+1}\big) \\ & \quad = \gamma_{1,\ldots,d} \cdot \alpha_1 \wedge \cdots \wedge \alpha_d + \sum_{i=1}^{d} (-1)^{d-i} c_i \gamma_{1,\ldots,\widehat{i},\ldots,d+1} \cdot \alpha_1 \wedge \cdots \wedge \widehat{\alpha_i} \wedge \cdots \wedge \alpha_{d+1}. \end{aligned}$$

This also equals

$$\begin{aligned} & s_{d+1} \cdot \varphi(\alpha_1 \wedge \cdots \wedge \alpha_d) \\ & \quad = \gamma_{1,\ldots,d} s_{d+1} \cdot (\alpha_1 \wedge \cdots \wedge \alpha_d) \\ & \quad = \gamma_{1,\ldots,d} \cdot \alpha_1 \wedge \cdots \wedge \alpha_d + \sum_{i=1}^{d} (-1)^{d-i} c_i \gamma_{1,\ldots,d} \cdot \alpha_1 \wedge \cdots \wedge \widehat{\alpha_i} \wedge \cdots \wedge \alpha_{d+1}. \end{aligned}$$

Note that $c_d \neq 0$, and that the vectors involved in the summations above are linearly independent. Thus, we have the desired equality $\gamma_{1,\ldots,d} = \gamma_{1,\ldots,d-1,d+1}$ by comparing the coefficients of $\alpha_1 \wedge \cdots \wedge \alpha_{d-1} \wedge \alpha_{d+1}$. ∎

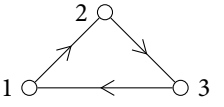
In general, for two subsets J and J' of I, if both of them consist of d elements, then, since G_I is weakly connected, one can be obtained from the other by finite steps of moves by Lemma 2.5. Therefore, the coefficients $\gamma_{i_1,\ldots,i_d}$ are constant among all choices of the distinct indices $1 \le i_1, \ldots, i_d \le n$.

The proof of Theorem 1.2 is completed.

Remark 4.4 We cannot expect the digraph G_I in Lemma 4.1 to be strongly connected. For example, let $S = \{s_1, s_2, s_3\}$ consist of 3 elements, and $V = \mathbb{F}\alpha_1 \oplus \mathbb{F}\alpha_2$ be a two-dimensional vector space. Define three reflections on V by

$$\begin{aligned} & s_1 \cdot \alpha_1 = -\alpha_1, && s_1 \cdot \alpha_2 = \alpha_2, \\ & s_2 \cdot \alpha_1 = \alpha_1 + 2\alpha_2, && s_2 \cdot \alpha_2 = -\alpha_2, \\ & s_3 \cdot \alpha_1 = \alpha_1, && s_3 \cdot \alpha_2 = -2\alpha_1 - \alpha_2. \end{aligned}$$

Then the corresponding reflection vectors are α_1, α_2, and $\alpha_3 := -\alpha_1 - \alpha_2$, respectively. The associated digraph $G_{[3]}$ is as follows:



In this digraph, each sub-digraph spanned by two vertices is not strongly connected.

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5 Proof of Theorem 1.3

This section is devoted to proving Theorem 1.3.

Recall that W is a group endowed with a set of generators $S = \{s_1, \dots, s_k\}$, and (V_ι, ρ_ι), $\iota = 1, 2$, are two irreducible reflection representations. We use the following notations.

$$\begin{array}{ll} n_\iota (\iota = 1,2): & \dim V_\iota \\ \alpha_i (i \in [k]): & \text{the chosen reflection vector of } s_i \text{ in} V_1 \\ \lambda_i (\neq 1): & \text{the corresponding eigenvalue, } s_i \cdot \alpha_i = \lambda_i \alpha_i \\ \beta_i (i \in [k]): & \text{the chosen reflection vector of } s_i \text{ in } V_2 \\ \mu_i (\neq 1): & \text{the corresponding eigenvalue, } s_i \cdot \beta_i = \mu_i \beta_i. \end{array}$$

Suppose

$$\psi : \bigwedge^{d_1} V_1 \xrightarrow{\sim} \bigwedge^{d_2} V_2$$

is an isomorphism of W-modules, where d_1, d_2 are certain integers satisfying $1 \le d_\iota \le n_\iota - 1$ $(\iota = 1, 2)$. As in Section 3, for each $i \in [k]$ we denote by

$$V^+_{1,d_1,i} := \left\{ v \in \bigwedge^{d_1} V_1 \,\middle|\, s_i \cdot v = v \right\}, \quad V^-_{1,d_1,i} := \left\{ v \in \bigwedge^{d_1} V_1 \,\middle|\, s_i \cdot v = \lambda_i v \right\},$$
$$V^+_{2,d_2,i} := \left\{ v \in \bigwedge^{d_2} V_2 \,\middle|\, s_i \cdot v = v \right\}, \quad V^-_{2,d_2,i} := \left\{ v \in \bigwedge^{d_2} V_2 \,\middle|\, s_i \cdot v = \mu_i v \right\}$$

the eigen-subspaces of s_i.

Before giving the rigorous proof, let us talk a little more about Theorem 1.3 informally. A priori, an isomorphism $f : V_1 \to V_2$ of reflection representations gives an isomorphism $\bigwedge^d f : \bigwedge^d V_1 \to \bigwedge^d V_2$ via

$$\left(\bigwedge^d f\right)(v_1 \wedge \cdots \wedge v_d) = f(v_1) \wedge \cdots \wedge f(v_d), \quad \forall v_1, \dots, v_d \in V_1.$$

It is not difficult to see that $f(\alpha_i) = z_i \beta_i$ for some $z_i \in \mathbb{F}^\times$. Then we have

$$\left(\bigwedge^d f\right)(\alpha_{i_1} \wedge \cdots \wedge \alpha_{i_d}) = z_{i_1} \cdots z_{i_d} \beta_{i_1} \wedge \cdots \wedge \beta_{i_d} \text{ for any } i_1, \dots, i_d \in [k].$$

Conversely suppose in Theorem 1.3 that $d = d_1 = d_2$, and that the isomorphism $\psi : \bigwedge^d V_1 \to \bigwedge^d V_2$ is given by an isomorphism $f : V_1 \to V_2$. Suppose further that we are able to show for any indices $i_1, \dots, i_d \in [k]$ that

$$\psi(\alpha_{i_1} \wedge \cdots \wedge \alpha_{i_d}) = \zeta_{i_1,\dots,i_d} \beta_{i_1} \wedge \cdots \wedge \beta_{i_d} \text{ for some } \zeta_{i_1,\dots,i_d} \in \mathbb{F}^\times.$$

(This is indeed the case, see Section 5.3.) Since the map f is of the form $f(\alpha_i) = z_i \beta_i$, we have $\zeta_{i_1,\dots,i_d} = z_{i_1} \cdots z_{i_d}$, and

$$\frac{z_i}{z_j} = \frac{\zeta_{i,i_2,\dots,i_d}}{\zeta_{j,i_2,\dots,i_d}} \text{ for any suitable indices } i, j, i_2, \dots, i_d \in [k].$$

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This indicates that

$$\text{the ratio } \frac{\zeta_{i,i_2,\dots,i_d}}{\zeta_{j,i_2,\dots,i_d}} \text{ only depends on } i \text{ and } j, \quad \text{but independent of the indices } i_2,\dots,i_d. \tag{5.1}$$

We would be close to find the desired isomorphism f if we can prove (5.1) (this is essentially Lemma 5.9).

We divide the proof of Theorem 1.3 into the following five steps, presented in Sections 5.1 to 5.5 respectively:

Step 1. Show that $d_1 = d_2$, $n_1 = n_2$, and $\lambda_i = \mu_i$ for each $i \in [k]$.
Step 2. Show that the linear independence of a set of reflection vectors in V_1 is equivalent to that in V_2.
Step 3. Show that the two reflection representations have the same associated digraphs.
Step 4. Define a linear isomorphism $f : V_1 \to V_2$ of vector spaces.
Step 5. Show that f is an isomorphism of W-modules.

5.1 A preliminary numerical result

Proposition 5.1 *$d_1 = d_2$, $n_1 = n_2$. Moreover, $\lambda_i = \mu_i$ for each $i \in [k]$.*

Proof Note that the element $s_1 \in S$ acts by reflections on both V_1 and V_2. Since $\bigwedge^{d_1} V_1 \simeq \bigwedge^{d_2} V_2$ as W-modules, we have

$$\dim V^+_{1,d_1,1} = \dim V^+_{2,d_2,1}, \quad \dim V^-_{1,d_1,1} = \dim V^-_{2,d_2,1}.$$

Then we have by Lemma 3.1(1)(2)

$$\binom{n_1-1}{d_1} = \binom{n_2-1}{d_2}, \quad \binom{n_1-1}{d_1-1} = \binom{n_2-1}{d_2-1}. \tag{5.2}$$

Notice that $1 \le d_\iota \le n_\iota - 1$ for $\iota = 1, 2$. By Lemma 2.9, Equations (5.2) imply $d_1 = d_2$, $n_1 = n_2$.

By Lemma 3.1(3), we have $\bigwedge^{d_1} V_1 = V^+_{1,d_1,i} \oplus V^-_{1,d_1,i}$ for each $i \in [k]$, and the only possible eigenvalues of s_i on $\bigwedge^{d_1} V_1$ are 1 and λ_i. But $\dim V^-_{1,d_1,i} = \binom{n_1-1}{d_1-1} \neq 0$ since $1 \le d_1 \le n_1 - 1$. Thus λ_i is indeed an eigenvalue. Similarly, the only eigenvalues of s_i on $\bigwedge^{d_2} V_2$ are 1 and μ_i. Thus we must have $\lambda_i = \mu_i$. ∎

Remark 5.2 From the proof of Proposition 5.1, we see that the results hold for two representations (V_1, ρ_1), (V_2, ρ_2) on which $\rho_1(s)$ and $\rho_2(s)$ are both reflections for some element $s \in W$, not necessary to be reflection representations.

In view of Proposition 5.1, we denote

$$d := d_1 = d_2 \quad \text{and} \quad n := \dim V_1 = \dim V_2$$

from now on. Note that we have $1 \le d \le n - 1$ by assumption. However, Theorem 1.3 for the case $d = 1$ is trivial. Thus we may assume $2 \le d \le n - 1$.

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5.2 Preliminary results on linear independence of reflection vectors

This subsection aims to prove Propositions 5.5 and 5.6, which transfer linear independence property of reflection vectors in V_1 to those with the same indices in V_2. Recall that $k = |S|$ is the number of chosen generators of the group W, and α_i, β_i $(i \in [k])$ are the reflection vectors of the generator s_i in the space V_1 and V_2, respectively. By Corollary 4.2, the n-dimensional vector space V_1 is spanned by $\alpha_1, \dots, \alpha_k$, and we have $n \le k$.

Lemma 5.3 *Suppose $\{\alpha_1, \dots, \alpha_n\}$ is a basis of V_1. Then we have decompositions of vector spaces*

$$\bigwedge^d V_1 = \bigoplus_{1 \le i_1 < \cdots < i_d \le n} \Big(\bigcap_{1 \le j \le d} V^-_{1,d,i_j} \Big)$$

and

$$\bigwedge^d V_2 = \bigoplus_{1 \le i_1 < \cdots < i_d \le n} \Big(\bigcap_{1 \le j \le d} V^-_{2,d,i_j} \Big).$$

Proof The vector space $\bigwedge^d V_1$ has a basis

$$\{\alpha_{i_1} \wedge \cdots \wedge \alpha_{i_d} \mid 1 \le i_1 < \cdots < i_d \le n\}.$$

Note that by Lemma 3.2 the vector $\alpha_{i_1} \wedge \cdots \wedge \alpha_{i_d}$ is a basis vector of the one-dimensional space $\bigcap_{1 \le j \le d} V^-_{1,d,i_j}$, which is the intersection of eigen-subspaces of s_{i_j}'s in $\bigwedge^d V_1$. Therefore, we have a decomposition of vector space

$$\begin{aligned} \bigwedge^d V_1 &= \bigoplus_{1 \le i_1 < \cdots < i_d \le n} \mathbb{F}\langle \alpha_{i_1} \wedge \cdots \wedge \alpha_{i_d} \rangle \\ &= \bigoplus_{1 \le i_1 < \cdots < i_d \le n} \Big(\bigcap_{1 \le j \le d} V^-_{1,d,i_j} \Big). \end{aligned}$$

Since $\psi : \bigwedge^d V_1 \xrightarrow{\sim} \bigwedge^d V_2$ is an isomorphism of W-modules, we have

$$\psi\Big(\bigcap_{1 \le j \le d} V^-_{1,d,i_j} \Big) = \bigcap_{1 \le j \le d} V^-_{2,d,i_j}$$

for any set of indices $1 \le i_1 < \cdots < i_d \le n$, and hence

$$\bigwedge^d V_2 = \bigoplus_{1 \le i_1 < \cdots < i_d \le n} \Big(\bigcap_{1 \le j \le d} V^-_{2,d,i_j} \Big)$$

as claimed. ∎

Recall that the number d satisfies $d + 1 \le n$. We have the following lemma which is a "weak version" of Proposition 5.6.

Lemma 5.4 *Suppose $1 \le j_1, \dots, j_{d+1} \le k$. If $\alpha_{j_1}, \dots, \alpha_{j_{d+1}}$ are linearly independent, then so are $\beta_{j_1}, \dots, \beta_{j_{d+1}}$.*

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Proof Suppose otherwise that $\beta_{j_1}, \dots, \beta_{j_{d+1}}$ are linearly dependent and the subset $\{\beta_{j_1}, \dots, \beta_{j_h}\}$ $(h \le d)$ is a maximal linearly independent set. Then there exists a nonzero vector v in $\bigwedge^d V_2$ of the form $v = \beta_{j_1} \wedge \cdots \wedge \beta_{j_h} \wedge v_{h+1} \wedge \cdots \wedge v_d$. By Lemma 3.2, we have

$$v \in \bigcap_{1 \le i \le h} V_{2,d,j_i}^-.$$

Note that for any index i such that $h+1 \le i \le d+1$, β_{j_i} is a linear combination of $\beta_{j_1}, \dots, \beta_{j_h}$. Then by Lemma 3.3, we have

$$v \in \bigcap_{1 \le i \le h} V_{2,d,j_i}^- = \bigcap_{1 \le i \le d+1} V_{2,d,j_i}^-.$$

As in the proof of Lemma 5.3, since $\psi : \bigwedge^d V_1 \xrightarrow{\sim} \bigwedge^d V_2$ is an isomorphism of W-modules, we have

$$\psi^{-1}(v) \in \bigcap_{1 \le i \le d+1} V_{1,d,j_i}^-.$$

Note that $\psi^{-1}(v)$ is a nonzero vector. However, the intersection $\bigcap_{1 \le i \le d+1} V_{1,d,j_i}^-$ is zero by Lemma 3.2. This is absurd. ∎

Proposition 5.5 *Suppose $1 \le j_1, \dots, j_n \le k$. If $\{\alpha_{j_1}, \dots, \alpha_{j_n}\}$ is a basis of V_1, then $\{\beta_{j_1}, \dots, \beta_{j_n}\}$ is a basis of V_2, and vice versa.*

Proof Without loss of generality, we may assume $j_1 = 1$, $j_2 = 2$, …, $j_n = n$. Then $\{\alpha_1, \dots, \alpha_n\}$ is a basis of V_1. Suppose the space $U := \mathbb{F}\langle \beta_1, \dots, \beta_n \rangle$ spanned by $\beta_1, \dots, \beta_n$ is a proper subspace of V_2, and $m := \dim U < n$. We may assume further that $\{\beta_1, \dots, \beta_m\}$ is a basis of U.

For any indices $1 \le i_1 < \cdots < i_d \le n$, there exists an index $i_{d+1} \in [n] \backslash \{i_1, \dots, i_d\}$ since $d \le n-1$. Note that the vectors $\alpha_{i_1}, \dots, \alpha_{i_d}, \alpha_{i_{d+1}}$ are linearly independent. By Lemma 5.4, $\beta_{i_1}, \dots, \beta_{i_d}, \beta_{i_{d+1}} \in U$ are linearly independent as well. In particular, $\beta_{i_1} \wedge \cdots \wedge \beta_{i_d} \ne 0$ and we have by Lemma 3.2 that

$$\bigcap_{1 \le j \le d} V_{2,d,i_j}^- = \mathbb{F}\langle \beta_{i_1} \wedge \cdots \wedge \beta_{i_d} \rangle \subseteq \bigwedge^d U.$$

But then by Lemma 5.3, we have

$$\bigwedge^d V_2 = \bigoplus_{1 \le i_1 < \cdots < i_d \le n} \left(\bigcap_{1 \le j \le d} V_{2,d,i_j}^- \right) \subseteq \bigwedge^d U \subsetneq \bigwedge^d V_2$$

which is a contradiction. Thus, we must have $U = V_2$, $m = n$, and $\{\beta_1, \dots, \beta_n\}$ is a basis of V_2. ∎

As a corollary, we have the following proposition.

Proposition 5.6 *Suppose $h \le n$ and $1 \le i_1, \dots, i_h \le k$. If $\alpha_{i_1}, \dots, \alpha_{i_h}$ are linearly independent, then so are $\beta_{i_1}, \dots, \beta_{i_h}$. In particular, if α_i and α_j are not proportional for some $1 \le i \ne j \le k$, then so are β_i, β_j.*

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Proof Recall Corollary 4.2 that V_1 is spanned by $\alpha_1, \dots, \alpha_k$. Thus there exist reflection vectors $\alpha_{i_{h+1}}, \alpha_{i_{h+2}}, \dots, \alpha_{i_n}$ such that $\alpha_{i_1}, \dots, \alpha_{i_h}, \alpha_{i_{h+1}}, \dots, \alpha_{i_n}$ form a basis of V_1. Then use Proposition 5.5. ∎

5.3 Coincidence of the associated digraphs

Recall in Definition 2.6 that a digraph is associated with any subset $I \subseteq [k]$ and any reflection representation. For $\iota = 1, 2$, we denote temporarily by G_ι the associated graph to the full set $[k]$ and the representation (V_ι, ρ_ι). In this subsection, we will prove that $G_1 = G_2$.

For two distinct indices $i, j \in [k]$, we set

$$s_i \cdot \alpha_j = \alpha_j + x_{ji}\alpha_i, \quad x_{ji} \in \mathbb{F},$$
$$s_i \cdot \beta_j = \beta_j + y_{ji}\beta_i, \quad y_{ji} \in \mathbb{F}.$$

Then $j \to i$ is an arrow in G_1, G_2 if and only if $x_{ji}, y_{ji} \neq 0$, respectively.

For distinct indices $1 \le i_1, \dots, i_d \le k$, if $\alpha_{i_1} \wedge \cdots \wedge \alpha_{i_d} \neq 0$, that is, if $\alpha_{i_1}, \dots, \alpha_{i_d}$ are linearly independent, then by Proposition 5.6, the vectors $\beta_{i_1}, \dots, \beta_{i_d}$ are linearly independent as well. Moreover, we have by Lemma 3.2

$$\bigcap_{1 \le j \le d} V_{1,d,i_j}^- = \mathbb{F}\langle \alpha_{i_1} \wedge \cdots \wedge \alpha_{i_d} \rangle, \quad \text{and} \quad \bigcap_{1 \le j \le d} V_{2,d,i_j}^- = \mathbb{F}\langle \beta_{i_1} \wedge \cdots \wedge \beta_{i_d} \rangle.$$

Therefore, since $\psi : \bigwedge^d V_1 \xrightarrow{\sim} \bigwedge^d V_2$ is an isomorphism of W-modules, it holds

$$\psi(\alpha_{i_1} \wedge \cdots \wedge \alpha_{i_d}) = \zeta_{i_1,\dots,i_d}\beta_{i_1} \wedge \cdots \wedge \beta_{i_d}, \quad \text{for some } \zeta_{i_1,\dots,i_d} \in \mathbb{F}^\times.$$

By convention, we define $\zeta_{i_1,\dots,i_d} := 0$ if $\alpha_{i_1} \wedge \cdots \wedge \alpha_{i_d} = 0$.

Remark 5.7 Note that the coefficients $\zeta_{i_1,\dots,i_d}$ are independent of the order of $i_1, \dots, i_d$, that is, $\zeta_{i_1,\dots,i_d} = \zeta_{i_{\sigma(1)},\dots,i_{\sigma(d)}}$ for any permutation $\sigma \in \mathfrak{S}_d$ (as $\gamma_{i_1,\dots,i_d}$ in Section 4).

Lemma 5.8 *Suppose $i, j \in [k]$ and $i \neq j$. There exist distinct indices $i_2, \dots, i_d \in [k]$ such that*

(1) *$\alpha_i, \alpha_j, \alpha_{i_2}, \dots, \alpha_{i_d}$ are linearly independent if α_i, α_j are not proportional;*
(2) *$\alpha_i, \alpha_{i_2}, \dots, \alpha_{i_d}$ are linearly independent if α_i, α_j are proportional (thus in this case the vectors $\alpha_j, \alpha_{i_2}, \dots, \alpha_{i_d}$ are linearly independent as well).*

Proof The existence of the required indices is ensured by the facts that $d + 1 \le n$ and that V_1 is spanned by all the reflection vectors (Corollary 4.2). ∎

The following lemma is the key in this subsection.

Lemma 5.9 *Suppose $i, j \in [k]$ and $i \neq j$. Let $i_2, \dots, i_d \in [k]$ be any indices satisfying the conditions (1)(2) in Lemma 5.8. Then we have*

$$\zeta_{i,i_2,\dots,i_d} y_{ij} = \zeta_{j,i_2,\dots,i_d} x_{ij}. \tag{5.3}$$

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Proof We consider

$$\begin{aligned}
&s_j \cdot (\psi(\alpha_i \wedge \alpha_{i_2} \wedge \cdots \wedge \alpha_{i_d})) \\
&\quad = \zeta_{i,i_2,\ldots,i_d} s_j \cdot (\beta_i \wedge \beta_{i_2} \wedge \cdots \wedge \beta_{i_d}) \\
&\quad = \zeta_{i,i_2,\ldots,i_d} (\beta_i + y_{ij}\beta_j) \wedge (\beta_{i_2} + y_{i_2 j}\beta_j) \wedge \cdots \wedge (\beta_{i_d} + y_{i_d j}\beta_j) \\
&\quad = \zeta_{i,i_2,\ldots,i_d} (\beta_i \wedge \beta_{i_2} \wedge \cdots \wedge \beta_{i_d} + y_{ij}\beta_j \wedge \beta_{i_2} \wedge \cdots \wedge \beta_{i_d}) \\
&\qquad + \sum_{2 \le l \le d} (-1)^{d-l} \zeta_{i,i_2,\ldots,i_d} y_{i_l j} \beta_i \wedge \beta_{i_2} \wedge \cdots \wedge \widehat{\beta}_{i_l} \wedge \cdots \wedge \beta_{i_d} \wedge \beta_j
\end{aligned} \tag{5.4}$$

which also equals

$$\begin{aligned}
&\psi(s_j \cdot (\alpha_i \wedge \alpha_{i_2} \wedge \cdots \wedge \alpha_{i_d})) \\
&\quad = \psi((\alpha_i + x_{ij}\alpha_j) \wedge (\alpha_{i_2} + x_{i_2 j}\alpha_j) \wedge \cdots \wedge (\alpha_{i_d} + x_{i_d j}\alpha_j)) \\
&\quad = \psi\Big(\alpha_i \wedge \alpha_{i_2} \wedge \cdots \wedge \alpha_{i_d} + x_{ij}\alpha_j \wedge \alpha_{i_2} \wedge \cdots \wedge \alpha_{i_d} \\
&\qquad + \sum_{2 \le l \le d} (-1)^{d-l} x_{i_l j} \alpha_i \wedge \alpha_{i_2} \wedge \cdots \wedge \widehat{\alpha}_{i_l} \wedge \cdots \wedge \alpha_{i_d} \wedge \alpha_j\Big) \\
&\quad = \zeta_{i,i_2,\ldots,i_d} \beta_i \wedge \beta_{i_2} \wedge \cdots \wedge \beta_{i_d} + \zeta_{j,i_2,\ldots,i_d} x_{ij} \beta_j \wedge \beta_{i_2} \wedge \cdots \wedge \beta_{i_d} \\
&\qquad + \sum_{2 \le l \le d} (-1)^{d-l} \zeta_{i,i_2,\ldots,\widehat{i_l},\ldots,i_d,j} x_{i_l j} \beta_i \wedge \beta_{i_2} \wedge \cdots \wedge \widehat{\beta}_{i_l} \wedge \cdots \wedge \beta_{i_d} \wedge \beta_j.
\end{aligned} \tag{5.5}$$

If α_i, α_j are not proportional, then $\alpha_i, \alpha_j, \alpha_{i_2}, \ldots, \alpha_{i_d}$ are linearly independent by our assumption, and then so are $\beta_i, \beta_j, \beta_{i_2}, \ldots, \beta_{i_d}$ by Proposition 5.6 (or Lemma 5.4). Therefore, the vectors occurring in (5.4) and (5.5) are nonzero and linearly independent. By comparing the coefficients of $\beta_j \wedge \beta_{i_2} \wedge \cdots \wedge \beta_{i_d}$ in (5.4) and (5.5) we see that the desired Equation (5.3) holds.

If α_i, α_j are proportional, then so are β_i, β_j by Proposition 5.6, and the summations in (5.4) and (5.5) vanish, that is,

$$\begin{aligned}
(5.4) &= \zeta_{i,i_2,\ldots,i_d} \beta_i \wedge \beta_{i_2} \wedge \cdots \wedge \beta_{i_d} + \zeta_{i,i_2,\ldots,i_d} y_{ij} \beta_j \wedge \beta_{i_2} \wedge \cdots \wedge \beta_{i_d}, \\
(5.5) &= \zeta_{i,i_2,\ldots,i_d} \beta_i \wedge \beta_{i_2} \wedge \cdots \wedge \beta_{i_d} + \zeta_{j,i_2,\ldots,i_d} x_{ij} \beta_j \wedge \beta_{i_2} \wedge \cdots \wedge \beta_{i_d}.
\end{aligned}$$

As pointed out in Lemma 5.8, the vectors $\alpha_j, \alpha_{i_2}, \ldots, \alpha_{i_d}$ are linearly independent, and so are $\beta_j, \beta_{i_2}, \ldots, \beta_{i_d}$ by Proposition 5.6. Therefore, we have Equation (5.3) again by comparing the two equations above. ∎

Note that in Equation (5.3) the coefficients $\zeta_{i,i_2,\ldots,i_d}$ and $\zeta_{j,i_2,\ldots,i_d}$ are nonzero. Therefore we have the following corollary.

Corollary 5.10 *Suppose $i, j \in [k]$ and $i \neq j$. Then x_{ij} and y_{ij} are equal or not equal to zero simultaneously, that is, either $x_{ij} = y_{ij} = 0$ or $x_{ij} y_{ij} \neq 0$.*

By the definition of the associated digraphs G_1 and G_2, Corollary 5.10 implies

Corollary 5.11 $G_1 = G_2$.

From now on, we recover the notation $G = G_{[k]}$ to indicate uniformly the digraphs G_1 and G_2, and G_I to be the sub-digraph spanned by a subset $I \subseteq [k]$ (see Definition 2.6).

We will also need the following corollary of Lemma 5.9.

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Corollary 5.12 *Let $i, j, i_2, \dots, i_d \in [k]$ be as in Lemma 5.9. Suppose $x_{ij} \neq 0$ and $y_{ji} \neq 0$ (equivalently, $y_{ij} \neq 0$ and $x_{ji} \neq 0$). Then we have*

$$\frac{y_{ij}}{x_{ij}} = \frac{\zeta_{j,i_2,\dots,i_d}}{\zeta_{i,i_2,\dots,i_d}} = \frac{x_{ji}}{y_{ji}}.$$

Proof The first desired equality is nothing but Equation (5.3). By swapping the indices i and j in Equation (5.3), we obtain the second equality. ∎

5.4 The linear isomorphism f from V_1 to V_2

Remember that our final goal is to find an isomorphism $f : V_1 \xrightarrow{\sim} V_2$ of W-modules. For this, let us introduce some notations.

By applying Lemma 4.1 to the reflection representation (V_1, ρ_1), we choose and fix a subset $I \subseteq [k]$ such that

(1) G_I is weakly connected, and
(2) $\{\alpha_i \mid i \in I\}$ is a basis of V_1.

Then by Proposition 5.5, $\{\beta_i \mid i \in I\}$ is a basis of V_2.

For two indices $i, j \in I$ such that $i \neq j$ and either $i \to j$ or $j \to i$ is an arrow in G_I, we define

$$z_{ij} := \begin{cases} \frac{y_{ij}}{x_{ij}}, & \text{if } i \to j \text{ is an arrow} \\ \frac{x_{ji}}{y_{ji}}, & \text{if } j \to i \text{ is an arrow.} \end{cases}$$

By Corollary 5.12, we have $\frac{y_{ij}}{x_{ij}} = \frac{x_{ji}}{y_{ji}}$ if both $i \to j$ and $j \to i$ are arrows. So the element $z_{ij} \in \mathbb{F}^\times$ is well defined. We have the following lemma.

Lemma 5.13 *Let $h \geq 1$ be an integer and*

$$i_0, \quad a_1, \quad i_1, \quad a_2, \quad i_2, \quad \dots \quad i_{h-1}, \quad a_h, \quad i_h$$

be an undirected walk in G_I. For any $p \in [h]$ and any distinct indices $j_2, \dots, j_d \in I\backslash\{i_0, i_p\}$ (if $i_0 = i_p$ then we regard $\{i_0, i_p\} = \{i_0\}$) we have

$$z_{i_0 i_1} \cdots z_{i_{p-1} i_p} = \frac{\zeta_{i_p, j_2, \dots, j_d}}{\zeta_{i_0, j_2, \dots, j_d}}. \tag{5.6}$$

Note that $|I\backslash\{i_0, i_p\}| \geq n - 2 \geq d - 1$. Therefore such indices $j_2, \dots, j_d$ exist. Note also that $\{\alpha_j \mid j \in I\}$ is a basis for V_1. Thus $\zeta_{i_0, j_2, \dots, j_d}$ and $\zeta_{i_p, j_2, \dots, j_d}$ are nonzero.

Proof We prove by induction on p. Suppose first that $p = 1$. Then the desired equality $z_{i_0 i_1} = \zeta_{i_1, j_2, \dots, j_d} / \zeta_{i_0, j_2, \dots, j_d}$ follows from Corollary 5.12.

Suppose now $p \geq 2$. The induction hypothesis reads

$$z_{i_0 i_1} \cdots z_{i_{p-2} i_{p-1}} = \frac{\zeta_{i_{p-1}, j_2, \dots, j_d}}{\zeta_{i_0, j_2, \dots, j_d}} \text{ for any distinct } j_2, \dots, j_d \in I\backslash\{i_0, i_{p-1}\}.$$

For distinct indices $j_2, \dots, j_d \in I\backslash\{i_0, i_p\}$ given arbitrarily, we have three cases.

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Case one: $i_{p-1} \notin \{j_2, \ldots, j_d\}$. By the same arguments as in the beginning case "$p = 1$", we have

$$z_{i_{p-1}i_p} = \frac{\zeta_{i_p, j_2, \ldots, j_d}}{\zeta_{i_{p-1}, j_2, \ldots, j_d}}$$

whenever $a_p = i_{p-1} \to i_p$ or $a_p = i_p \to i_{p-1}$. Therefore by induction hypothesis we have

$$z_{i_0 i_1} \cdots z_{i_{p-2} i_{p-1}} z_{i_{p-1} i_p} = \frac{\zeta_{i_{p-1}, j_2, \ldots, j_d}}{\zeta_{i_0, j_2, \ldots, j_d}} \cdot \frac{\zeta_{i_p, j_2, \ldots, j_d}}{\zeta_{i_{p-1}, j_2, \ldots, j_d}} = \frac{\zeta_{i_p, j_2, \ldots, j_d}}{\zeta_{i_0, j_2, \ldots, j_d}}$$

as claimed in Equation (5.6).

Case two: $i_{p-1} \in \{j_2, \ldots, j_d\}$ and $i_0 \neq i_p$. We may assume $i_{p-1} = j_2$. Then i_0, i_p, i_{p-1} $(= j_2), j_3, \ldots, j_d$ are distinct. We have

$$z_{i_0 i_1} \cdots z_{i_{p-2} i_{p-1}} = \frac{\zeta_{i_{p-1}, i_p, j_3, \ldots, j_d}}{\zeta_{i_0, i_p, j_3, \ldots, j_d}}$$

by applying induction hypothesis to the indices $i_p, j_3, \ldots, j_d$, and

$$z_{i_{p-1} i_p} = \frac{\zeta_{i_p, i_0, j_3, \ldots, j_d}}{\zeta_{i_{p-1}, i_0, j_3, \ldots, j_d}}$$

by the same arguments as in the beginning case "$p = 1$". Therefore,

$$z_{i_0 i_1} \cdots z_{i_{p-2} i_{p-1}} z_{i_{p-1} i_p} = \frac{\zeta_{i_{p-1}, i_p, j_3, \ldots, j_d}}{\zeta_{i_0, i_p, j_3, \ldots, j_d}} \cdot \frac{\zeta_{i_p, i_0, j_3, \ldots, j_d}}{\zeta_{i_{p-1}, i_0, j_3, \ldots, j_d}}. \tag{5.7}$$

Note that $\zeta_{i_0, i_p, j_3, \ldots, j_d} = \zeta_{i_p, i_0, j_3, \ldots, j_d}$ (see Remark 5.7). Therefore Equation (5.7) reduces to

$$z_{i_0 i_1} \cdots z_{i_{p-2} i_{p-1}} z_{i_{p-1} i_p} = \frac{\zeta_{i_{p-1}, i_p, j_3, \ldots, j_d}}{\zeta_{i_{p-1}, i_0, j_3, \ldots, j_d}} = \frac{\zeta_{i_p, j_2, j_3, \ldots, j_d}}{\zeta_{i_0, j_2, j_3, \ldots, j_d}}$$

which is what we want.

Case three: $i_{p-1} \in \{j_2, \ldots, j_d\}$ and $i_0 = i_p$. Then our goal Equation (5.6) becomes

$$z_{i_0 i_1} \cdots z_{i_{p-1} i_p} = 1. \tag{5.8}$$

We can still assume $i_{p-1} = j_2$. Note that in this case $i_0 (= i_p)$, $i_{p-1} (= j_2)$, $j_3, \ldots, j_d$ are distinct d indices. But $d \leq n - 1$, so there exists an extra index $j \in I \backslash \{i_0, j_2, j_3, \ldots, j_d\}$. Similar to the former cases, we have

$$z_{i_0 i_1} \cdots z_{i_{p-2} i_{p-1}} = \frac{\zeta_{i_{p-1}, j, j_3, \ldots, j_d}}{\zeta_{i_0, j, j_3, \ldots, j_d}}$$

and

$$z_{i_{p-1} i_p} = \frac{\zeta_{i_p, j, j_3, \ldots, j_d}}{\zeta_{i_{p-1}, j, j_3, \ldots, j_d}}.$$

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Therefore, we have

$$z_{i_0 i_1} \cdots z_{i_{p-2} i_{p-1}} z_{i_{p-1} i_p} = \frac{\zeta_{i_{p-1}, j, j_3, \ldots, j_d}}{\zeta_{i_0, j, j_3, \ldots, j_d}} \cdot \frac{\zeta_{i_p, j, j_3, \ldots, j_d}}{\zeta_{i_{p-1}, j, j_3, \ldots, j_d}} = \frac{\zeta_{i_p, j, j_3, \ldots, j_d}}{\zeta_{i_0, j, j_3, \ldots, j_d}} = 1$$

which is exactly Equation (5.8). ∎

Now we are ready to construct a linear map f from V_1 to V_2. Such a map is determined by vectors $\{f(\alpha_i) \in V_2 \mid i \in I\}$ since $\{\alpha_i \in V_1 \mid i \in I\}$ is a basis of V_1. Because α_i and β_i are the reflection vectors of s_i in V_1 and V_2, respectively, we are excepted to have

$$f(\alpha_i) = z_i \beta_i, \text{ for some } z_i \in \mathbb{F}^\times \text{ and each } i \in I.$$

Below we propose a choice of the coefficients z_i.

From now on we fix an index $i_0 \in I$ and set $z_{i_0} := 1$. For any other index $i \in I$, we choose an undirected walk in G_I from i_0 to i, say,

$$i_0, \quad a_1, \quad i_1, \quad a_2, \quad i_2, \quad \ldots, \quad i_{l-1}, \quad a_l, \quad i_l = i,$$

where $i_0, i_1, \ldots, i_l \in I$. Then z_i is defined to be

$$z_i := z_{i_0 i_1} z_{i_1 i_2} \cdots z_{i_{l-1} i_l} \in \mathbb{F}^\times.$$

We need to show that z_i such defined is independent of the choice of the undirected walk (but it does depend on the choice of the beginning vertex i_0).

Proposition 5.14 *Fix $i_0 \in I$ as above. For each $i \in I$, the value of z_i only depends on i, not on the choice of the undirected walk from i_0 to i.*

Proof Suppose there exist two undirected walks in G_I from 1 to i, say ($h > l$),

$$\text{(5.9)} \quad i_0, \quad a_1, \quad i_1, \quad a_2, \quad i_2, \quad \ldots, \quad i_{l-1}, \quad a_l, \quad i_l = i,$$

$$\text{(5.10)} \quad i_0, \quad a_h, \quad i_{h-1}, \quad a_{h-1}, \quad i_{h-2}, \quad \ldots, \quad i_{l+1}, \quad a_{l+1}, \quad i_l = i.$$

By convention, we also denote $i_h := i_0$. We need to show

$$z_{i_0 i_1} z_{i_1 i_2} \cdots z_{i_{l-1} i_l} = z_{i_h i_{h-1}} z_{i_{h-1} i_{h-2}} \cdots z_{i_{l+1} i_l}.$$

By definition we have $z_{i_{j-1} i_j} = z_{i_j i_{j-1}}^{-1}$. Thus our goal becomes

$$z_{i_0 i_1} z_{i_1 i_2} \cdots z_{i_{l-1} i_l} z_{i_l i_{l+1}} \cdots z_{i_{h-2} i_{h-1}} z_{i_{h-1} i_h} = 1.$$

By extending the first undirected walk (5.9) by the reverse of (5.10), we have an undirected walk in G_I from i_0 to i_0,

$$i_0, \quad a_1, \quad i_1, \quad \ldots, \quad a_{l-1}, \quad i_l, \quad a_{l+1}, \quad \ldots, \quad i_{h-1}, \quad a_h, \quad i_h = i_0.$$

By taking $p = h$ in Lemma 5.13, we have that

$$z_{i_0 i_1} \cdots z_{i_{h-1} i_h} = \frac{\zeta_{i_h, j_2, \ldots, j_d}}{\zeta_{i_0, j_2, \ldots, j_d}} = 1$$

for any distinct indices $j_1, \ldots, j_d \in I \backslash \{i_0, i_h\}$. This equality is what we want. ∎

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By Proposition 5.14, we have a well defined linear map $f : V_1 \to V_2$ by setting

$$f(\alpha_i) := z_i \beta_i, \quad \text{for each } i \in I.$$

Since $z_i \in \mathbb{F}^\times$ and $\{\alpha_i \mid i \in I\}$, $\{\beta_i \mid i \in I\}$ are bases of V_1, V_2 respectively, the map f is clearly an isomorphism of vector spaces. It remains to show that f is a homomorphism of W-modules.

5.5 The map f is a W-isomorphism

To show that f is a homomorphism of W-modules, it suffices to show

$$f(s_h \cdot \alpha_i) = s_h \cdot f(\alpha_i), \quad \text{for any } h \in [k] \text{ and } i \in I.$$

We split the proof into two parts (Propositions 5.15 and 5.16), depending on whether $h \in I$ or $h \notin I$.

Proposition 5.15 *For any $h, i \in I$, we have $f(s_h \cdot \alpha_i) = s_h \cdot f(\alpha_i)$.*

Proof In this case we have

$$f(s_h \cdot \alpha_i) = f(\alpha_i + x_{ih}\alpha_h) = z_i\beta_i + x_{ih} z_h \beta_h,$$

and

$$s_h \cdot f(\alpha_i) = s_h \cdot (z_i \beta_i) = z_i \beta_i + z_i y_{ih} \beta_h.$$

Therefore, we need to show

$$x_{ih} z_h = z_i y_{ih}. \tag{5.11}$$

If $x_{ih} = 0$, then $y_{ih} = 0$ by Corollary 5.10, and thus Equation (5.11) holds trivially. Suppose otherwise that $x_{ih} \neq 0$. Then $i \to h$ is an arrow. Recall that the coefficient z_i is computed by taking arbitrarily an undirected walk in G_I from the fixed i_0 ($\in I$) to i, say,

$$i_0, \quad a_1, \quad i_1, \quad \ldots, \quad a_l, \quad i_l = i.$$

If we set $i_{l+1} = h$ and $a_{l+1} = i \to h$, then the extended undirected walk

$$i_0, \quad a_1, \quad i_1, \quad \ldots, \quad a_l, \quad i_l = i, \quad a_{l+1}, \quad i_{l+1} = h$$

goes from i_0 to h. Therefore, by the definitions of z_i and z_{ih} in Section 5.4, we have $z_h = z_i z_{ih} = z_i y_{ih}/x_{ih}$ which is exactly Equation (5.11). ∎

Proposition 5.16 *For any $i \in I$ and $h \in [k] \backslash I$, we have $f(s_h \cdot \alpha_i) = s_h \cdot f(\alpha_i)$.*

The rest of this subsection is devoted to proving Proposition 5.16. Since $\{\alpha_l \mid l \in I\}$ is a basis for V_1, and so is $\{\beta_l \mid l \in I\}$ for V_2 by Proposition 5.5, we write

$$\alpha_h = \sum_{l \in I} a_l \alpha_l, \quad \beta_h = \sum_{l \in I} b_l \beta_l, \quad \text{where } a_l, b_l \in \mathbb{F}.$$

Then we have

$$f(s_h \cdot \alpha_i) = f(\alpha_i + x_{ih}\alpha_h) = f\Big(\alpha_i + x_{ih} \sum_{l \in I} a_l \alpha_l\Big) = z_i \beta_i + \sum_{l \in I} x_{ih} a_l z_l \beta_l$$

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and

$$s_h \cdot f(\alpha_i) = s_h \cdot (z_i \beta_i) = z_i \beta_i + z_i y_{ih} \beta_h = z_i \beta_i + \sum_{l \in I} z_i y_{ih} b_l \beta_l.$$

Therefore, to prove Proposition 5.16 it suffices to show for any $j \in I$ that

$$x_{ih} a_j z_j = y_{ih} b_j z_i. \tag{5.12}$$

For this, we need the following lemma.

Lemma 5.17 *Suppose $h \in [k] \backslash I$, $j \in I$. We write $\alpha_h = \sum_{l \in I} a_l \alpha_l$, $\beta_h = \sum_{l \in I} b_l \beta_l$ as above. If $a_j \neq 0$, then for any distinct $i_2, \ldots, i_d \in I \backslash \{j\}$, we have*

$$\zeta_{j, i_2, \ldots, i_d} a_j = \zeta_{h, i_2, \ldots, i_d} b_j.$$

Note that both $\zeta_{j, i_2, \ldots, i_d}$ and $\zeta_{h, i_2, \ldots, i_d}$ are nonzero, because both of the two sets $\{\alpha_j, \alpha_{i_2}, \ldots, \alpha_{i_d}\}$ and $\{\alpha_h, \alpha_{i_2}, \ldots, \alpha_{i_d}\}$ are linearly independent sets.

Proof Since $a_j \neq 0$, the vectors $\alpha_h, \alpha_{i_2}, \ldots, \alpha_{i_d}$ are linearly independent. We write $I = \{j, i_2, \ldots, i_d, i_{d+1}, \ldots, i_n\}$. Then

$$\alpha_h \wedge \alpha_{i_2} \wedge \cdots \wedge \alpha_{i_d} = \Big(a_j \alpha_j + \sum_{d+1 \leq l \leq n} a_{i_l} \alpha_{i_l} \Big) \wedge \alpha_{i_2} \wedge \cdots \wedge \alpha_{i_d}. \tag{5.13}$$

The image under ψ of the left hand side of Equation (5.13) is

$$\begin{aligned} \psi(\alpha_h \wedge \alpha_{i_2} \wedge \cdots \wedge \alpha_{i_d}) &= \zeta_{h, i_2, \ldots, i_d} \beta_h \wedge \beta_{i_2} \wedge \cdots \wedge \beta_{i_d} \\ &= \zeta_{h, i_2, \ldots, i_d} \Big(b_j \beta_j + \sum_{d+1 \leq l \leq n} b_{i_l} \beta_{i_l} \Big) \wedge \beta_{i_2} \wedge \cdots \wedge \beta_{i_d}. \end{aligned} \tag{5.14}$$

Moreover, the image under ψ of the right hand side of Equation (5.13) equals

$$\zeta_{j, i_2, \ldots, i_d} a_j \beta_j \wedge \beta_{i_2} \wedge \cdots \wedge \beta_{i_d} + \sum_{d+1 \leq l \leq n} \zeta_{i_l, i_2, \ldots, i_d} a_{i_l} \beta_{i_l} \wedge \beta_{i_2} \wedge \cdots \wedge \beta_{i_d}. \tag{5.15}$$

The equality of (5.14) and (5.15) gives $\zeta_{j, i_2, \ldots, i_d} a_j = \zeta_{h, i_2, \ldots, i_d} b_j$. ∎

Now we are ready to complete the proof of Proposition 5.16.

Proof of Proposition 5.16 As we mentioned, it suffices to prove Equation (5.12) for any $j \in I$. We have three cases.

Case one: $a_j = 0$. Then $\{\alpha_h\} \cup \{\alpha_l \mid l \in I \backslash \{j\}\}$ is a linearly dependent set. Then $\{\beta_h\} \cup \{\beta_l \mid l \in I \backslash \{j\}\}$ is also linearly dependent. Otherwise, it would be a basis for V_2, contradicting Proposition 5.5. Therefore, we have $b_j = 0$, and hence Equation (5.12) holds trivially.

Case two: $a_j \neq 0$ and $j = i$. In this case Equation (5.12) reduces to

$$x_{jh} a_j = y_{jh} b_j.$$

If α_h and α_j are proportional, then $\alpha_h, \alpha_{i_2}, \ldots, \alpha_{i_d}$ are linearly independent for any distinct $i_2, \ldots, i_d \in I \backslash \{j\}$. If α_h and α_j are not proportional, then $a_{j'} \neq 0$ for some $j' \in I \backslash \{j\}$. There exist $d-1$ distinct indices $i_2, \ldots, i_d \in I \backslash \{j, j'\}$ since $d \leq n-1$. In both cases, we have by Lemma 5.9 that

$$\zeta_{j, i_2, \ldots, i_d} y_{jh} = \zeta_{h, i_2, \ldots, i_d} x_{jh}.$$

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By Lemma 5.17, we also have

$$\zeta_{j,i_2,\ldots,i_d} a_j = \zeta_{h,i_2,\ldots,i_d} b_j.$$

Therefore $x_{jh} a_j = y_{jh} b_j$ as desired.

Case three: $a_j \neq 0$ and $i \neq j$. By the same arguments as in case one, we have $b_j \neq 0$. We may further assume $x_{ih} y_{ih} \neq 0$ by Corollary 5.10, otherwise Equation (5.12) reduces to "0 = 0". Suppose

$$i_0, \quad a_1, \quad i_1, \quad \ldots, \quad i_{p-1}, \quad a_p, \quad i_p = i$$

is an undirected walk in G_I from i_0 to i, and

$$i = i_p, \quad a_{p+1}, \quad i_{p+1}, \quad \ldots, \quad i_{q-1}, \quad a_q, \quad i_q = j$$

is an undirected walk in G_I from i to j. Their concatenation is an undirected walk from i_0 to j. Note that there exist $d-1$ distinct indices $i_2, \ldots, i_d \in I \backslash \{i, j\}$ since $d \leq n-1$. Then by definitions of z_i and z_j and Lemma 5.13, we have

$$\frac{z_j}{z_i} = z_{i_p i_{p+1}} \cdots z_{i_{q-1} i_q} = \frac{\zeta_{j,i_2,\ldots,i_d}}{\zeta_{i,i_2,\ldots,i_d}}. \tag{5.16}$$

Also note that $\alpha_h, \alpha_i, \alpha_{i_2}, \ldots, \alpha_{i_d}$ are linearly independent since $a_j \neq 0$. Then by Lemma 5.9, we have

$$\frac{x_{ih}}{y_{ih}} = \frac{\zeta_{i,i_2,\ldots,i_d}}{\zeta_{h,i_2,\ldots,i_d}}. \tag{5.17}$$

Moreover, by Lemma 5.17 we also have

$$\frac{a_j}{b_j} = \frac{\zeta_{h,i_2,\ldots,i_d}}{\zeta_{j,i_2,\ldots,i_d}}. \tag{5.18}$$

Multiplying Equations (5.16), (5.17), and (5.18) together, we obtain

$$\frac{z_j}{z_i} \cdot \frac{x_{ih}}{y_{ih}} \cdot \frac{a_j}{b_j} = 1.$$

This is exactly Equation (5.12). ∎

By Propositions 5.15 and 5.16, $f : V_1 \to V_2$ is a homomorphism of W-modules. We have finished the proof of Theorem 1.3.

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