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(∞, n)-limits I: Definition and first consistency results

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Abstract. We give a model-independent definition of limits for diagrams valued in an (∞, n)-category. We show that this definition is compatible with the existing notion of homotopy 2-limits for 2-categories, with the existing notion of $(\infty, 1)$-limits for $(\infty, 1)$-categories, and with itself across different values of n.

1 Introduction

1.1 The role of (co)limits in the literature

Given a diagram $K\colon \mathcal{J} \to \mathcal{C}$, the property for an object ℓ of $\mathcal{C}$ to be a (weighted/(op)lax/pseudo/conical) limit of K can be summarized by saying that ℓ comes with a (weighted/(op)lax/pseudo/conical) cone over the diagram K and is universal with this property. The notion of limit has been considered in a variety of contexts: for diagrams between strict n-categories – either in a strict or homotopical sense – as well as between (∞, n)-categories for all $n > 0$, and the various variances succeed at capturing many concepts of relevance. In order to motivate the study of limits in all their flavors, we list here various situations where limits play a crucial role.

As a first series of examples, we recall several constructions of interest whose universal property is that of an appropriate (co)limit:

» Given a ring (spectrum) R and an element p of R, the *localization* $R[p^{-1}]$ *of* R *at* p is the colimit (resp. $(\infty,1)$-colimit) of the diagram $\mathbb{N} \to \mathcal{R}ng$ (resp. $(\infty,1)$-diagram $\mathbb{N} \to \mathcal{R}ng(\mathcal{S}p)$) given by

$$R \xrightarrow{-\cdot p} R \xrightarrow{-\cdot p} R \xrightarrow{-\cdot p} \dots.$$

» The $(\infty,1)$-*category* $\mathcal{S}p$ *of spectra* is the $(\infty,1)$-colimit of the $(\infty,1)$-diagram $\mathbb{N} \to \infty\mathcal{C}at$ given by

$$\mathcal{S}_* \xrightarrow{\Sigma} \mathcal{S}_* \xrightarrow{\Sigma} \mathcal{S}_* \xrightarrow{\Sigma} \dots,$$

where $\mathcal{S}_*$ denotes the $(\infty,1)$-category of pointed spaces and Σ is the suspension ∞-functor.

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» Given a prime number p, the *ring of p-adic integers* $\mathbb{Z}_{(p)}$ is the limit of the diagram $\mathbb{N}^{\mathrm{op}} \to \mathcal{R}ng$ given by

$$\cdots \to \mathbb{Z}/p^n \to \mathbb{Z}/p^{n-1} \to \cdots \to \mathbb{Z}/p.$$

» Given a finite group G and a G-space $X \curvearrowleft G$, the *space of homotopy fixed points* X^G *of* X (resp. *homotopy orbit space* $X /\!\!/ G$) is the $(\infty,1)$-limit (resp. $(\infty,1)$-colimit) of the $(\infty,1)$-diagram $BG \to \mathcal{S}$ encoding the action of G on X.

» Let $\mathbb{T}$ be a type theory and $\mathcal{C}_{\mathbb{T}}$ the category associated with the type theory $\mathbb{T}$ (see [Joh02, Section D4]). Given a morphism of types $f\colon Y \to X$ in $\mathbb{T}$ and a proposition $x : X \vdash P[x]$ dependent on the type X, the *substitution* $y : Y \vdash P[f(y)/x]$ dependent on the type Y is the limit of the diagram $[\bullet \to \bullet \leftarrow \bullet] \to \mathcal{C}_{\mathbb{T}}$, that is, the pullback, given by

$$[\![Y]\!] \xrightarrow{[\![f]\!]} [\![X]\!] \xleftarrow{\pi} [\![\sum_{x:X} P[x]]\!],$$

where $[\![X]\!]$ denotes the object of $\mathcal{C}_{\mathbb{T}}$ associated with a given type X. As a result, these kinds of limits are a fundamental building block of categorical logic. An example of categories arising from type theories is topoi, which as part of their definition assume the existence of pullbacks (see [Joh02, Definition A2.1.1]).

Further, there are several constructions whose universal property is that of an appropriate lax or oplax (co)limit, or, more generally, a (co)limit that is *weighted* by a certain functor:

» Given an $(\infty$-)monad T, the *Eilenberg–Moore* $(\infty$-)*category* $\mathcal{A}lg(T)$ *of* $(\infty$-)*algebras over* T is a lax 2-limit (resp. lax $(\infty,2)$-limit) of the 2-diagram (resp. $(\infty,2)$-diagram) $\widetilde{T}\colon \mathcal{M}nd \to \mathcal{C}at$ (resp. $\widetilde{T}\colon \mathcal{M}nd \to \infty\mathcal{C}at$) represented by T, as described in [RV16, Section 6.1]. Similarly, the *Kleisli* $(\infty$-)*category* $\mathcal{K}l(T)$ of T is an oplax 2-colimit (resp. oplax $(\infty,2)$-colimit) of $\widetilde{T}$.

» The ∞-category of global spaces [Sch18], a central object in equivariant homotopy theory, can be computed as the partially lax $(\infty,2)$-limit of an $(\infty,2)$-diagram $\mathcal{G}lo^{\mathrm{op}} \to \infty\mathcal{C}at$, as described in [LNP25, Theorem 6.18].

» Given a presheaf $F\colon \mathcal{J}^{\mathrm{op}} \to \mathcal{C}at$ (resp. $F\colon \mathcal{J}^{\mathrm{op}} \to \infty\mathcal{C}at$), the *Grothendieck construction* $\int_{\mathcal{J}} F$ of F is the lax 2-colimit (resp. lax $(\infty,2)$-colimit) of F, as described in [Str76, Section 5] (resp. [GHN17, Theorem 1.1] or [Ber24, Theorem 4.4]).

» Given a Reedy category Θ and a presheaf $F\colon \Theta^{\mathrm{op}} \to \mathcal{C}$, the *m-th latching object* L_nF (resp. *matching object* M_nF) of F is a certain weighted (co)limit of the diagram $F\colon \Theta^{\mathrm{op}} \to \mathcal{C}$ as described in [RV14, Observation 3.15].

» Given a 2-topos $\mathcal{E}$ in the sense of [Web07] (which conjecturally encodes a directed type theory), one of its requirements is that, for any finite computad $\mathcal{G}$ (in the sense of [Str76, Section 2]), the 2-limit of a 2-diagram $F\colon \mathcal{G} \to \mathcal{E}$ exists.

Finally, (co)limits are used in various contexts to express how global information can be recovered from local information conditions, as in the following examples:

» Let $(\mathcal{C},\mathcal{J})$ be a site and $\mathcal{D}$ a category (resp. $(\infty,1)$-, 2-, $(\infty,2)$-category). Given a *sheaf* (resp. ∞-*sheaf*, *stack*, ∞-*stack*) $F\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$, its defining property is that, given a covering sieve $S = \{U_i \to U\}$ in $\mathcal{J}(U)$, the value $F(U)$ of F at U is the limit (resp.

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$(\infty,1)$-, 2-, $(\infty,2)$-limit) of the diagram (resp. $(\infty,1)$-, 2-, $(\infty,2)$-diagram)

$$(\mathcal{C}^{S}_{/U})^{\mathrm{op}} \xrightarrow{\pi^{\mathrm{op}}} \mathcal{C}^{\mathrm{op}} \xrightarrow{F} \mathcal{D},$$

where $\mathcal{C}^{S}_{/U}$ denotes the full subcategory of $\mathcal{C}_{/U}$ spanned by those objects that are in the covering sieve S (see [Lur09, Section 6.2.2]).

» Given a *good* ∞*-functor* $F\colon \mathcal{M}fld^{\mathrm{op}} \to \mathcal{C}$, its defining property is that, given any increasing sequence $\{U_i\}_{i=0}^{\infty}$ in the $(\infty,1)$-category of manifolds, the value of F at $\cup_{i=0}^{\infty} U_i$ is the $(\infty,1)$-limit of the $(\infty,1)$-diagram

$$\mathbb{N} \xrightarrow{U_-} \mathcal{M}fld^{\mathrm{op}} \xrightarrow{F} \mathcal{C}$$

(see [BdBW13, Section 7]).

» Let $n \geq 0$ and $\mathcal{A}\colon \mathcal{D}isk_n \to \mathcal{C}$ be an $\mathbb{E}_n$-algebra in an $(\infty,1)$-category $\mathcal{C}$. Given an n-manifold M, the *factorization homology* $\int_M \mathcal{A}$ *of* M *with coefficients in* $\mathcal{A}$ is the $(\infty,1)$-colimit of the $(\infty,1)$-diagram

$$\mathcal{D}isk_n \downarrow M \to \mathcal{D}isk_n \xrightarrow{\mathcal{A}} \mathcal{C}$$

(see [AF20, Section 3]). More generally, for an arbitrary (∞,n)-category $\mathcal{C}$ and a framed n-manifold M, we can define its factorization homology as a left Kan extension (see [AFR18] for more details).

The natural progression of these examples, as well as more recent conjectural advances, suggests the need for a proper theory of (∞,n)-(co)limits and (∞,n)-Kan extensions. For example, even the definition of (∞,n)-sheaves requires a definition of (∞,n)-limits. However, beyond that, in the case of $(\infty,1)$-categories, we can leverage a whole theory of $(\infty,1)$-limits to establish *Giraud's theorem* to classify all $(\infty,1)$-categorical sheaves [Lur09]. While recent advances in the theory of $(\infty,2)$-limits have made an $(\infty,2)$-categorical Giraud's theorem possible [AM24], generalizations thereof are beyond our current capabilities and require (∞,n)-limits.

Similarly, the generalization of factorization homology to (∞,n)-categories [AFR18] discussed above, by definition, relies on an $(\infty,1)$-categorical Kan extension. However, (∞,n)-categories naturally assemble into an $(\infty,n+1)$-category, which suggest the possibility of stronger functoriality in this setting, meaning the existence of an $(\infty,n+1)$-functors. Again, even analyzing such a question requires a proper theory of $(\infty,n+1)$-Kan extensions.

Beyond those examples, we are currently observing the growing importance of presentable (∞,n)-categories in the Langlands program and the study of motives [Sch24]. Currently, these methods rely on a recent development of presentable (∞,n)-category [Ste20] as a theory of modules. While this comes with certain benefits (it permits an inductive definition), in the $(\infty,1)$-categorical case, we have many equivalent characterizations of presentability, each one of which plays their own role in the theory and applications [Lur09, Chapter 5]. These equivalent characterizations fundamentally require a careful study of $(\infty,1)$-limits and $(\infty,1)$-Kan extensions, for example, if we want to prove that presentable $(\infty,1)$-categories are $(\infty,1)$-cocomplete. A similar advancement in the theory of presentable (∞,n)-categories, for example,

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trying to show that presentable (∞, n)-categories in the sense of [Ste20] are (∞, n)-cocomplete, requires a suitable theory of (∞, n)-limits.

In addition to these cases, we have recently seen a growing interest in a theory of stability for $(\infty, 2)$-categories [BC21]. While, as the references suggest, there has been some success studying stable $(\infty, 2)$-categorical phenomena directly, it is well-established that, in the $(\infty, 1)$-categorical setting, $(\infty, 1)$-(co)limits are essential for the study of stability. For example, we can characterize stable $(\infty, 1)$-categories via the property that finite limits and colimits commute. Hence, a proper advancement of stable (∞, n)-categories similarly needs a functioning theory of (∞, n)-(co)limits.

Finally, while we already have (∞, n)-categories of spans [Hau18] or (∞, n)-categories of $\mathbb{E}_n$-algebras [Hau17], the construction and properties of their (co)limits have only been analyzed in the $(\infty, 1)$-categorical setting [Lur17, Har20].

The goal of this article is to define a notion of (∞, n)-limit in the model of (∞, n)-categories given by complete Segal objects in $(\infty, n-1)$-categories, which is well-behaved and usable in practice. We further show that this definition recovers the theory of homotopy 2-limits for strict 2-categories and the established notion of $(\infty, 1)$-limit for $(\infty, 1)$-categories. For ease of exposition, we will focus on the treatment of limits and leave the dual treatment of *co*limits to the interested reader. In a follow-up paper [MRR24], we compare this new definition to the definition of (∞, n)-limits in the model of (∞, n)-categories given by categories strictly enriched over $(\infty, n-1)$-categories, which ultimately validates the consistency of the perspectives across different models.

While we primarily focus on conical limits, our framework can in fact accommodate weights. Indeed, in our follow-up paper [MRR24], we generalize our work and introduce weighted limits in the context of complete Segal objects in $(\infty, n-1)$-categories and prove it coincides with the established notion of weighted limits in the context of categories enriched in $(\infty, n-1)$-categories.

1.2 Limits in terms of categories of cones

Aiming at introducing a definition of (∞, n)-limit, we first review the notion of limit in the strict context, emphasizing a fibrational viewpoint. Going back to the notion of limit in the context of ordinary categories, one can recognize a limit object of a diagram $K\colon \mathcal{J} \to \mathcal{C}$ essentially as the terminal object in the category $\mathcal{C}\mathrm{one}_{\mathcal{J}} K$ of cones over K. This can be expressed in terms of its slice $\mathcal{C}\mathrm{one}_{\mathcal{J}} K/(\ell, \lambda)$ at a given cone (ℓ, λ) (see [Rie16, Definition 3.1.6], with ideas originated in [Gro71, Str74]), as follows.

Proposition (Folklore) *Given a diagram $K\colon \mathcal{J} \to \mathcal{C}$ between categories, an object ℓ of $\mathcal{C}$ is a limit object of K if and only if there is an object (ℓ, λ) in $\mathcal{C}\mathrm{one}_{\mathcal{J}} K$ such that the canonical projection induces an isomorphism of categories*

$$\mathcal{C}\mathrm{one}_{\mathcal{J}} K/(\ell, \lambda) \xrightarrow{\cong} \mathcal{C}\mathrm{one}_{\mathcal{J}} K.$$

When trying to formulate an analogous statement for general n, and focusing first on the case $n = 2$, Clingman and the first author proved in [cM22a] that 2-limits do not correspond to 2-terminal objects in any 2-category of cones. This negative answer implies that the world of n-categories is insufficient to formulate such a theorem,

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and instead one needs to pass to a larger framework. This is achieved by regarding each n-category as an internal category to $(n-1)$-categories, a.k.a. a *double* $(n-1)$-*category*, via the canonical embedding $\mathbb{H}$, and make the appropriate adjustments: the n-category of cones should be replaced with an appropriate double $(n-1)$-category $\mathbb{C}\mathrm{one}_{\mathbb{H}\mathcal{J}}(\mathbb{H}K)$ of cones over $\mathbb{H}K$, and the slice $\mathbb{C}\mathrm{one}_{\mathbb{H}\mathcal{J}}(\mathbb{H}K) /\!\!/ (\ell, \lambda)$ should be computed in the category of double $(n-1)$-categories, as opposed to n-categories.

We review some of these constructions in Section 3. With these adjustments, the desired statement was proven in [GP99, Section 4.2(c)] for $n = 2$ and in [MSV23a, Corollary 8.22] for general n.

Theorem (Grandis–Paré and Moser–Sarazola–Verdugo) *Given a diagram $K\colon \mathcal{J} \to \mathcal{C}$ between n-categories, an object ℓ of $\mathcal{C}$ is an n-limit object of K if and only if there is an object (ℓ, λ) in $\mathbb{C}\mathrm{one}_{\mathbb{H}\mathcal{J}}(\mathbb{H}K)$ such that the canonical projection induces an isomorphism of double $(n-1)$-categories*

$$\mathbb{C}\mathrm{one}_{\mathbb{H}\mathcal{J}}(\mathbb{H}K) /\!\!/ (\ell, \lambda) \xrightarrow{\cong} \mathbb{C}\mathrm{one}_{\mathbb{H}\mathcal{J}}(\mathbb{H}K).$$

This result was also proven as [cM22b, Corollary 7.23] in the context of the homotopy theory of 2-categories and homotopy 2-limits in the following form (and in this article, we also give a variant using a more homotopical version of $\mathbb{H}$ as Theorem 4.19).

Theorem (Clingman–Moser) *Given a diagram $K\colon \mathcal{J} \to \mathcal{C}$ between 2-categories, an object ℓ of $\mathcal{C}$ is a homotopy 2-limit object of K if and only if there is an object (ℓ, λ) in $\mathbb{C}\mathrm{one}^{\mathrm{ps}}_{\mathbb{H}\mathcal{J}}(\mathbb{H}K)$ such that the canonical projection is an equivalence of double categories*

$$\mathbb{C}\mathrm{one}^{\mathrm{ps}}_{\mathbb{H}\mathcal{J}}(\mathbb{H}K) /\!\!/^{\mathrm{ps}} (\ell, \lambda) \xrightarrow{\simeq} \mathbb{C}\mathrm{one}^{\mathrm{ps}}_{\mathbb{H}\mathcal{J}}(\mathbb{H}K).$$

Here, the main adjustments consist of defining appropriate versions of the double category of cones $\mathbb{C}\mathrm{one}^{\mathrm{ps}}_{\mathbb{H}\mathcal{J}}(\mathbb{H}K)$ and its slice which are based on the pseudo Gray tensor product from [Böh20] and the internal pseudo-hom for double categories, rather than the ordinary Cartesian product and internal hom (see Section 3 for more details).

Motivated by the evidence that we just described, we are led to introduce, as Definition 2.12, the following notion of (∞, n)-limit for an (∞, n)-diagram between (∞, n)-categories. For the purpose of this article, we define (∞, n)-categories as complete Segal objects in $(\infty, n-1)$-categories. This definition is convenient because it admits a canonical inclusion into double $(\infty, n-1)$-categories, which are Segal objects in $(\infty, n-1)$-categories. In particular, it allows for a simple construction for the double $(\infty, n-1)$-category of cones $\mathrm{Cone}_J^{(\infty,n)} K$ and its slice $\mathrm{Cone}_J^{(\infty,n)} K /\!\!/^{(\infty,n)} (\ell, \lambda)$. Furthermore, when $n = 1$, this definition retrieves the definition of $(\infty, 1)$-limits in the context of $(\infty, 1)$-categories regarded as complete Segal spaces, as in (the dual of) [Ras23, Section 5.2].

Definition 1 Given a diagram $K\colon J \to C$ between (∞, n)-categories, we say that an object ℓ of C is an (∞, n)-*limit object* of K if there is an object (ℓ, λ) in $\mathrm{Cone}_J^{(\infty,n)} K$ such that the canonical projection is an equivalence of double $(\infty, n-1)$-categories

$$\mathrm{Cone}_J^{(\infty,n)} K /\!\!/^{(\infty,n)} (\ell, \lambda) \xrightarrow{\simeq} \mathrm{Cone}_J^{(\infty,n)} K.$$

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After introducing the definition, we prove consistency results and provide proofs of concept for the correctness of this notion.

1.3 Overview of article's results

In this article, we provide the first evidence that the proposed definition of (∞, n)-limit is meaningful.

First, if I_n denotes the embedding of $(\infty, n-1)$-categories into (∞, n)-categories, and we prove, as Theorem 2.20, that the definition is self-consistent across different values of n. In particular, when $n > 1$, this entails that our notion of (∞, n)-limit is compatible with the established notions of $(\infty, 1)$-limits for $(\infty, 1)$-categories.

Theorem A *An object ℓ of an $(\infty, n-1)$-category C is an $(\infty, n-1)$-limit object of a diagram of $(\infty, n-1)$-categories $K\colon J \to C$ if and only if the object ℓ is an (∞, n)-limit object of the diagram of (∞, n)-categories $I_n K\colon I_n J \to I_n C$.*

Second, we consider an appropriate embedding $\mathbb{N}\widetilde{\mathbb{H}}$ of the homotopy theory of strict 2-categories into that of $(\infty, 2)$-categories constructed in [Mos20, Section 6], and we prove, as Theorem 3.14, the consistency of the definition of limit between the contexts of $(\infty, 2)$-categories and of strict 2-categories.

Theorem B *An object ℓ of a 2-category $\mathcal{C}$ is a homotopy 2-limit object of a diagram of 2-categories $K\colon \mathcal{J} \to \mathcal{C}$ if and only if the object ℓ is an $(\infty, 2)$-limit object of the diagram of $(\infty, 2)$-categories $\mathbb{N}\widetilde{\mathbb{H}}K\colon \mathbb{N}\widetilde{\mathbb{H}}\mathcal{J} \to \mathbb{N}\widetilde{\mathbb{H}}\mathcal{C}$.*

These results guarantee that the theory is strong enough to recover two rather orthogonal and nontrivial theories: that of homotopy 2-limits and that of $(\infty, 1)$-limits, and even more generally $(\infty, n-1)$-limits.

Given the importance of (∞, n)-limits in a variety of settings, it is unsurprising that there have been other efforts in this direction. Notable examples include $(\infty, 2)$-limits in [GHL20, Section 5], using an $(\infty, 2)$-categorical Gray tensor product, and (∞, n)-limits in [Lou23, Section 6.2.3], using a representable approach via the Yoneda lemma.

While every approach has their own merit, the crucial advantage of our result is the centrality of an elegant theory of fibrations in the study of (∞, n)-limits, which distinguishes itself from the aforementioned advancements, but is much closer to the original development of $(\infty, 1)$-limits. Indeed, in the $(\infty, 1)$-categorical setting, it is standard practice to construct explicit (right or Cartesian) fibrations that bundle coherence data of interest, this includes the study of $(\infty, 1)$-limits, but also, for example, $(\infty, 1)$-operads and $(\infty, 1)$-monoidal structure [Lur09, Lur17]. These computational settings benefit from the existence of a model structure for right fibrations, which makes certain proofs computationally accessible (an elegant example includes the analysis of $(\infty, 1)$-cofinality [Lur09, Section 4.1]).

Double $(\infty, n-1)$-categories of cones can be realized as fibrant objects in a model structure [Ras21], meaning we can use very analogous techniques already developed in the $(\infty, 1)$-categorical framework. This is in stark contrast to other approaches, which rely on more complicated tools (Gray tensor product) or simply do not provide particular additional methods (the Yoneda approach).

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1.4 ArXiv version

For the convenience of the interested reader, we note here that a previous version of this article, containing the same results, but more detailed and explicit arguments, can be found on the arXiv [MRR23].

2 Limits in an (∞, n)-category

In this section, we introduce the notion of (∞, n)-limit in Section 2.1 and prove that it is self-consistent across different values of n in Section 2.2.

2.1 Limits in an (∞, n)-category

We start by introducing the definitions of *double* $(\infty, n-1)$-*categories* and (∞, n)-*categories* that we will work with throughout the article. For this, recall the following.

Notation 2.1 Let $Cat_{(\infty,n-1)}$ denote the ∞-category of $(\infty, n-1)$-categories in the sense of [BSP21, Definition 7.2].

By [BSP21, Axiom C.3], the ∞-category $Cat_{(\infty,n-1)}$ is Cartesian closed. Moreover, since $Cat_{(\infty,n-1)}$ is presentable, there is a canonical inclusion $\mathcal{S} \hookrightarrow Cat_{(\infty,n-1)}$ from the ∞-category of spaces, sending the point to the terminal $(\infty, n-1)$-category.

Definition 2.2 A *simplicial* $(\infty, n-1)$-*category* is a functor $X\colon \Delta^{\mathrm{op}} \to Cat_{(\infty,n-1)}$, where Δ is the simplex category. We denote by $Fun(\Delta^{\mathrm{op}}, Cat_{(\infty,n-1)})$ the ∞-category of simplicial $(\infty, n-1)$-categories and their maps.

Since the ∞-category of $(\infty, n-1)$-categories is Cartesian closed, the ∞-category of simplicial $(\infty, n-1)$-categories is also Cartesian closed. Given simplicial $(\infty, n-1)$-categories X and X', we denote by $[X, X']$ the internal hom in $Fun(\Delta^{\mathrm{op}}, Cat_{(\infty,n-1)})$.

Definition 2.3 A simplicial $(\infty, n-1)$-category D is a *double* $(\infty, n-1)$-*category* if it is a Segal object in $Cat_{(\infty,n-1)}$. That is, for all $m \geq 2$, the Segal map

$$D_m \to D_1 \times_{D_0} \cdots \times_{D_0} D_1$$

is an equivalence of $(\infty, n-1)$-categories. We denote by $DblCat_{(\infty,n-1)}$ the full reflective subcategory of $Fun(\Delta^{\mathrm{op}}, Cat_{(\infty,n-1)})$ spanned by the double $(\infty, n-1)$-categories.

Remark 2.4 Given a simplicial $(\infty, n-1)$-category J and a double $(\infty, n-1)$-category D, the internal hom $[J, D]$ in $Fun(\Delta^{\mathrm{op}}, Cat_{(\infty,n-1)})$ is a double $(\infty, n-1)$-category. This follows from the argument, including the computation, in [Rez10, Section 6], there articulated using model categorical terminology for this fact.

Definition 2.5 A simplicial $(\infty, n-1)$-category C is an (∞, n)-*category* if it is a complete Segal object in $Cat_{(\infty,n-1)}$ and C_0 is a space. In other words, an (∞, n)-category C is a double $(\infty, n-1)$-category such that the completeness map $C_0 \to C_0 \times_{C_1} C_3 \times_{C_1} C_0$ is an equivalence of $(\infty, n-1)$-categories and the $(\infty, n-1)$-category C_0 is in the essential image of the canonical inclusion $\mathcal{S} \hookrightarrow Cat_{(\infty,n-1)}$.

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Remark 2.6 Given a simplicial $(\infty, n-1)$-category J and an (∞, n)-category C, the internal hom $[J, C]$ in $Fun(\Delta^{op}, Cat_{(\infty,n-1)})$ is *not* necessarily an (∞, n)-category, since the condition that level 0 is a space is not preserved. However, by Remark 2.4, the internal hom $[J, C]$ is a double $(\infty, n-1)$-category.

We are now ready to introduce the main definition of this article, namely, the notion of limit in an (∞, n)-category. With the following definition, we want to emphasize that the source of the diagram of which we want to take the limit does not need to be an (∞, n)-category but can in fact be any simplicial $(\infty, n-1)$-category. This makes the theory of (∞, n)-limits more general and will give a very natural way of formulating *weighted* (∞, n)-limits in the follow-up paper [MRR24].

Definition 2.7 An (∞, n)-*diagram* is a map of simplicial $(\infty, n-1)$-categories $K\colon J \to C$ with C an (∞, n)-category.

Notation 2.8 For $m \geq 0$, we also denote by $[m]$ the image of the object $[m] \in \Delta$ under the canonical inclusion $\Delta \hookrightarrow Fun(\Delta^{op}, \mathcal{S}) \hookrightarrow Fun(\Delta^{op}, Cat_{(\infty,n-1)})$, where the first map is Yoneda. Heuristically, the object $[m] \in Fun(\Delta^{op}, Cat_{(\infty,n-1)})$ represents the (∞, n)-category free on the diagram

$$0 \to 1 \to \ldots \to m-1 \to m.$$

Notation 2.9 Given a simplicial $(\infty, n-1)$-category J and an (∞, n)-category C, we denote by $\Delta^J_C\colon C \to [J, C]$ the diagonal map in $DblCat_{(\infty,n-1)}$ induced by the unique map $J \to [0]$.

Notation 2.10 Given an (∞, n)-diagram $K\colon J \to C$, the *double* $(\infty, n-1)$-*category of cones* $\mathrm{Cone}_J^{(\infty,n)} K$ is the following pullback in the ∞-category $DblCat_{(\infty,n-1)}$:

$$\begin{array}{ccc} \mathrm{Cone}_J^{(\infty,n)} K & \longrightarrow & [[1], [J, C]] \\ \downarrow & \lrcorner & \downarrow{\scriptstyle (ev_0, ev_1)} \\ C \times [0] & \xrightarrow[\Delta^J_C \times K]{} & [J, C] \times [J, C]. \end{array}$$

Notation 2.11 Given a double $(\infty, n-1)$-category D and an object d of D, the *slice double* $(\infty, n-1)$-*category* $D /\!\!/^{(\infty,n)} d$ of D over d is the following pullback in the ∞-category $DblCat_{(\infty,n-1)}$:

$$\begin{array}{ccc} D /\!\!/^{(\infty,n)} d & \longrightarrow & [[1], D] \\ \downarrow & \lrcorner & \downarrow{\scriptstyle (ev_0, ev_1)} \\ D \times [0] & \xrightarrow[D \times d]{} & D \times D. \end{array}$$

The following definition generalizes the viewpoint for the case $n = 1$ from [Joy02, Section 4], [Lur09, Section 4], and [Ras23, Section 5.2].

Definition 2.12 Given an (∞, n)-diagram $K\colon J \to C$, an object ℓ of C is an (∞, n)-*limit object* of K if there is an object (ℓ, λ) in $\mathrm{Cone}_J^{(\infty,n)} K$ such that the canonical

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projection is an equivalence of double $(\infty, n-1)$-categories

$$\mathrm{Cone}_J^{(\infty,n)} K \mathbin{/\!\!/^{(\infty,n)}} (\ell, \lambda) \xrightarrow{\simeq} \mathrm{Cone}_J^{(\infty,n)} K.$$

2.2 Self-consistency with respect to varying *n*

We show that our definition is consistent with respect to the established notion of limits in the context of $(\infty,1)$-categories. More generally, we show as Theorem 2.20 how, for $n > 1$, it is consistent across increasing values of n.

Remark 2.13 By combining [BSP21, Theorem 13.15] and [BR20, Corollary 7.1], the ∞-category $Cat_{(\infty,n-1)}$ is equivalent to the full reflective subcategory of $Fun(\Delta^{\mathrm{op}}, Cat_{(\infty,n-2)})$ spanned by the complete Segal objects in $Cat_{(\infty,n-2)}$ whose level 0 is a space.

Construction 2.14 We construct an inclusion $\iota_{n-1}\colon Cat_{(\infty,n-2)} \hookrightarrow Cat_{(\infty,n-1)}$ by induction that

(1) commutes with the canonical inclusions from $\mathcal{S}$;
(2) is a right adjoint; in particular, it preserves limits;
(3) is fully faithful.

- When $n = 1$, we set $\iota_0\colon \mathcal{S} \to Cat_{(\infty,1)}$ to be the canonical inclusion of the ∞-category of spaces into the ∞-category of $(\infty,1)$-categories, which satisfies (1)–(3).
- When $n > 0$, we define $\iota_{n-1}\colon Cat_{(\infty,n-2)} \hookrightarrow Cat_{(\infty,n-1)}$ to be the restriction to full reflective subcategories (using Remark 2.13) of the functor induced by postcomposing with ι_{n-2}:

$$\begin{array}{ccc} Cat_{(\infty,n-2)} & \xrightarrow{\iota_{n-1}} & Cat_{(\infty,n-1)} \\ \downarrow & & \downarrow \\ Fun(\Delta^{\mathrm{op}}, Cat_{(\infty,n-3)}) & \xrightarrow[(\iota_{n-2})_*]{} & Fun(\Delta^{\mathrm{op}}, Cat_{(\infty,n-2)}). \end{array}$$

This can be seen to restrict appropriately using (1) and (2) for ι_{n-2}. Moreover, (2) and (3) then hold for ι_{n-1} as they hold for ι_{n-2}. The fact that (1) holds for ι_{n-1} follows from the fact that the following square of functors commutes and (2) for ι_{n-2}:

$$\begin{array}{ccc} \mathcal{S} & \hookrightarrow & Cat_{(\infty,n-1)} \\ \downarrow & & \downarrow \\ Fun(\Delta^{\mathrm{op}}, \mathcal{S}) & \hookrightarrow & Fun(\Delta^{\mathrm{op}}, Cat_{(\infty,n-2)}). \end{array}$$

Proposition 2.15 *The left adjoint of $\iota_{n-1}\colon Cat_{(\infty,n-2)} \hookrightarrow Cat_{(\infty,n-1)}$ preserves products.*

Proof We prove this by induction on $n \geq 1$. When $n = 1$, this is clear.

So, let $n > 1$. Let $CS(Cat_{(\infty,n-2)})$ be the full reflective subcategory of $Fun(\Delta^{\mathrm{op}}, Cat_{(\infty,n-2)})$ spanned by the complete Segal objects in $Cat_{(\infty,n-2)}$, and denote by L_{CS} the left adjoint of the inclusion. As a first step, we show that if X is a simplicial object in $Cat_{(\infty,n-2)}$ such that X_0 is a space, then $(L_{CS}X)_0$ is a space as well.

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Define $P(Cat_{(\infty,n-2)})$ to be the full reflective subcategory of $Fun(\Delta^{op}, Cat_{(\infty,n-2)})$ spanned by those $X\colon \Delta^{op} \to Cat_{(\infty,n-2)}$ such that X_0 is a space. We then have the following adjunctions:

$$\begin{array}{ccc} Cat_{(\infty,n-1)} & \underset{R'}{\overset{L'_P}{\underset{\longleftarrow}{\overset{\longleftarrow}{\hookrightarrow}}}} & CS(Cat_{(\infty,n-2)}) \\ \Big\uparrow L'_{CS} \;\dashv\; \Big\downarrow & & \Big\uparrow L_{CS} \;\dashv\; \Big\downarrow \\ P(Cat_{(\infty,n-2)}) & \underset{R}{\overset{L_P}{\underset{\longleftarrow}{\overset{\longleftarrow}{\hookrightarrow}}}} & Fun(\Delta^{op}, Cat_{(\infty,n-2)}). \end{array}$$

The left adjoints of the inclusions are given by the corresponding localization functors, and the right adjoint $R\colon Fun(\Delta^{op}, Cat_{(\infty,n-2)}) \to P(Cat_{(\infty,n-2)})$ of the inclusion sends a simplicial object $X\colon \Delta^{op} \to Cat_{(\infty,n-2)}$ to the pullback

$$\begin{array}{ccc} RX & \longrightarrow & \mathrm{cosk}((X_0)^{\simeq}) \\ \downarrow & \lrcorner & \downarrow \\ X & \longrightarrow & \mathrm{cosk} X_0, \end{array}$$

where $\mathrm{cosk}\colon Cat_{(\infty,n-2)} \to Fun(\Delta^{op}, Cat_{(\infty,n-2)})$ is the right adjoint of the evaluation at 0 functor $(-)_0\colon Fun(\Delta^{op}, Cat_{(\infty,n-2)}) \to Cat_{(\infty,n-2)}$ and $(-)^{\simeq}\colon Cat_{(\infty,n-2)} \to \mathcal{S}$ is the right adjoint of the canonical inclusion $\mathcal{S} \hookrightarrow Cat_{(\infty,n-2)}$. In particular, since it preserves complete Segal objects and $Cat_{(\infty,n-1)}$ is the full subcategory of $P(Cat_{(\infty,n-2)})$ spanned by the complete Segal objects, it restricts to a right adjoint $R'\colon CS(Cat_{(\infty,n-2)}) \to Cat_{(\infty,n-1)}$ of the inclusion. Since R and R' commute with the inclusions, the corresponding left adjoints commute. Hence, for $X \in P(Cat_{(\infty,n-2)})$, there is an equivalence $L_{CS}X \simeq L'_{CS}X$ in $CS(Cat_{(\infty,n-2)})$, showing that $(L_{CS}X)_0$ is a space.

Now consider the following diagram of adjunctions:

$$\begin{array}{ccccc} Cat_{(\infty,n-2)} & \underset{\perp}{\overset{L'_P}{\underset{\longleftarrow}{\overset{\longleftarrow}{\hookrightarrow}}}} & CS(Cat_{(\infty,n-3)}) & \underset{\perp}{\overset{L_{CS}}{\underset{\longleftarrow}{\overset{\longleftarrow}{\hookrightarrow}}}} & Fun(\Delta^{op}, Cat_{(\infty,n-3)}) \\ \Big\uparrow L_{n-1} \;\dashv\; \Big\downarrow \iota_{n-1} & & \Big\uparrow L_{CS}(L_{n-2})_* \;\dashv\; \Big\downarrow (\iota_{n-2})_* & & \Big\uparrow (L_{n-2})_* \;\dashv\; \Big\downarrow (\iota_{n-2})_* \\ Cat_{(\infty,n-1)} & \underset{\perp}{\overset{L'_P}{\underset{\longleftarrow}{\overset{\longleftarrow}{\hookrightarrow}}}} & CS(Cat_{(\infty,n-2)}) & \underset{\perp}{\overset{L_{CS}}{\underset{\longleftarrow}{\overset{\longleftarrow}{\hookrightarrow}}}} & Fun(\Delta^{op}, Cat_{(\infty,n-2)}). \end{array}$$

By induction, the left adjoint

$$(L_{n-2})_*\colon Fun(\Delta^{op}, Cat_{(\infty,n-2)}) \to Fun(\Delta^{op}, Cat_{(\infty,n-3)})$$

preserves products. By [BR20, Proposition 5.9], the localization functor

$$L_{CS}\colon Fun(\Delta^{op}, Cat_{(\infty,n-2)}) \to CS(Cat_{(\infty,n-2)})$$

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preserves products, and so the left adjoint

$$L_{CS}(L_{n-2})_*\colon CS(Cat_{(\infty,n-2)}) \to CS(Cat_{(\infty,n-3)})$$

also preserves products. Finally, since L_{CS} and $(L_{n-2})_*$ preserve the property that level 0 is a space, the left adjoint $L_{n-1}\colon Cat_{(\infty,n-1)} \to Cat_{(\infty,n-2)}$ coincides with the restriction of $L_{CS}(L_{n-2})_*$ to full reflective subcategories, and so it preserves products. ∎

The inclusion ι_{n-1} induces by postcomposition an inclusion

$$(\iota_{n-1})_*\colon Fun(\Delta^{\mathrm{op}}, Cat_{(\infty,n-2)}) \hookrightarrow Fun(\Delta^{\mathrm{op}}, Cat_{(\infty,n-1)}).$$

Since ι_{n-1} preserves limits, it preserves Segal objects, and so $(\iota_{n-1})_*$ restricts to an inclusion

$$I_n\colon DblCat_{(\infty,n-2)} \to DblCat_{(\infty,n-1)}.$$

We list the properties of this inclusion that we will need below.

Proposition 2.16 *The inclusion $I_n\colon DblCat_{(\infty,n-2)} \to DblCat_{(\infty,n-1)}$*

(1) *sends $(\infty, n-1)$-categories to (∞, n)-categories;*
(2) *is fully faithful; in particular, it is conservative;*
(3) *is a right adjoint; in particular, it preserves limits.*

Moreover, we also have that

(4) *the left adjoint of I_n preserves products.*

Proof This follows directly from Construction 2.14 and Proposition 2.15. ∎

Moreover, the inclusion I_n is compatible with the internal hom as follows.

Proposition 2.17 *Given a simplicial $(\infty, n-2)$-category J and a double $(\infty, n-2)$-category D, there is an equivalence of double $(\infty, n-1)$-categories*

$$[I_n J, I_n D] \xrightarrow{\simeq} I_n[J, D],$$

natural in J and D.

Proof This follows directly from Proposition 2.16(4). ∎

With these, we can show that the inclusion I_n preserves cone and slice constructions.

Lemma 2.18 *Given an $(\infty, n-1)$-diagram $K\colon J \to C$, there is an equivalence of double $(\infty, n-1)$-categories*

$$\mathrm{Cone}^{(\infty,n)}_{I_n J}(I_n K) \xrightarrow{\simeq} I_n(\mathrm{Cone}^{(\infty,n-1)}_{J} K).$$

Proof By definition, we have a pullback in $DblCat_{(\infty,n-2)}$

$$\begin{array}{ccc} \mathrm{Cone}^{(\infty,n-1)}_{J} K & \longrightarrow & [[1],[J,C]] \\ \downarrow & \lrcorner & \downarrow{\scriptstyle (\mathrm{ev}_0,\mathrm{ev}_1)} \\ C \times [0] & \xrightarrow[\Delta^J_C \times K]{} & [J,C] \times [J,C]. \end{array}$$

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Hence, by Proposition 2.16(3), we also have a pullback in $DblCat_{(\infty,n-1)}$

$$\begin{array}{ccc} I_n(\mathrm{Cone}_J^{(\infty,n-1)}K) & \longrightarrow & I_n[[1],[J,C]] \\ \downarrow & \lrcorner & \downarrow{\scriptstyle I_n(\mathrm{ev}_0,\mathrm{ev}_1)} \\ I_nC\times[0] & \xrightarrow[I_n\Delta_C^J\times K]{} & I_n[J,C]\times I_n[J,C]. \end{array}$$

On the other hand, we have, by definition, a pullback in $DblCat_{(\infty,n-1)}$

$$\begin{array}{ccc} \mathrm{Cone}_{I_nJ}^{(\infty,n)}(I_nK) & \longrightarrow & [[1],[I_nJ,I_nC]] \\ \downarrow & \lrcorner & \downarrow{\scriptstyle (\mathrm{ev}_0,\mathrm{ev}_1)} \\ I_nC\times[0] & \xrightarrow[\Delta_{I_nC}^{I_nJ}\times I_nK]{} & [I_nJ,I_nC]\times[I_nJ,I_nC]. \end{array}$$

Using Proposition 2.17 and the fact that $I_n[1]\simeq[1]$, there is a natural comparison map between the two cospans defining the above pullbacks in $DblCat_{(\infty,n-1)}$, which is pointwise an equivalence of double $(\infty,n-1)$-categories. Hence, this induces a unique map between the two double $(\infty,n-1)$-categories presenting the pullback

$$\mathrm{Cone}_{I_nJ}^{(\infty,n)}(I_nK)\longrightarrow I_n(\mathrm{Cone}_J^{(\infty,n-1)}K),$$

and this map is an equivalence of double $(\infty,n-1)$-categories, as desired. ∎

Lemma 2.19 *Given a double $(\infty,n-2)$-category D and an object d of D, there is an equivalence of double $(\infty,n-1)$-categories over I_nD*

$$(I_nD)\mathbin{/\!\!/}^{(\infty,n)}d\xrightarrow{\simeq}I_n(D\mathbin{/\!\!/}^{(\infty,n-1)}d).$$

Proof The proof follows the same steps as the one of Lemma 2.18. ∎

We can now prove the main result.

Theorem 2.20 *Given an $(\infty,n-1)$-diagram $K\colon J\to C$, an object ℓ of C is an $(\infty,n-1)$-limit object of the $(\infty,n-1)$-diagram K if and only if the object ℓ is an (∞,n)-limit object of the (∞,n)-diagram $I_nK\colon I_nJ\to I_nC$.*

Proof We have that an object ℓ of C is an $(\infty,n-1)$-limit object of the $(\infty,n-1)$-diagram $K\colon J\to C$ if and only if, by definition, there is an object (ℓ,λ) in $\mathrm{Cone}_J^{(\infty,n-1)}K$ such that the canonical projection is an equivalence of double $(\infty,n-2)$-categories

$$\mathrm{Cone}_J^{(\infty,n-1)}K\mathbin{/\!\!/}^{(\infty,n-1)}(\ell,\lambda)\xrightarrow{\simeq}\mathrm{Cone}_J^{(\infty,n-1)}K$$

if and only if, by Proposition 2.16(2), its image under I_n is an equivalence of double $(\infty,n-1)$-categories

$$I_n(\mathrm{Cone}_J^{(\infty,n-1)}K\mathbin{/\!\!/}^{(\infty,n-1)}(\ell,\lambda))\xrightarrow{\simeq}I_n(\mathrm{Cone}_J^{(\infty,n-1)}K)$$

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if and only if, by Lemma 2.19, the canonical projection is an equivalence of double $(\infty, n-1)$-categories

$$I_n(\mathrm{Cone}_J^{(\infty,n-1)}K) \mathbin{/\!\!/^{(\infty,n)}} (\ell, \lambda) \xrightarrow{\simeq} I_n(\mathrm{Cone}_J^{(\infty,n-1)}K)$$

if and only if, by Lemma 2.18, the canonical projection is an equivalence of double $(\infty, n-1)$-categories

$$\mathrm{Cone}_{I_nJ}^{(\infty,n)}(I_nK) \mathbin{/\!\!/^{(\infty,n)}} (\ell, \lambda) \xrightarrow{\simeq} \mathrm{Cone}_{I_nJ}^{(\infty,n)}(I_nK)$$

if and only if, by definition, the object ℓ of C is an (∞, n)-limit object of the induced (∞, n)-diagram $I_nK\colon I_nJ \to I_nC$, as desired. ∎

3 Homotopy limits in a 2-category

We revisit, in Section 3.1, the notion of homotopy 2-limit presenting an alternative viewpoint that makes use of double categories as a richer and larger environment to study 2-categories. Then, in Theorem 3.14, we prove that our notion of $(\infty, 2)$-limit is compatible with the notion of homotopy 2-limit.

3.1 Homotopy limits in a 2-category

In this section, we state a characterization of homotopy 2-limits in a double categorical language. This result will be proven in Section 4.2. We start by recalling the definition of a homotopy 2-limit.

Definition 3.1 A *2-diagram* is a 2-functor $K\colon \mathcal{J} \to \mathcal{C}$.

Given two 2-categories $\mathcal{J}$ and $\mathcal{C}$, we denote by $[\mathcal{J}, \mathcal{C}]^{\mathrm{ps}}$ the 2-category of 2-functors $\mathcal{J} \to \mathcal{C}$, pseudo-natural transformations, and modifications.

Definition 3.2 Given a 2-diagram $K\colon \mathcal{J} \to \mathcal{C}$, an object ℓ of $\mathcal{C}$ is a *homotopy 2-limit object* of K if there is a pseudo-cone $\lambda\colon \Delta\ell \Rightarrow K$ that induces, for every object c of $\mathcal{C}$, an equivalence of categories

$$\lambda^*\colon \mathcal{C}(c, \ell) \xrightarrow{\simeq} [\mathcal{J}, \mathcal{C}]^{\mathrm{ps}}(\Delta c, K).$$

Remark 3.3 The notion of homotopy 2-limit above is sometimes referred to as *pseudo-bilimit*. It corresponds to the notion of homotopy limit in the canonical model structure on $2Cat$ from [Lac02] using the definition from [Shu06].

To state the desired characterization, we first recall the notion of a *double category*.

Definition 3.4 A *double category* is a category internal to – that is, a strict Segal object in – the category $\mathcal{C}at$, and a *double functor* is an internal functor. We denote by $Dbl\mathcal{C}at$ the category of double categories and double functors.

Notation 3.5 We denote by $[DblCat]_\infty$ the underlying ∞-category (in the sense of [Ber09, Theorem 6.2]) of the model structure on $DblCat$ from [MSV23b, Theorem 3.26] (see Section 4.1 for further details).

Every 2-category $\mathcal{C}$ can be seen as a double category $\widetilde{\mathbb{H}}\mathcal{C}$ in $[DblCat]_\infty$ and this assignment extends to 2-functors (see Construction 4.7).

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The ∞-category $[DblCat]_\infty$ is Cartesian closed, as we prove in Corollary 4.13. Given two double categories $\mathbb{D}$ and $\mathbb{D}'$ in $[DblCat]_\infty$, we denote by $[\![\mathbb{D}, \mathbb{D}']\!]^h$ the internal hom in $[DblCat]_\infty$.

Notation 3.6 Given a 2-diagram $K\colon \mathcal{J} \to \mathcal{C}$, the *double category* $\mathbb{C}\mathrm{one}^h_{\widetilde{\mathbb{H}}\mathcal{J}}(\widetilde{\mathbb{H}}K)$ *of cones* over K is the following pullback in $[Dbl\mathcal{C}at]_\infty$:

$$\begin{array}{ccc} \mathbb{C}\mathrm{one}^h_{\widetilde{\mathbb{H}}\mathcal{J}}(\widetilde{\mathbb{H}}K) & \longrightarrow & [\![\widetilde{\mathbb{H}}[1], [\![\widetilde{\mathbb{H}}\mathcal{J}, \widetilde{\mathbb{H}}\mathcal{C}]\!]^h]\!]^h \\ \downarrow & \lrcorner & \downarrow{\scriptstyle (\mathrm{ev}_0, \mathrm{ev}_1)} \\ \widetilde{\mathbb{H}}\mathcal{C} \times \widetilde{\mathbb{H}}[0] & \xrightarrow[\Delta^{\widetilde{\mathbb{H}}\mathcal{J}}_{\widetilde{\mathbb{H}}\mathcal{C}} \times \widetilde{\mathbb{H}}K]{} & [\![\widetilde{\mathbb{H}}\mathcal{J}, \widetilde{\mathbb{H}}\mathcal{C}]\!]^h \times [\![\widetilde{\mathbb{H}}\mathcal{J}, \widetilde{\mathbb{H}}\mathcal{C}]\!]^h. \end{array}$$

Notation 3.7 Given a double category $\mathbb{D}$ in $[DblCat]_\infty$ and an object d of $\mathbb{D}$, the *slice double category* $\mathbb{D} /\!\!/^h d$ of $\mathbb{D}$ over d is the following pullback in $[Dbl\mathcal{C}at]_\infty$:

$$\begin{array}{ccc} \mathbb{D} /\!\!/^h d & \longrightarrow & [\![\widetilde{\mathbb{H}}[1], \mathbb{D}]\!]^h \\ \downarrow & \lrcorner & \downarrow{\scriptstyle (\mathrm{ev}_0, \mathrm{ev}_1)} \\ \mathbb{D} \times \widetilde{\mathbb{H}}[0] & \xrightarrow[\mathbb{D} \times d]{} & \mathbb{D} \times \mathbb{D}. \end{array}$$

We prove the following as Theorem 4.19.

Theorem 3.8 *Given a* 2*-diagram* $K\colon \mathcal{J} \to \mathcal{C}$*, an object* ℓ *of* $\mathcal{C}$ *is a homotopy* 2*-limit object of* K *if and only if there is an object* (ℓ, λ) *in* $\mathbb{C}\mathrm{one}^h_{\widetilde{\mathbb{H}}\mathcal{J}}(\widetilde{\mathbb{H}}K)$ *such that the canonical projection is an equivalence in* $[DblCat]_\infty$

$$\mathbb{C}\mathrm{one}^h_{\widetilde{\mathbb{H}}\mathcal{J}}(\widetilde{\mathbb{H}}K) /\!\!/ (\ell, \lambda) \xrightarrow{\simeq} \mathbb{C}\mathrm{one}^h_{\widetilde{\mathbb{H}}\mathcal{J}}(\widetilde{\mathbb{H}}K).$$

3.2 Consistency with respect to strict context

In this section, we show as Theorem 3.14 how, for $n = 2$, the new notion of $(\infty, 2)$-limit is consistent with the strict context. We focus here on the main result and postpone the technical details to Section 4.3.

Notation 3.9 We denote by

$$\mathbb{N}\colon [Dbl\mathcal{C}at]_\infty \to Dbl\mathcal{C}at_{(\infty,1)}$$

the functor induced by the nerve functor constructed in [Mos20, Section 5.1] (see Section 4.3 for further details).

We collect here the properties of $\mathbb{N}$ needed for the main result.

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Proposition 3.10 *The functor* $\mathbb{N}\colon [DblCat]_\infty \to DblCat_{(\infty,1)}$
(1) *sends 2-categories to* $(\infty,2)$*-categories, that is, for every 2-category* $\mathcal{C}$*, the double* $(\infty,1)$*-category* $\mathbb{N}\widetilde{\mathbb{H}}\mathcal{C}$ *is an* $(\infty,2)$*-category;*
(2) *is fully faithful; in particular, it is conservative;*
(3) *is a right adjoint; in particular, it preserves limits.*
Moreover, we have that
(4) *the left adjoint of* $\mathbb{N}$ *preserves products.*

Proof First, note that (1) is a direct consequence of Theorem 4.36(3), while (2) and (3) follow from the fact that $\mathbb{N}\colon [DblCat]_\infty \to DblCat_{(\infty,1)}$ has a left adjoint and the counit of this adjunction is an equivalence in $[DblCat]_\infty$ by Theorem 4.36(1) and (2). Finally, (4) is proven in Corollary 4.44. ∎

Proposition 3.11 *Given a 2-category* $\mathcal{J}$ *and a double category* $\mathbb{D}$ *in* $[DblCat]_\infty$*, there is an equivalence of double* $(\infty,1)$*-categories*

$$\mathbb{N}[\![\widetilde{\mathbb{H}}\mathcal{J}, \mathbb{D}]\!]^{\mathrm{ps}} \xrightarrow{\simeq} [\mathbb{N}\widetilde{\mathbb{H}}\mathcal{J}, \mathbb{N}\mathbb{D}].$$

Proof This follows directly from Proposition 3.10(4). ∎

With these, we can show that the nerve $\mathbb{N}$ preserves cone and slice constructions.

Lemma 3.12 *Given a 2-diagram* $K\colon \mathcal{J} \to \mathcal{C}$*, there is an equivalence of double* $(\infty,1)$*-categories*

$$\mathbb{N}(\mathbb{C}\mathrm{one}^h_{\widetilde{\mathbb{H}}\mathcal{J}}(\widetilde{\mathbb{H}}K)) \xrightarrow{\simeq} \mathrm{Cone}^{(\infty,2)}_{\mathbb{N}\widetilde{\mathbb{H}}\mathcal{J}}(\mathbb{N}\widetilde{\mathbb{H}}K).$$

Proof By definition, we have a pullback in $[Dbl\mathcal{C}at]_\infty$

$$\begin{array}{ccc} \mathbb{C}\mathrm{one}^h_{\widetilde{\mathbb{H}}\mathcal{J}}(\widetilde{\mathbb{H}}K) & \longrightarrow & [\![\widetilde{\mathbb{H}}[1], [\![\widetilde{\mathbb{H}}\mathcal{J}, \widetilde{\mathbb{H}}\mathcal{C}]\!]^h]\!]^h \\ \downarrow & \lrcorner & \downarrow{\scriptstyle (\mathrm{ev}_0, \mathrm{ev}_1)} \\ \widetilde{\mathbb{H}}\mathcal{C} \times \widetilde{\mathbb{H}}[0] & \xrightarrow[\Delta^{\widetilde{\mathbb{H}}\mathcal{J}}_{\widetilde{\mathbb{H}}\mathcal{C}} \times \widetilde{\mathbb{H}}K]{} & [\![\widetilde{\mathbb{H}}\mathcal{J}, \widetilde{\mathbb{H}}\mathcal{C}]\!]^h \times [\![\widetilde{\mathbb{H}}\mathcal{J}, \widetilde{\mathbb{H}}\mathcal{C}]\!]^h. \end{array}$$

Hence, by Proposition 3.10(3), we also have a pullback in $DblCat_{(\infty,1)}$

$$\begin{array}{ccc} \mathbb{N}(\mathbb{C}\mathrm{one}^h_{\widetilde{\mathbb{H}}\mathcal{J}}(\widetilde{\mathbb{H}}K)) & \longrightarrow & \mathbb{N}[\![\widetilde{\mathbb{H}}[1], [\![\widetilde{\mathbb{H}}\mathcal{J}, \widetilde{\mathbb{H}}\mathcal{C}]\!]^h]\!]^h \\ \downarrow & \lrcorner & \downarrow{\scriptstyle \mathbb{N}(\mathrm{ev}_0, \mathrm{ev}_1)} \\ \mathbb{N}\widetilde{\mathbb{H}}\mathcal{C} \times \mathbb{N}\widetilde{\mathbb{H}}[0] & \xrightarrow[\mathbb{N}\Delta^{\widetilde{\mathbb{H}}\mathcal{J}}_{\widetilde{\mathbb{H}}\mathcal{C}} \times \mathbb{N}\widetilde{\mathbb{H}}K]{} & \mathbb{N}[\![\widetilde{\mathbb{H}}\mathcal{J}, \widetilde{\mathbb{H}}\mathcal{C}]\!]^h \times \mathbb{N}[\![\widetilde{\mathbb{H}}\mathcal{J}, \widetilde{\mathbb{H}}\mathcal{C}]\!]^h. \end{array}$$

On the other hand, we have, by definition, a pullback in $DblCat_{(\infty,1)}$

$$\begin{array}{ccc} \mathrm{Cone}^{(\infty,2)}_{\mathbb{N}\widetilde{\mathbb{H}}\mathcal{J}}(\mathbb{N}\widetilde{\mathbb{H}}K) & \longrightarrow & [[1], [\mathbb{N}\widetilde{\mathbb{H}}\mathcal{J}, \mathbb{N}\widetilde{\mathbb{H}}\mathcal{C}]] \\ \downarrow & \lrcorner & \downarrow{\scriptstyle (\mathrm{ev}_0, \mathrm{ev}_1)} \\ \mathbb{N}\widetilde{\mathbb{H}}\mathcal{C} \times [0] & \xrightarrow[\Delta^{\mathbb{N}\widetilde{\mathbb{H}}\mathcal{J}}_{\mathbb{N}\widetilde{\mathbb{H}}\mathcal{C}} \times \mathbb{N}\widetilde{\mathbb{H}}K]{} & [\mathbb{N}\widetilde{\mathbb{H}}\mathcal{J}, \mathbb{N}\widetilde{\mathbb{H}}\mathcal{C}] \times [\mathbb{N}\widetilde{\mathbb{H}}\mathcal{J}, \mathbb{N}\widetilde{\mathbb{H}}\mathcal{C}]. \end{array}$$

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Using Proposition 3.11 and the fact that $\mathbb{N}\widetilde{\mathbb{H}}[0] \cong [0]$ and $\mathbb{N}\widetilde{\mathbb{H}}[1] \cong [1]$, there is a natural comparison map between the two cospans defining the above pullbacks in $DblCat_{(\infty,1)}$, which is pointwise an equivalence of double $(\infty,1)$-categories. Hence, this induces a unique map between the two double $(\infty,1)$-categories presenting the pullback

$$\mathbb{N}(\mathbb{C}\mathrm{one}^{h}_{\widetilde{\mathbb{H}}\mathcal{J}}(\widetilde{\mathbb{H}}K)) \longrightarrow \mathrm{Cone}^{(\infty,2)}_{\mathbb{N}\widetilde{\mathbb{H}}\mathcal{J}}(\mathbb{N}\widetilde{\mathbb{H}}K),$$

and this map is an equivalence of double $(\infty,1)$-categories, as desired. ∎

Lemma 3.13 *Given a double category $\mathbb{D}$ in $[DblCat]_\infty$ and an object d of $\mathbb{D}$, there is an equivalence of double $(\infty,1)$-categories over $\mathbb{N}\mathbb{D}$*

$$\mathbb{N}(\mathbb{D} /\!\!/^{h} d) \xrightarrow{\simeq} (\mathbb{N}\mathbb{D}) /\!\!/^{(\infty,2)} d.$$

Proof The proof follows the same steps as the one of Lemma 3.12. ∎

We can now prove the main result.

Theorem 3.14 *Given a 2-diagram $K\colon \mathcal{J} \to \mathcal{C}$, an object ℓ of $\mathcal{C}$ is a homotopy 2-limit object of the 2-diagram $K\colon \mathcal{J} \to \mathcal{C}$ if and only if the object ℓ is an $(\infty,2)$-limit object of the $(\infty,2)$-diagram $\mathbb{N}\widetilde{\mathbb{H}}K\colon \mathbb{N}\widetilde{\mathbb{H}}\mathcal{J} \to \mathbb{N}\widetilde{\mathbb{H}}\mathcal{C}$.*

Proof We have that an object ℓ of $\mathcal{C}$ is a homotopy 2-limit object of the 2-diagram $K\colon \mathcal{J} \to \mathcal{C}$ if and only if, by Theorem 4.19, there is an object (ℓ,λ) in $\mathbb{C}\mathrm{one}^{\mathrm{ps}}_{\widetilde{\mathbb{H}}\mathcal{J}}(\widetilde{\mathbb{H}}K)$ such that the canonical projection is an equivalence in $[DblCat]_\infty$

$$\mathbb{C}\mathrm{one}^{h}_{\widetilde{\mathbb{H}}\mathcal{J}}(\widetilde{\mathbb{H}}K) /\!\!/^{h} (\ell,\lambda) \xrightarrow{\simeq} \mathbb{C}\mathrm{one}^{h}_{\widetilde{\mathbb{H}}\mathcal{J}}(\widetilde{\mathbb{H}}K)$$

if and only if, by Proposition 3.10(2), its image under $\mathbb{N}$ is an equivalence of double $(\infty,1)$-categories

$$\mathbb{N}(\mathbb{C}\mathrm{one}^{h}_{\widetilde{\mathbb{H}}\mathcal{J}}(\widetilde{\mathbb{H}}K) /\!\!/^{h} (\ell,\lambda)) \xrightarrow{\simeq} \mathbb{N}(\mathbb{C}\mathrm{one}^{h}_{\widetilde{\mathbb{H}}\mathcal{J}}(\widetilde{\mathbb{H}}K))$$

if and only if, using 2-out-of-3 and Lemma 3.13, the canonical projection is an equivalence of double $(\infty,1)$-categories

$$\mathbb{N}(\mathbb{C}\mathrm{one}^{h}_{\widetilde{\mathbb{H}}\mathcal{J}}(\widetilde{\mathbb{H}}K)) /\!\!/^{(\infty,2)} (\ell,\lambda) \xrightarrow{\simeq} \mathbb{N}(\mathbb{C}\mathrm{one}^{h}_{\widetilde{\mathbb{H}}\mathcal{J}}(\widetilde{\mathbb{H}}K))$$

if and only if, by Lemma 3.12, the canonical projection is an equivalence of double $(\infty,1)$-categories

$$\mathrm{Cone}^{(\infty,2)}_{\mathbb{N}\widetilde{\mathbb{H}}\mathcal{J}}(\mathbb{N}\widetilde{\mathbb{H}}K) /\!\!/^{(\infty,2)} (\ell,\lambda) \xrightarrow{\simeq} \mathrm{Cone}^{(\infty,2)}_{\mathbb{N}\widetilde{\mathbb{H}}\mathcal{J}}(\mathbb{N}\widetilde{\mathbb{H}}K)$$

if and only if, by definition, the object ℓ of $\mathcal{C}$ is an $(\infty,2)$-limit object of the induced $(\infty,2)$-diagram $\mathbb{N}\widetilde{\mathbb{H}}K\colon \mathbb{N}\widetilde{\mathbb{H}}\mathcal{J} \to \mathbb{N}\widetilde{\mathbb{H}}\mathcal{C}$, as desired. ∎

4 Auxiliary results

In this last section, we prove the technical details needed in Section 3. First, in Section 4.1, we recall the main features of the model structure on $DblCat$ for weakly

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horizontally invariant double categories from [MSV23b], needed to define the ∞-category $[\mathit{DblCat}]_\infty$. Moreover, we prove that $[\mathit{DblCat}]_\infty$ is Cartesian closed and provide tools to compute double categories of cones and slice double categories in $[\mathit{DblCat}]_\infty$. Then, in Section 4.2, we prove the double categorical characterization of homotopy 2-limits stated in Theorem 3.8.

Finally, in Section 4.3, we recall the nerve construction from double categories to double $(\infty,1)$-categories from [Mos20] and prove the properties of the nerve stated in Proposition 3.10.

4.1 Model structure for weakly horizontally invariant double categories

In this section, we recall from [MSV23b] the main definitions and facts about the homotopy theory of *weakly horizontally invariant* double categories up to *double biequivalence*.

When unpacking the definition of a double category, one sees that a double category $\mathbb{D}$ consists of a category $\mathbb{D}_0$ of objects and a category $\mathbb{D}_1$ of morphisms, together with appropriate domain and codomain maps, composition and identity maps. We refer to $\mathbb{D}_0$ as the category of objects and *vertical* morphisms and to $\mathbb{D}_1$ as the category of *horizontal* morphisms and *squares*. A double functor is an assignment that preserves strictly the whole structure.

Construction 4.1 There is a canonical fully faithful functor

$$\mathbb{H}\colon 2\mathcal{C}at \to \mathit{DblCat},$$

which assigns to a 2-category $\mathcal{C}$ the double category $\mathbb{H}\mathcal{C}$ whose objects are the objects of $\mathcal{C}$, whose horizontal morphisms are the morphisms of $\mathcal{C}$, whose vertical morphisms are all trivial, and whose squares are the 2-morphisms in $\mathcal{C}$.

The functor $\mathbb{H}\colon 2\mathcal{C}at \to \mathit{DblCat}$ admits a right adjoint $\mathbf{H}\colon \mathit{DblCat} \to 2\mathcal{C}at$, which sends a double category $\mathbb{D}$ to its underlying horizontal 2-category $\mathbf{H}\mathbb{D}$, essentially obtained by forgetting the vertical morphisms of $\mathbb{D}$.

There is a canonical inclusion $\mathbb{V}\colon \mathcal{C}at \to \mathit{DblCat}$, which sends a category $\mathcal{A}$ to the double category $\mathbb{V}\mathcal{A}$ whose objects are the objects of $\mathcal{A}$ and whose vertical morphisms are the morphisms of $\mathcal{A}$; the horizontal morphisms and squares in $\mathbb{V}\mathcal{A}$ are all trivial.

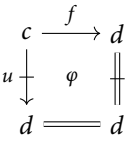
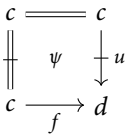
Moreover, the category DblCat is Cartesian closed. Given double categories $\mathbb{X}$ and $\mathbb{X}'$, we denote by $[\![\mathbb{X}, \mathbb{X}']\!]$ the internal hom in DblCat.

Definition 4.2 [MSV23b, Definition 2.9] and [MSV22, Proposition 3.11] A double functor $F\colon \mathbb{D} \to \mathbb{D}'$ is a *double biequivalence* if F induces biequivalences of 2-categories

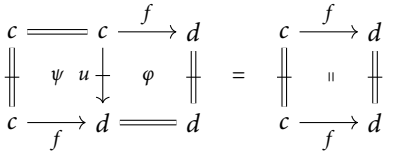
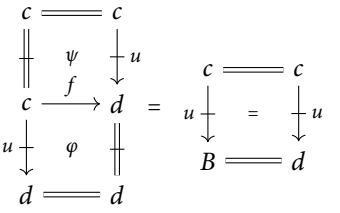
$$\mathbf{H}F\colon \mathbf{H}\mathbb{D} \to \mathbf{H}\mathbb{D}' \quad \text{and} \quad \mathbf{H}[\![\mathbb{V}[1], F]\!]\colon \mathbf{H}[\![\mathbb{V}[1], \mathbb{D}]\!] \to \mathbf{H}[\![\mathbb{V}[1], \mathbb{D}']\!].$$

Definition 4.3 A *companion pair* in a double category $\mathbb{D}$ is a tuple (f, u, φ, ψ) consisting of a horizontal morphism $f\colon c \to d$, a vertical morphism $u\colon c \nrightarrow d$, and two squares

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satisfying the following pasting equalities:

We say that a horizontal morphism f (resp. a vertical morphism u) *has a vertical (resp. horizontal) companion* if there is a companion pair (f, u, φ, ψ).

Definition 4.4 [MSV23b, Definition 2.10] and [GMSV23, Proposition 5.6] A double category $\mathbb{D}$ is *weakly horizontally invariant* if every horizontal equivalence in $\mathbb{D}$, that is, an equivalence in $\mathbf{H}\mathbb{D}$, has a vertical companion.

The homotopy theory of weakly horizontally invariant double categories and double biequivalences is supported by a model structure.

Theorem 4.5 [MSV23b, Theorem 3.26] and [GMSV23, Theorem 5.10] *There is a model structure on the category $Dbl\mathcal{C}at$ in which*

- *the fibrant objects are the weakly horizontally invariant double categories;*
- *the trivial fibrations are the double functors that are surjective on objects, full on horizontal and vertical morphisms, and fully faithful on squares;*
- *the weak equivalences between fibrant objects are precisely the double biequivalences.*

We denote this model structure by $Dbl\mathcal{C}at_{\mathrm{whi}}$.

Recall that the ∞-category $[DblCat]_\infty$ from Notation 3.5 is, by definition, the underlying ∞-category of the model structure $DblCat_{\mathrm{whi}}$, in the sense of [Ber09, Theorem 6.2].

Remark 4.6 The weakly horizontal invariance is a sort of Reedy fibrancy condition, which naturally appears when trying to build a model structure on $DblCat$, compatible with the canonical model structure on $2Cat$ and with "symmetric" trivial fibrations as described in Theorem 4.5. Indeed, by [GMSV23, Proposition 5.6], a double category $\mathbb{D}$ is weakly horizontally invariant if and only if the 2-functor

$$(\mathrm{ev}_0, \mathrm{ev}_1)\colon \mathbf{H}[\![\mathbb{V}[1], \mathbb{D}]\!] \to \mathbf{H}\mathbb{D} \times \mathbf{H}\mathbb{D}$$

is a fibration of 2-categories. Moreover, by [Mos20, Theorem 5.30], this is a necessary and sufficient condition for the nerve $\mathbb{N}\mathbb{D}$ of a double category $\mathbb{D}$ to be Reedy fibrant.

Given a 2-category $\mathcal{C}$, the double category $\mathbb{H}\mathcal{C}$ is generally *not* weakly horizontally invariant. Hence, in order to recover 2-categories as weakly horizontally invariant double categories, we shall consider the following refined construction.

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Construction 4.7 We denote by

$$\widetilde{\mathbb{H}}\colon 2\mathcal{C}at \to Dbl\mathcal{C}at,$$

the functor from [MSV23b, Definition 2.13]. The functor $\widetilde{\mathbb{H}}$ assigns to a 2-category $\mathcal{C}$ the double category $\widetilde{\mathbb{H}}\mathcal{C}$ whose objects are the objects of $\mathcal{C}$, whose horizontal morphisms are the morphisms of $\mathcal{C}$, whose vertical morphisms are the adjoint equivalence data in $\mathcal{C}$, and whose squares are the 2-morphisms in $\mathcal{C}$ of the form

$$\begin{array}{ccc} c & \longrightarrow & d \\ {\scriptstyle\simeq}\downarrow & \swarrow & \downarrow{\scriptstyle\simeq} \\ c' & \longrightarrow & d' \end{array}.$$

Remark 4.8 As every identity is an equivalence, for any 2-category $\mathcal{C}$, there is a canonical inclusion $I_{\mathcal{C}}\colon \mathbb{H}\mathcal{C} \to \widetilde{\mathbb{H}}\mathcal{C}$. If $\mathcal{C}$ has no non-identity equivalence, this inclusion is an isomorphism $\mathbb{H}\mathcal{C} \cong \widetilde{\mathbb{H}}\mathcal{C}$ in $Dbl\mathcal{C}at$. In particular, we have isomorphisms in $Dbl\mathcal{C}at$

$$\mathbb{H}[0] \cong \widetilde{\mathbb{H}}[0] \quad \text{and} \quad \mathbb{H}[1] \cong \widetilde{\mathbb{H}}[1].$$

Proposition 4.9 [MSV23b, Theorem 6.5] *Given a 2-category $\mathcal{C}$, the double category $\widetilde{\mathbb{H}}\mathcal{C}$ is weakly horizontally invariant, and the canonical inclusion $I_{\mathcal{C}}\colon \mathbb{H}\mathcal{C} \to \widetilde{\mathbb{H}}\mathcal{C}$ is a double biequivalence.*

Remark 4.10 Given a 2-category $\mathcal{C}$, for each equivalence $f\colon c \to d$ in $\mathcal{C}$, the corresponding horizontal morphism $f\colon c \to d$ and vertical morphism $f\colon c \nrightarrow d$ form a companion pair in $\widetilde{\mathbb{H}}\mathcal{C}$ with squares induced by the identity 2-morphism at f.

While the category $DblCat$ is Cartesian closed, the model structure $DblCat_{\mathrm{whi}}$ is not compatible with the Cartesian product (see [MSV22, Remark 5.1]). Instead, one needs to refine it and consider the double categorical analog of the pseudo Gray tensor product $\boxtimes^{\mathrm{ps}}\colon Dbl\mathcal{C}at \times Dbl\mathcal{C}at \to Dbl\mathcal{C}at$ introduced in [Böh20, Section 3].

Theorem 4.11 [MSV23b, Theorem 7.8] *The model category $Dbl\mathcal{C}at_{\mathrm{whi}}$ is a closed monoidal model category with respect to the pseudo Gray tensor product $\boxtimes^{\mathrm{ps}}$.*

Lemma 4.12 [MSV23b, Lemma 7.3] *Given double categories $\mathbb{J}$ and $\mathbb{J}'$, there is a canonical trivial fibration*

$$\mathbb{J} \boxtimes^{\mathrm{ps}} \mathbb{J}' \to \mathbb{J} \times \mathbb{J}'$$

natural in $\mathbb{J}$ and $\mathbb{J}'$.

By putting the above results together, we get the following corollary.

Corollary 4.13 *The ∞-category $[DblCat]_\infty$ is Cartesian closed.*

Proof By Theorem 4.11, the ∞-category $[DblCat]_\infty$ has a closed monoidal structure induced by the pseudo Gray tensor product $\boxtimes^{\mathrm{ps}}$, and by Lemma 4.12, this monoidal structure is equivalent to the one induced by the Cartesian product. ∎

Given double categories $\mathbb{J}$ and $\mathbb{D}$, we denote by $[\![\mathbb{J}, \mathbb{D}]\!]^{\mathrm{ps}}$ the internal pseudo-hom associated with the pseudo Gray tensor product on $DblCat$.

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Remark 4.14 Given double categories $\mathbb{J}$ and $\mathbb{D}$, we provide a short description of the double category given by the internal pseudo-hom $[\![\mathbb{J}, \mathbb{D}]\!]^{\mathrm{ps}}$. An object in $[\![\mathbb{J}, \mathbb{D}]\!]^{\mathrm{ps}}$ is a double functor $\mathbb{J} \to \mathbb{D}$. A horizontal morphism in $[\![\mathbb{J}, \mathbb{D}]\!]^{\mathrm{ps}}$ between double functors $F, G\colon \mathbb{J} \to \mathbb{D}$ is a *pseudo-natural horizontal transformation* $\varphi\colon F \Rightarrow G$, consisting of

- for each object j in $\mathbb{J}$, a horizontal morphism $\varphi_j\colon Fj \to Gj$ in $\mathbb{D}$,
- for each vertical morphism $u\colon j \nrightarrow j'$ in $\mathbb{J}$, a square φ_u in $\mathbb{D}$ of the form

$$\begin{array}{ccc} Fj & \xrightarrow{\varphi_j} & Gj \\ {\scriptstyle Fu}\downarrow & \varphi_u & \downarrow{\scriptstyle Gu} \\ Fj' & \xrightarrow[\varphi_{j'}]{} & Gj' \end{array}$$

- for each horizontal morphism $f\colon j \to k$ in $\mathbb{J}$, a vertically invertible square φ_f in $\mathbb{D}$ of the form

$$\begin{array}{ccccc} Fj & \xrightarrow{Ff} & Fk & \xrightarrow{\varphi_k} & Gk \\ \| & & \varphi_f \cong & & \| \\ Fj & \xrightarrow[\varphi_j]{} & Gj & \xrightarrow[Gf]{} & Gk \end{array}$$

such that the components φ_u are functorial in u, the components φ_f are functorial in f, and together they are natural with respect to squares in $\mathbb{J}$. Similarly, the vertical morphisms in $[\![\mathbb{J}, \mathbb{D}]\!]^{\mathrm{ps}}$ are the *pseudo-natural vertical transformations* obtained by interchanging the roles of the horizontal and vertical morphisms. Finally, a square α in $[\![\mathbb{J}, \mathbb{D}]\!]^{\mathrm{ps}}$ as depicted below left is a *modification* consisting of, for each object j in $\mathbb{J}$, a square α_j in $\mathbb{D}$ as depicted below right,

$$\begin{array}{ccc} F & \xRightarrow{\varphi} & G \\ {\scriptstyle \mu}\Downarrow & \alpha & \Downarrow{\scriptstyle \nu} \\ F' & \xRightarrow[\varphi']{} & G' \end{array} \qquad \begin{array}{ccc} Fj & \xrightarrow{\varphi_j} & Gj \\ {\scriptstyle \mu_j}\downarrow & \alpha_j & \downarrow{\scriptstyle \nu_j} \\ F'j & \xrightarrow[\varphi'_j]{} & G'j \end{array}$$

such that the components α_j are appropriately compatible with the data of the horizontal and vertical transformations.

Understanding these internal homs and fibrancy conditions is of utmost importance when studying the ∞-category $[DblCat]_\infty$. Indeed, in general, there is no easy way to compute the internal hom between two arbitrary objects in the underlying Cartesian closed ∞-category of a model category, as even simple mapping spaces are generally described via *Hammock localization* (see [Ber09, Definition 3.4]). However, in our case, as the model structure $Dbl\mathcal{C}at_{\mathrm{whi}}$ is closed monoidal with respect to the pseudo Gray tensor product, we get from the argument in [Hov99, Section 5.6], that, for a cofibrant double category $\mathbb{J}$ and a fibrant double category $\mathbb{D}$, that is, a weakly horizontally invariant one, the ∞-categorical internal hom $[\![\mathbb{J}, \mathbb{D}]\!]^h$ is equivalent to the internal pseudo-hom $[\![\mathbb{J}, \mathbb{D}]\!]^{\mathrm{ps}}$. Hence, for a cofibrant double category $\mathbb{J}$ and a weakly horizontally invariant double category $\mathbb{D}$, the internal hom in the ∞-category $[DblCat]_\infty$ is given by $[\![\mathbb{J}, \mathbb{D}]\!]^{\mathrm{ps}}$.

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The next technical lemma gives us access to the ∞-categorical internal hom in a broader range of situations.

Lemma 4.15 *Let $\mathbb{D}$ be a double category. Then the functor*

$$[\![\widetilde{\mathbb{H}}(-), \mathbb{D}]\!]^{\mathrm{ps}} \colon 2\mathcal{C}at \to DblCat$$

takes biequivalences to weak equivalences in the model structure $DblCat_{\mathrm{whi}}$.

Proof Let $F \colon \mathcal{J}' \to \mathcal{J}$ be a biequivalence of 2-categories. Then F admits a pseudo-inverse, that is, there is a pseudo-functor $G \colon \mathcal{J} \to \mathcal{J}'$ together with pseudo-natural equivalences $\eta \colon \mathrm{id}_{\mathcal{J}} \simeq F \circ G$ and $\varepsilon \colon G \circ F \simeq \mathrm{id}_{\mathcal{J}'}$. In particular, this induces a double biequivalence $\widetilde{\mathbb{H}}F \colon \widetilde{\mathbb{H}}\mathcal{J}' \to \widetilde{\mathbb{H}}\mathcal{J}$ that admits a horizontal pseudo-inverse given by the data of the induced double pseudo-functor $\widetilde{\mathbb{H}}G \colon \widetilde{\mathbb{H}}\mathcal{J} \to \widetilde{\mathbb{H}}\mathcal{J}'$ and the induced horizontal pseudo-natural equivalences $\widetilde{\mathbb{H}}\eta \colon \mathrm{id}_{\widetilde{\mathbb{H}}\mathcal{J}} \simeq \widetilde{\mathbb{H}}F \circ \widetilde{\mathbb{H}}G$ and $\widetilde{\mathbb{H}}\varepsilon \colon \widetilde{\mathbb{H}}G \circ \widetilde{\mathbb{H}}F \simeq \mathrm{id}_{\widetilde{\mathbb{H}}\mathcal{J}'}$. This in turn implies that, for every double category $\mathbb{D}$, the induced double functor $(\widetilde{\mathbb{H}}F)^* \colon [\![\widetilde{\mathbb{H}}\mathcal{J}', \mathbb{D}]\!]^{\mathrm{ps}} \to [\![\widetilde{\mathbb{H}}\mathcal{J}, \mathbb{D}]\!]^{\mathrm{ps}}$ has a horizontal pseudo-inverse given by the data $((\widetilde{\mathbb{H}}G)^*, (\widetilde{\mathbb{H}}\eta)^*, (\widetilde{\mathbb{H}}\varepsilon)^*)$. Hence, it is a horizontal biequivalence in the sense of [MSV23b, Definition 8.8]. Then it is a double biequivalence by [MSV23b, Proposition 8.11], and so a weak equivalence in the model structure $DblCat_{\mathrm{whi}}$ by [MSV23b, Proposition 3.25]. ∎

Remark 4.16 Let $\mathcal{J}$ be a 2-category, and $\widehat{\mathcal{J}} \to \mathcal{J}$ be a cofibrant replacement. Notice that both cofibrant 2-categories and double categories are those for which the underlying categories are free (for the case of 2-categories, see [Lac02, Theorem 4.8] and the case of double categories, see [MSV23b, Corollary 3.13]). By construction, this property is preserved by $\widetilde{\mathbb{H}}$, meaning $\widetilde{\mathbb{H}}\widehat{\mathcal{J}}$ is cofibrant. Moreover, for a weakly horizontally invariant double category $\mathbb{D}$, we have the following chain of equivalences in $[DblCat]_\infty$:

$$[\![\widetilde{\mathbb{H}}\mathcal{J}, \mathbb{D}]\!]^h \simeq [\![\widetilde{\mathbb{H}}\widehat{\mathcal{J}}, \mathbb{D}]\!]^h \simeq [\![\widetilde{\mathbb{H}}\widehat{\mathcal{J}}, \mathbb{D}]\!]^{\mathrm{ps}} \simeq [\![\widetilde{\mathbb{H}}\mathcal{J}, \mathbb{D}]\!]^{\mathrm{ps}},$$

where the last equivalence follows from Lemma 4.15. Hence, for $\mathcal{J}$ a 2-category and $\mathbb{D}$ a weakly horizontally invariant double category, the ∞-categorical internal hom $[\![\widetilde{\mathbb{H}}\mathcal{J}, \mathbb{D}]\!]^h$ is already equivalent to the internal pseudo-hom $[\![\widetilde{\mathbb{H}}\mathcal{J}, \mathbb{D}]\!]^{\mathrm{ps}}$.

Hence, throughout we always consider internal pseudo-homs with domain of the form $\widetilde{\mathbb{H}}\mathcal{J}$, for $\mathcal{J}$ a 2-category, and codomain a weakly horizontally invariant double category.

Remark 4.17 Let $K \colon \mathcal{J} \to \mathcal{C}$ be a 2-diagram. The double category $\mathbb{C}\mathrm{one}^h_{\widetilde{\mathbb{H}}\mathcal{J}}(\widetilde{\mathbb{H}}K)$ in $[DblCat]_\infty$ from Notation 3.6 is equivalent to the *double category* $\mathbb{C}\mathrm{one}^{\mathrm{ps}}_{\widetilde{\mathbb{H}}\mathcal{J}}(\widetilde{\mathbb{H}}K)$ *of pseudo-cones* over $\widetilde{\mathbb{H}}K$, given by the strict pullback in $DblCat$ (which is a homotopy pullback in $DblCat_{\mathrm{whi}}$)

$$\begin{array}{ccc} \mathbb{C}\mathrm{one}^{\mathrm{ps}}_{\widetilde{\mathbb{H}}\mathcal{J}}(\widetilde{\mathbb{H}}K) & \longrightarrow & [\![\mathbb{H}[1], [\![\widetilde{\mathbb{H}}\mathcal{J}, \widetilde{\mathbb{H}}\mathcal{C}]\!]^{\mathrm{ps}}]\!]^{\mathrm{ps}} \\ \downarrow & \lrcorner & \downarrow{\scriptstyle (\mathrm{ev}_0, \mathrm{ev}_1)} \\ \widetilde{\mathbb{H}}\mathcal{C} \times \mathbb{H}[0] & \xrightarrow[\Delta^{\widetilde{\mathbb{H}}\mathcal{J}}_{\widetilde{\mathbb{H}}\mathcal{C}} \times \widetilde{\mathbb{H}}K]{} & [\![\widetilde{\mathbb{H}}\mathcal{J}, \widetilde{\mathbb{H}}\mathcal{C}]\!]^{\mathrm{ps}} \times [\![\widetilde{\mathbb{H}}\mathcal{J}, \widetilde{\mathbb{H}}\mathcal{C}]\!]^{\mathrm{ps}}. \end{array}$$

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In particular, if $\mathcal{J}$ is a 2-category with no equivalences (which is usually the case of the diagrams we like to consider), the double category $\mathbb{C}\mathrm{one}^{\mathrm{ps}}_{\widetilde{\mathbb{H}}\mathcal{J}}(\widetilde{\mathbb{H}}K)$ can be described as follows:

- its objects are pairs (c,κ) of an object $c \in \mathcal{C}$ and a pseudo-cone $\kappa\colon \Delta c \Rightarrow K$;
- its vertical morphisms $(c,\kappa) \to (c',\kappa')$ are pairs (u,μ) of an equivalence $u\colon c \xrightarrow{\simeq} c'$ in $\mathcal{C}$ and a modification $\mu\colon \kappa \to \kappa' \circ \Delta u$;
- its horizontal morphisms $(c,\kappa) \to (d,\lambda)$ are pairs (f,φ) of a morphism $f\colon c \to d$ in $\mathcal{C}$ and an invertible modification $\varphi\colon \kappa \xrightarrow{\cong} \lambda \circ \Delta f$;
- its squares

$$\begin{array}{ccc} (c,\kappa) & \xrightarrow{(f,\varphi)} & (d,\lambda) \\ {\scriptstyle (u,\mu)}\downarrow & \Downarrow & \downarrow{\scriptstyle (v,\nu)} \\ (c',\kappa') & \xrightarrow[(f',\varphi')]{} & (d',\lambda') \end{array}$$

are 2-morphisms $\alpha\colon vf \Rightarrow f'u$ in $\mathcal{C}$ making the following diagram of modifications commute:

$$\begin{array}{ccccc} \kappa & & \xrightarrow{\quad\mu\quad} & & \kappa' \circ \Delta u \\ {\scriptstyle \varphi}\downarrow & & & & \downarrow{\scriptstyle \varphi' \circ \Delta u} \\ \lambda \circ \Delta f & \xrightarrow[\nu \circ \Delta f]{} & \lambda' \circ \Delta \nu \circ \Delta f & \xrightarrow[\lambda' \circ \Delta \alpha]{} & \lambda' \circ \Delta f' \circ \Delta u. \end{array}$$

Remark 4.18 Let $\mathbb{D}$ be a weakly horizontally invariant double category and d in $\mathbb{D}$ be an object. The double category $\mathbb{D} /\!\!/^h d$ in $[DblCat]_\infty$ from Notation 3.7 is equivalent to the strict pullback in $DblCat$ (which is a homotopy pullback in $DblCat_{\mathrm{whi}}$)

$$\begin{array}{ccc} \mathbb{D} /\!\!/^{\mathrm{ps}} d & \longrightarrow & [\![\mathbb{H}[1], \mathbb{D}]\!]^{\mathrm{ps}} \\ \downarrow & \lrcorner & \downarrow{\scriptstyle (\mathrm{ev}_0, \mathrm{ev}_1)} \\ \mathbb{D} \times \mathbb{H}[0] & \xrightarrow[\mathbb{D} \times d]{} & \mathbb{D} \times \mathbb{D}. \end{array}$$

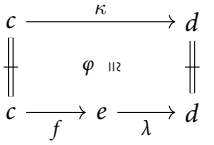
In particular, the double category $\mathbb{D} /\!\!/^{\mathrm{ps}} d$ can be described as follows:

- its objects are pairs (c,κ) of an object $c \in \mathbb{D}$ and a horizontal morphism $\kappa\colon c \to d$;
- its vertical morphisms $(c,\kappa) \to (c',\kappa')$ are pairs (u,μ) of a vertical morphism $u\colon c \to c'$ in $\mathbb{D}$ and a square μ in $\mathbb{D}$ of the form

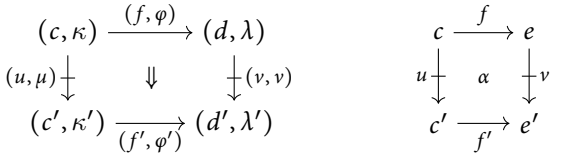
$$\begin{array}{ccc} c & \xrightarrow{\kappa} & d \\ {\scriptstyle u}\downarrow & \mu & \| \\ c' & \xrightarrow[\kappa']{} & d \end{array}$$

- its horizontal morphisms $(c,\kappa) \to (e,\lambda)$ are pairs (f,φ) of a horizontal morphism $f\colon c \to e$ in $\mathbb{D}$ and a vertically invertible square φ in $\mathbb{D}$ of the form

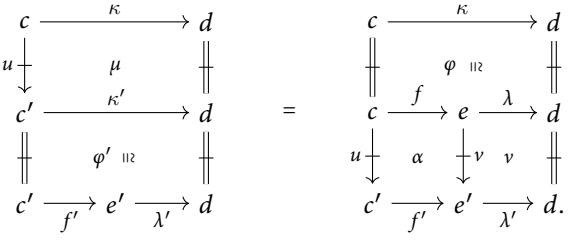
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- its squares as below right



are squares α in $\mathbb{D}$ as above right satisfying the following pasting equality:



4.2 Characterization of homotopy 2-limits

The goal of this section is to prove the characterization of homotopy 2-limits stated in Theorem 3.8. For this, we first reformulate the statement in the model categorical language.

Theorem 4.19 *Given a 2-diagram $K\colon \mathcal{J} \to \mathcal{C}$, an object ℓ of $\mathcal{C}$ is a homotopy 2-limit object of K if and only if there is an object (ℓ, λ) in $\mathbb{C}\mathrm{one}^{\mathrm{ps}}_{\widetilde{\mathbb{H}}\mathcal{J}}(\widetilde{\mathbb{H}}K)$ such that the canonical projection defines a double biequivalence*

$$\mathbb{C}\mathrm{one}^{\mathrm{ps}}_{\widetilde{\mathbb{H}}\mathcal{J}}(\widetilde{\mathbb{H}}K) \mathbin{/\!\!/^{\mathrm{ps}}} (\ell, \lambda) \xrightarrow{\simeq} \mathbb{C}\mathrm{one}^{\mathrm{ps}}_{\widetilde{\mathbb{H}}\mathcal{J}}(\widetilde{\mathbb{H}}K).$$

There is an analogous construction $\mathbb{C}\mathrm{one}^{\mathrm{ps}}_{\mathbb{H}\mathcal{J}}(\mathbb{H}K)$ obtained by replacing all instances of $\widetilde{\mathbb{H}}$ with $\mathbb{H}$, and together with the construction $\mathbb{D} \mathbin{/\!\!/^{\mathrm{ps}}} d$, they are fruitful to detect homotopy 2-limits.

Theorem 4.20 [cM22b, Corollary 7.22] *Given a 2-diagram $K\colon \mathcal{J} \to \mathcal{C}$, an object ℓ of $\mathcal{C}$ is a homotopy 2-limit object of K if and only if there is an object (ℓ, λ) in $\mathbb{C}\mathrm{one}^{\mathrm{ps}}_{\mathbb{H}\mathcal{J}}(\mathbb{H}K)$ such that the canonical projection defines a double biequivalence*

$$\mathbb{C}\mathrm{one}^{\mathrm{ps}}_{\mathbb{H}\mathcal{J}}(\mathbb{H}K) \mathbin{/\!\!/^{\mathrm{ps}}} (\ell, \lambda) \xrightarrow{\simeq} \mathbb{C}\mathrm{one}^{\mathrm{ps}}_{\mathbb{H}\mathcal{J}}(\mathbb{H}K).$$

To prove Theorem 4.19, we will use the characterization from Theorem 4.20 and show that, given a 2-diagram $K\colon \mathcal{J} \to \mathcal{C}$, an object ℓ of $\mathcal{C}$, and a pseudo-cone λ with

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summit ℓ over K, there is a zig-zag of double biequivalences connecting the canonical projection

$$\mathbb{C}\mathrm{one}^{\mathrm{ps}}_{\mathbb{H}\mathcal{J}}(\mathbb{H}K) \mathbin{/\!\!/^{\mathrm{ps}}} (\ell, \lambda) \longrightarrow \mathbb{C}\mathrm{one}^{\mathrm{ps}}_{\mathbb{H}\mathcal{J}}(\mathbb{H}K)$$

and the canonical projection

$$\mathbb{C}\mathrm{one}^{\mathrm{ps}}_{\widetilde{\mathbb{H}}\mathcal{J}}(\widetilde{\mathbb{H}}K) \mathbin{/\!\!/^{\mathrm{ps}}} (\ell, \lambda) \longrightarrow \mathbb{C}\mathrm{one}^{\mathrm{ps}}_{\widetilde{\mathbb{H}}\mathcal{J}}(\widetilde{\mathbb{H}}K).$$

We first collect some homotopical facts about the pseudo-hom that are not formal.

Lemma 4.21 *Given a double category $\mathbb{J}$ and a 2-category $\mathcal{C}$, the double category $[\![\mathbb{J}, \widetilde{\mathbb{H}}\mathcal{C}]\!]^{\mathrm{ps}}$ is weakly horizontally invariant.*

Proof Given a horizontal pseudo-natural equivalence $\alpha\colon F \xRightarrow{\simeq} G\colon \mathbb{J} \to \widetilde{\mathbb{H}}\mathcal{C}$, we construct a vertical pseudo-natural transformation $\beta\colon F \Rightarrow G$ as follows:

- for an object i in $\mathbb{J}$, set $\beta_i \coloneqq \alpha_i\colon Fi \xrightarrow{\simeq} Gi$ to be the component of α at the object i, which is an equivalence and hence a vertical morphism in $\widetilde{\mathbb{H}}\mathcal{C}$;
- for a horizontal morphism $f\colon i \to j$ in $\mathbb{J}$, set $\beta_f \coloneqq \alpha_f\colon \alpha_j \circ Ff \Rightarrow Gf \circ \alpha_i$ to be the pseudo-naturality constraint of α;
- for a vertical morphism $u\colon i \to j$ in $\mathbb{J}$, set $\beta_u \coloneqq \alpha_u\colon \alpha_j \circ Fu \Rightarrow Gu \circ \alpha_i$ to be the component on vertical morphism of α, which is an invertible 2-morphism by [Mos20, Lemma A.2.3] as α_u is a weakly horizontally invertible square.

The horizontal pseudo-naturality of α then implies the vertical pseudo-naturality of β. Moreover, it is not hard to see that (α, β) forms a companion pair. ∎

Lemma 4.22 *Let $F\colon \mathbb{J} \to \mathbb{J}'$ be a trivial fibration in $Dbl\mathcal{C}at_{\mathrm{whi}}$ and $\mathbb{D}$ be a double category. Then the induced double functor*

$$F^*\colon [\![\mathbb{J}', \mathbb{D}]\!]^{\mathrm{ps}} \to [\![\mathbb{J}, \mathbb{D}]\!]^{\mathrm{ps}}$$

is a double biequivalence.

Proof Since F is surjective on objects, full on horizontal and vertical morphisms, and fully faithful on squares, one can construct a normal pseudo double functor $G\colon \mathbb{J}' \to \mathbb{J}$ such that $FG = \mathrm{id}_{\mathbb{J}'}$ and a horizontal pseudo-natural equivalence $\eta\colon \mathrm{id}_{\mathbb{J}} \simeq GF$. Applying the functor $[\![-, \mathbb{D}]\!]^{\mathrm{ps}}$, this induces data (F^*, G^*, η^*) with $F^*\colon [\![\mathbb{J}', \mathbb{D}]\!]^{\mathrm{ps}} \to [\![\mathbb{J}, \mathbb{D}]\!]^{\mathrm{ps}}$ a double functor, $G^*\colon [\![\mathbb{J}, \mathbb{D}]\!]^{\mathrm{ps}} \to [\![\mathbb{J}', \mathbb{D}]\!]^{\mathrm{ps}}$ a normal pseudo double functor such that $G^*F^* = \mathrm{id}_{[\![\mathbb{J}', \mathbb{D}]\!]^{\mathrm{ps}}}$, and $\eta^*\colon \mathrm{id}_{[\![\mathbb{J}, \mathbb{D}]\!]^{\mathrm{ps}}} \simeq F^*G^*$ a horizontal pseudo-natural transformation. This shows that F^* is a horizontal biequivalence in the sense of [MSV23b, Definition 8.8], and so a double biequivalence by [MSV23b, Proposition 8.11]. ∎

Lemma 4.23 *Given 2-categories $\mathcal{J}$ and $\mathcal{C}$, the double functor*

$$I^*_{\mathcal{J}}\colon [\![\widetilde{\mathbb{H}}\mathcal{J}, \widetilde{\mathbb{H}}\mathcal{C}]\!]^{\mathrm{ps}} \to [\![\mathbb{H}\mathcal{J}, \widetilde{\mathbb{H}}\mathcal{C}]\!]^{\mathrm{ps}}$$

induced by the canonical inclusion $I_{\mathcal{J}}\colon \mathbb{H}\mathcal{J} \to \widetilde{\mathbb{H}}\mathcal{J}$ is a double biequivalence.

Proof Recall from Proposition 4.9 that the inclusion $I_{\mathcal{J}}\colon \mathbb{H}\mathcal{J} \to \widetilde{\mathbb{H}}\mathcal{J}$ is a double biequivalence. We factor $I_{\mathcal{J}}$ as a trivial cofibration followed by a trivial fibration in $Dbl\mathcal{C}at_{\mathrm{whi}}$

$$\mathbb{H}\mathcal{J} \xhookrightarrow{\simeq} (\mathbb{H}\mathcal{J})^{\mathrm{fib}} \xrightarrow{\simeq} \widetilde{\mathbb{H}}\mathcal{J}.$$

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By applying the functor $[\![-, \widetilde{\mathbb{H}}\mathcal{C}]\!]^{\mathrm{ps}}$, we get a factorization of $I_{\mathcal{J}}^*$

$$[\![\mathbb{H}\mathcal{J}, \widetilde{\mathbb{H}}\mathcal{C}]\!]^{\mathrm{ps}} \xleftarrow{\sim} [\![(\mathbb{H}\mathcal{J})^{\mathrm{fib}}, \widetilde{\mathbb{H}}\mathcal{C}]\!]^{\mathrm{ps}} \xleftarrow{\sim} [\![\widetilde{\mathbb{H}}\mathcal{J}, \widetilde{\mathbb{H}}\mathcal{C}]\!]^{\mathrm{ps}},$$

where the left-hand double functor is a trivial fibration in $Dbl\mathcal{C}at_{\mathrm{whi}}$ by Theorem 4.11, and the right-hand one is a double biequivalence by Lemma 4.22. This shows that $I_{\mathcal{J}}^*$ is a weak equivalence in $Dbl\mathcal{C}at_{\mathrm{whi}}$ and so a double biequivalence as all objects involved are weakly horizontally invariant by Lemma 4.21. ∎

We now turn to the proof of Theorem 4.19. For this, we start by studying the cone constructions $\mathbb{C}\mathrm{one}^{\mathrm{ps}}_{\widetilde{\mathbb{H}}\mathcal{J}}(\widetilde{\mathbb{H}}K)$ and $\mathbb{C}\mathrm{one}^{\mathrm{ps}}_{\mathbb{H}\mathcal{J}}(\mathbb{H}K)$ and see how they are related. First, we modify $\mathbb{C}\mathrm{one}^{\mathrm{ps}}_{\widetilde{\mathbb{H}}\mathcal{J}}(\widetilde{\mathbb{H}}K)$ by precomposing the diagram $\widetilde{\mathbb{H}}K$ with the canonical inclusion $I_{\mathcal{J}}\colon \mathbb{H}\mathcal{J} \to \widetilde{\mathbb{H}}\mathcal{J}$.

Notation 4.24 Given a double functor $K\colon \mathbb{J} \to \mathbb{D}$, let $\mathbb{C}\mathrm{one}^{\mathrm{ps}}_{\mathbb{J}}(K)$ denote the pullback in $Dbl\mathcal{C}at$

$$\begin{array}{ccc} \mathbb{C}\mathrm{one}^{\mathrm{ps}}_{\mathbb{J}}(K) & \longrightarrow & [\![\mathbb{H}[1], [\![\mathbb{J}, \mathbb{D}]\!]^{\mathrm{ps}}]\!]^{\mathrm{ps}} \\ \downarrow & \lrcorner & \downarrow{\scriptstyle (\mathrm{ev}_0, \mathrm{ev}_1)} \\ \mathbb{D} \times \mathbb{H}[0] & \xrightarrow[\Delta^{\mathbb{J}}_{\mathbb{D}} \times K]{} & [\![\mathbb{J}, \mathbb{D}]\!]^{\mathrm{ps}} \times [\![\mathbb{J}, \mathbb{D}]\!]^{\mathrm{ps}}. \end{array}$$

Proposition 4.25 *Given a 2-diagram $K\colon \mathcal{J} \to \mathcal{C}$, the double functor*

$$\mathbb{C}\mathrm{one}^{\mathrm{ps}}_{\widetilde{\mathbb{H}}\mathcal{J}}(\widetilde{\mathbb{H}}K) \longrightarrow \mathbb{C}\mathrm{one}^{\mathrm{ps}}_{\mathbb{H}\mathcal{J}}(\widetilde{\mathbb{H}}K \circ I_{\mathcal{J}})$$

induced by the canonical inclusion $I_{\mathcal{J}}\colon \mathbb{H}\mathcal{J} \to \widetilde{\mathbb{H}}\mathcal{J}$ is a double biequivalence between weakly horizontally invariant double categories.

Proof We have, by definition, a pullback in the model category $Dbl\mathcal{C}at_{\mathrm{whi}}$

$$\begin{array}{ccc} \mathbb{C}\mathrm{one}^{\mathrm{ps}}_{\widetilde{\mathbb{H}}\mathcal{J}}(\widetilde{\mathbb{H}}K) & \longrightarrow & [\![\mathbb{H}[1], [\![\widetilde{\mathbb{H}}\mathcal{J}, \widetilde{\mathbb{H}}\mathcal{C}]\!]^{\mathrm{ps}}]\!]^{\mathrm{ps}} \\ \downarrow & \lrcorner & \downarrow{\scriptstyle (\mathrm{ev}_0, \mathrm{ev}_1)} \\ \widetilde{\mathbb{H}}\mathcal{C} \times \mathbb{H}[0] & \xrightarrow[\Delta^{\widetilde{\mathbb{H}}\mathcal{J}}_{\widetilde{\mathbb{H}}\mathcal{C}} \times \widetilde{\mathbb{H}}K]{} & [\![\widetilde{\mathbb{H}}\mathcal{J}, \widetilde{\mathbb{H}}\mathcal{C}]\!]^{\mathrm{ps}} \times [\![\widetilde{\mathbb{H}}\mathcal{J}, \widetilde{\mathbb{H}}\mathcal{C}]\!]^{\mathrm{ps}}. \end{array}$$

On the other hand, using Lemma 4.21, we have, by definition, a pullback in the model category $Dbl\mathcal{C}at_{\mathrm{whi}}$

$$\begin{array}{ccc} \mathbb{C}\mathrm{one}^{\mathrm{ps}}_{\mathbb{H}\mathcal{J}}(\widetilde{\mathbb{H}}K \circ I_{\mathcal{J}}) & \longrightarrow & [\![\mathbb{H}[1], [\![\mathbb{H}\mathcal{J}, \widetilde{\mathbb{H}}\mathcal{C}]\!]^{\mathrm{ps}}]\!]^{\mathrm{ps}} \\ \downarrow & \lrcorner & \downarrow{\scriptstyle (\mathrm{ev}_0, \mathrm{ev}_1)} \\ \widetilde{\mathbb{H}}\mathcal{C} \times \mathbb{H}[0] & \xrightarrow[\Delta^{\mathbb{H}\mathcal{J}}_{\widetilde{\mathbb{H}}\mathcal{C}} \times (\widetilde{\mathbb{H}}K \circ I_{\mathcal{J}})]{} & [\![\mathbb{H}\mathcal{J}, \widetilde{\mathbb{H}}\mathcal{C}]\!]^{\mathrm{ps}} \times [\![\mathbb{H}\mathcal{J}, \widetilde{\mathbb{H}}\mathcal{C}]\!]^{\mathrm{ps}}. \end{array}$$

Using Lemma 4.23, there is a natural comparison map between the two cospans defining the above pullbacks in $Dbl\mathcal{C}at_{\mathrm{whi}}$, which is levelwise a double biequivalence.

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Hence, it induces a unique map between the two weakly horizontally invariant double categories presenting the pullback

$$\mathbb{C}\mathrm{one}^{\mathrm{ps}}_{\widetilde{\mathbb{H}}\mathcal{J}}(\widetilde{\mathbb{H}}K) \longrightarrow \mathbb{C}\mathrm{one}^{\mathrm{ps}}_{\mathbb{H}\mathcal{J}}(\widetilde{\mathbb{H}}K \circ I_{\mathcal{J}})$$

and this double functor is a double biequivalence, as desired. ∎

Next, we modify $\mathbb{C}\mathrm{one}^{\mathrm{ps}}_{\mathbb{H}\mathcal{J}}(\mathbb{H}K)$ by postcomposing the diagram $\mathbb{H}K$ with the canonical inclusion $I_{\mathcal{C}}\colon \mathbb{H}\mathcal{C} \to \widetilde{\mathbb{H}}\mathcal{C}$. First, notice the following link with the preceding result.

Remark 4.26 Given a 2-diagram $K\colon \mathcal{J} \to \mathcal{C}$, we have the relation

$$\widetilde{\mathbb{H}}K \circ I_{\mathcal{J}} = I_{\mathcal{C}} \circ \mathbb{H}K,$$

where $I_{\mathcal{J}}\colon \mathbb{H}\mathcal{J} \to \widetilde{\mathbb{H}}\mathcal{J}$ and $I_{\mathcal{C}}\colon \mathbb{H}\mathcal{C} \to \widetilde{\mathbb{H}}\mathcal{C}$ denote the canonical inclusions. In particular, we have that

$$\mathbb{C}\mathrm{one}^{\mathrm{ps}}_{\mathbb{H}\mathcal{J}}(\widetilde{\mathbb{H}}K \circ I_{\mathcal{J}}) = \mathbb{C}\mathrm{one}^{\mathrm{ps}}_{\mathbb{H}\mathcal{J}}(I_{\mathcal{C}} \circ \mathbb{H}K).$$

Proposition 4.27 *Given a 2-diagram $K\colon \mathcal{J} \to \mathcal{C}$, the double functor*

$$\mathbb{C}\mathrm{one}^{\mathrm{ps}}_{\mathbb{H}\mathcal{J}}(\mathbb{H}K) \longrightarrow \mathbb{C}\mathrm{one}^{\mathrm{ps}}_{\mathbb{H}\mathcal{J}}(I_{\mathcal{C}} \circ \mathbb{H}K)$$

induced by the canonical inclusion $I_{\mathcal{C}}\colon \mathbb{H}\mathcal{C} \to \widetilde{\mathbb{H}}\mathcal{C}$ is a double biequivalence.

This result is less formal as the double category $\mathbb{C}\mathrm{one}^{\mathrm{ps}}_{\mathbb{H}\mathcal{J}}(\mathbb{H}K)$ is not necessarily weakly horizontally invariant. Hence, to show this result, we first prove some auxiliary lemmas. We start by introducing a 2-categorical analog of the cone construction.

Notation 4.28 Given a 2-functor $K\colon \mathcal{J} \to \mathcal{C}$, the 2-*category* $\mathcal{C}\mathrm{one}^{\mathrm{ps}}_{\mathcal{J}}(K)$ *of pseudo-cones* over K is the following pullback in $2\mathcal{C}at$:

$$\begin{array}{ccc} \mathcal{C}\mathrm{one}^{\mathrm{ps}}_{\mathcal{J}}(K) & \longrightarrow & [[1],[\mathcal{J},\mathcal{C}]^{\mathrm{ps}}]^{\mathrm{ps}} \\ \downarrow & \lrcorner & \downarrow{\scriptstyle (\mathrm{ev}_0,\mathrm{ev}_1)} \\ \mathcal{C}\times[0] & \xrightarrow[\Delta^{\mathcal{J}}_{\mathcal{C}}\times K]{} & [\mathcal{J},\mathcal{C}]^{\mathrm{ps}}\times[\mathcal{J},\mathcal{C}]^{\mathrm{ps}}. \end{array}$$

Next, we study the compatibility of the functors $\mathbf{H}, \mathbf{H}[\![\mathbb{V}[1], -]\!]\colon Dbl\mathcal{C}at \to 2\mathcal{C}at$ with the cone constructions.

Lemma 4.29 *Given a 2-category $\mathcal{J}$ and a double functor $K\colon \mathbb{H}\mathcal{J} \to \mathbb{D}$, there is a 2-isomorphism*

$$\mathbf{H}\mathbb{C}\mathrm{one}^{\mathrm{ps}}_{\mathbb{H}\mathcal{J}}K \cong \mathcal{C}\mathrm{one}^{\mathrm{ps}}_{\mathcal{J}}(K^{\sharp}),$$

where $K^{\sharp}\colon \mathcal{J} \to \boldsymbol{H}\mathbb{D}$ is the 2-functor corresponding to K under the adjunction $\mathbb{H} \dashv \boldsymbol{H}$.

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Proof By applying the limit-preserving functor $\mathbf{H}$ to the pullback in $Dbl\mathcal{C}at$ from Notation 4.24 applied to $K\colon \mathbb{H}\mathcal{J} \to \mathbb{D}$, we get the following pullback diagram in $2\mathcal{C}at$:

$$\begin{array}{ccc} \mathbf{H}\mathbb{C}\mathrm{one}^{\mathrm{ps}}_{\mathbb{H}\mathcal{J}}(K) & \longrightarrow & \mathbf{H}[\![\mathbb{H}[1], [\![\mathbb{H}\mathcal{J}, \mathbb{D}]\!]^{\mathrm{ps}}]\!]^{\mathrm{ps}} \\ \downarrow & \lrcorner & \downarrow{\scriptstyle \mathbf{H}(\mathrm{ev}_0, \mathrm{ev}_1)} \\ \mathbf{H}\mathbb{D} \times [0] & \xrightarrow[\mathbf{H}\Delta^{\mathbb{H}\mathcal{J}}_{\mathbb{D}} \times K]{} & \mathbf{H}[\![\mathbb{H}\mathcal{J}, \mathbb{D}]\!]^{\mathrm{ps}} \times \mathbf{H}[\![\mathbb{H}\mathcal{J}, \mathbb{D}]\!]^{\mathrm{ps}} \end{array}$$

We can then compute the top and bottom right-hand 2-categories as follows. By [MSV22, Lemma 2.14], there is a natural 2-isomorphism

$$\mathbf{H}[\![\mathbb{H}\mathcal{J}, \mathbb{D}]\!]^{\mathrm{ps}} \cong [\mathcal{J}, \mathbf{H}\mathbb{D}]^{\mathrm{ps}}.$$

Next, there are also natural 2-isomorphisms

$$\begin{aligned} \mathbf{H}[\![\mathbb{H}[1], [\![\mathbb{H}\mathcal{J}, \mathbb{D}]\!]^{\mathrm{ps}}]\!]^{\mathrm{ps}} &\cong \mathbf{H}[\![\mathbb{H}[1] \boxtimes^{\mathrm{ps}} \mathbb{H}\mathcal{J}, \mathbb{D}]\!]^{\mathrm{ps}} && \text{internal pseudo-hom} \\ &\cong \mathbf{H}[\![\mathbb{H}([1] \otimes^{\mathrm{ps}} \mathcal{J}), \mathbb{D}]\!]^{\mathrm{ps}} && [\text{MSV22, Lemma 7.8}] \\ &\cong [[1] \otimes^{\mathrm{ps}} \mathcal{J}, \mathbf{H}\mathbb{D}]^{\mathrm{ps}} && [\text{MSV22, Lemma 2.14}] \\ &\cong [[1], [\mathcal{J}, \mathbf{H}\mathbb{D}]^{\mathrm{ps}}]^{\mathrm{ps}} && \text{internal pseudo-hom.} \end{aligned}$$

Hence, the above pullback square in $2\mathcal{C}at$ becomes

$$\begin{array}{ccc} \mathbf{H}\mathbb{C}\mathrm{one}^{\mathrm{ps}}_{\mathbb{H}\mathcal{J}}(K) & \longrightarrow & [[1], [\mathcal{J}, \mathbf{H}\mathbb{D}]^{\mathrm{ps}}]^{\mathrm{ps}} \\ \downarrow & \lrcorner & \downarrow{\scriptstyle (\mathrm{ev}_0, \mathrm{ev}_1)} \\ \mathbf{H}\mathbb{D} \times [0] & \xrightarrow[\Delta^{\mathcal{J}}_{\mathbf{H}\mathbb{D}} \times K^\sharp]{} & [\mathcal{J}, \mathbf{H}\mathbb{D}]^{\mathrm{ps}} \times [\mathcal{J}, \mathbf{H}\mathbb{D}]^{\mathrm{ps}}. \end{array}$$

But, by Notation 4.28, this pullback is exactly $\mathcal{C}\mathrm{one}^{\mathrm{ps}}_{\mathcal{J}}(K^\sharp)$. ∎

Lemma 4.30 *Given a 2-category $\mathcal{J}$ and a double functor $K\colon \mathbb{H}\mathcal{J} \to \mathbb{D}$, there is a 2-isomorphism*

$$\mathbf{H}[\![\mathbb{V}[1], \mathbb{C}\mathrm{one}^{\mathrm{ps}}_{\mathbb{H}\mathcal{J}}(K)]\!] \cong \mathcal{C}\mathrm{one}^{\mathrm{ps}}_{\mathcal{J}}(\mathbf{H}E_{\mathbb{D}} \circ K^\sharp),$$

where $E_{\mathbb{D}}\colon \mathbb{D} \to [\![\mathbb{V}[1], \mathbb{D}]\!]$ is the double functor induced by the unique map $\mathbb{V}[1] \to [0]$.

Proof By applying the limit-preserving functor $\mathbf{H}[\![\mathbb{V}[1], -]\!]$ to the pullback in $Dbl\mathcal{C}at$ from Notation 4.24 applied to $K\colon \mathbb{H}\mathcal{J} \to \mathbb{D}$, we get the following pullback diagram in $2\mathcal{C}at$:

$$\begin{array}{ccc} \mathbf{H}[\![\mathbb{V}[1], \mathbb{C}\mathrm{one}^{\mathrm{ps}}_{\mathbb{H}\mathcal{J}}(K)]\!] & \longrightarrow & \mathbf{H}[\![\mathbb{V}[1], [\![\mathbb{H}[1], [\![\mathbb{H}\mathcal{J}, \mathbb{D}]\!]^{\mathrm{ps}}]\!]^{\mathrm{ps}}]\!] \\ \downarrow & \lrcorner & \downarrow{\scriptstyle \mathbf{H}[\![\mathbb{V}[1], (\mathrm{ev}_0, \mathrm{ev}_1)]\!]} \\ \mathbf{H}[\![\mathbb{V}[1], \mathbb{D}]\!] \times [0] & \xrightarrow[\mathbf{H}[\![\mathbb{V}[1], \Delta^{\mathbb{H}\mathcal{J}}_{\mathbb{D}}]\!] \times e_K]{} & \mathbf{H}[\![\mathbb{V}[1], [\![\mathbb{H}\mathcal{J}, \mathbb{D}]\!]^{\mathrm{ps}}]\!] \times \mathbf{H}[\![\mathbb{V}[1], [\![\mathbb{H}\mathcal{J}, \mathbb{D}]\!]^{\mathrm{ps}}]\!], \end{array}$$

where e_K denotes the vertical identity at K. We can then compute the top and bottom right-hand 2-categories as follows. By [MSV22, Lemma 2.14], there is a natural 2-isomorphism

$$\mathbf{H}[\![\mathbb{V}[1], [\![\mathbb{H}\mathcal{J}, \mathbb{D}]\!]^{\mathrm{ps}}]\!] \cong [\mathcal{J}, \mathbf{H}[\![\mathbb{V}[1], \mathbb{D}]\!]]^{\mathrm{ps}}$$

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and we also have natural 2-isomorphisms

$$\begin{aligned}
&\mathbf{H}[\![\mathbb{V}[1],[\![\mathbb{H}[1],[\![\mathbb{H}\mathcal{J},\mathbb{D}]\!]^{\mathrm{ps}}]\!]^{\mathrm{ps}}]\!] \\
&\quad\cong \mathbf{H}[\![\mathbb{V}[1],[\![\mathbb{H}[1]\boxtimes^{\mathrm{ps}}\mathbb{H}\mathcal{J},\mathbb{D}]\!]^{\mathrm{ps}}]\!] && \text{internal pseudo-hom} \\
&\quad\cong \mathbf{H}[\![\mathbb{V}[1],[\![\mathbb{H}([1]\otimes^{\mathrm{ps}}\mathcal{J}),\mathbb{D}]\!]^{\mathrm{ps}}]\!] && [\text{MSV22, Lemma 7.8}] \\
&\quad\cong [[1]\otimes^{\mathrm{ps}}\mathcal{J},\mathbf{H}[\![\mathbb{V}[1],\mathbb{D}]\!]]^{\mathrm{ps}} && [\text{MSV22, Lemma 2.14}] \\
&\quad\cong [[1],[\mathcal{J},\mathbf{H}[\![\mathbb{V}[1],\mathbb{D}]\!]]^{\mathrm{ps}}]^{\mathrm{ps}} && \text{internal pseudo-hom.}
\end{aligned}$$

Hence, the above pullback square in $2\mathcal{C}at$ becomes

$$\begin{array}{ccc}
\mathbf{H}[\![\mathbb{V}[1],\mathbb{C}\mathrm{one}^{\mathrm{ps}}_{\mathbb{H}\mathcal{J}}(K)]\!] & \longrightarrow & [[1],[\mathcal{J},\mathbf{H}[\![\mathbb{V}[1],\mathbb{D}]\!]]^{\mathrm{ps}}]^{\mathrm{ps}} \\
\downarrow & \lrcorner & \downarrow{\scriptstyle (\mathrm{ev}_0,\mathrm{ev}_1)} \\
\mathbf{H}[\![\mathbb{V}[1],\mathbb{D}]\!]\times[0] & \xrightarrow[\Delta^{\mathcal{J}}_{\mathbf{H}[\![\mathbb{V}[1],\mathbb{D}]\!]}\times\mathbf{H}E_{\mathbb{D}}\circ K^\sharp]{} & [\mathcal{J},\mathbf{H}[\![\mathbb{V}[1],\mathbb{D}]\!]]^{\mathrm{ps}}\times[\mathcal{J},\mathbf{H}[\![\mathbb{V}[1],\mathbb{D}]\!]]^{\mathrm{ps}}.
\end{array}$$

But, by Notation 4.28, this pullback is exactly $\mathcal{C}\mathrm{one}^{\mathrm{ps}}_{\mathcal{J}}(\mathbf{H}E_{\mathbb{D}}\circ K^\sharp)$. ∎

We can now prove the proposition.

Proof of Proposition 4.27 We need to show that applying the functors $\mathbf{H}$ and $\mathbf{H}[\![\mathbb{V}[1],-]\!]$ to the double functor

$$\mathbb{C}\mathrm{one}^{\mathrm{ps}}_{\mathbb{H}\mathcal{J}}(\mathbb{H}K)\to\mathbb{C}\mathrm{one}^{\mathrm{ps}}_{\mathbb{H}\mathcal{J}}(I_{\mathcal{C}}\circ\mathbb{H}K)$$

yield two biequivalences.

First, note that $\mathbf{H}\mathbb{H}\mathcal{C}=\mathcal{C}=\mathbf{H}\widetilde{\mathbb{H}}\mathcal{C}$ so that we have

$$(\mathbb{H}K)^\sharp=K=(I_{\mathcal{C}}\circ\mathbb{H}K)^\sharp.$$

Hence, by Lemma 4.29, we have 2-isomorphisms

$$\mathbf{H}\mathbb{C}\mathrm{one}^{\mathrm{ps}}_{\mathbb{H}\mathcal{J}}(\mathbb{H}K)\cong\mathcal{C}\mathrm{one}^{\mathrm{ps}}_{\mathcal{J}}(K)\cong\mathbf{H}\mathbb{C}\mathrm{one}^{\mathrm{ps}}_{\mathbb{H}\mathcal{J}}(I_{\mathcal{C}}\circ\mathbb{H}K)$$

and so the 2-functor $\mathbf{H}\mathbb{C}\mathrm{one}^{\mathrm{ps}}_{\mathbb{H}\mathcal{J}}(\mathbb{H}K)\to\mathbf{H}\mathbb{C}\mathrm{one}^{\mathrm{ps}}_{\mathbb{H}\mathcal{J}}(I_{\mathcal{C}}\circ\mathbb{H}K)$ is a 2-isomorphism and hence a biequivalence.

Now, by Lemma 4.30, we have that the 2-functor

$$\mathbf{H}[\![\mathbb{V}[1],\mathbb{C}\mathrm{one}^{\mathrm{ps}}_{\mathbb{H}\mathcal{J}}(\mathbb{H}K)]\!]\to\mathbf{H}[\![\mathbb{V}[1],\mathbb{C}\mathrm{one}^{\mathrm{ps}}_{\mathbb{H}\mathcal{J}}(I_{\mathcal{C}}\circ\mathbb{H}K))]\!]$$

is given by the 2-functor

$$\mathcal{C}\mathrm{one}^{\mathrm{ps}}_{\mathcal{J}}(\mathbf{H}E_{\mathbb{H}\mathcal{C}}\circ K)\to\mathcal{C}\mathrm{one}^{\mathrm{ps}}_{\mathcal{J}}(\mathbf{H}E_{\widetilde{\mathbb{H}}\mathcal{C}}\circ K).$$

Since $\mathbf{H}E_{\widetilde{\mathbb{H}}\mathcal{C}}$ can be seen as the composite of 2-functors

$$\mathbf{H}E_{\widetilde{\mathbb{H}}\mathcal{C}}\colon\mathcal{C}=\mathbf{H}\mathbb{H}\mathcal{C}\xrightarrow{\mathbf{H}E_{\mathbb{H}\mathcal{C}}}\mathbf{H}[\![\mathbb{V}[1],\mathbb{H}\mathcal{C}]\!]\xrightarrow{\mathbf{H}[\![\mathbb{V}[1],I_{\mathcal{C}}]\!]}\mathbf{H}[\![\mathbb{V}[1],\widetilde{\mathbb{H}}\mathcal{C}]\!],$$

the above 2-functor corresponds to the 2-functor

$$\mathcal{C}\mathrm{one}^{\mathrm{ps}}_{\mathcal{J}}(\mathbf{H}E_{\mathbb{H}\mathcal{C}}\circ K)\to\mathcal{C}\mathrm{one}^{\mathrm{ps}}_{\mathcal{J}}(\mathbf{H}[\![\mathbb{V}[1],I_{\mathcal{C}}]\!]\circ\mathbf{H}E_{\mathbb{H}\mathcal{C}}\circ K)$$

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induced by $\mathbf{H}[\![\mathbb{V}[1], I_{\mathcal{C}}]\!]$. As $I_{\mathcal{C}}$ is a double biequivalence by Proposition 4.9, $\mathbf{H}[\![\mathbb{V}[1], I_{\mathcal{C}}]\!]$ is a biequivalence. Hence,

$$\mathcal{C}\mathrm{one}^{\mathrm{ps}}_{\mathcal{J}}(\mathbf{H}E_{\mathbb{H}\mathcal{C}} \circ K) \to \mathcal{C}\mathrm{one}^{\mathrm{ps}}_{\mathcal{J}}(\mathbf{H}[\![\mathbb{V}[1], I_{\mathcal{C}}]\!] \circ \mathbf{H}E_{\mathbb{H}\mathcal{C}} \circ K)$$

is a biequivalence since the two spans defining these pullbacks are levelwise equivalent, and the pullbacks are in fact homotopy pullbacks as every 2-category is fibrant. ∎

We now study the slice constructions of $\mathbb{C}\mathrm{one}^{\mathrm{ps}}_{\widetilde{\mathbb{H}}\mathcal{J}}(\widetilde{\mathbb{H}}K)$ and $\mathbb{C}\mathrm{one}^{\mathrm{ps}}_{\mathbb{H}\mathcal{J}}(\mathbb{H}K)$ over an object (ℓ, λ). We first modify $\mathbb{C}\mathrm{one}^{\mathrm{ps}}_{\widetilde{\mathbb{H}}\mathcal{J}}(\widetilde{\mathbb{H}}K) /\!\!/^{\mathrm{ps}} (\ell, \lambda)$ as before.

Proposition 4.31 *Given a 2-diagram $K\colon \mathcal{J} \to \mathcal{C}$, the double functor between slices over (ℓ, λ) induced by the double functor from Proposition 4.25 is a double biequivalence, and fits into a commutative diagram of the following form:*

$$\begin{array}{ccc}
\mathbb{C}\mathrm{one}^{\mathrm{ps}}_{\widetilde{\mathbb{H}}\mathcal{J}}(\widetilde{\mathbb{H}}K) /\!\!/^{\mathrm{ps}} (\ell, \lambda) & \xrightarrow{\simeq} & \mathbb{C}\mathrm{one}^{\mathrm{ps}}_{\mathbb{H}\mathcal{J}}(\widetilde{\mathbb{H}}K \circ I_{\mathcal{J}}) /\!\!/^{\mathrm{ps}} (\ell, \lambda) \\
\downarrow & & \downarrow \\
\mathbb{C}\mathrm{one}^{\mathrm{ps}}_{\widetilde{\mathbb{H}}\mathcal{J}}(\widetilde{\mathbb{H}}K) & \xrightarrow{\simeq} & \mathbb{C}\mathrm{one}^{\mathrm{ps}}_{\mathbb{H}\mathcal{J}}(\widetilde{\mathbb{H}}K \circ I_{\mathcal{J}}).
\end{array}$$

Proof This follows from Proposition 4.25. ∎

Next, we modify $\mathbb{C}\mathrm{one}^{\mathrm{ps}}_{\mathbb{H}\mathcal{J}}(\mathbb{H}K) /\!\!/^{\mathrm{ps}} (\ell, \lambda)$ as before.

Proposition 4.32 *Given a 2-diagram $K\colon \mathcal{J} \to \mathcal{C}$, the double functor between slices over (ℓ, λ) induced by the double functor from Proposition 4.27 is a double biequivalence, and fits into a commutative diagram of the following form:*

$$\begin{array}{ccc}
\mathbb{C}\mathrm{one}^{\mathrm{ps}}_{\mathbb{H}\mathcal{J}}(\mathbb{H}K) /\!\!/^{\mathrm{ps}} (\ell, \lambda) & \xrightarrow{\simeq} & \mathbb{C}\mathrm{one}^{\mathrm{ps}}_{\mathbb{H}\mathcal{J}}(I_{\mathcal{C}} \circ \mathbb{H}K) /\!\!/^{\mathrm{ps}} (\ell, \lambda) \\
\downarrow & & \downarrow \\
\mathbb{C}\mathrm{one}^{\mathrm{ps}}_{\mathbb{H}\mathcal{J}}(\mathbb{H}K) & \xrightarrow{\simeq} & \mathbb{C}\mathrm{one}^{\mathrm{ps}}_{\mathbb{H}\mathcal{J}}(I_{\mathcal{C}} \circ \mathbb{H}K).
\end{array}$$

This result is again less formal as the double category $\mathbb{C}\mathrm{one}^{\mathrm{ps}}_{\mathbb{H}\mathcal{J}}(\mathbb{H}K)$ is not necessarily weakly horizontally invariant. As before, we start by introducing a 2-categorical analog of the slice construction.

Notation 4.33 Given a 2-category $\mathcal{C}$ and an object c of $\mathcal{C}$, the *pseudo-slice 2-category* $\mathcal{C}/^{\mathrm{ps}}c$ of $\mathcal{C}$ over c is the following pullback in $2\mathcal{C}at$:

$$\begin{array}{ccc}
\mathcal{C}/^{\mathrm{ps}}c & \longrightarrow & [[1], \mathcal{C}]^{\mathrm{ps}} \\
\downarrow & \lrcorner & \downarrow{\scriptstyle (\mathrm{ev}_0, \mathrm{ev}_1)} \\
\mathcal{C} \times [0] & \xrightarrow[\mathcal{C} \times c]{} & \mathcal{C} \times \mathcal{C}.
\end{array}$$

Next, we study the compatibility of the functors $\mathbf{H}, \mathbf{H}[\![\mathbb{V}[1], -]\!]\colon Dbl\mathcal{C}at \to 2\mathcal{C}at$ with the slice constructions.

Lemma 4.34 *Given a double category $\mathbb{D}$ and an object d of $\mathbb{D}$, there are 2-isomorphism*

$$\mathbf{H}(\mathbb{D} /\!\!/^{\mathrm{ps}} d) \cong (\mathbf{H}\mathbb{D})/^{\mathrm{ps}}d \quad \textit{and} \quad \mathbf{H}[\![\mathbb{V}[1], \mathbb{D} /\!\!/^{\mathrm{ps}} d]\!] \cong (\mathbf{H}[\![\mathbb{V}[1], \mathbb{D}]\!])/^{\mathrm{ps}}e_d,$$

where e_d denotes the vertical identity at d.

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Proof Note that, by considering an object d of a double category $\mathbb{D}$ as a diagram $d\colon \mathbb{H}[0] \to \mathbb{D}$, we have an isomorphism in $Dbl\mathcal{C}at$

$$\mathbb{D} /\!\!/^{\mathrm{ps}} d \cong \mathbb{C}\mathrm{one}^{\mathrm{ps}}_{\mathbb{H}[0]} d$$

and, by considering an object c of $\mathcal{C}$ as a 2-diagram $c\colon [0] \to \mathcal{C}$, we have a 2-isomorphism

$$\mathcal{C}/^{\mathrm{ps}} c \cong \mathcal{C}\mathrm{one}^{\mathrm{ps}}_{[0]} c.$$

The result is then obtained by specializing Lemmas 4.29 and 4.30 to the case $K = d\colon \mathbb{H}[0] \to \mathbb{D}$. ∎

We can now prove the proposition.

Proof of Proposition 4.32 We need to show that applying the functors $\mathbf{H}$ and $\mathbf{H}[\![\mathbb{V}[1], -]\!]$ to the double functor

$$\mathbb{C}\mathrm{one}^{\mathrm{ps}}_{\mathbb{H}\mathcal{J}}(\mathbb{H}K) /\!\!/^{\mathrm{ps}} (\ell, \lambda) \to \mathbb{C}\mathrm{one}^{\mathrm{ps}}_{\mathbb{H}\mathcal{J}}(I_{\mathcal{C}} \circ \mathbb{H}K) /\!\!/^{\mathrm{ps}} (\ell, \lambda)$$

yields two biequivalences.

First, by Lemma 4.34, we have that the 2-functor

$$\mathbf{H}(\mathbb{C}\mathrm{one}^{\mathrm{ps}}_{\mathbb{H}\mathcal{J}}(\mathbb{H}K) /\!\!/^{\mathrm{ps}} (\ell, \lambda)) \to \mathbf{H}(\mathbb{C}\mathrm{one}^{\mathrm{ps}}_{\mathbb{H}\mathcal{J}}(I_{\mathcal{C}} \circ \mathbb{H}K) /\!\!/^{\mathrm{ps}} (\ell, \lambda))$$

is given by the 2-functor

$$(\mathbf{H}\mathbb{C}\mathrm{one}^{\mathrm{ps}}_{\mathbb{H}\mathcal{J}}(\mathbb{H}K))/^{\mathrm{ps}}(\ell, \lambda) \to (\mathbf{H}\mathbb{C}\mathrm{one}^{\mathrm{ps}}_{\mathbb{H}\mathcal{J}}(I_{\mathcal{C}} \circ \mathbb{H}K))/^{\mathrm{ps}}(\ell, \lambda).$$

Since the 2-functor $\mathbf{H}\mathbb{C}\mathrm{one}^{\mathrm{ps}}_{\mathbb{H}\mathcal{J}}(\mathbb{H}K) \to \mathbf{H}\mathbb{C}\mathrm{one}^{\mathrm{ps}}_{\mathbb{H}\mathcal{J}}(I_{\mathcal{C}} \circ \mathbb{H}K)$ is a biequivalence by Proposition 4.27, so is the above 2-functor between slices.

Now, by Lemma 4.34, we have that the 2-functor

$$\mathbf{H}[\![\mathbb{V}[1], \mathbb{C}\mathrm{one}^{\mathrm{ps}}_{\mathbb{H}\mathcal{J}}(\mathbb{H}K) /\!\!/^{\mathrm{ps}} (\ell, \lambda)]\!] \to \mathbf{H}[\![\mathbb{V}[1], \mathbb{C}\mathrm{one}^{\mathrm{ps}}_{\mathbb{H}\mathcal{J}}(I_{\mathcal{C}} \circ \mathbb{H}K) /\!\!/^{\mathrm{ps}} (\ell, \lambda)]\!]$$

is given by the 2-functor

$$(\mathbf{H}[\![\mathbb{V}[1], \mathbb{C}\mathrm{one}^{\mathrm{ps}}_{\mathbb{H}\mathcal{J}}(\mathbb{H}K)]\!])/^{\mathrm{ps}}(\ell, \mathrm{id}_\lambda) \to (\mathbf{H}[\![\mathbb{V}[1], \mathbb{C}\mathrm{one}^{\mathrm{ps}}_{\mathbb{H}\mathcal{J}}(I_{\mathcal{C}} \circ \mathbb{H}K)]\!])/^{\mathrm{ps}}(\ell, \mathrm{id}_\lambda).$$

Since the 2-functor $\mathbf{H}[\![\mathbb{V}[1], \mathbb{C}\mathrm{one}^{\mathrm{ps}}_{\mathbb{H}\mathcal{J}}(\mathbb{H}K)]\!] \to \mathbf{H}[\![\mathbb{V}[1], \mathbb{C}\mathrm{one}^{\mathrm{ps}}_{\mathbb{H}\mathcal{J}}(I_{\mathcal{C}} \circ \mathbb{H}K)]\!]$ is a biequivalence by Proposition 4.27, so is the above 2-functor between slices. ∎

We can now prove the desired characterization of homotopy 2-limits.

Proof of Theorem 4.19 Combining Propositions 4.31 and 4.32, there are zig-zags of double biequivalences fitting into a commutative diagram of the following form:

$$\begin{array}{ccccc}
\mathbb{C}\mathrm{one}^{\mathrm{ps}}_{\mathbb{H}\mathcal{J}}(\mathbb{H}K) /\!\!/^{\mathrm{ps}} (\ell, \lambda) & \xrightarrow{\simeq} & \mathbb{C}\mathrm{one}^{\mathrm{ps}}_{\mathbb{H}\mathcal{J}}(F) /\!\!/^{\mathrm{ps}} (\ell, \lambda) & \xleftarrow{\simeq} & \mathbb{C}\mathrm{one}^{\mathrm{ps}}_{\widetilde{\mathbb{H}}\mathcal{J}}(\widetilde{\mathbb{H}}K) /\!\!/^{\mathrm{ps}} (\ell, \lambda) \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{C}\mathrm{one}^{\mathrm{ps}}_{\mathbb{H}\mathcal{J}}(\mathbb{H}K) & \xrightarrow{\simeq} & \mathbb{C}\mathrm{one}^{\mathrm{ps}}_{\mathbb{H}\mathcal{J}}(F) & \xleftarrow{\simeq} & \mathbb{C}\mathrm{one}^{\mathrm{ps}}_{\widetilde{\mathbb{H}}\mathcal{J}}(\widetilde{\mathbb{H}}K),
\end{array}$$

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where $F = I_{\mathcal{C}} \circ \mathbb{H}K = \widetilde{\mathbb{H}}K \circ I_{\mathcal{J}} \colon \mathbb{H}\mathcal{J} \to \widetilde{\mathbb{H}}\mathcal{C}$ using Remark 4.26. Hence, by 2-out-of-3, the double functor

$$\mathbb{C}\mathrm{one}^{\mathrm{ps}}_{\mathbb{H}\mathcal{J}}(\mathbb{H}K) \mathbin{/\!\!/^{\mathrm{ps}}} (\ell, \lambda) \longrightarrow \mathbb{C}\mathrm{one}^{\mathrm{ps}}_{\mathbb{H}\mathcal{J}}(\mathbb{H}K)$$

is a double biequivalence if and only if the double functor

$$\mathbb{C}\mathrm{one}^{\mathrm{ps}}_{\widetilde{\mathbb{H}}\mathcal{J}}(\widetilde{\mathbb{H}}K) \mathbin{/\!\!/^{\mathrm{ps}}} (\ell, \lambda) \longrightarrow \mathbb{C}\mathrm{one}^{\mathrm{ps}}_{\widetilde{\mathbb{H}}\mathcal{J}}(\widetilde{\mathbb{H}}K)$$

is a double biequivalence. Hence, the desired result follows from Theorem 4.20. ∎

4.3 Properties of double $(\infty, 1)$-nerve of double categories

In this section, we provide proofs for the properties of the nerve

$$\mathbb{N} \colon [\mathit{DblCat}]_\infty \to \mathit{DblCat}_{(\infty,1)}$$

stated in Proposition 3.10. Consider the adjunction

$$\mathbb{C} \colon s\mathcal{S}\mathit{et}^{\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}} \rightleftarrows \mathit{DblCat} \colon \mathbb{N}$$

constructed in [Mos20, Section 5.1].

Notation 4.35 We denote by $s\mathcal{S}\mathit{et}^{\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}}_{\mathrm{dbl}(\infty,1)}$ the model structure on the category $s\mathcal{S}\mathit{et}^{\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}}$ for Segal objects in complete Segal spaces. It follows from [BSP21, Theorem 13.15] and the definition of $\mathit{DblCat}_{(\infty,1)}$ as the ∞-category of Segal objects in $\mathit{Cat}_{(\infty,1)}$ that there is an equivalence between the underlying ∞-category of $s\mathcal{S}\mathit{et}^{\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}}_{\mathrm{dbl}(\infty,1)}$ and $\mathit{DblCat}_{(\infty,1)}$.

The adjunction $\mathbb{C} \dashv \mathbb{N}$ has the following homotopical properties.

Theorem 4.36 [Mos20, Theorems 5.2.8, 5.3.1, and 6.6.1] *The adjunction*

$$\mathbb{C} \colon s\mathcal{S}\mathit{et}^{\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}}_{\mathrm{dbl}(\infty,1)} \rightleftarrows \mathit{DblCat}_{\mathrm{whi}} \colon \mathbb{N}$$

(1) *is a Quillen pair;*
(2) *is a Quillen localization pair, that is, the derived counit is a weak equivalence in* $\mathit{DblCat}_{\mathrm{whi}}$.

Moreover, we have that

(3) *for every 2-category* $\mathcal{C}$*, the nerve* $\mathbb{N}\widetilde{\mathbb{H}}\mathcal{C}$ *is a two-fold complete Segal space, that is, an* $(\infty, 2)$*-category.*

Recall that the nerve $\mathbb{N} \colon [\mathit{DblCat}]_\infty \to \mathit{DblCat}_{(\infty,1)}$ from Notation 3.9 is, by definition, the composite of the induced functor between underlying ∞-categories by $\mathbb{N} \colon \mathit{DblCat}_{\mathrm{whi}} \to s\mathcal{S}\mathit{et}^{\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}}_{\mathrm{dbl}(\infty,1)}$ followed by the equivalence of the underlying ∞-category of $s\mathcal{S}\mathit{et}^{\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}}_{\mathrm{dbl}(\infty,1)}$ and $\mathit{DblCat}_{(\infty,1)}$. Hence, the results (1)–(3) of Proposition 3.10 are now formal consequences of the above result.

It remains to show (4) of Proposition 3.10. For this, we first study the compatibility of the functor

$$\mathbb{C} \colon s\mathcal{S}\mathit{et}^{\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}}_{\mathrm{dbl}(\infty,1)} \longrightarrow \mathit{DblCat}_{\mathrm{whi}}$$

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with the monoidal structures on both sides. We can restrict the verification to objects in the image of the canonical inclusion $\mathcal{S}et^{\Delta^{\mathrm{op}}\times\Delta^{\mathrm{op}}} \hookrightarrow s\mathcal{S}et^{\Delta^{\mathrm{op}}\times\Delta^{\mathrm{op}}}$ since the functor $\mathbb{C}\colon s\mathcal{S}et^{\Delta^{\mathrm{op}}\times\Delta^{\mathrm{op}}} \to Dbl\mathcal{C}at$ is a simplicial left Kan extension, where $Dbl\mathcal{C}at$ is tensored over $s\mathcal{S}et$ via the functor $- \boxtimes^{\mathrm{ps}} \mathbb{C}K$, for every $K \in s\mathcal{S}et$, seen as a bisimplicial space through the canonical inclusion $s\mathcal{S}et \hookrightarrow s\mathcal{S}et^{\Delta^{\mathrm{op}}\times\Delta^{\mathrm{op}}}$.

Hence, given bisimplicial sets X and Y, that is, objects in $\mathcal{S}et^{\Delta^{\mathrm{op}}\times\Delta^{\mathrm{op}}}$ regarded as bisimplicial spaces through the canonical inclusion $\mathcal{S}et^{\Delta^{\mathrm{op}}\times\Delta^{\mathrm{op}}} \hookrightarrow s\mathcal{S}et^{\Delta^{\mathrm{op}}\times\Delta^{\mathrm{op}}}$, we construct in Proposition 4.41 a double functor

$$\mathbb{C}(X \times Y) \to \mathbb{C}X \boxtimes^{\mathrm{ps}} \mathbb{C}Y$$

natural in X and Y, and then show as Theorem 4.43 that it is a weak equivalence in $Dbl\mathcal{C}at_{\mathrm{whi}}$.

Notation 4.37 Given $m, t \geq 0$, we denote by $[m,t]$ the double category $\mathbb{H}[m] \times \mathbb{V}[t]$ and we denote by $F[m,t]$ the representable in $s\mathcal{S}et^{\Delta^{\mathrm{op}}\times\Delta^{\mathrm{op}}}$ at the object $([m],[t]) \in \Delta \times \Delta$. We have an identification $\mathbb{N}[m,t] \cong F[m,t]$.

We start by constructing a double functor

$$\mathbb{C}(F[m,t] \times F[m',t']) \to \mathbb{C}F[m,t] \boxtimes^{\mathrm{ps}} \mathbb{C}F[m',t']$$

natural in $m, t, m', t' \geq 0$.

By the universal property of the product, for all $m, t, m', t' \geq 0$, there are natural double functors

$$\mathbb{C}(F[m,t] \times F[m',t']) \to \mathbb{C}F[m,t] \times \mathbb{C}F[m',t']$$

that assemble into a natural transformation of functors $\Delta^{\times 4} \to Dbl\mathcal{C}at$

$$\mathbb{C}(F[-,-] \times F[-,-]) \Rightarrow \mathbb{C}F[-,-] \times \mathbb{C}F[-,-].$$

Moreover, for all $m, t, m', t' \geq 0$, there are natural projection double functors

$$\mathbb{C}F[m,t] \boxtimes^{\mathrm{ps}} \mathbb{C}F[m',t'] \to \mathbb{C}F[m,t] \times \mathbb{C}F[m',t']$$

that assemble into a natural transformation of functors $\Delta^{\times 4} \to Dbl\mathcal{C}at$

$$\mathbb{C}F[-,-] \boxtimes^{\mathrm{ps}} \mathbb{C}F[-,-] \Rightarrow \mathbb{C}F[-,-] \times \mathbb{C}F[-,-].$$

Moreover, these projection double functors are all trivial fibrations in $Dbl\mathcal{C}at_{\mathrm{whi}}$ by Lemma 4.12, and so the resulting natural transformation is levelwise a trivial fibration in $Dbl\mathcal{C}at_{\mathrm{whi}}$.

We want to find a lift in the following diagram in the projective model structure on $(Dbl\mathcal{C}at_{\mathrm{whi}})^{\Delta^{\times 4}}$:

(4.1)
$$\begin{array}{ccc} & & \mathbb{C}F[-,-] \boxtimes^{\mathrm{ps}} \mathbb{C}F[-,-] \\ & \nearrow & \downarrow \\ \mathbb{C}(F[-,-] \times F[-,-]) & \longrightarrow & \mathbb{C}F[-,-] \times \mathbb{C}F[-,-]. \end{array}$$

For this, it is sufficient to show that $\mathbb{C}(F[-,-] \times F[-,-])\colon \Delta^{\times 4} \to Dbl\mathcal{C}at_{\mathrm{whi}}$ is projectively cofibrant.

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Lemma 4.38 *The functor* $F[-,-] \times F[-,-] \colon \Delta^{\times 4} \to s\mathcal{S}et^{\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}}_{\mathrm{dbl}(\infty,1)}$ *given at* $m, t, m', t' \geq 0$ *by the product* $F[m,t] \times F[m',t']$ *is projectively cofibrant.*

Proof First, recall that $F[-,-] \colon \Delta^{\times 2} \to s\mathcal{S}et^{\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}}_{\mathrm{dbl}(\infty,1)}$ given at $m, t \geq 0$ by the representable $F[m,t]$ is projectively cofibrant. As the projection $\pi_{1,2} \colon \Delta^{\times 4} \to \Delta^{\times 2}$ onto the first two components is left adjoint to the inclusion $\Delta^{\times 2} \to \Delta^{\times 4}$ given at $m, t \geq 0$ by $([m],[t],[0],[0])$, it induces by pre-composition a left Quillen functor between projective model structures

$$\pi_{1,2}^* \colon \left(s\mathcal{S}et^{\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}}_{\mathrm{dbl}(\infty,1)}\right)^{\Delta^{\times 2}} \to \left(s\mathcal{S}et^{\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}}_{\mathrm{dbl}(\infty,1)}\right)^{\Delta^{\times 4}}.$$

In particular, this left Quillen functor preserves cofibrant objects and so the functor $F[-,-] \colon \Delta^{\times 4} \to s\mathcal{S}et^{\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}}_{\mathrm{dbl}(\infty,1)}$ given at $m, t, m', t' \geq 0$ by the representable $F[m,t]$ is also projectively cofibrant. Similarly, the functor $F[-,-] \colon \Delta^{\times 4} \to s\mathcal{S}et^{\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}}_{\mathrm{dbl}(\infty,1)}$ given at $m, t, m', t' \geq 0$ by the representable $F[m',t']$ is projectively cofibrant.

Next, since the product $- \times - \colon s\mathcal{S}et^{\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}}_{\mathrm{dbl}(\infty,1)} \times s\mathcal{S}et^{\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}}_{\mathrm{dbl}(\infty,1)} \to s\mathcal{S}et^{\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}}_{\mathrm{dbl}(\infty,1)}$ is a left Quillen bifunctor, it is also a left Quillen functor as all objects are cofibrant in $s\mathcal{S}et^{\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}}_{\mathrm{dbl}(\infty,1)}$. Therefore, it induces by postcomposition a left Quillen functor between projective model structures

$$\begin{array}{ccc}
\left(s\mathcal{S}et^{\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}}_{\mathrm{dbl}(\infty,1)}\right)^{\Delta^{\times 4}} \times \left(s\mathcal{S}et^{\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}}_{\mathrm{dbl}(\infty,1)}\right)^{\Delta^{\times 4}} & & \\
\cong \downarrow & \searrow & \\
\left(s\mathcal{S}et^{\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}}_{\mathrm{dbl}(\infty,1)} \times s\mathcal{S}et^{\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}}_{\mathrm{dbl}(\infty,1)}\right)^{\Delta^{\times 4}} & \xrightarrow[(-\times-)_*]{} & \left(s\mathcal{S}et^{\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}}_{\mathrm{dbl}(\infty,1)}\right)^{\Delta^{\times 4}}.
\end{array}$$

In particular, this left Quillen functor preserves cofibrant objects and so the functor $F[-,-] \times F[-,-] \colon \Delta^{\times 4} \to s\mathcal{S}et^{\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}}_{\mathrm{dbl}(\infty,1)}$ given at $m, t, m', t' \geq 0$ by the product $F[m,t] \times F[m',t']$ is also projectively cofibrant. ∎

Corollary 4.39 *The functor* $\mathbb{C}(F[-,-] \times F[-,-]) \colon \Delta^{\times 4} \to \mathit{Dbl}\mathcal{C}\mathit{at}_{\mathrm{whi}}$ *given at* $m, t, m', t' \geq 0$ *by the double category* $\mathbb{C}(F[m,t] \times F[m',t'])$ *is projectively cofibrant.*

Proof This follows directly from Lemma 4.38 and the fact that

$$\mathbb{C} \colon s\mathcal{S}et^{\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}}_{\mathrm{dbl}(\infty,1)} \to \mathit{Dbl}\mathcal{C}\mathit{at}_{\mathrm{whi}}$$

is left Quillen. ∎

Lemma 4.40 *For* $m, t, m', t' \geq 0$*, there is a double functor*

$$\mathbb{C}(F[m,t] \times F[m',t']) \to \mathbb{C}F[m,t] \boxtimes^{\mathrm{ps}} \mathbb{C}F[m',t']$$

natural in $m, t, m', t' \geq 0$*, which provides a lift in* (4.1)*.*

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Proof This natural map is obtained as a lift in the following diagram in the projective model structure on $(Dbl\mathcal{C}at_{\mathrm{whi}})^{\Delta^{\times 4}}$:

$$\begin{array}{ccc} & & \mathbb{C}F[-,-] \boxtimes^{\mathrm{ps}} \mathbb{C}F[-,-] \\ & \nearrow & \downarrow \\ \mathbb{C}(F[-,-] \times F[-,-]) & \longrightarrow & \mathbb{C}F[-,-] \times \mathbb{C}F[-,-]. \end{array}$$

using Corollary 4.39 and the fact that the right-hand natural transformation is levelwise a trivial fibration in $Dbl\mathcal{C}at_{\mathrm{whi}}$ by Lemma 4.12. ∎

As a consequence, we obtain the desired map.

Proposition 4.41 *Given bisimplicial sets X and Y, there is a double functor*

$$\mathbb{C}(X \times Y) \to \mathbb{C}X \boxtimes^{\mathrm{ps}} \mathbb{C}Y$$

natural in X and Y.

Proof We write

$$X \cong \mathrm{colim}_{F[m,t] \to X} F[m,t] \quad \text{and} \quad Y \cong \mathrm{colim}_{F[m',t'] \to Y} F[m',t'].$$

Then there are natural isomorphisms in $Dbl\mathcal{C}at$

$$\begin{aligned} \mathbb{C}(X \times Y) &\cong \mathbb{C}((\mathrm{colim}_{F[m,t] \to X} F[m,t]) \times (\mathrm{colim}_{F[m',t'] \to Y} F[m',t'])) \\ &\cong \mathbb{C}(\mathrm{colim}_{F[m,t] \to X} \mathrm{colim}_{F[m',t'] \to Y} (F[m,t] \times F[m',t'])) \\ &\cong \mathrm{colim}_{F[m,t] \to X} \mathrm{colim}_{F[m',t'] \to Y} \mathbb{C}(F[m,t] \times F[m',t']) \end{aligned}$$

using that $\times$ and $\mathbb{C}$ preserve colimits. Moreover, there are natural isomorphisms in $Dbl\mathcal{C}at$

$$\begin{aligned} \mathbb{C}X \boxtimes^{\mathrm{ps}} \mathbb{C}Y &\cong \mathbb{C}(\mathrm{colim}_{F[m,t] \to X} F[m,t]) \boxtimes^{\mathrm{ps}} \mathbb{C}(\mathrm{colim}_{F[m',t'] \to Y} F[m',t']) \\ &\cong (\mathrm{colim}_{F[m,t] \to X} \mathbb{C}F[m,t]) \boxtimes^{\mathrm{ps}} (\mathrm{colim}_{F[m',t'] \to Y} \mathbb{C}F[m',t']) \\ &\cong \mathrm{colim}_{F[m,t] \to X} \mathrm{colim}_{F[m',t'] \to Y} (\mathbb{C}F[m,t] \boxtimes^{\mathrm{ps}} \mathbb{C}F[m',t']) \end{aligned}$$

using that $\mathbb{C}$ and $\boxtimes^{\mathrm{ps}}$ preserve colimits. Then the double functor

$$\mathbb{C}(F[m,t] \times F[m',t']) \to \mathbb{C}F[m,t] \boxtimes^{\mathrm{ps}} \mathbb{C}F[m',t']$$

from Lemma 4.40, which is natural in m, t, m', t', induces a double functor between colimits

$$\mathbb{C}(X \times Y) \to \mathbb{C}X \boxtimes^{\mathrm{ps}} \mathbb{C}Y.$$

Naturality in X and Y follows from the naturality in the indexing shapes of the colimits. ∎

Given bisimplicial sets X and Y, we now aim to show that the double functor

$$\mathbb{C}(X \times Y) \to \mathbb{C}X \boxtimes^{\mathrm{ps}} \mathbb{C}Y$$

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is a weak equivalence in $Dbl\mathcal{C}at_{\mathrm{whi}}$. Again, we start with the case where $X = F[m,t]$ and $Y = F[m',t']$ are representables.

Lemma 4.42 *Given $m, t, m', t' \geq 0$, the double functor from Lemma 4.40 is a double biequivalence*

$$\mathbb{C}(F[m,t] \times F[m',t']) \xrightarrow{\simeq} \mathbb{C}F[m,t] \boxtimes^{\mathrm{ps}} \mathbb{C}F[m',t'].$$

Proof First recall from Theorem 4.36(2) that the components of the counit

$$\varepsilon_{[m,t]} \colon \mathbb{C}F[m,t] \cong \mathbb{C}\mathbb{N}[m,t] \xrightarrow{\simeq} [m,t]$$

and

$$\varepsilon_{[m,t]\times[m',t']} \colon \mathbb{C}(F[m,t] \times F[m',t']) \cong \mathbb{C}\mathbb{N}([m,t] \times [m',t']) \to [m,t] \times [m',t']$$

are double biequivalences. Hence, by [MSV23b, Remark 7.5], so is the product

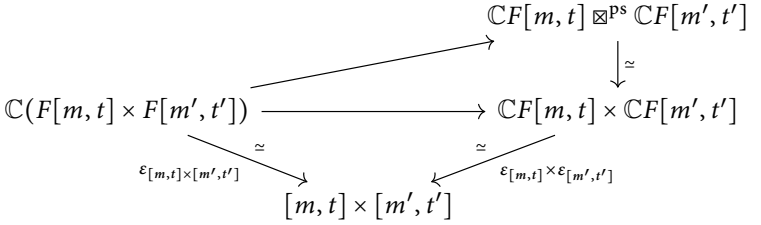
$$\varepsilon_{[m,t]} \times \varepsilon_{[m',t']} \colon \mathbb{C}F[m,t] \times \mathbb{C}F[m',t'] \xrightarrow{\simeq} [m,t] \times [m',t'].$$

Moreover, recall from [MSV23b, Lemma 7.3] that the natural projection double functor

$$\mathbb{C}F[m,t] \boxtimes^{\mathrm{ps}} \mathbb{C}F[m',t'] \xrightarrow{\simeq} \mathbb{C}F[m,t] \times \mathbb{C}F[m',t']$$

is also a double biequivalence.

Hence, we have the following commutative diagram in $Dbl\mathcal{C}at_{\mathrm{whi}}$:



Note that the diagram indeed commutes by naturality of the counit and by construction of the double functor from Lemma 4.40. So the result follows by 2-out-of-3. ■

We can finally prove that the desired double functor is also a weak equivalence in $Dbl\mathcal{C}at_{\mathrm{whi}}$.

Theorem 4.43 *Given bisimplicial sets X and Y, the map from Proposition 4.41 is a weak equivalence in the model structure $Dbl\mathcal{C}at_{\mathrm{whi}}$*

$$\mathbb{C}(X \times Y) \xrightarrow{\simeq} \mathbb{C}X \boxtimes^{\mathrm{ps}} \mathbb{C}Y.$$

Proof Given $m, t \geq 0$, we first show that the double functor from Proposition 4.41 is a weak equivalence in the model structure $Dbl\mathcal{C}at_{\mathrm{whi}}$

$$\mathbb{C}(F[m,t] \times Y) \xrightarrow{\simeq} \mathbb{C}F[m,t] \boxtimes^{\mathrm{ps}} \mathbb{C}Y, \tag{4.2}$$

for all bisimplicial sets Y. By Theorems 4.11 and 4.36(1) and Remark 2.4, the functors

$$\mathbb{C}(F[m,t] \times -),\ \mathbb{C}F[m,t] \boxtimes^{\mathrm{ps}} \mathbb{C}(-) \colon DblCat_{(\infty,1)} \to Dbl\mathcal{C}at_{\mathrm{whi}}$$

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are left Quillen, so the set of $(\Delta \times \Delta)$-sets Y for which the double functor (4.2) is a weak equivalence in $\mathit{DblCat}_{\mathrm{whi}}$ is saturated by monomorphisms in the sense of [Cis19, Definition 1.3.9]. Furthermore, by Lemma 4.42, this holds when $Y = F[m', t']$ is a representable in $\mathit{Set}^{\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}}$. Hence, by [Cis19, Corollary 1.3.10], we obtain that the double functor (4.2) is a weak equivalence in $\mathit{DblCat}_{\mathrm{whi}}$ for all bisimplicial sets Y.

Now, given a bisimplicial set Y, a similar argument using the above result shows that the double functor from Proposition 4.41 is a weak equivalence in the model structure $\mathit{DblCat}_{\mathrm{whi}}$

$$\mathbb{C}(X \times Y) \xrightarrow{\simeq} \mathbb{C}X \boxtimes^{\mathrm{ps}} \mathbb{C}Y,$$

for all bisimplicial sets X, as desired. ∎

By putting this result with Lemma 4.12 together, we get the following corollary.

Corollary 4.44 *The left adjoint of* $\mathbb{N}\colon [\mathit{DblCat}]_\infty \to \mathit{DblCat}_{(\infty,1)}$ *preserves products.*

Proof By Theorem 4.43, we have that the left adjoint of $\mathbb{N}$ sends Cartesian products to pseudo Gray tensor products. Then, by Lemma 4.12, the monoidal structure induced by the pseudo Gray tensor product in $[\mathit{DblCat}]_\infty$ is equivalent to the one given by the Cartesian product. Hence, the left adjoint of $\mathbb{N}$ preserves products. ∎

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