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Asymptotic entropy of transformed random walks

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Abstract. We consider general transformations of random walks on groups determined by Markov stopping times and prove that the asymptotic entropy (respectively, rate of escape) of the transformed random walks is equal to the asymptotic entropy (respectively, rate of escape) of the original random walk multiplied by the expectation of the corresponding stopping time. This is an analogue of the well-known Abramov formula from ergodic theory; its particular cases were established earlier by Kaimanovich [Differential entropy of the boundary of a random walk on a group. *Uspekhi Mat. Nauk* **38**(5(233)) (1983), 187–188] and Hartman *et al* [An Abramov formula for stationary spaces of discrete groups. *Ergod. Th. & Dynam. Sys.* **34**(3) (2014), 837–853].

1. *Introduction*

The notion of entropy of a countable probability space was introduced by Shannon in 1948. He used it to define the asymptotic entropy (entropy rate) in order to quantify the amount of information for a stationary stochastic process [**Sha48**]. Later in the mid 1950s Kolmogorov developed the notion of entropy of a measure-preserving dynamical system [**Kol58**], and his work was completed by Sinai [**Sin59**]. But it was only in 1972 that Avez defined the asymptotic entropy of a random walk on a group [**Ave72**]. Despite a formal similarity, the contexts of these definitions are different, and so far there is no common approach which would unify them.

The asymptotic entropy is an important quantity which describes the behavior of a random walk at infinity. For instance, the triviality of the Poisson boundary of a random walk is equivalent to vanishing of the asymptotic entropy [**Ave74**, **Der80**, **KV83**].

There are various formulas for the asymptotic entropy of a random walk on a group.

(i) In terms of the entropy of convolution powers [**KV83**], see equality (3.1) below.
(ii) Shannon's formula [**Der80**, **KV83**], see Theorem 3.1 below.
(iii) As the average of *Kullback–Liebler deviations* between the harmonic measure and its translates [**Kai83**, **KV83**], see equality (3.8) below.

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(iv) As the exponential growth rate of the Radon–Nikodym derivatives of the translates of the harmonic measure along sample paths of the random walk [**KV83**], see equality (3.2) below.

In the last two formulas, the asymptotic entropy is expressed in terms of the Poisson boundary of the random walk, which suggests considering a possible relationship between asymptotic entropies for random walks on the same group which share a common Poisson boundary.

Earlier, this relationship was studied in two particular situations.

(j) Convex combinations of convolutions of a given probability measure [**Kai83**].

(jj) The induced random walk on a recurrent subgroup [**Fur71**, **Kai91**, **Kai92**, **HLT14**].

In case (j), the asymptotic entropy can be obtained by a direct calculation based on formula (iii) [**Kai83**].

In case (jj), Furstenberg [**Fur71**] introduced the induced random walk on a recurrent subgroup and proved that its Poisson boundary is the same as the original Poisson boundary. Kaimanovich [**Kai92**] used a similar model to study harmonic functions on a Riemannian manifold and to compare the asymptotic entropies in this context. Although his setup was somewhat different, by the same approach one can also find the asymptotic entropy of the induced random walk on a recurrent subgroup [**Kai91**]. Recently, Hartman *et al* [**HLT14**] calculated the asymptotic entropy of the random walk induced on a finite index subgroup in an alternative way by using formula (iii) (although, apparently, they were not aware of [**Kai92**, **Kai91**]).

The probability measures arising in the above situations are examples of transformations of probability measures which do not change the Poisson boundary. Finding all probability measures with the same Poisson boundary is important for understanding the structure of a group G. For instance, Furstenberg [**Fur71**] proved that the Poisson boundary of $SL(n, \mathbb{R})$ is the same as the Poisson boundary of a lattice of $SL(n, \mathbb{R})$ endowed with an appropriate measure. By using this fact, he concluded that the lattices of $SL(n, \mathbb{R})$ for $n \geq 3$ are different from discrete subgroups of $SL(2, \mathbb{R})$.

Despite the existence of several examples, currently there is no general way to find all probability measures on a group G which have the same Poisson boundary as a given probability measure μ. However, Kaimanovich and the author [**FK**] (see [**For15**, Ch. 3]) proposed a method to construct many probability measures whose Poisson boundary coincides with that of μ. This method consists of applying a Markov stopping time to the original random walk (or to its randomization) and all currently known examples can be obtained by this method.

The purpose of this article is to show how these transformations affect the asymptotic entropy. We will show that the asymptotic entropy h' of the transformed random walk is the result of rescaling the asymptotic entropy h of the original random walk by the expectation $\boldsymbol{\tau}$ of the stopping time (Theorem 3.6):

$$h' = \boldsymbol{\tau} h. \tag{1.1}$$

The aforementioned examples (j) and (jj) are contained in this result as particular cases.

Equation (1.1), the rescaling of the asymptotic entropy under a ‘time change’, is analogous to Abramov’s formula [**Abr59**] for the entropy of induced dynamical systems

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(Theorem 3.3). However, as we have already pointed out, we are not aware of any common context which would unify these two formulas.

Our proof consists of three steps. Firstly, by using martingale theory, we prove that finiteness of the entropy of a probability measure is preserved after applying a Markov stopping time with a finite expectation (Lemma 3.5). Secondly, by taking into account formula (iv) for the asymptotic entropy, we will establish the main result. And, finally, applying the same method, we will prove that this result holds for randomized Markov stopping times as well (Theorem 3.6).

We would like to emphasize that in our general setup finiteness of the expectation $\boldsymbol{\tau}$ is not related to finiteness of any associated space (which can already be observed in the case of convolution powers, see above (j)). On the other hand, the technique used by Hartman *et al* [**HLT14**] crucially depends on the fact that for the induced random walks on recurrent subgroups $\boldsymbol{\tau} < \infty$ if and only if the subgroup has finite index, in combination with a number of properties of finite-state Markov chains (formulated in the Appendix to [**HLT14**]).

The rate of escape is another quantity which describes behavior of a random walk at infinity. There are some interrelations between the rate of escape and asymptotic entropy [**Var85**, **KL07**]. We will show that the rate of escape of a transformed random walk under a randomized Markov stopping time is also transformed according to formula (1.1).

2. *Preliminaries*

In this section, we will recall the basic definitions related to random walks on a group, the associated Poisson boundary and transformations of a random walk via a Markov stopping time.

2.1. *Random walks on groups.*

Throughout this paper, we assume that G is a countable group with the identity element e endowed with a probability measure μ. The random walk (G, μ) is the Markov chain on G with the transition probabilities

$$p(g_1, g_2) = \mu(g_1^{-1} g_2).$$

The *space of increments* of the random walk (G, μ) is denoted by $(G^{\mathbb{N}}, \bigotimes_1^\infty \mu)$ ($\bigotimes$ denotes the product measure); equivalently, it is the set of sequences $(h_n)_{n\geq 1}$ such that the h_n are independent and identically μ-distributed. The space of sample paths of the random walk (G, μ) is the probability space $(G^{\mathbb{Z}+}, \mathbf{P})$, where $G^{\mathbb{Z}+} = \{e\} \times G^{\mathbb{N}}$ (we only consider random walks issued from the identity), and the probability measure $\mathbf{P}$ is the image of the measure $\bigotimes_1^\infty \mu$ under the map

$$(h_1, h_2, \ldots) \mapsto \boldsymbol{x} = (e, x_1, x_2, \ldots),$$

where $x_n = h_1 \ldots h_n$ is the position of the random walk (G, μ) at time n.

2.2. *Poisson boundary.*

Let T be the time shift on the space of sample paths, i.e., $T(x_n) = (x_{n+1})$. Let $\mathcal{A}_T$ be the invariant σ-algebra of the time shift T. In other words,

$$\mathcal{A}_T = \{A \text{ measurable } : \ A = T^{-1}A\}.$$

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Rokhlin's correspondence theorem [**Roh52**] implies that there exists a quotient map **bnd** from $(G^{\mathbb{Z}+}, \mathbf{P})$ onto a Lebesgue space (Γ, ν) such that the pre-image of the σ-algebra of (Γ, ν) under the map **bnd** coincides (mod 0) with $\mathcal{A}_T$. The image of the measure **P** under the map **bnd** is called the *harmonic measure*. The Lebesgue space (Γ, ν) is called the *Poisson boundary* of the random walk (G, μ).

The group G acts naturally on the Poisson boundary and ν is a μ-stationary measure, that is,

$$\mu * \nu = \sum_g \mu(g) g\nu = \nu.$$

Let μ' be another probability measure on the group G whose Poisson boundary is (Γ', ν'). We say that Poisson boundaries (Γ, ν) and (Γ', ν') are the same whenever these boundaries are isomorphic as two measure G-spaces. More precisely, there exists a bijective measurable map $\phi : (\Gamma, \nu) \to (\Gamma', \nu')$ such that ν' is the image of the probability measure ν and $g\phi(\gamma) = \phi(g\gamma)$ for every $g \in G$ and almost every $\gamma \in \Gamma$.

2.3. *Transformed random walks.* Let $\tau : (G^{\mathbb{Z}+}, \mathbf{P}) \mapsto \mathbb{N}$ be a Markov stopping time on the space of sample paths, i.e., τ is a measurable map such that for every natural number s the set $\tau^{-1}\{s\}$ belongs to $\mathcal{A}_0^s$, the σ-algebra generated by the position of the random walk between times 0 and s of the random walk (G, μ). (We only consider Markov stopping times which are finite almost everywhere.)

Random walks on groups are time and space homogenous, which allows us to iterate the Markov stopping time τ and produce a new random walk [**FK**] (see [**For15**, Ch. 3]). More precisely, let $\tau_1 = \tau$ and define by induction

$$\tau_{n+1} = \tau_n + \tau(U^\tau), \tag{2.1}$$

where

$$U(\boldsymbol{x}) = (x_1^{-1} x_{n+1})_{n \geq 0} \tag{2.2}$$

is the transformation of the path space induced by the time shift in the space of increments.

Then (x_{τ_n}) is a sample path of the random walk (G, μ_τ), where μ_τ is the distribution of x_τ, i.e.,

$$\mu_\tau(g) = \mathbf{P}\{\boldsymbol{x} : x_\tau = g\}.$$

Obviously, each τ_n is also a Markov stopping time and, moreover, the distribution of x_{τ_n} is the n-fold convolution of μ_τ, i.e., $\mu_{\tau_n} = (\mu_\tau)^{*n}$.

Straightforward examples of this construction are now given.

Example 2.1. Let k be a positive integer and let τ be the constant function k. Then μ_τ is the k-fold convolution of measure μ, i.e., μ^{*k}.

Example 2.2. Let τ_1 and τ_2 be two Markov stopping times for the random walk (G, μ). Then $\tau = \tau_1 + \tau_2(U^{\tau_1})$ is a Markov stopping time and $\mu_\tau = \mu_{\tau_1} * \mu_{\tau_2}$.

Example 2.3. Let B be a subset of G with $\mu(B) > 0$. For a sample path $\boldsymbol{x} = (x_n)$, define

$$\tau(\boldsymbol{x}) = \min\{i \geq 1 : h_i \in B\}$$

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to be the minimal time i such that the increment h_i belongs to B. Then τ is a Markov stopping time. If $\mu(B) = 1$, then trivially $\mu_\tau = \mu$, otherwise

$$\mu_\tau = \beta + \sum_{i=1}^{\infty} \alpha^{*i} * \beta,$$

where β is the restriction of μ into B, i.e., $\beta(A) = \mu(A \cap B)$ for a subset A of G, and $\alpha = \mu - \beta$.

Example 2.4. Let H be a recurrent subset of G, i.e., $\mathbf{P}\{\boldsymbol{x} : x_n \in H \text{ for some } n\} = 1$. Define

$$\tau(\boldsymbol{x}) = \min\{i \geq 1 \; x_i \in H\}.$$

Then τ is a Markov stopping time called the first *hitting time* of H. Note that τ_2 is not generally a hitting time of H, that is, x_{τ_2} does not necessarily belong to the subset H. Although, if, in addition, H is a subgroup of G, then τ_n is precisely the moment when the random walk returns to H for the nth time. Hence, in this case, μ_τ is the probability measure induced on the recurrent subgroup H. Consequently, (G, μ_τ) is the induced random walk on the recurrent subgroup H.

By [**FK**] (see [**For15**, Ch. 3]), the Poisson boundary of (G, μ_τ) is the same as the Poisson boundary of (G, μ). However, Markov stopping times cannot produce all random walks with the same Poisson boundary, as the following trivial example shows.

Example 2.5. Let $G = \mathbb{Z}_2 = \{0, 1\}$; then the Poisson boundary of $(\mathbb{Z}_2, \mu)$ is trivial for any probability measure μ. Consider $\mu = \delta_1$; then $(0, 1, 0, 1, 0, \ldots)$ is the only sample path. Hence, in this situation, any Markov stopping time must be a constant function. Consequently, the only probability measures that can be obtained by the above procedure for the random walk $(\mathbb{Z}_2, \delta_1)$ are convolution powers of δ_1, which are δ_0 and δ_1. Therefore, the above procedure cannot produce the probability measure $\frac{1}{2}(\delta_0 + \delta_1)$.

In the next section, we will extend our random walk to a larger space and apply Markov stopping times to the new space to obtain even more random walks with the same Poisson boundary.

2.4. *Randomized Markov stopping times.* Let (Ω, θ) be a Lebesgue space. Define transition probabilities $\pi = \{\pi_{(g,\omega)}\}$ on the probability space $(G \times \Omega, \mu \otimes \theta)$ as follows:

$$\pi_{(g,\omega)} = \sum_h \mu(h)\delta_{gh} \otimes \theta.$$

Denote by $\mathbf{P}_{\delta_e \otimes m}$ the probability measure on the product space of the extended chain with the initial distribution $\delta_e \otimes m$.

In this construction, we replace a sample path $\boldsymbol{x} = (x_n)$ with its 'randomization'

$$(\boldsymbol{x}, \boldsymbol{\omega}) = ((e, \omega_0), (x_1, \omega_1), \ldots),$$

where the ω_i are independent and identically θ-distributed random variables.

In particular, the distribution of this Markov chain at time n is $\pi^n_{(g,\omega)} = \sum_h \mu^{*n}(h)$ $\delta_{gh} \otimes \theta$.

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Now let $\tau : G^{\mathbb{N}} \times \Omega^{\mathbb{N}} \to \mathbb{N}$ be a Markov stopping time for the extended Markov chain. Then we shall call τ a *randomized Markov stopping time* of the random walk (G, μ). Note that, if m is concentrated on one point, then any randomized Markov stopping time is just a usual Markov stopping time.

The transformation U (see equation 2.2) in the path space of the original random walk naturally extends to the following transformation of the path space of the randomized random walk:

$$\mathcal{U}(\boldsymbol{x}, \boldsymbol{\omega}) = (x_1^{-1} x_{n+1}, \omega_{n+1})_{n \geq 0}$$

and then we construct τ_n on the extended Markov chain as in the previous section. By using $\mathcal{U}$, one can define the iterated stopping times τ_n for the randomized random walk in the same way as in formula (2.1) above.

By projecting $(x_{\tau_n}, \omega_{\tau_n})_{n \geq 0}$ onto the first component, $(x_{\tau_n})_{n \geq 0}$, we obtain a new random walk (G, μ_τ) with

$$\mu_\tau(g) = \mathbf{P}_{\delta_e \otimes m}\{(\boldsymbol{x}, \boldsymbol{\omega}) : \boldsymbol{x}_\tau = g\}.$$

THEOREM 2.6. ([**FK**] (see [**For15**, Ch. 3])) *Let τ be a randomized Markov stopping time on the random walk (G, μ). If $\boldsymbol{E}(\log \tau)$ is finite, then $\Gamma(G, \mu) = \Gamma(G, \mu_\tau)$.*

This randomization allows one to construct even more random walks such that their Poisson boundaries are the same as for the original random walk. For example, now we can obtain convex combinations of convolution powers (cf. Example 2.5).

Example 2.7. ([**FK**] (see [**For15**, Ch. 3])) Let $\Omega = \mathbb{N}$ with probability measure θ. Define a Markov stopping time τ as

$$\tau((h_1, \omega_1), (h_2, \omega_2), \ldots) = \omega_1.$$

Then $\mu_\tau = \sum_{n \geq 0} \theta(n) \mu^{*n}$.

Example 2.8. ([**FK**] (see [**For15**, Ch. 3])) Let α and β be two subprobability measures with $\mu = \alpha + \beta$. Then there is a randomized Markov stopping time τ such that

$$\mu_\tau = \beta + \sum_{i \geq 1} \alpha^{*i} * \beta.$$

This transformation, $\mu = \alpha + \beta \mapsto \mu' = \beta + \sum_{i \geq 1} \alpha^{*i} * \beta$, was introduced by Willis [**Wil90**]. He also proved that $\Gamma(G, \mu) = \Gamma(G, \mu')$. If, in addition, α and β are mutually singular, then this example reduces to Example 2.3.

3. *Entropy of transformed measures*

The aim of this section is to study the relation between the asymptotic entropies of the transformed random walks defined as above and of the original random walk.

3.1. *Asymptotic entropy of a random walk.*

The quantity

$$H(\mu) = -\sum_g \mu(g) \log \mu(g)$$

is called the *entropy* of μ. The sequence $\{H_n\}$ is subadditive, i.e., $H_{n+m} \leq H_n + H_m$, where $H_n = H(\mu^{*n})$. Therefore, the limit of H_n/n exists. The *asymptotic entropy*

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of (G, μ) is defined as

$$h(\mu) = \lim_n \frac{H_n}{n}. \tag{3.1}$$

Throughout the rest of the paper, we always assume that $H(\mu) = H_1$ is finite, in which case $h(\mu) \leq H_1 < \infty$.

The asymptotic entropy can also be obtained by Shannon's formula.

THEOREM 3.1. [**KV83**, **Der80**] *For* $\mathbf{P}$*-almost every sample path* (x_n)*,*

$$-h(\mu) = \lim \frac{1}{n} \log \mu^{*n}(x_n).$$

Moreover, the convergence holds in $L^1(\mathbf{P})$.

Shannon's formula leads to the following description of the asymptotic entropy of (G, μ) as well.

THEOREM 3.2. [**KV83**] *For* $\mathbf{P}$*-almost every sample path* $\boldsymbol{x} = (x_n)$*,*

$$h(\mu) = \lim_n \frac{1}{n} \log \frac{dx_n \nu}{d\nu} \mathbf{bnd}(\boldsymbol{x}).$$

3.2. *Main results.* As a motivation, let us consider the asymptotic entropy of the random walk determined by the k-fold convolution of μ, i.e., (G, μ^{*k}). By definition of the asymptotic entropy,

$$h(\mu^{*k}) = \lim_n \frac{1}{n} H_{kn} = \lim_n k \frac{H_{kn}}{kn} = kh(\mu), \tag{3.2}$$

which can be rewritten as

$$h(\mu_\tau) = \boldsymbol{E}(\tau) h(\mu), \tag{3.3}$$

where τ is the constant Markov stopping time k (see Example 2.1).

The aim of the next theorem is to show that equality (3.3) can be generalized to all Markov stopping times (and to all randomized Markov stopping times, as we shall do later; see Theorem 3.6) with finite expectation. This is analogous to Abramov's theorem for the entropy of induced dynamical systems.

THEOREM 3.3. [**Abr59**] *Let* (X, ϕ, μ) *be an ergodic measure-preserving dynamical system. If* A *is a measurable subset of* X *with* $\mu(A) > 0$*, then*

$$h(\mu_A, \phi_A) = \frac{1}{\mu(A)} h(\mu, \phi).$$

In order to complete the analogy, note that the constant $1/\mu(A)$ is equal to the expectation of the return time to the set A (Kac formula) [**Kac47**].

Since the sequence $\{\mu_{\tau_n}\}$ is not, generally speaking, a subsequence of the sequence of convolution powers of μ, the generalization of the equality (3.3) cannot be done by the same trick as in (3.2).

THEOREM 3.4. *Let* τ *be a Markov stopping time with a finite expectation* $\boldsymbol{E}(\tau)$*. Then* $H(\mu_\tau)$ *is also finite and*

$$h(\mu_\tau) = \boldsymbol{E}(\tau) h(\mu).$$

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The proof is based on the fact that each sample path of the random walk transformed by a Markov stopping time is a subsequence of the corresponding sample path of the original random walk, and the description of the asymptotic entropy as the exponential growth rate of the Radon–Nikodym derivatives of the translates of the harmonic measure along the sample path (Theorem 3.2). In order to apply Theorem 3.2, we need the Poisson boundaries of the random walks (G, μ) and (G, μ_τ) to be the same (which follows from Theorem 2.6) and the entropy of the random walk (G, μ_τ) to be finite.

First, we show that the entropy of μ_τ is finite.

LEMMA 3.5. *If $\boldsymbol{E}(\tau) < \infty$, then the entropy of μ_τ is also finite and*

$$H(\mu_\tau) \le \boldsymbol{E}(\tau) H_1.$$

Proof. Let

$$M_n(\boldsymbol{x}) = n H_1 + \log \mu^{*n}(x_n).$$

Since H_1 is finite, the M_n are integrable. If $\mathcal{A}_0^n$ is the σ-algebra generated by the position of the random walk (G, μ) between times 0 and n, then

$$\boldsymbol{E}(M_{n+1}|\mathcal{A}_0^n)(\boldsymbol{x}) = (n+1)H_1 + \sum_h \mu(h) \log \mu^{*(n+1)}(x_n h).$$

The sequence $\{M_n, \mathcal{A}_0^n\}$ is a submartingale, i.e.,

$$\boldsymbol{E}(M_{n+1}|\mathcal{A}_0^n)(\boldsymbol{x}) \ge (n+1)H_1 + \sum_h \mu(h)(\log \mu^{*n}(x_n) + \log \mu(h)) = M_n(\boldsymbol{x}),$$

because

$$\mu^{*(n+1)}(x_n h) \ge \mu^{*n}(x_n)\mu(h). \tag{3.4}$$

Let $\tau \wedge n = \min\{\tau, n\}$. Then Doob's optional theorem (see [**Doo90**, p. 300]) implies that

$$\boldsymbol{E}(M_{\tau\wedge n}) \ge \boldsymbol{E}(M_0) = 0$$

and, consequently,

$$\boldsymbol{E}(\tau \wedge n) H_1 + \sum_g \mathbf{P}(x_{\tau\wedge n} = g) \log \mathbf{P}(x_{\tau\wedge n} = g) \ge 0.$$

Since $\boldsymbol{E}(\tau \wedge n) \le \boldsymbol{E}(\tau)$, we can write

$$-\sum_g \mathbf{P}(x_{\tau\wedge n} = g) \log \mathbf{P}(x_{\tau\wedge n} = g) \le \boldsymbol{E}(\tau) H_1. \tag{3.5}$$

Applying Fatou's lemma to inequality (3.5) gives

$$-\sum_g \liminf_n \mathbf{P}(x_{\tau\wedge n} = g) \log \mathbf{P}(x_{\tau\wedge n} = g) \le \boldsymbol{E}(\tau) H_1. \tag{3.6}$$

On the other hand, the fact that $\lim_n \mathbf{P}(x_{\tau\wedge n} = g) = \mu_\tau(g)$ in combination with the continuity of the function $x \log x$ implies that

$$\lim_n \mathbf{P}(x_{\tau\wedge n} = g) \log \mathbf{P}(x_{\tau\wedge n} = g) = \mu_\tau(g) \log \mu_\tau(g). \tag{3.7}$$

Now, by (3.6) and (3.7), we obtain

$$H(\mu_\tau) = -\sum_g \lim_n \mathbf{P}(x_{\tau\wedge n} = g) \log \mathbf{P}(x_{\tau\wedge n} = g) \le \boldsymbol{E}(\tau) H_1.$$

Therefore, we have proved that

$$\lim_{n\to\infty} H(\mu_{\tau\wedge n}) = H(\mu_\tau). \qquad \square$$

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3.3. *Proof of Theorem 3.4.* Now we can find the asymptotic entropy of the random walk (G, μ_τ). Since the expectation of the Markov stopping time τ is finite, Lemma 3.5 implies that $H(\mu_\tau)$ is also finite. Therefore, Theorem 3.2 implies that

$$h(\mu_\tau) = \lim_n \frac{1}{n} \log \frac{dx_{\tau_n}\nu}{d\nu}\mathbf{bnd}(\boldsymbol{x})$$

for almost every sample path $\boldsymbol{x} = (x_n)$. Since U is a measure-preserving transformation, $\boldsymbol{E}(\tau_n) = n\boldsymbol{E}(\tau)$ and, moreover, $\lim_n (\tau_n(\boldsymbol{x})/n) = \boldsymbol{E}(\tau)$ for almost every sample path $\boldsymbol{x}$. Therefore, it is obvious that

$$h(\mu_\tau) = \lim_n \frac{\tau_n(\boldsymbol{x})}{n} \frac{1}{\tau_n(\boldsymbol{x})} \log \frac{dx_{\tau_n}\nu}{d\nu}\mathbf{bnd}(\boldsymbol{x}).$$

By applying Theorem 3.2 to the random walk (G, μ), we will have

$$h(\mu_\tau) = \boldsymbol{E}(\tau)h(\mu).$$

3.4. *Entropy of random walks transformed via a randomized Markov stopping time.* Let τ be a randomized Markov stopping time with finite expectation. If we replace the submartingale in Lemma 3.5 with

$$M_n(\boldsymbol{x}, \boldsymbol{\omega}) = nH_1 + \log \mu^{*n}(x_n),$$

then finiteness of the entropy of μ_τ can be obtained by reproducing the proof of Lemma 3.5. Since the Poisson boundary of (G, μ_τ) is the same as the Poisson boundary of (G, μ), the proof of Theorem 3.3 applies. Hence, we have the following result.

THEOREM 3.6. *Let τ be a randomized Markov stopping time for the random walk (G, μ). If $\boldsymbol{E}(\tau)$ is finite, then*

$$h(\mu_\tau) = \boldsymbol{E}(\tau)h(\mu).$$

The combination of Example 2.7 and the preceding theorem implies the following result of Kaimanovich [**Kai83**].

Example 3.7. Let $\mu' = \sum_{k\geq 0} a_k\mu_k$, where $\sum_{k\geq 0} a_k = 1$ and $a_k \geq 0$. Then

$$h(\mu') = \left(\sum_{k\geq 0} ka_k\right)h(\mu).$$

Example 3.8. Let $\mu = \alpha + \beta$ be as in Example 2.8. Let $\mu' = \beta + \sum_{i\geq 1} \alpha^{*i} * \beta$; then

$$h(\mu') = \frac{1}{\|\beta\|}h(\mu),$$

where $\|\beta\|$ is the total mass of β.

3.5. *μ-boundary.* A *μ-boundary* (Γ_ξ, ν_ξ) is the quotient of the Poisson boundary with respect to a G-invariant measurable partition ξ (see [**Fur71**, **Kai00**]). The *differential entropy* of a μ-boundary is defined as

$$E_\mu(\Gamma_\xi, \nu_\xi) = \sum_g \mu(g) \int \log \frac{dg\nu_\xi}{d\nu_\xi}(g\gamma_\xi)\, d\nu_\xi(\gamma_\xi). \tag{3.8}$$

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Kaimanovich [**Kai83**] showed that the asymptotic entropy $h(\mu)$ is the upper bound for the asymptotic entropies of μ-boundaries, i.e.,

$$E_\mu(\Gamma_\xi, \nu_\xi) \leq h(\mu).$$

Moreover, Kaimanovich [**Kai00**] proved an analogue of Theorem 3.2 for μ-boundaries. Therefore, the claim of Theorem 3.6 is also valid for the differential entropy of μ-boundaries.

THEOREM 3.9. *Let ξ be a measurable G-invariant partition of the Poisson boundary (G, μ). If τ is a randomized Markov stopping time with finite expectation, then*

$$E_{\mu_\tau}(\Gamma_\xi, \nu_\xi) = \boldsymbol{E}(\tau) E_\mu(\Gamma_\xi, \nu_\xi).$$

4. *Rate of escape*

In this section, we establish a relationship between the escape rates of the random walk transformed via a randomized Markov stopping time and of the original one.

Definition 4.1. A gauge $\mathcal{G} = (\mathcal{G}_n)$ on a group G is an increasing sequence of subsets of G such that $G = \bigcup_n \mathcal{G}_n$. A gauge function $|.|_\mathcal{G}$ on G is defined as

$$|g| = |g|_\mathcal{G} = \min\{n : g \in \mathcal{G}_n\}.$$

A gauge $\mathcal{G}$ is called *subadditive* whenever its gauge function is subadditive.

A measure μ has a finite *first moment* with respect to a gauge $\mathcal{G}$ if

$$|\mu| = \sum_g |g| \mu(g)$$

is finite. If $\mathcal{G}$ is a subadditive gauge, then obviously

$$|x_{n+m}| \leq |x_n| + |(U^n \boldsymbol{x})_m|,$$

so that Kingman's subadditive theorem implies the following result.

THEOREM 4.2. [**Der80**, **Gui80**] *Let $\mathcal{G}$ be a subadditive gauge and μ have a finite first moment with respect to $\mathcal{G}$. Then*

$$\ell(G, \mu, \mathcal{G}) = \ell(\mu) = \lim_n \frac{|x_n|}{n} \tag{4.1}$$

exists for **P***-almost every sample path $\boldsymbol{x} = (x_n)$ and also in $L^1(\mathbf{P})$.*

The quantity $\ell(\mu)$ is called the rate of escape (drift) of the random walk (G, μ) with respect to the gauge $\mathcal{G}$.

Example 4.3. Let G be a group generated by a finite set $S = S^{-1}$. Then $(S^n)_{n\geq 1}$ is a subadditive gauge.

Example 4.4. Let (G, μ) be a *transient* random walk. Denote by $F(g) < 1$ the probability that the random walk (G, μ) ever visits a point $g \in G$. Let $\mathcal{G}_n = \{g \in G : -\ln F(g) \leq n\}$; then $\mathcal{G} = (\mathcal{G}_n)_{n\geq 0}$ is a subadditive gauge. Actually, $-\ln F(g)$ is the 'distance' of g from the identity element in the so-called *Green metric* [**BB07**]. The rate of escape with respect to the Green metric is equal to the asymptotic entropy [**BHM08**].

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THEOREM 4.5. *Let $\mathcal{G}$ be a subadditive gauge for a group G and μ have a finite first moment. Let τ be a randomized Markov stopping time with finite expectation $\boldsymbol{E}(\tau)$.*

Then the probability measure μ_τ has a finite first moment with respect to the gauge $\mathcal{G}$ and

$$\ell(\mu_\tau) = \boldsymbol{E}(\tau)\ell(\mu). \tag{4.2}$$

Proof. Let $L_n(\boldsymbol{x}, \boldsymbol{\omega}) = n\boldsymbol{E}(|x_1|) - |x_n|$. Then

$$\boldsymbol{E}(L_{n+1}|\mathcal{A}_0^n) = (n+1)\boldsymbol{E}(\tau) - \sum_h |x_n h|\mu(h).$$

The subadditivity of the gauge $\mathcal{G}$ implies that $\{L_n, \mathcal{A}_0^n\}$ is a submartingale. Now applying the same argument as in Lemma 3.5 and Theorem 3.4, we get the equality (4.2). □

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