Probability in the Engineering and Informational Sciences, **22**, 2008, 213–219. Printed in the U.S.A.
doi: 10.1017/S0269964808000120

INSURANCE RISK WITH VARIABLE NUMBER OF POLICIES

IVO ADAN
Department of Mathematics and Computer Science
Technische Universiteit Eindhoven
5600 MB, Eindhoven, The Netherlands
E-mail: iadan@win.tue.nl

VIDYADHAR KULKARNI
Department of Statistics and Operations Research
University of North Carolina
Chapel Hill, NC, 27599
E-mail: vkulkarn@email.unc.edu

In this article we consider an insurance company selling life insurance policies. New policies are sold at random points in time, and each policy stays active for an exponential amount of time with rate μ, during which the policyholder pays premiums continuously at rate r. When the policy expires, the insurance company pays a claim of random size. The aim is to compute the probability of eventual ruin starting with a given number of policies and a given level of insurance fund. We establish the remarkable result that the ruin probability is identical to the one in the standard compound Poisson model where the insurance fund increases at constant rate r and claims occur according to a Poisson process with rate μ.

1. INTRODUCTION

The most studied model in the literature on risk processes is the *compound Poisson model*. In this model, the insurance fund increases at constant rate r and claims arrive according to a Poisson process with rate μ. The claim sizes are independent and identically distributed (i.i.d.) with common distribution G and mean τ and are independent of the arrival process. Thus, the insurance fund jumps down by random amounts at Poisson event times, and it increases linearly between these events. Let $\psi_P(x)$ denote the probability that the insurance fund eventually reaches a level of

© 2008 Cambridge University Press 0269-9648/08 $25.00

https://doi.org/10.1017/S0269964808000120 Published online by Cambridge University Press

less than zero, given that the initial fund is x. The probability $\psi_P(x)$ is called the infinite horizon *ruin probability* of the compound Poisson model. If $\tau \geq r/\mu$, the ruin probability is 1 (i.e., ruin is certain). When $\tau < r/\mu$, it is well known (see, e.g., Rolski, Schmidli, Schmidt, and Teugels [1, Them. 5.3.3]) that the ruin probability is less than 1 and its Laplace transform is given by the *Pollackzek–Khinchin formula*

$$\psi_P^*(s) = \int_0^\infty e^{-sx}\psi_P(x)\,dx = \frac{\rho}{s}\frac{1-\tilde{G}_e(s)}{1-\rho\tilde{G}_e(s)}, \qquad \mathrm{Re}(s) > 0, \tag{1}$$

where $\rho = \mu\tau/r$ and $\tilde{G}_e(s)$ is the Laplace–Stieltjes transform of the equilibrium distribution of G; that is,

$$\tilde{G}_e(s) = \int_0^\infty e^{-sx}\frac{1-G(x)}{\tau}dx, \qquad \mathrm{Re}(s) \geq 0.$$

The motivation for the model studied in this article comes from a more realistic model of life insurance business. We assume that new life insurance policies are sold at random epochs and stay active for an exponential amount of time with rate μ. Let $X(t)$ be the number of policies active at time t. Initially, $X(0)$ policies are active, which were issued before or at time 0. We further assume that each active policy pays premiums continuously at constant rate r to the insurance company. The instantaneous total rate at which premiums are paid to the company at time t is thus equal to $rX(t)$. When a policy expires, the insurance company pays out a claim. The claim size distribution is G with mean τ. Policy lifetimes and payouts are i.i.d. and independent of the sales process. Assume that $\rho = \mu\tau/r < 1$ or, $r/\mu > \tau$; i.e., the expected premiums from a policy during its lifetime is greater than payout when the policy expires. This condition is therefore the usual *net profit condition*.

Let $Z(t)$ be the insurance fund at time t. Thus, $Z(t)$ is equal to the initial fund $Z(0)$ plus the total premiums collected over $[0, t]$ minus the total claims paid out over $[0, t]$. Let T be the ruin time, defined by

$$T = \inf\{t > 0 : Z(t) < 0\}.$$

Sample paths of $X = \{X(t),\, t \geq 0\}$ and $Z = \{Z(t),\, t \geq 0\}$ are displayed in Figure 1.

In this article, we compute the infinite horizon ruin probability

$$\psi(x, i) = P(T < \infty | Z(0) = x, X(0) = i), \qquad x \geq 0, i = 0, 1, \ldots.$$

In the next section we start the analysis under the assumption of Poisson sales.

2. POISSON SALES

Assume that new policies are sold according to a Poisson process with rate λ. We establish the following result.

https://doi.org/10.1017/S0269964808000120 Published online by Cambridge University Press

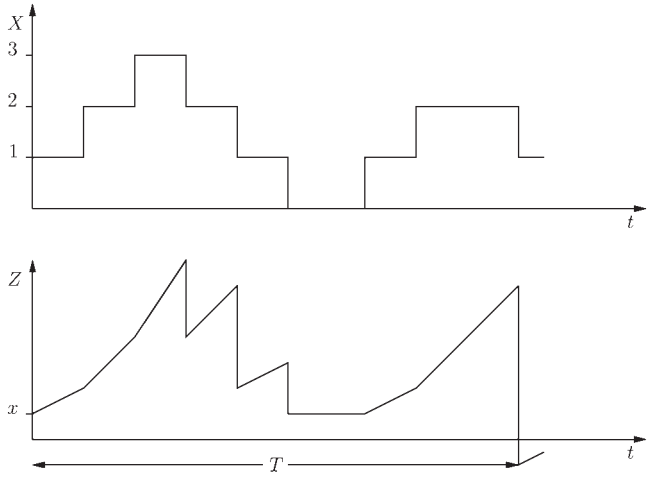


FIGURE 1. Sample paths of X and Z.

THEOREM 1: *The Laplace transform of the ruin probability $\psi(x, i)$ is given by*

$$\psi^*(s, i) = \int_0^\infty e^{-sx}\psi(x, i)\,dx = \frac{\rho}{s}\frac{1 - \tilde{G}_e(s)}{1 - \rho\tilde{G}_e(s)}, \qquad \mathrm{Re}(s) > 0, i = 0, 1, \ldots \qquad \textbf{(2)}$$

PROOF: We first derive the integral-differential equations satisfied by $\psi(x, i)$. For $x \geq 0$, $i \geq 1$, and small $h > 0$, we have

$$\begin{aligned}\psi(x, i) = {} & (1 - (\lambda + i\mu)h)\psi(x + irh, i) + \lambda h\psi(x + irh, i + 1) \\ & + i\mu h(1 - G(x + irh)) + i\mu h\int_0^{x+irh} \psi(x + irh - u, i - 1)\,dG(u) + o(h).\end{aligned}$$

Rearranging the terms, we get

$$\begin{aligned}-ir\frac{\psi(x + irh, i) - \psi(x, i)}{irh} = {} & -(\lambda + i\mu)\psi(x + irh, i) + \lambda\psi(x + irh, i + 1) \\ & + i\mu(1 - G(x + irh)) \\ & + i\mu\int_0^{x+irh} \psi(x + irh - u, i - 1)\,dG(u) + o(1).\end{aligned}$$

https://doi.org/10.1017/S0269964808000120 Published online by Cambridge University Press

Letting $h \to 0$ yields

$$ir\,\frac{d}{dx}\psi(x,i) - (\lambda + i\mu)\psi(x,i) + \lambda\psi(x,i+1) + i\mu(1-G(x)) + i\mu\int_0^x \psi(x-u,i-1)\,dG(u) = 0, \tag{3}$$

valid for $x \geq 0$ and $i \geq 1$. Further, $\psi(x,0) = \psi(x,1)$, since the fund level stays constant while $i = 0$. Together with the boundary conditions $\psi(x,i) \to 0$ as $x \to \infty$, (3) has a unique solution. To solve this system, we take as the trial solution

$$\psi(x,i) = \phi(x), \qquad x \geq 0, i = 0, 1, \ldots.$$

Substitution of this solution into (3) yields

$$r\,\frac{d}{dx}\phi(x) - \mu\phi(x) + \mu(1-G(x)) + \mu\int_0^x \phi(x-u)\,dG(u) = 0.$$

Taking the Lapalce transform

$$\phi^*(s) = \int_0^\infty e^{-sx}\phi(x)\,dx, \qquad \mathrm{Re}(s) > 0,$$

we obtain

$$r(-\phi(0) + s\phi^*(s)) - \mu\phi^*(s) + \mu\tau\tilde{G}_e(s) + \mu\phi^*(s)\tilde{G}(s) = 0$$

and, thus,

$$\phi^*(s) = \frac{r\phi(0) - \mu\tau\tilde{G}_e(s)}{rs - \mu + \mu\tilde{G}(s)} = \frac{1}{s}\frac{\phi(0) - \rho\tilde{G}_e(s)}{1 - \rho(1-\tilde{G}(s))/(\tau s)}.$$

Since $\tilde{G}_e(s) = (1-\tilde{G}(s))/(\tau s)$, the above expression reduces to

$$\phi^*(s) = \frac{1}{s}\frac{\phi(0) - \rho\tilde{G}_e(s)}{1 - \rho\tilde{G}_e(s)}. \tag{4}$$

Finally, $\phi(0)$ follows from the condition that $s\phi^*(s) \to 0$ as $s \to 0$. From (4), this limits is equal to $(\phi(0) - \rho)/(1-\rho)$. Hence, $\phi(0) = \rho$, which completes the proof. ■

The result in Theorem 1 is remarkable: The ruin function $\psi(x, i)$ is independent of i and is also independent of the arrival rate $\lambda > 0$! Moreover, (2) is exactly the same formula as (1). An explanation will be provided in the next section.

3. GENERAL SALES

The absence of λ in the formula for the ruin probability suggests that the result may be valid for *any arrival process*. Therefore, we now assume that new policies arrive

https://doi.org/10.1017/S0269964808000120 Published online by Cambridge University Press

according to an arbitrary arrival process, independent of policy lifetimes and payouts; all other assumptions remain the same.

Consider a realization of the sequence of sales epochs $0 < t_1 < t_2 < \cdots$, where t_j is the time of sale of the j-th new policy, and assume that t_j tends to infinity. Now we will show that the fund level process Z can be transformed into the fund level process of the standard compound Poisson model, without changing the ruin probability. The transformation is first explained intuitively. When there are i active policies, the fund increases with rate ir, and after an exponential time with rate $i\mu$, one of the policies will expire. The idea is to make the clock tick i times slower until the next event, i.e., an arrival of a new policy or an expiration of an active policy. Then the fund increases with rate r and a policy will expire after an exponential time with rate μ. Note that the rates no longer depend on i. After the next event the clock is adapted again, and so on. If zero policies remain, then we remove the time period up to the first arrival (the fund remains constant over this period). The resulting process is the same as in the compound Poisson model: the fund level always increases at rate r, i.e., it does not notice any changes in the number of active policies, and, by the memoryless property of exponentials, the times in between subsequent claims are always exponentially distributed with rate μ. This explains why the arrival process and the initial number are not relevant to the ruin probability. By formalizing the transformation of time we prove the following theorem.

THEOREM 2: *For any realization of the sequence of sales epochs, the ruin probability $\psi(x, i)$ is exactly the same as the ruin probability $\psi_P(x)$ in the compound Poisson model.*

To prove Theorem 2, we first define the time transformation formally. We map each time point t to a new point $\eta(t)$ defined by

$$\eta(t) = \int_0^t X(u)\, du.$$

Clearly, all time points in an interval during which X is zero are mapped to the same point (i.e., we remove such intervals). Now, define the (generalized) inverse of η as

$$\sigma(t) = \sup\left\{ s : \int_0^s X(u)\, du = t \right\}.$$

So, if η maps multiple time points to the same point, the inverse of η takes the right-most time point (for which X is equal to 1). The mapping η transforms (Z, X) into $(Z(\sigma), X(\sigma))$. We first establish the following lemma.

LEMMA 1: *The fund level process $Z(\sigma)$ is stochastically identical to the fund level process in the compound Poisson model.*

https://doi.org/10.1017/S0269964808000120 Published online by Cambridge University Press

PROOF: Although Z is regulated by X, we show that $Z(\sigma)$ no longer depends on $X(\sigma)$. Given that $X(\sigma(t)) = i$, the fund level increases during $[t, t + dt]$ by an amount

$$ir\,\frac{dt}{i} = r\,dt$$

and the probability that a claim occurs in $[t, t + dt]$ is equal to

$$\mu i\,\frac{dt}{i} = \mu\,dt.$$

Hence, the rate at which the fund level increases is always r and the rate at which a claim occurs is always μ, independent of $X(\sigma)$. In other words, $Z(\sigma)$ is *stochastically identical* to the fund level process in the compound Poisson model. ■

We now turn to the proof of Theorem 2. Since the time transformation η, which dynamically slows down the speed of time and removes time periods during which the fund level stays constant, has no effect on the infinite horizon ruin probability, we can conclude that the ruin probability is the same as in the compound Poisson model. This completes the proof of Theorem 2.

The result in Theorem 2 can also be seen by the following arguments.

LEMMA 2: *The amount received between successive claims is exponentially distributed with rate* μ/r.

PROOF: No matter how much has so far been received since the last claim, the probability that the additional amount will be less than h, given that the current number of active policies is i, is equal to

$$P\left(\exp(i\mu) < \frac{h}{ir}\right) + o(h) = \frac{h\mu}{r} + o(h).$$

So the hazard rate of the distribution of the amount received between claims is μ/r. ■

Since ruin can only occur when a claim is made, Lemma 2 implies that the ruin probability $\psi(x, i)$ is the same as in the compound Poisson model, where the fund increases at constant rate r and claims arrive according to a Poisson process with rate μ, the sizes of which are i.i.d. with distribution G.

Since $\psi(x, i)$ is the same for every realization of the sales process, we immediately obtain the following.

COROLLARY 1: *For an arbitrary arrival process of new policies, the ruin probability* $\psi(x, i)$ *is exactly the same as the ruin probability* $\psi_P(x)$ *in the compound Poisson model.*

https://doi.org/10.1017/S0269964808000120 Published online by Cambridge University Press

The assumption of i.i.d. exponential lifetimes is critical to Theorem 2: The result is no longer true for nonexponential lifetimes or for the *finite* horizon ruin probability. Further research on relaxing these assumptions seems warranted.

Acknowledgment

The authors thank Sheldon Ross for pointing out the alternative proof of Theorem 2 based on Lemma 2.

Reference

1. Rolski, T., Schmidli, H., Schmidt, V., & Teugels, J. (1999). *Stochastic Processes for Insurance and Finance*. Chichester: Wiley.

https://doi.org/10.1017/S0269964808000120 Published online by Cambridge University Press