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Gaps and steps for the sequence $n\theta$ mod 1

BY NOEL B. SLATER
Department of Applied Mathematics, University of Hull

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1. *Introduction.* For a given θ $(0 < \theta < 1)$ and $r = 0, 1, 2, 3, \ldots$ there are two related problems on the fractional parts $\{r\theta\}$, namely:

the gap problem: for any ϕ $(0 < \phi < 1)$ to determine the gaps between the successive r for which

$$\{r\theta\} < \phi; \tag{1}$$

the step problem: for the set

$$\{1\theta\}, \{2\theta\}, \ldots, \{N\theta\} \tag{2}$$

rearranged in ascending order, to determine the steps into which the interval $[0, 1]$ is thereby partitioned. There are subsidiary problems of the numbers of gaps or steps of the various lengths, and also of the permutation of $1, \ldots, N$ when (2) is arranged in order.

The simple results are that for general ϕ (or N) the gaps (or steps) have three lengths, one being the sum of the other two. After my heavy investigation (4) of gap lengths and numbers in terms of the continued fraction for θ, Florek (1) remarked that the three-gap result for irrational θ could be seen from a quite elementary argument, and I found that this was the case. After some years Świerczkowski (9), Surányi (8) and Sós [Mrs Turán] (6), (7) produced interestingly varied approaches to the step problem, and Sós (6) and recently Halton (2) developed continued fraction methods. These authors ignored the previous results on gaps, and I had omitted from my early paper a translation (which had been suggested by Professor Davenport) of gap into step results for rational θ. Thus it seems worth-while to put the problems together, choosing the simplest methods and adding some further results. The step problem seems in retrospect the more natural starting point, but the gap problem could have more interesting extensions to higher dimensions (5).

The simple results for steps are discussed in section 2 and for gaps in section 3. These results facilitate the approach to continued fraction formulae, which are given in section 4. In section 5 the step problem for a rational θ is identified with the gap problem for a related θ' which in the continued fraction form is the reverse of θ.

The three-step result has been called (2, 6–9) 'Steinhaus's conjecture'; his conjectures on the limiting behaviour of the step lengths for large N (see (3, 7, 2) will not be included here. The 'permutation problem' mentioned after (2) above will appear briefly. This has been treated by Sós (7) and particularly by Halton (2); I had discussed an analogous problem of ordering for gaps (4).

2. *Simple results for steps.* Greek letters denote real numbers between 0 and 1, and italic letters non-negative integers. Fractional parts will be used only for positive arguments, for example $\{(s-r)\theta\}$ with $s > r$. The integral part of $r\theta$ is written $[r\theta]$.

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In the step problem of (2) let θ and N be such that $\{1\theta\}, \ldots \{N\theta\}$ are distinct and non-zero. Let $\{a\theta\}$ and $\{b\theta\}$ be the *smallest* and *largest* of these, and write

$$\alpha = \{a\theta\}, \quad \beta = 1-\{b\theta\}, \quad A = [a\theta], \quad B = 1+[b\theta]. \tag{3}$$

As $\{(a+b)\theta\}$ is either $\{a\theta\}+\{b\theta\}$ or $\{a\theta\}+\{b\theta\}-1$, it either exceeds $\{b\theta\}$ or is less than $\{a\theta\}$, so that $a+b$ is not one of $1, 2, \ldots, N$. Thus

$$N \geqslant \max(a, b) \tag{4}$$

$$N \leqslant a+b-1; \tag{5}$$

the $\{r\theta\}$ are in fact distinct for $0 \leqslant r < a+b$.

We define points P_r within the line segment $[0, 1]$ by $\mathrm{OP}_r = \{r\theta\}$, and confine attention to *positive* segments $\mathrm{P}_r\mathrm{P}_s = \{s\theta\}-\{r\theta\} > 0$. The interval $\mathrm{P}_r\mathrm{P}_s$ is $\{(s-r)\theta\}$ if $s > r$ and $1-\{(r-s)\theta\}$ if $s < r$. It follows from the definition of a and b and (3) that if r and s lie in $0, 1, \ldots, N$, then

$$\left.\begin{aligned} \mathrm{P}_r\mathrm{P}_s &\geqslant \alpha \quad \text{if} \quad s > r \\ &\geqslant \beta \quad \text{if} \quad s < r, \end{aligned}\right\} \tag{6}$$

with equality in the first case when $s-r = a$ and in the second when $r-s = b$.

The interval $[0, 1]$ is divided into $N+1$ consecutive 'steps' by the points

$$\mathrm{P}_r (r = 1, \ldots, N)$$

arranged in order (the $\{r\theta\}$ in increasing magnitude); if $\mathrm{P}_r\mathrm{P}_s$ is such a 'step' we say that P_r i.p. (*immediately precedes*) P_s. From (6), some of these steps (including the initial step OP_a) are of length α and some (including the final step P_b to 1) of length β. There are in fact at most three step lengths, following points P_r of three types which we shall denote by $r = s, t, u$; these are

$$\left.\begin{aligned} &\text{(i)} \quad 0 \leqslant s \leqslant N-a; \quad \mathrm{P}_s \text{ i.p. } \mathrm{P}_{s+a}; \quad N+1-a \text{ steps, length } \alpha, \\ &\text{(ii)} \quad N-a < t < b; \quad \mathrm{P}_t \text{ i.p. } \mathrm{P}_{t+a-b}; \quad a+b-N-1 \text{ steps } \alpha+\beta, \\ &\text{(iii)} \quad b \leqslant u \leqslant N; \quad \mathrm{P}_u \text{ i.p. } \mathrm{P}_{u-b}; \quad N+1-b \text{ steps } \beta. \end{aligned}\right\} \tag{7}$$

The results in cases (i) and (iii) follow from (6). The range of t in case (ii) occurs when there is *inequality* in (5). To verify the result (ii), suppose that $\{v\theta\} > \{t\theta\}$ with at first $v > t$. For t as in (ii) and $v \leqslant N$, then $v-t < a$, so that $\{(v-t)\theta\} > \alpha$. Hence

$$\{(v-t)\theta\} = \{a\theta-(a-v+t)\theta\} = \alpha+1-\{(a-v+t)\theta\}$$

$$\gtreqless \alpha+\beta \text{ according as } v \gtreqless t+a-b.$$

This gives the result (7)(ii) if $a > b$; if $a < b$ the result comes from cases $\{v\theta\} > \{t\theta\}$ with $v < t$. The results (7) on the three step lengths and the permutation or 'i.p.' relations are contained in Świerczkowski (9) and Sós (7).

The $N+1$ steps in (7) fill the interval $[0, 1]$; so simple addition of their lengths gives the *basic formula*:

$$b\{a\theta\}+a(1-\{b\theta\}) \equiv b\alpha+a\beta = 1. \tag{8}$$

This can be written, by (3),

$$aB-bA = 1; \tag{9}$$

and (3) also gives

$$A/a < \theta < B/b. \tag{10}$$

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Thus in a Cartesian (x, y) plane the line $y = \theta x$ cuts through the triangle with vertices $(0, 0)$, (a, A), (b, B); but by (9) this triangle has no other integer lattice points in or on it. This geometrical view will not be pursued here.

By (4), (5) and (9), A/a and B/b are consecutive fractions in the Farey series $\mathscr{F}_N$, and and by (10) they enclose θ. Surányi (8) begins here, namely by defining A/a and B/b as the consecutive $\mathscr{F}_N$ fractions enclosing θ; this definition implies (4), (5) and (9). His basic theorem is then that the $\{r\theta\}$ have the same ordering as $\{rA/a\}$, or, in a case of equality there, as $\{rB/b\}$. He deduces that $\{a\theta\}$ and $\{b\theta\}$ are the extremes, at which point the present discussion could follow on, although he proceeds to (7) by way of his ordering theorem.

The following points are of interest:

(I) It is seen that $\{a\theta\}$ and $\{b\theta\}$, as initially defined for a given N, are also the extremes of $\{1\theta\}, \ldots, \{N'\theta\}$ for *any* N' satisfying (4) and (5), and the results (7) hold for any such N'. (Compare also the beginning of section 4 below.)

(II) For a set $\{1\theta\}, \ldots, \{N\theta\}$, the position of $\{N\theta\}$ or P_N is given by (7) (i), (iii) as P_{N-a} i.p. P_N i.p. P_{N-b}. If we *omit* $\{N\theta\}$, then P_{N-a} i.p. P_{N-b}, the intervening steps α and β coalescing into an $\alpha+\beta$. Thus for the set $\{1\theta\}, \ldots, \{(N-1)\theta\}$ as investigated by Halton by continued fraction methods (2), the step lengths are still as in (7), with now the numbers

$$\text{(i) } N-a, \quad \text{(ii) } a+b-N, \quad \text{(iii) } N-b. \tag{7'}$$

This change from N to $N-1$ is not always of the type in note (I) above; for example if $N = a$ then $N' = N-1$ violates (4) and (as (7') (i) shows) the step α does not then occur for N'.

(III) As Surányi observes, the ordering of the $\{r\theta\}$ is the same as that of $\{r\theta_1\}$ with *any* θ_1 in the range (10). An interesting θ_1 for ordering purposes is the mediant $(A+B)/(a+b)$ of (10). For this θ_1, by (9), $\{a\theta_1\} = 1-\{b\theta_1\} = 1/(a+b)$; so the step lengths in $\{r\theta_1\}$ are $1/(a+b)$ and $2/(a+b)$ only.

(IV) If θ is *rational*, say

$$\theta = p/q, \quad p < q, \quad p, q \text{ coprime}, \tag{11}$$

then the requirement of distinctness of the $\{r\theta\}$ limits us to $N < q$.

3. *Simple results for gaps.* The gap problem of (1) is dual to the step problem, ϕ replacing N as the determining parameter. At first θ will be assumed *irrational*; the rational case will be added and discussed again later.

For the inequality $\{r\theta\} < \phi$, a new a and b are defined as the least positive integers such that

$$\{a\theta\} < \phi, \quad \{b\theta\} > 1-\phi; \tag{12}$$

it will usually be assumed that $\phi \leqslant \frac{1}{2}$, so that $a \neq b$. It follows from the present definition that

$$\{a\theta\} < \{r\theta\} < \{b\theta\} \quad \text{for} \quad 0 < r < a+b, \quad r \neq a, b, \tag{13}$$

so that $\{a\theta\}$ and $\{b\theta\}$ are the extremes of $\{1\theta\}, \ldots, \{N\theta\}$ for any N satisfying (4) and (5) with the present a and b. This hints at a unified treatment; but we proceed here with the gap problem by an argument parallel to that in section 2.

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With α and β defined as in (3) for the present a and b, (12) may be written

$$\phi > \max(\alpha, \beta). \tag{14}$$

By examining $\{|a-b|\,\theta\}$ and the definitions (12), we see that

$$\phi \leqslant \alpha+\beta. \tag{15}$$

These inequalities (14) and (15) are dual to (4) and (5). The dual of (6) is that if r and s satisfy (1) and $r < s$, then

$$s-r \geqslant a \quad \text{if} \quad \{s\theta\} > \{r\theta\}; \quad s-r \geqslant b \quad \text{if} \quad \{s\theta\} < \{r\theta\}, \tag{16}$$

equality being possible only if no integer between r and s satisfies (1). Let successive integers satisfying (1) be $r, r+d_r$, so that d_r is the 'gap' after r. It follows from (16) (with details in case (ii) similar to the argument after (7)) that the r are of three types, say s, t, u, followed by characteristic gaps d_s, d_t, d_u, namely

$$\left.\begin{array}{lll} \text{(i)} & \text{if} \quad 0 \leqslant \{s\theta\} < \phi-\alpha, & d_s = a; \\ \text{(ii)} & \phi-\alpha \leqslant \{t\theta\} < \beta, & d_t = a+b; \\ \text{(iii)} & \beta \leqslant \{u\theta\} < \phi, & d_u = b. \end{array}\right\} \tag{17}$$

Thus the gaps d_r are of three lengths, the sum gap $a+b$ occurring when there is *inequality* in (15). Florek (1) stated this result (presumably on the basis of the present type of argument) for irrational θ, with (1) generalized to the form $\mu \leqslant \{r\theta\} < \mu+\phi$.

With θ irrational, the points $\{r\theta\}$ are asymptotically uniformly dense in the unit interval. Thus for a large range $0 \leqslant r \leqslant R$ the numbers S, T, U of gaps d_s, d_t, d_u, that is of integers of types s, t, u, are asymptotic to R times the ranges of $\{r\theta\}$ in (17):

$$S, T, U \sim R(\phi-\alpha, \quad a+\beta-\phi, \quad \phi-\beta). \tag{18}$$

Equating the sum $Sd_s+Td_t+Ud_u$ of all the gaps to the range R gives again the basic formula (8) for the present definition of a and b. The result is of course guaranteed by the applicability of the step theory to (13).

If θ is *rational*, the results (17) still hold, but not (18) unless we impose a simple restriction on ϕ. Suppose that $\theta = p/q$ as in (11), so that the $\{r\theta\}$, and of course α and β, are multiples of $1/q$. Then we take ϕ to be of the form

$$\phi = (f+1)/q, \tag{19}$$

so that the inequality (1) becomes

$$\{r\theta\} \leqslant f/q \tag{20}$$

(or equivalently $(rp) \bmod q \leqslant f$). Now $\{r\theta)$ has period q in r; and the gap numbers S, T, U are again given by (18), asymptotically for large R and precisely for R a multiple of q. In particular, for $0 \leqslant r \leqslant q$ the numbers of gaps $a, a+b, b$ are

$$f+1-\alpha q, \quad (\alpha+\beta)q-(f+1), \quad f+1-\beta q. \tag{21}$$

For general θ, two points are of interest:

(I) The gaps $a, a+b, b$ determined above for a given ϕ are also the gaps for any ϕ' satisfying (14) and (15).

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(II) The number of gaps for $0 \leqslant r \leqslant R$ is asymptotic to $R\phi$ (with ϕ as in (19) for rational θ), so that the average gap length is $1/\phi$. Thus by (17)

$$\min(a, b) < 1/\phi < a+b. \tag{22}$$

This relation also follows, without asymptotic considerations, from (14) and (15) with (8) treated as a step result from (13).

4. *Complete* (*continued fraction*) *results.* Explicit gap and step solutions will now be deduced for a given θ with general ϕ in (1) and N in (2). The starting-point is the 'basic formula' (8) or (9), which leads us to consider the continued fraction for θ. The full results then follow directly from (7) and (17).

For a given θ let a and b be a pair of integers satisfying (8), or equivalently (9) with A and B defined as in (3). Then (compare Surányi's approach described in section 2) $\{a\theta\}$ and $\{b\theta\}$ are the least and greatest of $\{1\theta\}, \ldots, \{N\theta\}$ for any N satisfying (4) and (5), so that (7) is the solution of the step problem for such an N. Likewise this a and b are the smallest gaps d_r between solutions of $\{r\theta\} < \phi$, for any ϕ satisfying (14) and (15), so that (17) is the gap solution for such a ϕ. If θ is irrational the gap numbers are as in (18); for a rational $\theta = p/q$ the gap numbers for $\{r\theta\} \leqslant f/q$ are given by (21) for any f satisfying $\max(\alpha, \beta) \leqslant f/q < \alpha+\beta$ (i.e. (14) and (15)).

Thus we can best start from the solutions of (8) or (9). As the inequalities (10) suggest, these solutions are provided by the best rational approximations to θ, namely the principal and intermediate convergents of its continued fraction.

Continued fraction (*c.f.*) *notations.* Let the simple c.f. for an *irrational* $\theta < 1$ be

$$\theta = [0, x_1, x_2, \ldots] = \frac{1}{x_1+}\,\frac{1}{x_2+}\cdots, \tag{23}$$

so that its principal convergents are $p_m/q_m = [0, x_1, \ldots, x_m]$ where q_m is the simple continuant $q_m = (x_1, \ldots, x_m) = (x_m, \ldots, x_1)$ and $p_m = (x_2, \ldots, x_m)$. Halton's η_m (2) (the θ_{m+1} of Slater (4)) is

$$\eta_m = (-1)^m (q_m\theta - p_m) \quad (>0). \tag{24}$$

As $q_{m+1} = x_{m+1}q_m + q_{m-1}$, and similarly for p_{m+1}, then

$$\eta_{m-1} = x_{m+1}\eta_m + \eta_{m+1};$$

also

$$q_m p_{m-1} - p_m q_{m-1} = (-1)^m, \quad q_m\eta_{m-1} + \eta_m q_{m-1} = 1. \tag{25}$$

The lowest value required for formal completeness will be $m = -1$; and consistently with the above identities we take

$$q_{-1} = 0, \quad q_0 = 1, \quad p_{-1} = 1, \quad p_0 = 0, \quad \eta_{-1} = 1. \tag{26}$$

For a *rational* θ, namely

$$\theta = [0, x_1, \ldots, x_M] = p_M/q_M \equiv p/q, \tag{27}$$

there is a choice of two forms of the continued fraction, namely one with last denominator $x_M = 1$, the other with last denominator greater than unity and with M one less. For a chosen form of θ we introduce the continuant of the last r denominators:

$$Q_r = (x_M, x_{M-1}, \ldots, x_{M-r+1}) \tag{28}$$

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so that in (27) $Q_M = q,\ Q_{M-1} = p,\ Q_0 = 1$; also (24) becomes

$$\eta_m = Q_{M-m-1}/q \quad (m = -1, \ldots, M). \tag{29}$$

Step results. For an irrational θ in the form (23), it is seen from (24) and (25) that the complete set of solutions of (8) or (9) is

$$\left.\begin{array}{l} a, b = q_m, (n+1)\,q_m + q_{m-1}; \quad \alpha, \beta = \eta_m, \eta_{m-1} - (n+1)\,\eta_m; \\ A, B = p_m, (n+1)\,p_m + p_{m-1}; \end{array}\right\} \tag{30}$$

where $$m \geqslant 0 \quad \text{and} \quad \text{for given } m, \quad 0 \leqslant n \leqslant x_{m+1} - 1. \tag{31}$$

The order in (30) is for m even; for m odd, (a, α, A) are interchanged with (b, β, B). A/a and B/b are the principal and intermediate convergents to θ, mentioned earlier.

For a given pair a, b in (30) the N satisfying (4) and (5) are

$$N = (n+1)\,q_m + q_{m-1} + k \quad (k = 0, 1, \ldots, q_m - 1), \tag{32}$$

so that the steps in $\{1\theta\}, \ldots, \{N\theta\}$ are given by (7) with a, b, α, β as in (30). Any positive N can be expressed uniquely in the form (32), (31), so determining m, n and k. Thus by (7) and (30) the steps for a given N are

$$\left.\begin{array}{lllll} \text{lengths} & \eta_m, & \eta_{m-1} - n\eta_m, & \eta_{m-1} - (n+1)\,\eta_m, \\ \text{numbers} & nq_m + q_{m-1} + k + 1, & q_m - (k+1), & (k+1), \end{array}\right\} \tag{33}$$

and this is the complete solution.

If θ is the *rational* (27), the modifications in (31) are:

$$\left.\begin{array}{l} \text{if} \quad x_M = 1, \quad m \leqslant M - 2; \\ \text{if} \quad x_M > 1, \quad m \leqslant M - 1, \quad \text{with} \quad 0 \leqslant n \leqslant x_M - 2 \quad \text{for} \quad m = M - 1. \end{array}\right\} \tag{34}$$

The formulae (33) now give the complete set of steps for values of N up to $N = q - 1$ (compare (32)).

In accordance with section 2 (II) the step numbers for the set $\{1\theta\}, \ldots, \{(N-1)\,\theta\}$ are given not by (7) but by (7′), which corresponds to (33) with $k+1$ replaced by k. This confirms Halton's results (2), except that (34) embodies a correction (with which Professor Halton agrees) to his formula for rational θ. It is also seen from section 2 (II) that the point $\{N\theta\}$ lies in a step $\alpha + \beta = \eta_{m-1} - n\eta_m$ of the set $\{1\theta\}, \ldots, \{(N-1)\,\theta\}$, as was shown by Halton.

Gap results (*irrational* θ). It is necessary for the gap problem to rearrange the solutions (30) of (8) in the form (with a, α interchanged with b, β if l is odd)

$$a, b = q_l, q_{l+1} - (n+1)\,q_l; \quad \alpha, \beta = \eta_l, (n+1)\,\eta_l + \eta_{l+1}, \tag{35}$$

where in the expansion (23) of θ

$$\left.\begin{array}{l} \text{if} \quad x_1 = 1 \quad \text{then} \quad l \geqslant 1 \quad \text{and} \quad 0 \leqslant n \leqslant x_{l+1} - 1, \\ \text{if} \quad x_1 > 1, \quad \text{also} \quad l = 0 \quad \text{with} \quad 0 \leqslant n \leqslant x_1 - 2. \end{array}\right\} \tag{36}$$

For a given solution pair a, b the ϕ satisfying (14) and (15) are (decreasing as l increases)

$$\phi = (n+1)\,\eta_l + \eta_{l+1} + \psi \quad (0 < \psi \leqslant \eta_l). \tag{37}$$

Any ϕ in $[0, 1]$ can be expressed uniquely in this form, determining l, n, ψ. Thus by (17) and (18) the gaps d_r for $\{r\theta\} < \phi$ are the corresponding $a, a+b, b$ of (35) with asymp-

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totic relative frequency $n\eta_l+\eta_{l+1}+\psi, \eta_l-\psi, \psi$. Earlier (4) I also gave precise gap numbers for r ranging up to any q_u with $u > l$.

Gap results (rational θ). For a θ of the form (27), the highest l in (36) is $M-1$. As in (19), we take $\phi = (f+1)/q$. To rearrange the f $(0 < f < q)$ in ascending order (for later comparisons) we write in (35)–(37)

$$m = M-1-l, \tag{38}$$

so that by (29) $\eta_l = Q_m/q$. The f values corresponding to (37) are then

$$f = (n+1)\,Q_m + Q_{m-1} + k \quad (k = 0, 1, \ldots, Q_m - 1), \tag{39}$$

where by (36), (38)

$$\left.\begin{array}{ll} \text{if} & x_1 = 1, \quad 0 \leqslant m \leqslant M-2 \quad \text{with} \quad 0 \leqslant n \leqslant x_{M-m}-1, \\ \text{if} & x_1 > 1, \quad \text{also} \quad m = M-1 \quad \text{with} \quad 0 \leqslant n \leqslant x_1 - 2. \end{array}\right\} \tag{40}$$

Thus for $\{r\theta\} \leqslant f/q$ and $0 \leqslant r \leqslant q$ the gaps d_r are, by (35),

$$\left.\begin{array}{lccc} \text{lengths} & q_{M-m-1}, & q_{M-m} - nq_{M-m-1}, & q_{M-m} - (n+1)\,q_{M-m-1} \\ \text{numbers} & nQ_m + Q_{m-1} + k + 1, & Q_m - (k+1), & k+1. \end{array}\right\} \tag{41}$$

This is the complete solution for $f < q$.

5. *Gaps and steps for related rationals.* So far the same θ's have been used for both problems. The step problem for a rational θ, however, is identifiable with the gap problem for a related rational θ'. At first we do not use c.f.'s. If $\theta = p/q$ in lowest terms, we solve the congruence $pp' \equiv 1 \bmod q$ for p', and so have the symmetrical relation

$$\theta = p/q, \quad \theta' = p'/q; \quad pp' \equiv 1 \bmod q \ (p, p' < q). \tag{42}$$

Correspondingly, associates r, r' in the ring of integers mod q are given by

$$\{r\theta\} = r'/q, \quad \{r'\theta'\} = r/q. \tag{43}$$

If the step problem for θ is as in (2), the corresponding gap problem is for the inequality $\{r'\theta'\} \leqslant N/q$; for by (43) the ascending $r'(0 < r' < q)$ for which $\{r'\theta'\} \leqslant N/q$ are q times the $\{r\theta\}$ $(0 < r \leqslant N)$ arranged in ascending order. Thus

$$\left.\begin{array}{l} \textit{gaps } d_{r'} \textit{ for } \{r'\theta'\} \leqslant N/q \textit{ are } q \times (\textit{steps in } \{1\theta\}, \ldots, \{N\theta\}) \\ \textit{gap numbers for } 0 \leqslant r' \leqslant q \textit{ equal corresponding step numbers.} \end{array}\right\} \tag{44}$$

This result converts the step formulae (7) (with a, b, α, β defined as in section 2) into the gap formulae (17) and (21) for $\{r'\theta'\} \leqslant N/q$, with the gap quantities defined in section 3 now relabelled, say as a', b', $\alpha' = \{a'\theta'\}$, $\beta' = 1-\{b'\theta'\}$, and N replacing f. The definitions correspond in detail; for example the least of (2) is $\{a\theta\} = \alpha$, and $a' = q\alpha$ is the least r' for $\{r'\theta'\} \leqslant N/q$; thus by (43) $a = q\alpha'$, and the $N+1-a$ steps α of (7)(i) give $N+1-q\alpha'$ gaps a' as in (21). Similarly, $b = q\beta'$ (although $(q-b)/q = \{b'\theta'\}$, against the notation (43)). The basic formula (8) (which applies now for both θ and θ') can be put in the symmetrical form

$$ab' + ba' = q. \tag{45}$$

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Continued fraction results. The relation (42) is easily solved for c.f.'s, and this leads also to a complete check of some previous results. Suppose first that $\theta = p/q$ in (27) is arranged to have M *odd*, with q_m, p_m, η_m, Q_r the functions of the denominators

$$x_1, \ldots, x_M$$

as in section 4. Let another rational θ' be

$$\theta' = p'/q' (\equiv p'_{M'}/q'_{M'}) = [0, x'_1, \ldots, x'_{M'}] \tag{46}$$

with corresponding $q'_m, p'_m, \eta'_m, Q'_r$ functions of $x'_1, \ldots, x'_{M'}$. If θ and θ' are related by (42) then in the first place $q' = q$, or in continuant notation

$$(x'_{M'}, \ldots, x'_1) = (x_1, \ldots, x_M). \tag{47}$$

By (25), since M is odd, pq_{M-1} $(= p_M q_{M-1}) \equiv 1 \bmod q$; the relation $pp' \equiv 1 \bmod q$ of (42) then makes $p' = q_{M-1}$, that is

$$(x'_{M'}, \ldots, x'_2) = (x_1, \ldots, x_{M-1}). \tag{48}$$

The solution of (47) and (48) (unique apart from a trivial variant of the c.f. form of θ') is

$$M' = M, \quad x'_r = x_{M+1-r} \quad (r = 1, 2, \ldots, M). \tag{49}$$

Thus the c.f. of θ' is that of θ with the (non-zero) denominators in reverse order. From (49) there are dual relations (compare (29))

$$q_m = Q'_m, \quad Q_m = q'_m, \quad \eta_m = q'_{M-m-1}/q. \tag{50}$$

We can now use (44) to convert the step results (31)–(34) for $\{1\theta\}, \ldots, \{N\theta\}$ into the gap results (39)–(41) for $\{r'\theta'\} \leqslant f/q$ with $f = N$ and with the primed quantities (x'_r, q'_r etc.) in the gap results. It is seen that the classification of N in (31), (32), (34) agrees with the f of (39), (40), and the step lengths and numbers give the gap results with primes added.

Notes. (I) The results are of course not affected by having chosen M odd in the form (27) of θ. If, for example, this form has $x_M = 1$, then the same θ can also be written as $[0, x_1, \ldots, x_{M-2}, x_{M-1} + 1]$ with an even number, $M - 1$, of non-zero denominators. If the reverse $[0, x_{M-1} + 1, \ldots, x_1]$ of this new form of the c.f. is $\theta'' = p''/q$, then it is seen that (contrast (42)) $pp'' \equiv -1 \bmod q$; or equivalently by direct addition of c.f.'s $\theta' + \theta'' = 1$, where θ' is the previous form (49). Consistently, the steps in $\{r\theta\}$ now give the gaps for $\{r'\theta''\} \leqslant N/q$ $(0 \leqslant r' \leqslant q)$ in reverse order.

(II) If a θ has a *symmetrical* c.f. $(x_r = x_{M+1-r})$, then the sequence of gaps for

$$\{r\theta\} \leqslant N/q$$

is the same as the sequence of steps in $\{1\theta\}, \ldots, \{N\theta\}$.

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