Ergod. Th. & Dynam. Sys. (2012), **32**, 1752–1782 © *Cambridge University Press, 2011*
doi:10.1017/S0143385711000484

Rigidity of commutative non-hyperbolic actions by toral automorphisms

ZHIREN WANG

Department of Mathematics, Princeton University, Princeton, NJ 08544, USA
(e-mail: zhirenw@math.princeton.edu)

(*Received* 10 *January* 2011 *and accepted in revised form* 1 *June* 2011)

Abstract. Berend [Multi-invariant sets on tori. *Trans. Amer. Math. Soc.* **280**(2) (1983), 509–532] obtained necessary and sufficient conditions on a $\mathbb{Z}^r$-action α on a torus $\mathbb{T}^d$ by toral automorphisms in order for every orbit to be either finite or dense. One of these conditions is that for every common eigendirection of the $\mathbb{Z}^r$-action there is an element $\mathbf{n} \in \mathbb{Z}^r$ such that $\alpha^{\mathbf{n}}$ expands this direction. In this paper, we investigate what happens when this condition is removed; more generally, we consider a partial orbit $\{\alpha^{\mathbf{n}} \cdot x : \mathbf{n} \in \Omega\}$ where Ω is a set of elements which acts in an approximately isometric way on a given set of eigendirections. This analysis is used in an essential way in the work of the author with E. Lindenstrauss [Topological self-joinings of Cartan actions by toral automorphisms. *Preprint*, 2010] classifying topological self-joinings of maximal $\mathbb{Z}^r$-actions on tori for $r \geq 3$.

1. *Introduction*

1.1. *Background.* In the landmark paper [**Fur67**], Furstenberg showed that any closed subset of $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ which is invariant under both $x \mapsto 2x$ and $x \mapsto 3x$ is either $\mathbb{T}$ itself or a finite collection of rational points. This theorem was extended by Berend [**Ber83**] to commutative semigroup actions on higher-dimensional tori. A special case is the following theorem.

THEOREM 1.1. [**Ber83**] *Let α be a faithful $\mathbb{Z}^r$-action on $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ by toral automorphisms given by a group embedding $\alpha : \mathbf{n} \mapsto \alpha^{\mathbf{n}}$ of $\mathbb{Z}^r$ into $\mathrm{GL}(d, \mathbb{Z}) = \mathrm{Aut}(\mathbb{T}^d)$ satisfying the following three conditions:*

(1) *$r \geq 2$;*
(2) *there exists $\mathbf{n} \in \mathbb{Z}^r$ such that $\alpha^{\mathbf{n}}$ is a totally irreducible toral automorphism;*
(3) *for any common eigenvector $v \in \mathbb{C}^d$ of $\alpha(\mathbb{Z}^r)$, there exists $\mathbf{n} \in \mathbb{Z}^r$ such that $|\alpha^{\mathbf{n}} \cdot v| > |v|$.*

Then for all $x \in \mathbb{T}^d$, $\{\alpha^{\mathbf{n}} \cdot x : \mathbf{n} \in \mathbb{Z}^r\}$ is dense in $\mathbb{T}^d$ unless x is rational.

Here a toral automorphism $\phi \in \mathrm{GL}(d, \mathbb{Z})$ is said to be *irreducible* if there is no proper ϕ-invariant subtorus of positive dimension in $\mathbb{T}^d$, and it is *totally irreducible* if any power ϕ^k, $k \neq 0$, is irreducible.

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The aim of this paper is to investigate what happens when the hyperbolicity assumption (3) in Theorem 1.1 fails.

Our result will cover two situations. The first simply deals with $\mathbb{Z}^r$-actions that do not satisfy assumption (3), i.e. there are one or several common eigenvectors whose norms are preserved by the group action.

In the second setup, which is more general, we take a $\mathbb{Z}^r$-action and fix several common eigenvectors $v_i \in \mathbb{C}^d$, $i \in S$. Instead of studying the full $\mathbb{Z}^r$-action, we only allow ourselves to apply those $\alpha^{\mathbf{n}}$s satisfying $|\alpha^{\mathbf{n}} \cdot v_i|/|v_i| \in (e^{-\epsilon}, e^{\epsilon})$, for all $i \in S$, to a point $x \in \mathbb{T}^d$ and ask how the resulting orbit distributes in $\mathbb{T}^d$. This case is more delicate as typically the elements we apply do not form a subgroup of $\mathbb{Z}^r$.

In Theorem 1.8 we will give an analogue to Theorem 1.1 in these situations with assumptions (1) and (2) properly reformulated.

Another question surrounding Berend's theorem is what happens when the action is no longer irreducible. In particular, it is interesting to investigate the diagonal action $\alpha_\triangle : \mathbb{Z}^r \curvearrowright \mathbb{T}^d \times \mathbb{T}^d$ by α, where $\alpha_\triangle^{\mathbf{n}} \cdot (x, y) = (\alpha^{\mathbf{n}} \cdot x, \alpha^{\mathbf{n}} \cdot y)$, and ask what the orbit closures are. In a forthcoming joint paper [**LW10**] with Lindenstrauss, we give a classification of orbit closures when $r \geq 3$ and the action α is Cartan. Theorem 1.8 is used as a key lemma in [**LW10**].

Before going further we give a more explicit form of the group actions described in Theorem 1.1.

1.2. *Number-theoretical description of the group action.* The irreducibility assumption on α has a number-theoretical interpretation.

Consider a number field K of degree d. Suppose that K has r_1 real embeddings $\sigma_1, \ldots, \sigma_{r_1}$ and r_2 conjugate pairs of complex embeddings $(\sigma_{r_1+1}, \sigma_{r_1+r_2+1})$, $(\sigma_{r_1+2}, \sigma_{r_1+r_2+2}), \ldots, (\sigma_{r_1+r_2}, \sigma_{r_1+2r_2})$; then $r_1 + 2r_2 = d$ and the group of units U_K has rank $r_1 + r_2 - 1$. Recall that $K \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{R}^{r_1} \oplus \mathbb{C}^{r_2}$, where the embedding of K into $\mathbb{R}^{r_1} \oplus \mathbb{C}^{r_2}$ is given by

$$\sigma : \theta \mapsto (\sigma_1(\theta), \ldots, \sigma_{r_1}(\theta), \sigma_{r_1+1}(\theta), \ldots, \sigma_{r_1+r_2}(\theta)). \tag{1.1}$$

K acts multiplicatively on $K \otimes_{\mathbb{Q}} \mathbb{R}$ by $\theta \cdot (\mu \otimes x) = \theta\mu \otimes x$, for all $\theta, \mu \in K$, $x \in \mathbb{R}$; or equivalently, on $\mathbb{R}^{r_1} \oplus \mathbb{C}^{r_2}$ by

$$\theta \cdot (x_1, \ldots, x_{r_1+r_2}) = (\sigma_1(\theta)x_1, \ldots, \sigma_{r_1+r_2}(\theta)x_{r_1+r_2}). \tag{1.2}$$

This multiplicative action is compatible with σ in the sense that

$$\theta \cdot \sigma(\mu) = \sigma(\theta\mu) \quad \text{for all } t, s \in K. \tag{1.3}$$

The following result makes the translation from the $\mathbb{T}^d$ setting to a number-theoretical one. It is a special case of a more general fact, for which we refer to Schmidt [**Sch95**, §§7 and 29] and Einsiedler and Lind [**EL04**] (see also Einsiedler and Lindenstrauss [**EL03**, Proposition 2.1]).

PROPOSITION 1.2. (See [**Wan10**, Proposition 2.13]) *Let $\alpha : \mathbb{Z}^r \hookrightarrow \mathrm{GL}(d, \mathbb{Z})$ be a $\mathbb{Z}^r$-action on $\mathbb{T}^d$ such that $\alpha(\mathbb{Z}^r)$ contains an irreducible toral automorphism. Then there exist:*

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- *a number field K of degree d with r_1 real embeddings $\sigma_1 \ldots, \sigma_{r_1}$ and r_2 conjugate pairs of complex embeddings $(\sigma_{r_1+1}, \sigma_{r_1+r_2+1})$, $(\sigma_{r_1+2}, \sigma_{r_1+r_2+2}), \ldots, (\sigma_{r_1+r_2}, \sigma_d)$ where $r_1 + 2r_2 = d$;*
- *a common eigenbasis in $\mathbb{C}^d$ with respect to which, for all $\mathbf{n} \in \mathbb{Z}^r$, $\alpha^{\mathbf{n}}$ can be diagonalized as $\operatorname{diag}(\zeta_1^{\mathbf{n}}, \zeta_2^{\mathbf{n}}, \ldots, \zeta_d^{\mathbf{n}})$, where $\zeta_1^{\mathbf{n}}, \ldots, \zeta_{r_1}^{\mathbf{n}} \in \mathbb{R}$, $\zeta_{r_1+1}^{\mathbf{n}}, \ldots, \zeta_d^{\mathbf{n}} \in \mathbb{C}$ and $\zeta_{r_1+j}^{\mathbf{n}} = \overline{\zeta_{r_1+r_2+j}^{\mathbf{n}}}$, for all $j = 1, \ldots, r_2$;*
- *a group embedding $\zeta : \mathbf{n} \mapsto \zeta^{\mathbf{n}}$ of $\mathbb{Z}^r$ into the group of units U_K;*
- *a cocompact lattice Γ in $\mathbb{R}^{r_1} \oplus \mathbb{C}^{r_2}$ that is contained in $\sigma(K)$, where σ is given in (1.1);*

such that:

- *$\zeta_i^{\mathbf{n}} = \sigma_i(\zeta^{\mathbf{n}})$, for all $i \in \{1, \ldots, d\}$, for all $\mathbf{n} \in \mathbb{Z}^r$;*
- *for all $\mathbf{n} \in \mathbb{Z}^r$, multiplication by $\zeta^{\mathbf{n}}$ on $\mathbb{R}^{r_1} \oplus \mathbb{C}^{r_2}$ as in (1.2) preserves Γ, hence induces an action on $(\mathbb{R}^{r_1} \oplus \mathbb{C}^{r_2})/\Gamma$ by*

$$\zeta^{\mathbf{n}} \cdot (\tilde{x} \bmod \Gamma) = (\zeta^{\mathbf{n}} \cdot \tilde{x} \bmod \Gamma) \quad \text{for all } \tilde{x} \in \mathbb{R}^{r_1} \oplus \mathbb{C}^{r_2};$$

- *the action α on $\mathbb{T}^d$ is algebraically conjugate to the multiplicative action $\zeta : \mathbb{Z}^r \curvearrowright (\mathbb{R}^{r_1} \oplus \mathbb{C}^{r_2})/\Gamma$, i.e. there is a continuous group isomorphism $\psi : \mathbb{T}^d \xrightarrow{\sim} (\mathbb{R}^{r_1} \oplus \mathbb{C}^{r_2})/\Gamma$ such that $\alpha^{\mathbf{n}} \cdot x = \psi^{-1}(\zeta^{\mathbf{n}} \cdot \psi(x))$, for all $\mathbf{n} \in \mathbb{Z}^r$ and $x \in \mathbb{T}^d$.*

Remark 1.3. For $\mathbf{n} \in \mathbb{Z}^r$, $\alpha^{\mathbf{n}}$ is an irreducible toral automorphism if and only if $\zeta^{\mathbf{n}}$ does not belong to any non-trivial proper subfield of K. This is simply because $\zeta_i^{\mathbf{n}}$ is an eigenvalue of $\alpha^{\mathbf{n}}$. In particular, $\alpha^{\mathbf{n}}$ is totally irreducible if and only if $(\zeta^{\mathbf{n}})^k \notin L$ for all non-trivial proper subfields L of K and $k \in \mathbb{N}$.

1.3. *Notation.* Throughout the rest of the paper we consider a $\mathbb{Z}^r$-action α as in Proposition 1.2. We will write G for the acting group $\mathbb{Z}^r$, and let K, Γ, ψ and ζ be as in Proposition 1.2.

Write $I = \{1, \ldots, r_1 + r_2\}$.

For $1 \le i \le r_1$, let V_i be the ith copy of $\mathbb{R}$ in $\mathbb{R}^{r_1} \oplus \mathbb{C}^{r_2}$; while for $r_1 < i \le r_1 + r_2$, let V_i be the $(i - r_1)$th copy of $\mathbb{C}$. Then $\mathbb{R}^{r_1} \oplus \mathbb{C}^{r_2}$ can be rewritten as $\bigoplus_{i \in I} V_i$. We write

$$X = (\mathbb{R}^{r_1} \oplus \mathbb{C}^{r_2})/\Gamma = \left(\bigoplus_{i \in I} V_i\right) \Big/ \Gamma, \tag{1.4}$$

and let π be the canonical projection from $\mathbb{R}^{r_1} \oplus \mathbb{C}^{r_2}$ to X. X is a compact abelian group.

Moreover, as ψ is a continuous group morphism from $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ to X, it can be easily seen that it lifts to a non-degenerate real linear map $\tilde{\psi}$ from $\mathbb{R}^d$ to $(\mathbb{R}^{r_1} \oplus \mathbb{C}^{r_2}) \cong \mathbb{R}^d$ such that $\tilde{\psi}(\mathbb{Z}^d) = \Gamma$ and $\alpha^n \cdot \tilde{x} = \tilde{\psi}^{-1}(\zeta^{\mathbf{n}} \cdot \tilde{\psi}(\tilde{x}))$, for all $\mathbf{n} \in \mathbb{Z}^r$, for all $\tilde{x} \in \mathbb{R}^d$.

The pushforward of the usual Euclidean distance by π induces a Riemannian metric on X and makes X a locally Euclidean metric space. With this metric, the distance from a point x to the origin 0 in X is given by

$$\|x\| = \min_{\substack{\tilde{x} \in \mathbb{R}^{r_1} \oplus \mathbb{C}^{r_2} \\ \pi(\tilde{x}) = x}} |\tilde{x}|, \tag{1.5}$$

where $|\tilde{x}| = (\sum_{i \in I} |\tilde{x}_i|^2)^{1/2}$, $\tilde{x}_i$ denoting the V_i coordinate of $\tilde{x}$. The distance between $x, x' \in X$ is just $\|x - x'\|$.

For $x \in X$ and $D \subset G$, write $D \cdot x = \{\zeta^{\mathbf{n}} \cdot x : \mathbf{n} \in D\}$.

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Definition 1.4. For a real linear subspace V of $\mathbb{R}^{r_1} \oplus \mathbb{C}^{r_2}$, a point $z \in X$ is a *V-translated torsion point* if there are a torsion point $x_* \in X$ and a vector $v \in V$ such that $x = x_* + v$.

A *V-disk centered at a torsion point* in X is a set of the form $\{x_* + v : v \in V, |v| \leq R\}$ where x_* is a given torsion point and R is a given radius.

Definition 1.5. For $i \in I$, the ith *Lyapunov functional* $\lambda_i : \mathbb{Z}^r \mapsto \mathbb{R}$ is given by

$$\lambda_i(\mathbf{n}) = \log |\zeta_i^{\mathbf{n}}|. \tag{1.6}$$

Moreover, define $\beta_i : \mathbb{Z}^r \mapsto \mathbb{R}/2\pi\mathbb{Z}$ by

$$\beta_i(\mathbf{n}) = \arg \zeta_i^{\mathbf{n}}. \tag{1.7}$$

Remark 1.6. It is not hard to see λ_i and β_i are group morphisms. In particular, λ_i extends uniquely to a linear map from $(\mathbb{R}^r)^*$, which we still denote by λ_i.

In addition, although β_i is defined for all $i \in I$, when $i \leq r_1$, it only takes values in $\{0, \pi\}$ modulo $2\pi\mathbb{Z}$.

Write $\|\beta\|$ for the distance from $\beta \in \mathbb{R}/2\pi\mathbb{Z}$ to the trivial element 0.

1.4. *Non-hyperbolic foliations.* The hyperbolicity condition (3) in Theorem 1.1 is actually equivalent to assuming for all $i \in I$ that λ_i is not the zero map from $(\mathbb{R}^r)^*$.

In this paper, we try to assume less hyperbolicity than [**Ber83**] did. One way is to allow the zero map to appear as Lyapunov functionals. This gives an 'isometric subspace',

$$V_{\text{Isom}} = \bigoplus_{i \in I : \lambda_i \equiv 0} V_i, \tag{1.8}$$

that is invariant under the multiplicative action ζ and cannot be expanded or contracted by any element as $|\zeta_i^{\mathbf{n}}| = 1$, for all $\mathbf{n}$ whenever $\lambda_i \equiv 0$.

Another way to lose hyperbolicity is to pose extra restrictions on which group elements from the action one may apply. More precisely, the question is: given $S \subset I$, for a generic $x \in \mathbb{T}^d$ and $\epsilon > 0$, is the partial orbit $\{\alpha^{\mathbf{n}} \cdot x : \mathbf{n} \in \mathbb{Z}^r \text{ s.t. } |\lambda_i(\mathbf{n})| < \epsilon, \forall i \in S\}$ dense in $\mathbb{T}^d$?

In fact we will consider even smaller truncations of $\mathbb{Z}^r$ as in the next definition.

Definition 1.7. If $S \subset I$, $\epsilon > 0$ and $\sigma + H$ is a coset of some subgroup $H \leq G$, the *ϵ-slice of $\sigma + H$ with respect to S* is

$$H^{\sigma}_{\epsilon,S} = \{\mathbf{n} \in \sigma + H : |\lambda_i(\mathbf{n})| < \epsilon, \forall i \in S \text{ and } \|\beta_j(\mathbf{n})\| < \epsilon, \forall j \in I\}. \tag{1.9}$$

When $\sigma + H$ is H itself, simply write $H_{\epsilon,S}$ for $H^{\sigma}_{\epsilon,S}$. $\sigma + H$ is said to be *compatible* with S if $H^{\sigma}_{\epsilon,S} \neq \emptyset$, for all $\epsilon > 0$.

Observe that $H_{\epsilon,S}$ contains the identity, hence H itself is always compatible. In particular, one wishes to understand the action of the non-empty set

$$G_{\epsilon,S} = \{\mathbf{n} \in \mathbb{Z}^r : |\lambda_i(\mathbf{n})| < \epsilon, \forall i \in S \text{ and } \|\beta_j(\mathbf{n})\| < \epsilon, \forall j \in I\}. \tag{1.10}$$

We emphasize that $G_{\epsilon,S}$, and $H_{\epsilon,S}$ in general, usually do not form a subgroup of $G = \mathbb{Z}^r$.

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1.5. *Main result.* The following theorem is the main result of this paper.

THEOREM 1.8. *Let α be a $\mathbb{Z}^r$-action on $\mathbb{T}^d$ by toral automorphisms. Assume that $\alpha^{\mathbf{n}}$ is irreducible for at least one $\mathbf{n} \in \mathbb{Z}^r$, hence the notation from §1.3 is applicable. For any subset $S \subset I$, let $L_S \subset (\mathbb{R}^r)^*$ be the $\mathbb{R}$-span of the set $\{\lambda_j : j \in S\}$ (or $L_S = \{0\}$ if $S = \emptyset$). Let $\langle S \rangle \supset S$ be the set of all $i \in I$ such that $\lambda_i \in L_S$. Denote $V_{\langle S \rangle} = \bigoplus_{i \in \langle S \rangle} V_i$. Assume further that:*

(1) $\dim L_S \leq r - 2$;

(2) *for all $\epsilon > 0$, there exists $\mathbf{n} \in \mathbb{Z}^r$ such that $\alpha^{\mathbf{n}}$ is a totally irreducible toral automorphism and $|\lambda_i(\mathbf{n})| < \epsilon$, for all $i \in S$.*

Then for any $\epsilon > 0$, a point $x \in X$ satisfies

$$\{\zeta^{\mathbf{n}} \cdot x : \mathbf{n} \in \mathbb{Z}^r \text{ s.t. } |\lambda_i(\mathbf{n})| < \epsilon, \forall i \in S\} \text{ is dense in } X \tag{1.11}$$

if and only if x is not a $V_{\langle S \rangle}$-translated torsion point.

Equivalently, for all $\epsilon > 0$, $y \in \mathbb{T}^d$,

$$\{\alpha^{\mathbf{n}} \cdot y : \mathbf{n} \in \mathbb{Z}^r \text{ s.t. } |\lambda_i(\mathbf{n})| < \epsilon, \forall i \in S\} \text{ is dense in } \mathbb{T}^d \tag{1.12}$$

if and only if y cannot be written as $y_* + v$, where $y_* \in \mathbb{T}^d$ is a rational point and $v \in \tilde{\psi}^{-1}(V_{\langle S \rangle})$.

Recall that $\tilde{\psi}$ is the lift of ψ described in §1.3.

The restricted dynamics studied here are special cases of non-expansive subdynamics of $\mathbb{Z}^d$-actions defined and investigated by Boyle and Lind [**BL97**].

Remark 1.9. In the special case $S = \emptyset$, the set (1.11) is just the full orbit $G \cdot x$ for all $\epsilon > 0$ and the theorem studies exactly the same situation as in Theorem 1.1 except that assumption (3) is removed. In this case $V_{\langle S \rangle}$ is exactly the isometric subspaces V_{Isom} defined in (1.8). Moreover, the subspace $\tilde{\psi}^{-1}(V_{\langle S \rangle}) = \tilde{\psi}^{-1}(V_{\text{Isom}})$ can be defined as $\{v \in \mathbb{R}^d : \exists C > 0,\ C^{-1}|v| \leq |\alpha^{\mathbf{n}} \cdot v| \leq C|v|,\ \forall \mathbf{n} \in \mathbb{Z}^r\}$.

Instead of Theorem 1.8, we will prove the following slightly stronger statement.

THEOREM 1.10. *Suppose that $S \subset I$ satisfies both conditions (1) and (2) in Theorem 1.8. Then for any point $x \in X$ we have the following dichotomy.*

(i) *If x is not a $V_{\langle S \rangle}$-translated torsion point, then $\bigcap_{\epsilon > 0} \overline{G_{\epsilon,S} \cdot x} = X$.*

(ii) *Otherwise $\bigcap_{\epsilon > 0} \overline{G_{\epsilon,S} \cdot x}$ is a finite set of $V_{\langle S \rangle}$-translated torsion points.*

The spirit of the argument is that though usually not a subgroup, the subset $G_{\epsilon,S}$ looks like an abelian group of rank $r - \dim L_S$. As this rank is at least two by assumption, techniques from [**Ber83**] still apply.

1.6. *Necessity of total irreducibility.*

Remark 1.11. Assumption (2) in Theorem 1.8 is necessary. In fact one can construct a $\mathbb{Z}^r$-action α by toral automorphisms together with a set $S \subset I$ that meet all the other requirements in Theorem 1.8, and a number $\epsilon_0 > 0$ such that:

(2′) there exists $\mathbf{n}$ such that $|\lambda_i(\mathbf{n})| < \epsilon_0$ for all $i \in S$, and $\alpha^{\mathbf{n}}$ is totally irreducible;

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(⋆) there exists $x \in X$ such that x is not a $V_{\langle S\rangle}$-translated torsion point but the subset

$$\{\zeta^{\mathbf{n}} \cdot x : \mathbf{n} \in \mathbb{Z}^r \text{ s.t. } |\lambda_i(\mathbf{n})| < \epsilon_0, \forall i \in S\} \tag{1.13}$$

is not dense in X either.

A counterexample can be constructed in the following way. The idea is that if the irreducibility assumption is not strong enough, then when one gets rid of hyperbolicity, irreducibility may be lost as well.

Let

$$\theta = \sqrt{\frac{\sqrt{6}+\sqrt{2}}{2} - 1} - 1 \tag{1.14}$$

and K be the octic number field $\mathbb{Q}(\theta)$. It is easy to check that $\deg K = 8$, and K has two real embeddings σ_1, σ_2, as well as three pairs of conjugate imaginary embeddings (σ_i, σ_{i+3}), where $i = 3, 4, 5$. Moreover,

$$\sigma_1(\theta) = \sqrt{\frac{\sqrt{6}+\sqrt{2}}{2} - 1} - 1 \quad \text{and} \quad \sigma_2(\theta) = -\sqrt{\frac{\sqrt{6}+\sqrt{2}}{2} - 1} - 1.$$

θ is an algebraic unit as it solves the polynomial

$$\theta^8 + 8\theta^7 + 32\theta^6 + 80\theta^5 + 132\theta^4 + 144\theta^3 + 96\theta^2 + 32\theta + 1.$$

It is not hard to check that $\sigma_i(\theta)/\sigma_j(\theta)$ is not a root of unity for all $i \neq j$ and it follows that for any $k \in N$ and any proper subfield $L \subset K$, $\theta^k \notin L$.

K contains a totally real quartic subfield $F = \mathbb{Q}(\sqrt{2}, \sqrt{3})$. Note that $\sigma_1|_F = \sigma_2|_F$. By Dirichlet's unit theorem, $\operatorname{rank}(U_K) = 4$ and $\operatorname{rank}(U_F) = 3$. $\sqrt{3}-\sqrt{2}$, $\sqrt{2}-1$ and $(\sqrt{6}+\sqrt{2})/2$ form a set of fundamental units of F; furthermore, together with θ, they generate a finite-index subgroup of U_K as θ is not a root of unity. Define a group embedding $\zeta : \mathbb{Z}^4 \mapsto U_K$ by setting

$$\zeta^{\mathbf{e}_1} = \theta,\ \zeta^{\mathbf{e}_2} = \sqrt{3}-\sqrt{2},\ \zeta^{\mathbf{e}_3} = \sqrt{2}-1,\ \zeta^{\mathbf{e}_4} = \frac{\sqrt{6}+\sqrt{2}}{2}, \tag{1.15}$$

where $\mathbf{e}_1, \ldots, \mathbf{e}_4$ is a basis of $\mathbb{Z}^4$. As explained earlier, ζ naturally defines a multiplicative $\mathbb{Z}^4$-action on $X = (\mathbb{R}^2 \oplus \mathbb{C}^3)/\sigma(\mathcal{O}_K)$ where $\mathcal{O}_K$ is the ring of integers in K and σ is the canonical embedding of K into $\mathbb{R}^2 \oplus \mathbb{C}^3$ given by (1.1). By choosing an arbitrary $\mathbb{Z}$-basis of $\mathcal{O}_K$, we can easily identify X with $\mathbb{T}^8$ and get a conjugate $\mathbb{Z}^r$-action α by toral automorphisms as in Proposition 1.2.

Let $S = \{1, 2\}$. One may explicitly compute the Lyapunov exponent $\lambda_i(\mathbf{n})$ for each i from $I = \{1, \ldots, 5\}$ and verify that λ_1 and λ_2 are not proportional, and none of $\lambda_3, \lambda_4, \lambda_5$ is inside the linear span L_S of λ_1, λ_2. Hence $\langle S\rangle = S$ and $\dim L_S = 2$. So condition (1) from Theorem 1.8 is satisfied as $r = 4$ in this case.

Note that $|\sigma_1(\theta)| \neq |\sigma_2(\theta)|$, hence

$$\lambda_1(\mathbf{e}_1) - \lambda_2(\mathbf{e}_1) = \log|\sigma_1(\theta)| - \log|\sigma_2(\theta)| \neq 0. \tag{1.16}$$

Take $\epsilon_0 > \max(\log|\sigma_1(\theta)|, \log|\sigma_2(\theta)|)$. Then $\mathbf{e}_1$ satisfies $|\lambda_i(\mathbf{e}_1)| < \epsilon_0$ for both $i = 1, 2$. By Remark 1.3, the fact that $(\zeta^{\mathbf{e}_1})^k = \theta^k$ does not belong to any proper subfield of K

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implies that $\alpha^{\mathbf{e}_1}$ is a totally irreducible automorphism on $\mathbb{T}^8$. Therefore we have verified condition (2′) in Remark 1.11.

Finally, we will establish the claim ($\star$). Consider all $\mathbf{n} \in \mathbb{Z}^4$ such that

$$|\lambda_i(\mathbf{n})| < \epsilon_0, \quad i = 1, 2; \tag{1.17}$$

write $\mathbf{n} = a\mathbf{e}_1 + \mathbf{n}'$, where $a \in \mathbb{Z}$ and $\mathbf{n}' \in \mathbb{Z}\mathbf{e}_2 \oplus \mathbb{Z}\mathbf{e}_3 \oplus \mathbb{Z}\mathbf{e}_4$. It follows that $\zeta^{\mathbf{n}'} \in U_F$. Since σ_1 and σ_2 coincide on F, we see that $\lambda_1(\mathbf{n}') = \lambda_2(\mathbf{n}')$. Thus $\lambda_1(\mathbf{n}) - \lambda_2(\mathbf{n}) = \lambda_1(a\mathbf{e}_1) - \lambda_2(a\mathbf{e}_1) = a(\lambda_1(\mathbf{e}_1) - \lambda_2(\mathbf{e}_1))$, and it follows from (1.17) that $|a|$ is bounded by $|2\epsilon_0/(\lambda_1(\mathbf{e}_1) - \lambda_2(\mathbf{e}_1))|$. Hence a can take only finitely many possible values.

Remark that under the natural projection π from $\mathbb{R}^2 \oplus \mathbb{C}^3$ to X, the four-dimensional real subspace $\overline{\sigma(F)} \subset \mathbb{R}^2 \oplus \mathbb{C}^3$ has a closed image $\pi(\overline{\sigma(F)}) = \overline{\sigma(F)}/\sigma(\mathcal{O}_F)$, which is a four-dimensional subtorus in the eight-dimensional twisted torus X. We denote this subtorus by Y. As U_F preserves F under multiplication, the restriction of the action by U_K on X to U_F stabilizes Y. In particular, $\zeta^{\mathbf{n}'} \cdot Y = Y$. Hence $\zeta^{\mathbf{n}} \cdot Y = \zeta^{a\mathbf{e}_1}\zeta^{\mathbf{n}'} \cdot Y = \zeta^{a\mathbf{e}_1} \cdot Y$. Note that, for each fixed a, $\zeta^{a\mathbf{e}_1} \cdot Y$ is a four-dimensional subtorus of X. Because, under assumption (1.17), there are only finitely many choices of a, for any $x \in Y$ the set (1.13) is contained in a given finite union of four-dimensional subtori and hence cannot be dense in X. But it is not hard to show that since $V_{\langle S \rangle} = V_S$ is a proper two-dimensional subspace of $\mathbb{R}^2 \oplus \mathbb{C}^3$ in this case, and thus not every $x \in Y$ can be represented as a $V_{\langle S \rangle}$-translated torsion point. This produces a counterexample as described in Remark 1.11.

2. (H, S)-*invariant sets*

In order to show Theorem 1.10 it is necessary to have some extra formulations, mainly to overcome the obstacle that a typical $G_{\epsilon,S}$ does not form a group.

Definition 2.1. For $S \subset I$ and a subgroup $H \leq G$, a closed subset A of X is said to be (H, S)-invariant if, for all $x \in A$, $\bigcap_{\epsilon>0} \overline{H_{\epsilon,S} \cdot x} \subset A$.

2.1. *Relation to Katok and Spatzier's suspension construction.*

The (H, S)-invariant sets in this paper are introduced in the spirit of subdynamics along foliations defined by Boyle and Lind [**BL97**]. They are also inspired by, and closely related to, the suspension construction of Katok and Spatzier.

Given a $\mathbb{Z}^r$-action ρ on a space N, sometimes it is desirable to pass to an $\mathbb{R}^r$-action. For this purpose, in [**KS96**] Katok and Spatzier introduced a suspension space $\mathcal{N} = (\mathbb{R}^r \times N)/\mathbb{Z}^r$, where the quotient is defined by the following $\mathbb{Z}^r$-action on $\mathbb{R}^r \times N$:

$$\mathbf{n} \cdot (\eta, x) = (\eta - \mathbf{n}, \rho(\mathbf{n}) \cdot x) \quad \text{for all } \mathbf{n} \in \mathbb{Z}^r, \text{ for all } (\eta, x) \in \mathbb{R}^r \times N. \tag{2.1}$$

For all $\eta \in \mathbb{R}^r$ and $x \in N$, denote by $\overline{(\eta, x)}$ the equivalence class from $\mathcal{N}$ that contains (η, x).

Note that the additive action $\mathbb{R}^r \curvearrowright \mathbb{R}^r \times N$ given by $\eta' \cdot (\eta, x) = (\eta + \eta', x)$ commutes with the quotient structure and induces an $\mathbb{R}^r$-action on $\mathcal{N}$, which is denoted by $\tilde{\rho}$.

The space $\mathcal{N}$ has a natural fiber structure over $\mathbb{T}^r = \mathbb{R}^r/\mathbb{Z}^r$, where the fiber above the equivalent class $\bar{\eta} = \eta + \mathbb{Z}^r$ is $\{\overline{(\eta, x)} : x \in N\}$ and is homeomorphic to N. For $\eta \in \mathbb{R}^r$ and $x \in X$, suppose that $\mathcal{O}$ is the orbit of $(0, x)$ under the $\mathbb{R}^r$-action $\tilde{\rho}$; then its intersection

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with the fiber above $\bar{\eta}$ is written as $\{\overline{(\eta, y)} : \exists \mathbf{n} \in \mathbb{Z}^r,\ y = \rho(\mathbf{n}) \cdot x\}$. In particular, for any $\tilde{\rho}$-orbit $\mathcal{O} \subset \mathcal{N}$, the intersection of $\mathcal{O}$ with any fiber, which we identify with N, is an orbit of the $\mathbb{Z}^r$-action ρ. Moreover, since $\mathcal{O}$ is $\tilde{\rho}$-invariant, its intersections with different fibers are homeomorphic to each other. Hence by classifying $\tilde{\rho}$-orbits in $\mathcal{N}$, one also classifies ρ-orbits in N.

In our case, let $N = X$ and ρ be the multiplicative $\mathbb{Z}^r$-action ζ, and construct the suspension system $(\mathcal{X}, \tilde{\zeta})$ as above. It follows from Berend's Theorem 1.1 and the previous discussion that the $\tilde{\zeta}$-orbit of any point $\overline{(\eta, x)} \in \mathcal{X}$ is dense in $\mathcal{X}$ unless x is a torsion point in X.

Definition 2.2. For any subset $S \subset I$, define a hyperplane

$$P_S = \{\eta \in \mathbb{R}^r : \lambda_i(\eta) = 0,\ \forall i \in S\}. \tag{2.2}$$

Notice that $P_S = L_S^{\perp}$ and $L_S = P_S^{\perp}$, where $L_S \subset (\mathbb{R}^r)^*$ is defined in the statement of Theorem 1.8. By assumption (1) in Theorem 1.8,

$$\dim P_S = r - \dim L_S \geq 2. \tag{2.3}$$

In Theorem 1.8 we study the partial orbit of a point $x \in X$ under the action of all elements from $\mathbb{Z}^r$ that are very close to the hyperplane $P_S \subset \mathbb{R}^r$. This is linked to the partial orbit $\mathcal{O}_{\tilde{\zeta}}(P_S, \overline{(0, x)})$ of $\overline{(0, x)} \in \mathcal{X}$ under the restriction of the action $\tilde{\zeta}$ to $P_S \subset \mathbb{R}^r$. Actually, it is possible to show that Theorem 1.8 is equivalent to the claim that $\mathcal{O}_{\tilde{\zeta}}(P_S, \overline{(0, x)})$ is dense in the fiber above $\overline{0}$ unless x is a $V_{\langle S \rangle}$-translated torsion point.

Theorem 1.10 is more precise than Theorem 1.8 and corresponds to a slightly more complicated kind of suspension system that takes complex conjugates into account.

Let $\mathbb{Z}^r$ act on the space $\mathbb{R}^r \times (\mathbb{R}/2\pi\mathbb{Z})^{r_1+r_2} \times X$ by

$$\mathbf{n} \cdot (\eta, \omega, x) = (\eta - \mathbf{n},\ \omega - (\beta_j(\mathbf{n}))_{j \in I},\ \zeta^{\mathbf{n}} \cdot x), \tag{2.4}$$

which induces a compact quotient $(\mathbb{R}^r \times (\mathbb{R}/2\pi\mathbb{Z})^{r_1+r_2} \times X)/\mathbb{Z}^r$ that we denote by $\mathcal{X}^\circ$. For $(\eta, \omega, x) \in \mathbb{R}^r \times (\mathbb{R}/2\pi\mathbb{Z})^{r_1+r_2} \times X$, denote by $\overline{(\eta, \omega, x)}$ its projection in $\mathcal{X}^\circ$.

Remark that

$$\Omega = \{(\mathbf{n}, (\tilde{\beta}_j(\mathbf{n}))_{j \in I}) : \mathbf{n} \in \mathbb{Z}^r\} \tag{2.5}$$

is a discrete cocompact subgroup in $\mathbb{R}^r \times (\mathbb{R}/2\pi\mathbb{Z})^{r_1+r_2}$. Note that the quotient $(\mathbb{R}^r \times (\mathbb{R}/2\pi\mathbb{Z})^{r_1+r_2})/\Omega$ is isomorphic to $\mathbb{T}^{r+r_1+r_2}$. For $(\eta, \omega) \in \mathbb{R}^r \times (\mathbb{R}/2\pi\mathbb{Z})^{r_1+r_2}$, let $\overline{(\eta, \omega)}$ be its projection in $(\mathbb{R}^r \times (\mathbb{R}/2\pi\mathbb{Z})^{r_1+r_2})/\Omega$.

Then $\mathcal{X}^\circ$ fibers naturally over $(\mathbb{R}^r \times (\mathbb{R}/2\pi\mathbb{Z})^{r_1+r_2})/\Omega$, each fiber being a copy of X.

The group $\mathbb{R}^r \times (\mathbb{R}/2\pi\mathbb{Z})^{r_1+r_2}$ acts naturally on $\mathbb{R}^r \times (\mathbb{R}/2\pi\mathbb{Z})^{r_1+r_2} \times X$ by translation on the first two factors. Remark that this action commutes with the $\mathbb{Z}^r$-action (2.4), hence passes to a $\mathbb{R}^r \times (\mathbb{R}/2\pi\mathbb{Z})^{r_1+r_2}$-action on the quotient space $\mathcal{X}^\circ$, which we still denote by $\tilde{\zeta}$ without risk of ambiguity.

The group $H_{\epsilon,S}$ consists of elements $\mathbf{n} \in H$ such that $(\mathbf{n}, (\beta_j(\mathbf{n}))_{j \in I})$ is sufficiently close to the subgroup $P_S \times \{0\}$ of $\mathbb{R}^r \times (\mathbb{R}/2\pi\mathbb{Z})^{r_1+r_2}$. Therefore the set $\bigcap_\epsilon \overline{H_{\epsilon,S} \cdot x}$ is closely related to the restriction of the action $\tilde{\zeta}$ to $P_S \times \{0\}$. In fact, if $\mathcal{O}_{\tilde{\zeta}}(P_S \times \{0\}, \overline{(0, 0, x)})$ denotes the orbit of $\overline{(0, 0, x)} \in \mathcal{X}^\circ$ under the restriction of $\tilde{\zeta}$ to the subgroup

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$\overline{P_S \times \{0\}}$, then one can show that the intersection between its closure and the fiber above $\overline{(0,0)}$ is exactly

$$\left\{\overline{(0,0,y)} : y \in \bigcap_{\epsilon>0} \overline{G_{\epsilon,S} \cdot x}\right\}. \tag{2.6}$$

Thus another way to formulate Theorem 1.10 is that $\mathcal{O}_{\tilde{\zeta}}(P_S \times \{0\}, \overline{(0,0,x)})$ *is dense in the fiber above* $\overline{(0,0)}$ *unless* x *is a* $V_{\langle S\rangle}$*-translated torsion point.*

It should be emphasized that though the results in this paper are proved using (H, S)-invariant sets, they can also be obtained by studying the suspension systems described above.

2.2. *Basic properties of invariant sets.* Despite the fact that the (H, S)-invariant sets are defined using the $H_{\epsilon,S}$s which are not groups in general, to some extent they have similar properties to invariant sets under group actions.

LEMMA 2.3.

(i) *Suppose that* $H \leq G$ *and two cosets* $\sigma + H$, $\tau + H$ *are both compatible with* S*; then so is* $\sigma + \tau + H$.

(ii) *For all* $x \in X$*, for all* $\epsilon > 0$, $H^{\tau}_{\delta,S} \cdot \overline{H^{\sigma}_{\epsilon,S} \cdot x} \subset \overline{H^{\sigma+\tau}_{\epsilon+\delta,S} \cdot x}$.

Proof. (i) It suffices to show for any $\epsilon, \delta > 0$, $H^{\sigma+\tau}_{\epsilon+\delta,S}$ is non-empty. By assumption both $H^{\sigma}_{\epsilon,S}$ and $H^{\tau}_{\delta,S}$ are non-empty, from which we respectively take elements $\mathbf{m}$ and $\mathbf{n}$. Then $\mathbf{m} + \mathbf{n} \in \sigma + \tau + H$ as $\mathbf{m} \in \sigma + H$, $\mathbf{n} \in \tau + H$. Furthermore by Definition 1.7,

$$|\lambda_i(\mathbf{m} + \mathbf{n})| \leq |\lambda_i(\mathbf{m})| + |\lambda_i(\mathbf{n})| < \epsilon + \delta \quad \text{for all } i \in S, \tag{2.7}$$

and

$$|\beta_j(\mathbf{m} + \mathbf{n})| \leq |\beta_j(\mathbf{m})| + |\beta_j(\mathbf{n})| < \epsilon + \delta \quad \text{for all } j \in I. \tag{2.8}$$

Hence $\mathbf{m} + \mathbf{n} \in H^{\sigma+\tau}_{\epsilon+\delta,S}$.

(ii) It is enough to prove for all $\mathbf{n} \in H^{\tau}_{\delta,S}$ that $\zeta^{\mathbf{n}} \cdot \overline{H^{\sigma}_{\epsilon,S} \cdot x} \subset \overline{H^{\sigma+\tau}_{\epsilon+\delta,S} \cdot x}$. Since $\zeta^{\mathbf{n}}$ is a continuous map, it suffices to show that $\zeta^{\mathbf{n}} \cdot (H^{\sigma}_{\epsilon,S} \cdot x) \subset H^{\sigma+\tau}_{\epsilon+\delta,S} \cdot x$. However, by the proof of part (i), for any $\mathbf{m} \in H^{\sigma}_{\epsilon,S}$, $\mathbf{m} + \mathbf{n} \in H^{\sigma+\tau}_{\epsilon+\delta,S}$, which completes the proof. □

COROLLARY 2.4. *Suppose that* $H \leq \tilde{H} \leq G$ *and* H *is of finite index in* $\tilde{H}$*; then the cosets* $\{\sigma + H : \sigma \in \tilde{H} \text{ s.t. } \sigma + H \text{ is compatible with } S\}$ *form a subgroup of* $\tilde{H}/H$.

Proof. By Lemma 2.3, the family of such cosets is stable under addition and contains the trivial element in the finite additive group $\tilde{H}/H$, hence is a subgroup. □

COROLLARY 2.5.

(i) *Suppose that two cosets* $\sigma + H$, $\tau + H$ *are both compatible with* S*; then for all* $y \in \bigcap_{\epsilon>0} \overline{H^{\sigma}_{\epsilon,S} \cdot x}$, $\bigcap_{\epsilon>0} \overline{H^{\tau}_{\epsilon,S} \cdot y} \subset \bigcap_{\epsilon>0} \overline{H^{\sigma+\tau}_{\epsilon,S} \cdot x}$.

(ii) *If* $\sigma + H$ *is compatible with* S*, then for all* $x \in X$*, the closed set* $\bigcap_{\epsilon>0} \overline{H^{\sigma}_{\epsilon,S} \cdot x}$ *is non-empty and* (H, S)*-invariant.*

In particular, $\bigcap_{\epsilon>0} \overline{H_{\epsilon,S} \cdot x}$ always contains x and is (H, S)-invariant, which is analogous to the fact that orbit closures are invariant in the setting of group actions.

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Proof. (i) By Lemma 2.3(ii),

$$\bigcap_{\epsilon>0} \overline{H^{\tau}_{\epsilon,S} \cdot y} \subset \bigcap_{\epsilon>0} \overline{H^{\tau}_{\epsilon,S} \cdot \overline{H^{\sigma}_{\epsilon,S} \cdot x}} \subset \bigcap_{\epsilon>0} \overline{H^{\sigma+\tau}_{2\epsilon,S} \cdot x} = \bigcap_{\epsilon>0} \overline{H^{\sigma+\tau}_{\epsilon,S} \cdot x}. \tag{2.9}$$

(ii) Invariance follows from part (i) by taking $\tau = 0$. Since $\sigma + H$ is compatible with S, $\overline{H^{\sigma}_{\epsilon,S} \cdot x}$ is non-empty for all ϵ. Thus by compactness of X, the limit $\bigcap_{\epsilon>0} \overline{H^{\sigma}_{\epsilon,S} \cdot x}$ is non-empty. □

Remark 2.6. Clearly $\bigcap_{\epsilon>0} \overline{H_{\epsilon,S} \cdot x}$ is the smallest (H, S)-invariant closed set containing x. Moreover, it is not hard to see that $x' \in \bigcap_{\epsilon>0} \overline{H^{\sigma}_{\epsilon,S} \cdot x}$ if and only if there is a sequence $\{\mathbf{n}_k\}_{k=1}^{\infty}$ such that $\mathbf{n}_k \in H^{\sigma}_{\epsilon_k,S}$ where $\epsilon_k \to 0$ and $\lim_{k\to\infty} \zeta^{\mathbf{n}_k} \cdot x = x'$.

The next property is that the family of invariant sets is stable under addition and subtraction.

LEMMA 2.7. *Suppose that two closed sets A and B are both (H, S)-invariant; then so are $A + B = \{x + y : x \in A,\ y \in B\}$ and $A - B = \{x - y : x \in A,\ y \in B\}$.*

Proof. Suppose that $z = x + y \in A + B$ where $x \in A,\ y \in B$. Take $z' \in \bigcap_{\epsilon>0} \overline{H_{\epsilon,S} \cdot z}$; then by Remark 2.6, there exist a sequence $\epsilon_k \to 0$ and elements $\mathbf{n}_k \in H_{\epsilon_k,S}$ such that $\lim_{k\to\infty} \zeta^{\mathbf{n}_k} \cdot z = z'$. As X is compact, by passing to a subsequence we may assume that $\lim_{k\to\infty} \zeta^{\mathbf{n}_k} \cdot x = x'$ and $\lim_{k\to\infty} \zeta^{\mathbf{n}_k} \cdot y = y'$, which belong respectively to $\bigcap_{\epsilon>0} \overline{H_{\epsilon,S} \cdot x}$ and $\bigcap_{\epsilon>0} \overline{H_{\epsilon,S} \cdot y}$. Hence $x' \in A,\ y' \in B$ and $z' = x' + y' \in A + B$. So $A + B$ is invariant. The proof for $A - B$ proceeds in the same way. □

The next lemma allows us to talk about 'finitely generated' invariant sets.

LEMMA 2.8. *For a finite set of points $x_1, \ldots, x_N \in X$ the closed set*

$$\bigcap_{\epsilon>0} \overline{H_{\epsilon,S} \cdot x_1} + \cdots + \bigcap_{\epsilon>0} \overline{H_{\epsilon,S} \cdot x_N} \tag{2.10}$$

is (H, S)-invariant. Moreover:

(i) *the set (2.10) is equal to $\bigcap_{\epsilon>0}(\overline{H_{\epsilon,S} \cdot x_1} + \cdots + \overline{H_{\epsilon,S} \cdot x_N})$;*

(ii) $H_{\delta,S} \cdot (\overline{H_{\epsilon,S} \cdot x_1} + \cdots + \overline{H_{\epsilon,S} \cdot x_N}) \subset \overline{H_{\epsilon+\delta,S} \cdot x_1} + \cdots + \overline{H_{\epsilon+\delta,S} \cdot x_N}$ *for all* $\epsilon > 0$.

Proof. The invariance follows from the previous lemma.

It is clear that (2.10) $\subset \bigcap_{\epsilon>0}(\overline{H_{\epsilon,S} \cdot x_1} + \cdots + \overline{H_{\epsilon,S} \cdot x_N})$; while the proof of the other direction of part (i) is basically the same as that of the previous lemma.

Part (ii) is an immediate corollary to Lemma 2.3(ii). □

2.3. *Minimal invariant sets.* It is easy to see that (H, S)-invariant closed sets satisfy the descending chain condition: if $A_1 \supset A_2 \supset \cdots$ is a sequence of decreasing non-empty (H, S)-invariant closed sets, then the limit set $A = \bigcap_{n=1}^{\infty} A_n$ is also non-empty and (H, S)-invariant. Therefore it follows directly from Zorn's lemma that it makes sense to talk about minimal invariant closed sets.

https://doi.org/10.1017/S0143385711000484 Published online by Cambridge University Press

LEMMA 2.9. *For a subgroup $H \leq G$ and $S \subset I$, any non-empty (H, S)-invariant closed set A contains a minimal (H, S)-invariant closed set M, i.e. M is non-empty and (H, S)-invariant, and has no non-empty proper closed subset which is also (H, S)-invariant.*

Proof. Apply Zorn's lemma. □

For a minimal set M, any point $x \in M$ 'generates' M.

LEMMA 2.10. *Let M be a minimal (H, S)-invariant closed set; then for all $x \in M$, $\bigcap_{\epsilon>0} \overline{H_{\epsilon,S} \cdot x} = M$.*

Proof. By definition of (H, S)-invariant closed sets, $\bigcap_{\epsilon>0} \overline{H_{\epsilon,S} \cdot x} \subset M$. By Remark 2.6, $\bigcap_{\epsilon>0} \overline{H_{\epsilon,S} \cdot x}$ is non-empty and (H, S)-invariant. So M cannot be minimal unless $\bigcap_{\epsilon>0} \overline{H_{\epsilon,S} \cdot x} = M$. □

Let $H \leq \tilde{H} \leq G$ be two subgroups such that $|\tilde{H}/H| < \infty$. An $(\tilde{H}, S)$-invariant closed set is necessarily (H, S)-invariant; however, it is not obvious whether a minimal $(\tilde{H}, S)$-invariant closed set should also be minimal in the (H, S)-sense. The following result gives a relationship between these two classes of minimal invariant sets.

PROPOSITION 2.11. *Let $H \leq \tilde{H} \leq G$ with $|\tilde{H}/H| < \infty$ and $M \subset X$ be a minimal $(\tilde{H}, S)$-invariant closed set. Denote by N the number of cosets from $\tilde{H}/H$ that are compatible with S. Then there exist N minimal (H, S)-invariant closed subsets $M_1, \ldots, M_N \subset M$ whose union is M.*

Proof. Write the N compatible cosets as $\sigma_1 + H, \ldots, \sigma_N + H$ where $\sigma_1 = \mathbf{0}$.

Step 1. First of all, we claim that

$$\bigcup_{n=1}^{N} \bigcap_{\epsilon>0} \overline{H_{\epsilon,S}^{\sigma_n} \cdot x} = M \quad \text{for all } x \in M. \tag{2.11}$$

Actually for any $x \in M$, by Lemma 2.10, $\bigcap_{\epsilon>0} \overline{\tilde{H}_{\epsilon,S} \cdot x} = M$. Since,

$$\text{for all } n, \quad \bigcap_{\epsilon>0} \overline{H_{\epsilon,S}^{\sigma_n} \cdot x} \subset \bigcap_{\epsilon>0} \overline{\tilde{H}_{\epsilon,S} \cdot x},$$

the left-hand side in (2.11) is contained in M. So it suffices to show that

$$\bigcup_{n=1}^{N} \bigcap_{\epsilon>0} \overline{H_{\epsilon,S}^{\sigma_n} \cdot x} \supset M.$$

As $\tilde{H}_{\epsilon,S} = \bigcup_{n=1}^{N} H_{\epsilon,S}^{\sigma_n}$, $\tilde{H}_{\epsilon,S} \cdot x$ is equal to the finite union $\bigcup_{n=1}^{N} H_{\epsilon,S}^{\sigma_n} \cdot x$, hence by finiteness $\overline{\tilde{H}_{\epsilon,S} \cdot x} = \bigcup_{n=1}^{N} \overline{H_{\epsilon,S}^{\sigma_n} \cdot x}$.

So for all $z \in M = \bigcap_{\epsilon>0} \overline{\tilde{H}_{\epsilon,S} \cdot x} = \bigcap_{\epsilon>0} \bigcup_{n=1}^{N} \overline{H_{\epsilon,S}^{\sigma_n} \cdot x}$ and all $k \in \mathbb{N}$, there exists $n_k \in \{1, \ldots, N\}$ such that $z \in \overline{H_{k^{-1},S}^{\sigma_{n_k}} \cdot x}$. Thus there is a subsequence $k_l \to \infty$ such that all the n_{k_l}s are the same, denoted by $n(z)$. Hence $z \in \overline{H_{k_l^{-1},S}^{\sigma_{n(z)}} \cdot x}$, for all l. Since $\overline{H_{\epsilon,S}^{\sigma_{n(z)}} \cdot x}$ is decreasing as ϵ decreases, z is in the limit set $\bigcap_{\epsilon>0} \overline{H_{\epsilon,S}^{\sigma_{n(z)}} \cdot x}$. Since $z \in M$ is chosen arbitrarily, this proves that $\bigcup_{n=1}^{N} \bigcap_{\epsilon>0} \overline{H_{\epsilon,S}^{\sigma_n} \cdot x} \supset M$. So (2.11) holds.

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Step 2. Denote by $m \leq N$ the largest number such that there are (H, S)-invariant closed subsets $\Omega_1, \ldots, \Omega_N$ of M that satisfy the following three conditions:

$$\text{at least } m \text{ of the } \Omega_n\text{s are minimal } (H, S)\text{-invariant sets;} \tag{2.12}$$

$$\bigcup_{n=1}^{N} \Omega_n = M; \tag{2.13}$$

$$\exists l, \exists x \in \Omega_l \text{ s.t. for all } n, \quad \Omega_n = \bigcap_{\epsilon>0} \overline{H_{\epsilon,S}^{\sigma_n-\sigma_l} \cdot x}. \tag{2.14}$$

To obtain the proposition, one needs to show $m = N$. We first show that $m \geq 1$.

Since M is $(\tilde{H}, S)$-invariant, it is also (H, S)-invariant. By Lemma 2.9 there is a minimal (H, S)-invariant closed set $\Omega_1 \subset M$. Take an arbitrary point $x \in \Omega_1$; then by Lemma 2.10, $\bigcap_{\epsilon>0} \overline{H_{\epsilon,S} \cdot x} = \Omega_1$. Take $l = 1$ and set $\Omega_n = \bigcap_{\epsilon>0} \overline{H_{\epsilon,S}^{\sigma_n} \cdot x}$, which does not change the meaning of Ω_1; this establishes (2.14). Then by Corollary 2.5, $\Omega_1, \ldots, \Omega_N$ are all non-empty (H, S)-invariant closed sets. Equation (2.13) follows from (2.11). As Ω_1 is minimal, we see that $m \geq 1$.

Step 3. We now prove that $m = N$.

Suppose that $m < N$. Then among the corresponding (H, S)-invariant sets $\Omega_1, \ldots, \Omega_N$, there is at least one Ω_k which is not minimal. By Lemma 2.9 we may take a minimal (H, S)-invariant closed subset $\Omega_k' \subsetneq \Omega_k$. Pick $x' \in \Omega_k'$; then by Lemma 2.10, $\bigcap_{\epsilon>0} \overline{H_{\epsilon,S} \cdot x'} = \Omega_k'$. In accordance with (2.14), define

$$\Omega_n' = \bigcap_{\epsilon>0} \overline{H_{\epsilon,S}^{\sigma_n-\sigma_k} \cdot x'} \quad \text{for all } n \in \{1, \ldots, N\}. \tag{2.15}$$

Remark that this does not change the definition of Ω_k'.

Since $x' \in \Omega_k = \bigcap_{\epsilon>0} \overline{H_{\epsilon,S}^{\sigma_k-\sigma_l} \cdot x}$, it follows from Corollary 2.5(i) that

$$\Omega_n' \subset \bigcap_{\epsilon>0} \overline{H_{\epsilon,S}^{(\sigma_n-\sigma_k)+(\sigma_k-\sigma_l)} \cdot x} = \bigcap_{\epsilon>0} \overline{H_{\epsilon,S}^{\sigma_n-\sigma_l} \cdot x} = \Omega_n. \tag{2.16}$$

On the other hand, by Corollary 2.5(ii), all the Ω_n's are non-empty and (H, S)-invariant. By Corollary 2.4, $\sigma_n - \sigma_k + H$ runs through $\sigma_1 + H, \ldots, \sigma_N + H$ as n varies, so

$$\bigcup_{n=1}^{N} \Omega_n' = \bigcup_{n=1}^{N} \bigcap_{\epsilon>0} \overline{H_{\epsilon,S}^{\sigma_n} \cdot x'} = M, \tag{2.17}$$

where the second equality follows from (2.11). Thus the Ω_n's satisfy (2.13).

For those n such that Ω_n is a minimal (H, S)-invariant closed set, by minimality Ω_n' is equal to Ω_n hence is still minimal. However, the minimal set Ω_k' is not one of these. Therefore there are at least $m + 1$ minimal sets among the Ω_n's, which contradicts the maximality of m. Hence $m = N$, and the proof of the proposition is complete. □

3. *Sets that accumulate at a $V_{\langle S\rangle}$-translated torsion point*

From now on let α, ζ, X, S, $\langle S\rangle$ and $V_{\langle S\rangle}$ be as in Theorem 1.8 and H be a subgroup of finite index in $G = \mathbb{Z}^r$.

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Definition 3.1. For a real subspace V of $\mathbb{R}^{r_1} \oplus \mathbb{C}^{r_2}$, we say that a closed subset $\Omega \subset X$ contains a *V-pattern* if there is a sequence of points $\{z_k\}_{k=1}^{\infty} \subset \Omega$ converging to a V-translated torsion point $z \in \Omega$, such that $z_k - z \notin V$ for all k.

Here and from now on in similar situations, the difference $z_k - z$ is viewed as a vector in $\mathbb{R}^{r_1} \oplus \mathbb{C}^{r_2}$ whose length tends to 0 as $k \to \infty$, as X is locally isomorphic to $\mathbb{R}^{r_1} \oplus \mathbb{C}^{r_2}$.

Throughout this section we consider a 'finitely generated' (H, S)-invariant set

$$A = \bigcap_{\epsilon>0} \overline{H_{\epsilon,S} \cdot x_1} + \cdots + \bigcap_{\epsilon>0} \overline{H_{\epsilon,S} \cdot x_l} \subset X, \tag{3.1}$$

whose invariance follows from Corollary 2.5 and Lemma 2.7. By Lemma 2.8, $A = \bigcap_{\epsilon>0} A_\epsilon$ where

$$A_\epsilon = \overline{H_{\epsilon,S} \cdot x_1} + \cdots + \overline{H_{\epsilon,S} \cdot x_l}. \tag{3.2}$$

Furthermore, $H_{\epsilon-\epsilon_0\, S} \cdot A_{\epsilon_0} \subset A_\epsilon$ for all $\epsilon > \epsilon_0 > 0$.

The rest of this section will be devoted to the proof of the following proposition.

PROPOSITION 3.2. *Let A and A_ϵ be as above. If A_{ϵ_0} contains a $V_{\langle S\rangle}$-pattern for some ϵ_0, then $A_\epsilon = X$ for all $\epsilon > \epsilon_0$. In particular, if A_ϵ contains a $V_{\langle S\rangle}$-pattern for all positive ϵ, then $A_\epsilon = X$, for all $\epsilon > 0$, and thus $A = X$ as well.*

3.1. *Orbits of $V_{\langle S\rangle}$-translated torsion points.* To begin with, we remark that the 'only if' part of Theorem 1.8 is not difficult to see.

Actually, suppose that $x = x_* + v$ where $x_* \in X$ is a torsion point and $v \in V_{\langle S\rangle}$. Then the orbit $G \cdot x_*$ is a finite set of torsion points (because $qx' = 0$ for any point $x' \in G \cdot x_*$ where q denotes the order of x_* and there are only finitely many torsion points of order q in X).

By definition of $\langle S\rangle$, there are constants $c_{ij} \in \mathbb{R}$ for all $j \in \langle S\rangle$ and $i \in S$, such that $\lambda_j = \sum_{i\in S} c_{ij}\lambda_i$. Denote $c = \max_{j\in\langle S\rangle} \sum_{i\in S} |c_{ij}|$. Then for all $j \in \langle S\rangle$,

$$|\lambda_j(\mathbf{n})| \le \sum_{i\in S} |c_{ij}\lambda_i(\mathbf{n})| \le c \max_{i\in S} |\lambda_i(\mathbf{n})|. \tag{3.3}$$

Assume that $|\lambda_i(\mathbf{n})| < \epsilon$ for all $i \in S$; then

$$|\zeta_j^{\mathbf{n}}| \in (e^{-c\epsilon}, e^{c\epsilon}) \quad \text{for all } j \in \langle S\rangle. \tag{3.4}$$

We can write $v = \sum_{j\in\langle S\rangle} v_j$ with $v_j \in V_j$. Then $\zeta^{\mathbf{n}} \cdot v$ is in $V_{\langle S\rangle}$ and its V_j-coordinate is $\zeta_j^{\mathbf{n}} v_j$. Hence by (3.4), $e^{-c\epsilon}|v| < |\zeta^{\mathbf{n}} \cdot v| < e^{c\epsilon}|v|$.

So $\zeta^{\mathbf{n}} \cdot x = \zeta^{\mathbf{n}} \cdot x_* + \zeta^{\mathbf{n}} \cdot v$ belongs to D_ϵ where

$$D_\epsilon = \{x' + v' : x' \in G \cdot x_*,\, v' \in V_{\langle S\rangle},\, |v'| \le e^{c\epsilon}|v|\} \tag{3.5}$$

is a finite union of $V_{\langle S\rangle}$-disks centered at torsion points.

Furthermore, let $\epsilon < 1/(c+1)$. If $\mathbf{n} \in G_{\epsilon,S}$, then in addition to (3.4), $\|\beta_j(\mathbf{n})\| < \epsilon$, for all $j \in \langle S\rangle$. Note that by definition $\|\beta_j(\mathbf{n})\| = \operatorname{Arg} \zeta_j^{\mathbf{n}}$ where Arg denotes the principal value of complex argument, so

$$\begin{aligned} |\zeta_j^{\mathbf{n}} - 1| &= |e^{\lambda_j(\mathbf{n}) + i \operatorname{Arg} \zeta_j^{\mathbf{n}}} - 1| \le 2|\lambda_j(\mathbf{n}) + i \operatorname{Arg} \zeta_j^{\mathbf{n}}| \\ &\le 2(c\epsilon + \epsilon) = 2(c+1)\epsilon, \end{aligned} \tag{3.6}$$

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where we have used the facts that $|\lambda_j(\mathbf{n}) + i \operatorname{Arg} \zeta_j^{\mathbf{n}}| \le (c+1)\epsilon < 1$ and $|e^z - 1| \le 2|z|$ as long as $|z| < 1$. Therefore $|\zeta_j^{\mathbf{n}} \cdot v_j - v_j| \le 2(c+1)\epsilon |v_j|$, for all $j \in \langle S \rangle$. In consequence, for all $\mathbf{n} \in G_{\epsilon,S}$,

$$|\zeta^{\mathbf{n}} \cdot v - v| \le 2(c+1)\epsilon |v|, \tag{3.7}$$

and thus $\zeta^{\mathbf{n}} \cdot x \in N_\epsilon$ where

$$N_\epsilon = \{x' + v' : x' \in G \cdot \epsilon_*,\ v' \in V_{\langle S \rangle},\ |v' - v| < 2(c+1)\epsilon |v|\}. \tag{3.8}$$

Because of the finiteness of $G \cdot x_*$, both D_ϵ and N_ϵ are closed in X. Moreover, notice that $\bigcap_{\epsilon>0} N_\epsilon = \{x' + v : x' \in G \cdot x_*\}$, which is finite.

Therefore, we have actually shown the following lemma.

LEMMA 3.3. *Let ζ, X, I, S, $\langle S \rangle$ and $V_{\langle S \rangle}$ be as in Theorem 1.8 and x be a $V_{\langle S \rangle}$-translated torsion point in X. Then, for all $\epsilon > 0$:*

(i) *the set (1.11) is contained in a finite union of $V_{\langle S \rangle}$-disks centered at torsion points;*

(ii) *in addition, $\bigcap_{\epsilon>0} \overline{G_{\epsilon,S} \cdot x}$ is a finite set of $V_{\langle S \rangle}$-translated torsion points.*

It can be shown that if the conditions in Theorem 1.8 are satisfied, then $\dim V_{\langle S \rangle} < d$.

LEMMA 3.4. *In the setting of Theorem 1.8, $V_{\langle S \rangle}$ must be a proper subspace of $\mathbb{R}^{r_1} \oplus \mathbb{C}^{r_2}$.*

Proof. Suppose the lemma fails, then $V_j \subset V_{\langle S \rangle}$, for all $j \in I$, or equivalently, $\langle S \rangle = I$. So the functionals $\{\lambda_i : i \in I\}$ span L_S, which, by assumption in Theorem 1.8, is a subspace of dimension at most $r - 2$ in $(\mathbb{R}^r)^*$. If $\mathbf{e}_1, \ldots, \mathbf{e}_r$ are a basis of $\mathbb{Z}^r$, then it is equivalent to say that the rank of the matrix

$$M = (\lambda_i(\mathbf{e}_k))_{i \in I; 1 \le k \le r} = (\log |\sigma_i(\eta^{\mathbf{e}_k})|)_{i \in I; 1 \le k \le r} \tag{3.9}$$

is at most $r - 2$.

Recall U_K is a finitely generated abelian group of rank $r_1 + r_2 - 1$, so we can always extend η to a group embedding of $\mathbb{Z}^{r_1+r_2-1}$ into U_K. In other words, if we regard $\mathbb{Z}^r$ as a subgroup of $\mathbb{Z}^{r_1+r_2-1}$ and supplement $\mathbf{e}_1, \ldots, \mathbf{e}_r$ by $\mathbf{e}_{r+1}, \ldots, \mathbf{e}_{r_1+r_2-1}$ to form a basis of $\mathbb{Z}^{r_1+r_2-1}$, then there are elements $\eta^{\mathbf{e}_{r+1}}, \ldots, \eta^{\mathbf{e}_{r_1+r_2-1}} \in U_K$ such that $\eta^{\mathbf{e}_1}, \ldots, \eta^{\mathbf{e}_{r_1+r_2-1}}$ generate a subgroup of rank $r_1 + r_2 - 1$, i.e. of finite index, in U_K.

Consider the $(r_1 + r_2) \times (r_1 + r_2 - 1)$ matrix

$$\tilde{M} = (\log |\sigma_i(\eta^{\mathbf{e}_k})|)_{i \in I; 1 \le k \le r_1+r_2-1}. \tag{3.10}$$

Since $\tilde{M}$ has M as a $(r_1 + r_2) \times r$ submatrix,

$$\begin{aligned} \operatorname{rank} \tilde{M} &\le \operatorname{rank} M + (r_1 + r_2 - 1) - r \\ &\le (r - 2) + (r_1 + r_2 - 1) - r \\ &= r_1 + r_2 - 3. \end{aligned} \tag{3.11}$$

However, because $\eta^{\mathbf{e}_1}, \ldots, \eta^{\mathbf{e}_{r_1+r_2-1}}$ generate a finite-index subgroup of U_K, the $\mathbb{Z}$-span of the rows of $\tilde{M}$ has finite index in $\{(\log |\sigma_i(\theta)|)_{i \in I} : \theta \in U_K\}$, which is a discrete subgroup of rank $r_1 + r_2 - 1$ in $\mathbb{R}^I$ by Dirichlet's unit theorem. It follows that $\operatorname{rank} \tilde{M} = r_1 + r_2 - 1$, which contradicts (3.11). This completes the proof. □

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COROLLARY 3.5. *In the setting of Theorem 1.8, if $x \in X$ is a $V_{\langle S\rangle}$-translated torsion point, then the set (1.11) is not dense in X.*

Proof. By Lemma 3.3, this set is in a finite union of compact $V_{\langle S\rangle}$-disks hence so is its closure. Thus the dimension of the closure is strictly less than $\dim X = d$ by Lemma 3.4, which implies the corollary. □

Theorem 1.10(ii) is covered by Lemma 3.3. The rest of the paper will focus on the proof of Theorem 1.10(i).

3.2. *Concentration on a coarse Lyapunov subspace.* Before working on X, we study first how elements of $H_{\epsilon,S}$ act on the linear space $\mathbb{R}^{r_1} \oplus \mathbb{C}^{r_2} = \bigoplus_{i\in I} V_i$.

We show that $H_{\epsilon,S}$ is a good enough approximation to the hyperplane P_S in (2.2). This uses the ideas already presented in §2.1.

LEMMA 3.6. *For all $\epsilon > 0$, there is a constant $C = C(\epsilon, H)$ such that for all $\eta \in P_S$, there exists $\mathbf{n} \in H_{\epsilon,S}$ such that $|\mathbf{n} - \eta| < C$.*

Proof. Let Ω be as in (2.5), then $(\mathbb{R}^r \times (\mathbb{R}/2\pi\mathbb{Z})^{r_1+r_2})/\Omega \cong \mathbb{T}^{r+r_1+r_2}$. Denote the natural projection by

$$p : \mathbb{R}^r \times (\mathbb{R}/2\pi\mathbb{Z})^{r_1+r_2} \mapsto (\mathbb{R}^r \times (\mathbb{R}/2\pi\mathbb{Z})^{r_1+r_2})/\Omega. \tag{3.12}$$

Take the product $P_S \times \{0\}$ where 0 denotes the trivial vector in $(\mathbb{R}/2\pi\mathbb{Z})^{r_1+r_2}$. Then $p(P_S \times \{0\})$ is a connected subgroup of $(\mathbb{R}^r \times (\mathbb{R}/2\pi\mathbb{Z})^{r_1+r_2})/\Omega$ and therefore $\overline{p(P_S \times \{0\})}$ is a connected closed subgroup, which has to be a subtorus containing 0. For any $\epsilon > 0$, take an $\epsilon/2$-dense subset E of $\overline{p(P_S \times \{0\})}$ (i.e. for all $z \in \overline{p(P_S \times \{0\})}$, there exists $z' \in E$, $\|z - z'\| < \epsilon/2$). By compactness of $\overline{p(P_S \times \{0\})}$, we can choose E to be finite. By density of $p(P_S \times \{0\})$, we may slightly modify E so that it is inside $p(P_S \times \{0\})$ and is ϵ-dense in $\overline{p(P_S \times \{0\})}$. Suppose that

$$E = \{p((\eta_k, 0)) : k = 1, \ldots, m\} \tag{3.13}$$

where $\eta_k \in P_S$. Let $C = \max_{k=1}^m |\eta_k| + \epsilon$. Then for all $\eta \in P_S$, there is an η_k such that

$$\|p((\eta - \eta_k, 0))\| = \|p((\eta, 0)) - p((\eta_k, 0))\| < \epsilon.$$

By construction of Ω, there exists $\mathbf{n} \in H$ such that $|\eta - \eta_k - \mathbf{n}| < \epsilon$ and $\beta_j(\mathbf{n})$ is within distance ϵ from zero modulo 2π for all $j \in I$.

It follows first that $|\eta - \mathbf{n}| < |\eta_k| + \epsilon \le C$. Hence for all $i \in S$, by construction of P_S, $\lambda_i(\eta - \eta_k) = 0$ and

$$|\lambda_i(\mathbf{n})| \le \|\lambda_i\| \cdot |\eta - \eta_k - \mathbf{n}| < \|\lambda_i\|\epsilon. \tag{3.14}$$

Moreover, $\|\beta_j(\mathbf{n})\| < \epsilon$, for all $j \in I$. Therefore $\mathbf{n} \in H_{\max(\max_{i\in S}\|\lambda_i\|,1)\epsilon,S}$. By replacing ϵ with $\epsilon/\max(\max_{i\in S}\|\lambda_i\|, 1)$, we obtain the lemma. □

Now for every $i \in S$, denote by $\lambda_i^S \in (P_S)^*$ the restriction of the Lyapunov functional λ_i to P_S. First, notice

$$\lambda_i^S \equiv 0 \Leftrightarrow \lambda_i \in P_S^\perp = L_S \Leftrightarrow i \in \langle S\rangle, \tag{3.15}$$

which was the reason for introducing $\langle S\rangle$ in Theorem 1.8.

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By grouping together λ_i^Ss that are positively proportional to each other, the set of indices I decomposes into a disjoint union of subsets of the form

$$I_{[\lambda]} = \{i \in I : \lambda_i^S \in \mathbb{R}_+\lambda\}, \tag{3.16}$$

where $\lambda \in (P_S)^*$. There is a finite collection Λ^S of equivalence classes of the form $[\lambda] = \mathbb{R}_+\lambda$ such that for any $[\lambda] \in \Lambda^S$, $I_{[\lambda]}$ is not empty and

$$I = \bigsqcup_{[\lambda]\in\Lambda^S} I_{[\lambda]}. \tag{3.17}$$

Definition 3.7. The *coarse Lyapunov subspace* associated to $[\lambda] \in \Lambda^S$ is

$$V_{[\lambda]} = \bigoplus_{i\in I_{[\lambda]}} V_i. \tag{3.18}$$

For a subset $\Lambda \subset \Lambda^S$, let

$$V_\Lambda = \bigoplus_{[\lambda]\in\Lambda} V_{[\lambda]}. \tag{3.19}$$

By (3.15), the Lyapunov subspace $V_{[0]}$ corresponding to the constant zero map in $(P_S)^*$ is exactly the central foliation $V_{\langle S\rangle}$. By Lemma 3.4, Λ^S contains equivalence classes other than [0].

PROPOSITION 3.8. *If A_{ϵ_0} contains a $V_{\langle S\rangle}$-pattern for some ϵ_0, then there exists $[\lambda] \in \Lambda^S\backslash\{[0]\}$ such that for all $\epsilon > \epsilon_0$, there is a sequence $\{y_k\}_{k=1}^\infty \subset A_\epsilon$ converging to a $V_{\langle S\rangle}$-translated torsion point y such that $y_k - y \in (V_{\langle S\rangle} \oplus V_{[\lambda]})\backslash V_{\langle S\rangle}$ for all k.*

Here again $y_k - y$ is viewed as a very short vector in $\mathbb{R}^{r_1} \oplus \mathbb{C}^{r_2}$.

Proof. Let Λ be a minimal non-empty subset of $\Lambda^S\backslash\{[0]\}$ satisfying the following condition.

($\star$) *For all $\epsilon > \epsilon_0$, there is a sequence of points $\{y_k\}_{n=1}^\infty$ from A_ϵ that converges to a $V_{\langle S\rangle}$-translated torsion point y, which may depend on ϵ, such that $y_k - y \in (V_{\langle S\rangle} \oplus V_\Lambda)\backslash V_{\langle S\rangle}$.*

The definition of Λ makes sense since by definition of a $V_{\langle S\rangle}$-pattern, $\Lambda^S\backslash\{[0]\}$ satisfies condition ($\star$). To establish the proposition it suffices to prove any minimal set Λ satisfying ($\star$) consists of a single element $[\lambda] \in \Lambda^S\backslash\{[0]\}$.

Assume for the sake of obtaining a contradiction that Λ consists of more than one element.

Observe for any two non-zero linear maps $\lambda_1, \lambda_2 \in (P_S)^*$, that if $\lambda_2 \notin \mathbb{R}_+\lambda_1$ then there is $\eta \in P_S$ such that $\lambda_1(\eta) > 0$, $\lambda_2(\eta) < 0$. Thus Λ decomposes as a disjoint union of two non-empty parts Λ_+ and Λ_-, and there exists $\eta \in P_S$ such that

$$\lambda(\eta) > 0 \text{ (respectively } < 0) \quad \text{for all } [\lambda] \in \Lambda_+ \text{ (respectively } \Lambda_-). \tag{3.20}$$

Let $\epsilon' > \epsilon > \epsilon_0$. Take the sequence $\{y_k\}_{k=1}^\infty \in A_\epsilon$ and its limit y in assumption ($\star$) with respect to parameter ϵ, such that $y_k - y \in (V_{\langle S\rangle} \oplus V_\Lambda)\backslash V_{\langle S\rangle}$ and $\theta_k = \|y_k - y\| \to 0$. Write y_k as $y + v_k^0 + v_k^+ + v_k^-$ where:

- v_k^0, v_k^+ and v_k^- are respectively vectors from $V_{\langle S\rangle}$, V_{Λ_+} and V_{Λ_-};

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- $|v_k^0|$, $|v_k^+|$ and $|v_k^-|$ are all bounded by θ_k;
- at least one of v_k^+ and v_k^- is not equal to zero.

By passing to a sequence of $\epsilon \to 0$ and a subsequence of $\{y_k\}$ for each of these ϵs, we may assume without loss of generality that for all $\epsilon > \epsilon_0$, the sequence $\{y_k\}_{k=1}^{\infty}$ is chosen in such a way that the corresponding v_k^+ does not vanish for any k.

For each k, consider the map

$$b_k^+(t) = \left(\sum_{i \in \bigcup_{[\lambda] \in \Lambda_+} I_{[\lambda]}} (e^{\lambda_i(t\eta)} |(v_k^+)_i|)^2 \right)^{1/2} \tag{3.21}$$

from $\mathbb{R}$ to $\mathbb{R}_k^+$, where $(v_k^+)_i$ is the projection of v_k^+ in V_i. If $i \in \bigcup_{[\lambda] \in \Lambda_+} I_{[\lambda]}$ then by the construction of coarse Lyapunov subspaces and (3.20), $\lambda_i(\eta) = \lambda_i^S(\eta) > 0$, hence t is strictly increasing for $v_k^+ \neq 0$. Similarly, the map

$$b_k^-(t) = \left(\sum_{i \in \bigcup_{[\lambda] \in \Lambda_-} I_{[\lambda]}} (e^{\lambda_i(t\eta)} |(v_k^-)_i|)^2 \right)^{1/2}$$

is decreasing (strictly decreasing if $v_k^- \neq 0$) and

$$b_k^0(t) = \left(\sum_{i \in \langle S \rangle} (e^{\lambda_i(t\eta)} |(v_k^0)_i|)^2 \right)^{1/2} \equiv |v_k^0|.$$

Now fix a positive number δ. For sufficiently large k, $\theta_k < \delta$ and $b_k^+(0) = |v_k^+| \leq \theta_k < \delta$. Hence there exists $t_k > 0$ such that $b_k^+(t_k) = \delta$. Take $\mathbf{n}_k \in H_{\epsilon'-\epsilon,S}$ within distance C from $t_k\eta$ where $C = C(\epsilon' - \epsilon, H)$ is the constant given by Lemma 3.6.

Consider

$$\zeta^{\mathbf{n}_k} \cdot y_k = \zeta^{\mathbf{n}_k} \cdot y + \zeta^{\mathbf{n}_k} \cdot v_k^0 + \zeta^{\mathbf{n}_k} \cdot v_k^+ + \zeta^{\mathbf{n}_k} \cdot v_k^-. \tag{3.22}$$

As X is compact, by passing to a subsequence one may assume that $\zeta^{\mathbf{n}_k} \cdot y$ converges as $k \to \infty$.

Note that the length of $\zeta^{\mathbf{n}_k} v_k^+$ is

$$\left(\sum_{i \in \bigcup_{[\lambda] \in \Lambda_+} I_{[\lambda]}} (e^{\lambda_i(\mathbf{n}_k)} |(v_k^+)_i|)^2 \right)^{1/2}.$$

Since $|\lambda_i(\mathbf{n}_k) - \lambda_i(t_k\eta)| \leq \|\lambda_i\| |\mathbf{n}_k - t_k\eta| < aC$ where $a = \max_{i \in I} \|\lambda_i\|$, we have the relation $|\zeta^{\mathbf{n}_k} \cdot v_k^+| / b_k^+(t_k) \in (e^{-aC}, e^{aC})$. Similarly, $|\zeta^{\mathbf{n}_k} \cdot v_k^-| / b_k^-(t_k)$, $|\zeta^{\mathbf{n}_k} \cdot v_k^0| / b_k^0(t_k)$ fall into the same range. Hence

$$|\zeta^{\mathbf{n}_k} \cdot v_k^+| \in (e^{-aC} b_k^+(t_k), e^{aC} b_k^+(t_k)) = (e^{-aC}\delta, e^{aC}\delta). \tag{3.23}$$

Moreover,

$$|\zeta^{\mathbf{n}_k} \cdot v_k^-| \leq e^{aC} b_k^-(t_k) \leq e^{aC} b_k^-(0) = e^{aC} |v_k^-| \leq e^{aC}\theta_k, \tag{3.24}$$

and similarly,

$$|\zeta^{\mathbf{n}_k} \cdot v_k^0| \leq e^{aC} b_k^0(t_k) = e^{aC} |v_k^0| \leq e^{aC}\theta_k. \tag{3.25}$$

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Note that a, C are independent of δ and θ_k. So since $\theta_k \to 0$, $\zeta^{\mathbf{n}_k} \cdot v_k^-$ and $\zeta^{\mathbf{n}_k} \cdot v_k^0$ converge to 0 as $k \to \infty$. By Lemma 3.3, the limit of $\zeta^{\mathbf{n}_k} \cdot y$ is a $V_{\langle S\rangle}$-translated torsion point $z_\delta \in \overline{G_{\epsilon'-\epsilon,S} \cdot y}$ which depends on δ. Again as $\zeta^{\mathbf{n}_k} \cdot v_k^+$ is in the ball of fixed radius $e^{aC}\delta$ inside V_{Λ_+}, it is all right to suppose that $\lim_{k\to\infty} \zeta^{\mathbf{n}_k} \cdot v_k^+$ exists by passing to a subsequence if necessary. Denoting this limit by w_δ, then by (3.23),

$$w_\delta \in V_{\Lambda_+}, \quad |w_\delta| \in [e^{-aC}\delta, e^{aC}\delta]. \tag{3.26}$$

Taking the limit of (3.22) by summing the terms, we see that

$$z_\delta + w_\delta = \lim_{k\to\infty} \zeta^{\mathbf{n}_k} \cdot y_k \in \overline{H_{\epsilon'-\epsilon,S} \cdot A_\epsilon} \subset A_{\epsilon'}, \tag{3.27}$$

where Lemma 2.8 is used in the last step.

It was shown in Lemma 3.3 that $\overline{G_{\epsilon'-\epsilon,S} \cdot y}$ is inside a finite union of compact $V_{\langle S\rangle}$-disks centered at torsion points. Take a sequence of positive numbers $\{\delta_h\}_{h=1}^\infty$ decaying to 0 such that the z_{δ_h}s are in the same $V_{\langle S\rangle}$-disk and converge to some point y'. Then y' belongs to the same disk and is thus a $V_{\langle S\rangle}$-translated torsion point. As the constant aC is independent of δ, $w_{\delta_h} \to 0$ by (3.26). In consequence, the points $y'_h = z_{\delta_h} + w_{\delta_h}$ converge to y as well. Moreover, $z_{\delta_h} - y' \in V_{\langle S\rangle}$ and $w_{\delta_h} \in V_{\Lambda_+}\backslash\{0\}$, so the difference $y'_h - y$ is in $(V_{\langle S\rangle} \oplus V_{\Lambda_+})\backslash V_{\langle S\rangle}$.

By (3.27), $y'_h \in A_{\epsilon'}$. So we have proved that, for any $\epsilon' > \epsilon_0$, $A_{\epsilon'}$ contains a sequence $\{y'_h\}_{h=1}^\infty$ and a $V_{\langle S\rangle}$-translated point y' satisfying condition ($\star$) with respect to the non-empty subset $\Lambda_+ \subsetneq \Lambda$, in contradiction to the minimality of Λ. This completes the proof. □

The proposition will be used later in the form of the next corollary.

COROLLARY 3.9. *If A_{ϵ_0} contains a $V_{\langle S\rangle}$-pattern for some ϵ_0, then there exists $[\lambda] \in \Lambda^S\backslash\{[0]\}$ such that for all $\epsilon > \epsilon_0$, there is a constant $C = C(\epsilon_0, \epsilon, H, A)$ such that for all $R > 0$, A_ϵ contains a point of the form $y_* + v_{\langle S\rangle} + v$, where y_* is a torsion point, $v_{\langle S\rangle} \in V_{\langle S\rangle}$, $v \in V_{[\lambda]}$ and $|v_{\langle S\rangle}| \le C$, $v \in [R, CR]$.*

Proof. By the proposition, there is a class $[\lambda] \in \Lambda^S\backslash\{[0]\}$ such that we may pick from $A_{(\epsilon_0+\epsilon)/2}$ a point of the form $y'_* + v'_{\langle S\rangle} + v'$ where y'_* is a torsion point, $v'_{\langle S\rangle} \in V_{\langle S\rangle}$, $v' \in V_{[\lambda]}$ with $|v'_{\langle S\rangle}| \le C_1(\epsilon_0, \epsilon, H, A)$ and $0 < |v'| \le R$ where C_1 is a constant independent of R.

Since $\lambda \neq 0$, there exists $\eta \in P_S$ such that $\lambda(\eta) > 0$. Similar to (3.21), define

$$b(t) = \left(\sum_{i\in I_{[\lambda]}} (e^{\lambda_i(t\eta)}|(v')_i|)^2\right)^{1/2}. \tag{3.28}$$

Then $b(0) = |v|$. And since each λ_i involved is positively proportional to λ on P_S, $\lambda_i(t\eta) > 0$; so $b(t)$ is strictly increasing. Fix t such that $b(t) = e^{aC_2}R$, where $a = \max_{i\in I} \|\lambda_i\|$ and $C_2 = C_2((\epsilon - \epsilon_0)/2, H)$ is the constant defined in Lemma 3.6, according to which there exists $\mathbf{n} \in H_{(\epsilon-\epsilon_0)/2,S} \subset \mathbb{Z}^r$ within distance C_2 from η.

Consider the point

$$\zeta^{\mathbf{n}} \cdot y'_* + \zeta^{\mathbf{n}} \cdot v'_{\langle S\rangle} + \zeta^{\mathbf{n}} \cdot v' = \zeta^{\mathbf{n}} \cdot (y'_* + v'_{\langle S\rangle} + v') \in H_{(\epsilon-\epsilon_0)/2,S} \cdot A_{(\epsilon_0+\epsilon)/2} \subset A_\epsilon.$$

Obviously $\zeta^{\mathbf{n}} \cdot y'_*$ is of torsion, $\zeta^{\mathbf{n}} \cdot v'_{\langle S\rangle} \in V_{\langle S\rangle}$ and $\zeta^{\mathbf{n}} \cdot v' \in V_{[\lambda]}$.

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For any $j \in \langle S \rangle$, by inequality (3.4) in the proof of Lemma 3.3, $|\zeta_j^{\mathbf{n}}| \le e^{c(\epsilon-\epsilon_0)/2}$ where c is a constant depending only on the λ_is. Hence $|\zeta^{\mathbf{n}} \cdot v'_{\langle S \rangle}| \le e^{c(\epsilon-\epsilon_0)/2} C_1$.

Moreover,

$$|\zeta^{\mathbf{n}} \cdot v'| = \left(\sum_{i \in I_{[\lambda]}} (e^{\lambda_i(\mathbf{n})} |(v')_i|)^2 \right)^{1/2}$$

so, similar to (3.23), we have

$$\frac{|\zeta^{\mathbf{n}} \cdot v'|}{b(t)} \in (e^{-\max_{i \in I} |\lambda_i(\mathbf{n}-t\eta)|}, e^{\max_{i \in I} |\lambda_i(\mathbf{n}-t\eta)|}) \subset [e^{-aC_2}, e^{aC_2}].$$

Thus since $b(t) = e^{aC_2} R$, $|\zeta^{\mathbf{n}} \cdot v'| \in [R, e^{2aC_2} R]$.

The lemma follows by setting

$$C = \max(e^{c(\epsilon-\epsilon_0)/2} C_1, e^{2aC_2}). \qquad \square$$

3.3. *Existence of arbitrarily long line segments.* We aim to prove the following proposition.

PROPOSITION 3.10. *If A_{ϵ_0} contains a $V_{\langle S \rangle}$-pattern then there is a coarse Lyapunov subspace $V_{[\lambda]}$, where $[\lambda] \in \Lambda^S \backslash \{[0]\}$, such that for all $\epsilon > \epsilon_0$, there exist a real line $L \subset V_{[\lambda]}$ that passes through the origin and a point $y \in X$ such that A_ϵ contains $y + L = \{y + v : v \in L\} \subset X$.*

To begin with, we remark that to establish the proposition, it suffices to show the next lemma.

Definition 3.11. In a metric space, a subset Ω' is a *δ-net* of another subset Ω if for all $x \in \Omega$, there is $x' \in \Omega'$ within distance δ from x.

LEMMA 3.12. *If A_{ϵ_0} contains a $V_{\langle S \rangle}$-pattern then there is a coarse Lyapunov subspace $V_{[\lambda]}$, where $[\lambda] \in \Lambda^S \backslash \{[0]\}$, such that, for all $\epsilon > \epsilon_0$ and for all δ, $R > 0$, there is a line segment $E \subset V_{[\lambda]}$ of length R through the origin and a point $y \in X$ such that A_ϵ contains a δ-net of $y + E = \{y + v : v \in E\}$.*

Proof of the implication Lemma 3.12 $\Rightarrow$ Proposition 3.10. Fix $\epsilon > \epsilon_0$ and any increasing sequence of positive numbers $R_k \to \infty$. By Lemma 3.12 there is a sequence of pairs $\{(y_k, w_k)\}_{k=1}^{\infty}$ where $y_k \in X$ and $w_k \in \mathbb{S}V_{[\lambda]} := \{w \in V_{[\lambda]} : |w| = 1\}$ such that for all $\rho \in [-R_k, R_k]$, there is a point $x \in A_\epsilon$ within distance $1/R_k$ from $y_k + \rho w_k$.

Passing to a subsequence if necessary, one may assume that $y_k \to y \in X$ and $w_k \to w \in \mathbb{S}V_{[\lambda]}$ since X and $\mathbb{S}V_{[\lambda]}$ are both compact. Let $L = \{\rho w : \rho \in \mathbb{R}\}$.

For all $\rho \in \mathbb{R}$ and $\delta > 0$, pick a sufficiently large k so that $R_k > \max(|\rho|, 3/\delta)$, $\|y_k - y\| < \delta/3$ and $|w_k - w| < \delta/3|\rho|$. There is a point $x \in A_\epsilon$ within distance $1/R_k$ from $y_k + \rho w_k$. Then

$$\begin{aligned} \|x - (y + \rho w)\| &\le \|x - (y_k + \rho w_k)\| + \|(y_k + \rho w_k) - (y + \rho w)\| \\ &\le \frac{1}{R_k} + \|y - y_k\| + \rho |w_k - w| \le \frac{\delta}{3} + \frac{\delta}{3} + |\rho| \cdot \frac{\delta}{3|\rho|} \\ &= \delta. \end{aligned} \tag{3.29}$$

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Because A_ϵ is closed, by letting δ approach 0 we see that $y + \rho w \in A_\epsilon$ for all $\rho \in \mathbb{R}$. This proves Proposition 3.10. □

Before proving Lemma 3.12, we explore some consequences of the rank assumption in Theorem 1.8.

Take the subspace $L_S \subset (\mathbb{R}^r)^*$ spanned by $\{\lambda_i : i \in S\}$, whose dimension we denote by r_S. Then $r_S \leq r - 2$ by assumption (1) in Theorem 1.8. Choose $i_1, \ldots, i_{r_S} \in S$ such that $\lambda_{i_1}, \ldots, \lambda_{i_{r_S}}$ span L_S.

Let $V_{[\lambda]}$ be the coarse Lyapunov subspace in Proposition 3.8 and Corollary 3.9. Notice that though $\langle S \rangle$ may be empty, $I_{[\lambda]}$ is not. Fix an arbitrary index $i_0 \in I_{[\lambda]}$.

LEMMA 3.13. *For an arbitrarily fixed $i_0 \in I_{[\lambda]}$, $\lambda_{i_1}, \ldots, \lambda_{i_{r_S}}$ and λ_{i_0} are linearly independent. Moreover, for any $i \in \langle S \rangle \cup I_{[\lambda]}$, the Lyapunov functional λ_i is a linear combination of $\lambda_{i_1}, \ldots, \lambda_{i_{r_S}}$ and λ_{i_0}.*

Proof. The linear independence is clear by the choice of $\lambda_{i_1}, \ldots, \lambda_{i_{r_S}}$ and the fact that $i_0 \notin \langle S \rangle$. For $i \in \langle S \rangle$, the lemma follows from the construction of $\langle S \rangle$ in Theorem 1.8. Suppose that $i \in V_{[\lambda]}$. By definition of coarse Lyapunov subspaces, there exists $c > 0$ such that $\lambda_i|_{P_S} = \lambda_i^S = c\lambda_{i_0}^S = c\lambda_{i_0}|_{P_S}$. In other words, $\lambda_i - c\lambda_{i_0}$ vanishes when restricted to P_S, or $\lambda_i - c\lambda_{i_0} \in P_S^\perp = L_S$. So $\lambda_i - c\lambda_{i_0}$ is in the linear span of $\lambda_{i_1}, \ldots, \lambda_{i_{r_S}}$, which completes the proof. □

Consider the following group morphism from $G = \mathbb{Z}^r$ to the additive group $\mathbb{R}^{r_S+1} \oplus (\mathbb{R}/2\pi\mathbb{Z})^{r_1+r_2}$:

$$\mathcal{L}(\mathbf{n}) = (\lambda_{i_1}(\mathbf{n}), \ldots, \lambda_{i_{r_S}}(\mathbf{n}), \lambda_{i_0}(\mathbf{n})) \oplus (\beta_j(\mathbf{n}))_{j \in I}. \tag{3.30}$$

The first observation is that $\mathcal{L}$ is injective. Actually, suppose that $\mathcal{L}(\mathbf{n}) = 0$ for some $\mathbf{n}$. Then both $\lambda_{i_0}(\mathbf{n}) = \log |\zeta_{i_0}^{\mathbf{n}}| \in \mathbb{R}$ and $\beta_{i_0}(\mathbf{n}) \in \mathbb{R}/2\pi\mathbb{Z}$ vanish, so $\zeta_{i_0}^{\mathbf{n}} = 1$. However, $\zeta_{i_0}^{\mathbf{n}}$ is an algebraic conjugate to $\zeta^{\mathbf{n}}$. So $\zeta^{\mathbf{n}} = 1$. But because ζ is a group embedding from $\mathbb{Z}^r$ into U_K this implies that $\mathbf{n} = 0$, which shows the injectivity of $\mathcal{L}$.

In addition, we claim that 0 is a non-isolated point in $\mathcal{L}(H') \subset \mathbb{R}^{r_S+1} \oplus (\mathbb{R}/2\pi\mathbb{Z})^{r_1+r_2}$ for any subgroup $H' < G$ of finite index. Suppose, for the sake of obtaining a contradiction, that there is a ball $B_{2\theta}(0)$ of small radius 2θ centered at 0 such that $B_{2\theta}(0) \cap \mathcal{L}(H') = \{0\}$. Then since $\mathcal{L}(H')$ is an additive group, for all $\omega \in \mathbb{R}^{r_S+1} \oplus (\mathbb{R}/2\pi\mathbb{Z})^{r_1+r_2}$ the ball $B_\theta(\omega)$ centered at ω contains at most one element of $\mathcal{L}(H')$. Observe that the compact region $B_T = [-T, T]^{r_S+1} \oplus (\mathbb{R}/2\pi\mathbb{Z})^{r_1+r_2} \subset \mathbb{R}^{r_S+1} \oplus (\mathbb{R}/2\pi\mathbb{Z})^{r_1+r_2}$ can be covered by $O_\theta(T^{r_S+1})$ balls of radius θ. On the other hand, however, since all the λ_is are linear, the image $\mathcal{L}(\mathbf{n})$ is inside $B_T(0)$ if $T \geq a|\mathbf{n}|$ where $a = \max_{i \in I} \|\lambda_i\|$. By the pigeonhole principle, there are at most $O_\theta(T^{r_S+1})$ vectors $\mathbf{n} \in H$ of length $|\mathbf{n}| \leq T/a$. But this becomes false for sufficiently large T since, by assumption (1) in Theorem 1.8, H is of rank $r \geq r_S + 2$. Hence we have obtained a contradiction and proved that 0 must be non-isolated.

Because $\mathcal{L}(H')$ is a subgroup, so is its closure $\overline{\mathcal{L}(H')}$. The identity component $\mathcal{L}^0$ of the closure is a connected closed subgroup of the abelian Lie group $\mathbb{R}^{r_S+1} \oplus (\mathbb{R}/2\pi\mathbb{Z})^{r_1+r_2}$. It is known that such a subgroup must be isomorphic to some $\mathbb{R}^{d_1} \oplus \mathbb{T}^{d_2}$. By non-isolatedness, $\mathcal{L}^0$ is locally isomorphic to $\mathbb{R}^q$ for some $q \geq 1$.

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Moreover, any neighborhood of identity in $\mathcal{L}^0$ is not contained in the closed subgroup $\{\lambda_{i_0}=0,\ \beta_{i_0}=0\}\subset\mathbb{R}^{r_S+1}\oplus(\mathbb{R}/2\pi\mathbb{Z})^{r_1+r_2}$, where λ_{i_0} and β_{i_0} refer respectively to the two coordinates corresponding to $\lambda_{i_0}(\mathbf{n})$ and $\beta_{i_0}(\mathbf{n})$ under the map $\mathcal{L}$. This is because otherwise, by the non-isolatedness of 0 in $\mathcal{L}(H')$, there is some $\mathbf{n}\neq\mathbf{0}$ with $\lambda_{i_0}(\mathbf{n})=0$, $\beta_{i_0}(\mathbf{n})=0$, which would again imply that $\zeta_{i_0}^{\mathbf{n}}=1$. In consequence $\mathbf{n}=\mathbf{0}$, contradiction.

Combining these, there must be a real vector $\dot\lambda=(\dot\lambda_{i_1},\dots,\dot\lambda_{i_{r_S}},\dot\lambda_{i_0})\oplus(\dot\beta_j)_{j\in I}$ such that at least one of $\dot\lambda_{i_0}$ and $\dot\beta_{i_0}$ does not vanish, and $\mathcal{L}^0$ contains the projection of the line $\mathbb{R}\dot\lambda\subset\mathbb{R}^{r_S+1}\oplus\mathbb{R}^{r_1+r_2}$ into $\mathbb{R}^{r_S+1}\oplus(\mathbb{R}/2\pi\mathbb{Z})^{r_1+r_2}$.

Recall that if $V_j\cong\mathbb{R}$ then $\beta_j(\mathbf{n})$ is either 0 or π from $\mathbb{R}/2\pi\mathbb{Z}$ and thus $\mathcal{L}(\mathbb{Z}^r)$ is contained in two disjoint sets $\{\beta_j=\pi\}$ and $\{\beta_j=0\}$. As the identity component of $\overline{\mathcal{L}(H')}$, $\mathcal{L}^0$ must be contained in $\{\beta_j=0\}$. Thus

$$\dot\beta_j=0\quad\text{if } V_j\cong\mathbb{R}. \tag{3.31}$$

It is all right to suppose that

$$\dot\lambda_{i_0}^2+\dot\beta_{i_0}^2=1\quad\text{and}\quad|\dot\lambda|\le C, \tag{3.32}$$

where $C=C(H')$ can be made to be dependent only on the group action and the subgroup H' because there are only finitely many possible choices of $[\lambda]$ and $i_1,\dots,i_{r_S},i_0$.

We now give a proof of Lemma 3.12.

Proof of Lemma 3.12. Suppose that A_{ϵ_0} contains a $V_{\langle S\rangle}$-pattern. Fix $\epsilon>\epsilon_0$ and $R>0$, $\delta>0$.

Without loss of generality, we may assume that

$$\delta<\epsilon-\epsilon_0<1,\qquad R\ge1. \tag{3.33}$$

Let $[\lambda]$ be given by Corollary 3.9.

By Corollary 3.9, inside $A_{(\epsilon_0+\epsilon)/2}$ there is a point of the form $y=y_*+v_{\langle S\rangle}+v$ where y_* is of torsion, $v_{\langle S\rangle}\in V_{\langle S\rangle}$ is of length no more than C_1, and $v\in V_{[\lambda]}$ with

$$|v|\in\left[\frac{3d^2C_1C_2^2R^2}{\delta},\ \frac{3d^2C_1^2C_2^2R^2}{\delta}\right]. \tag{3.34}$$

Here $C_1=C_1(\epsilon_0,\epsilon,H,A)\ge1$ and C_2 is a constant to be specified later.

Since $\langle S\rangle\cap I_{[\lambda]}=\emptyset$, without risk of ambiguity, decompose

$$v_{\langle S\rangle}=\sum_{i\in\langle S\rangle}v_i;\qquad v=\sum_{i\in I_{[\lambda]}}v_i \tag{3.35}$$

where $v_i\in V_i$.

There is an index $i_0\in I_{[\lambda]}$ such that v_{i_0} has length greater than any other v_i with $i\in I_{[\lambda]}$. Then

$$|v_{i_0}|\in\left[\frac{3dC_1C_2^2R^2}{\delta},\ \frac{3d^2C_1^2C_2^2R^2}{\delta}\right], \tag{3.36}$$

as $|I_{[\lambda]}|<d$ where d is the dimension of the torus we work on.

Choose $i_1,\dots,i_{r_S}\in S$ such that $\lambda_{i_1},\dots,\lambda_{i_{r_S}}$ generate L_S. We may apply Lemma 3.13 and make the construction (3.30). Let

$$H'=\{\mathbf{n}\in H:\zeta^{\mathbf{n}}\cdot y_*=y_*\}, \tag{3.37}$$

which has finite index in H, hence in $\mathbb{Z}^r$ as well.

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By the discussion preceding this proof, there exists

$$\dot{\lambda} = (\dot{\lambda}_{i_1}, \dots, \dot{\lambda}_{i_{r_S}}, \dot{\lambda}_{i_0}) \oplus (\dot{\beta}_j)_{j\in I} \in \mathbb{R}^{r_S+1} \oplus \mathbb{R}^{r_1+r_2} \tag{3.38}$$

with $\dot{\lambda}_{i_0}^2 + \dot{\beta}_{i_0}^2 = 1$ and $|\dot{\lambda}| \le C_3$, such that $\overline{\mathcal{L}(H')}$ contains the projection of the line $\mathbb{R}\dot{\lambda}$ to $\mathbb{R}^{r_S+1} \oplus (\mathbb{R}/2\pi\mathbb{Z})^{r_1+r_2}$. Here $C_3 \ge 1$ depends only on H', which is the stabilizer of y_* in H, hence eventually on ϵ_0, ϵ, H and A.

By Lemma 3.13, there is a unique decomposition

$$\lambda_j = \sum_{i\in i_1,\dots,i_{r_S},i_0} c_{ij}\lambda_i \quad \text{for all } j \in \langle S\rangle \cup I_{[\lambda]}, \tag{3.39}$$

for some constants $c_{ij} \in \mathbb{R}$.

Corresponding to (3.39), for each $j \in \langle S\rangle \cup I_{[\lambda]}$, set

$$\dot{\lambda}_j = \sum_{i=i_1,\dots,i_{r_S},i_0} c_{ij}\dot{\lambda}_i, \tag{3.40}$$

which is compatible with the original value of $\dot{\lambda}_j$ if $j \in \{i_1, \dots, i_{r_S}, i_0\}$.

Notice that if $j \in \langle S\rangle \cup I_{[\lambda]}$, then

$$|\dot{\lambda}_j| \le \Bigg(\sum_{i\in\{i_1,\dots,i_{r_S},i_0\}} |c_{ij}|\Bigg) \cdot \max_{i\in\{i_1,\dots,i_{r_S},i_0\}} |\dot{\lambda}_i|. \tag{3.41}$$

Therefore, for all $j \in \langle S\rangle \cup I_{[\lambda]}$,

$$\begin{aligned} |\dot{\lambda}_j + \mathrm{i}\dot{\beta}_j| &\le \max\Bigg(\sum_{i\in\{i_1,\dots,i_{r_S},i_0\}} |c_{ij}|, 1\Bigg)\Bigg(\max_{i\in\{i_1,\dots,i_{r_S},i_0\}} |\dot{\lambda}_i|^2 + |\dot{\beta}_j|^2\Bigg)^{1/2} \\ &\le \Bigg(\max_{j\in\langle S\rangle\cup I_{[\lambda]}} \sum_{i\in\{i_1,\dots,i_{r_S},i_0\}} |c_{ij}|\Bigg) \cdot |\dot{\lambda}|. \end{aligned} \tag{3.42}$$

Here in the second step we have used the fact that

$$\max_{j\in\langle S\rangle\cup I_{[\lambda]}} \sum_{i\in\{i_1,\dots,i_{r_S},i_0\}} |c_{ij}| \ge 1, \tag{3.43}$$

which is true because when $j \in \{i_0, i_1, \dots, i_{r_S}\}$ it is easy to see that c_{ij} equals 1 if $i = j$ and vanishes otherwise.

Denote

$$C_2 = C_3 \cdot \max_{j\in\langle S\rangle\cup I_{[\lambda]}} \sum_{i\in\{i_1,\dots,i_{r_S},i_0\}} |c_{ij}|, \tag{3.44}$$

then $C_2 \ge C_3 \ge 1$ and it is determined by ϵ_0, ϵ, H and A. Furthermore, by the argument above,

$$|\dot{\lambda}_j + \mathrm{i}\dot{\beta}_j| \le C_2 \quad \text{for all } j \in \langle S\rangle \cup I_{[\lambda]}. \tag{3.45}$$

Now set

$$w = \sum_{j\in I_{[\lambda]}} (\dot{\lambda}_j + \mathrm{i}\cdot\dot{\beta}_j)v_j \in V_{[\lambda]}. \tag{3.46}$$

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Notice that this definition makes sense because of (3.31). Let E be the line segment

$$\left\{y+\rho w : \rho \in \left[-\frac{\delta}{6dC_1C_2^2R}, \frac{\delta}{6dC_1C_2^2R}\right]\right\}.$$

We claim that:

(i) E has length greater than or equal to R;

(ii) A_ϵ contains a δ-net of E.

Proof of (i). The length of E is $(\delta/3dC_1C_2^2R)|w|$. But

$$|w| \geq |(\dot{\lambda}_{i_0} + \mathrm{i}\cdot\dot{\beta}_{i_0})v_{i_0}| = |\dot{\lambda}_{i_0} + \mathrm{i}\cdot\dot{\beta}_{i_0}|\cdot|v_{i_0}| = |v_{i_0}|. \tag{3.47}$$

By (3.36),

$$\text{length}(E) \geq \frac{\delta}{3dC_1C_2^2R}\cdot\frac{3dC_1C_2^2R^2}{\delta} = R.$$

Proof of (ii). For all $\rho \in [-\delta/6dC_1C_2^2R, \delta/6dC_1C_2^2R]$, we hope to find in the subset

$$H_{(\epsilon-\epsilon_0)/2,S}\cdot(y_* + v_{\langle S\rangle} + v) \subset G_{(\epsilon-\epsilon_0)/2,S}\cdot A_{(\epsilon_2+\epsilon)/2} \subset A_\epsilon \tag{3.48}$$

a point within distance δ from $y+\rho w$. By the construction of $\dot{\lambda}$, for all $\theta > 0$, we may choose $\mathbf{n} \in H'$ such that $\mathcal{L}(\mathbf{n})$ is within distance θ from the projection of $\rho\dot{\lambda} \in \mathbb{R}^{rs+1}\oplus\mathbb{R}^{r_1+r_2}$ to $\mathbb{R}^{rs+1}\oplus(\mathbb{R}/2\pi\mathbb{Z})^{r_1+r_2}$. Therefore for all $j \in \langle S\rangle \cup I_{[\lambda]}$, $\|\beta_j(\mathbf{n}) - \rho\dot{\beta}_j\| \leq \theta$; furthermore, it follows from (3.39) and (3.40) that $\log|\zeta_j^{\mathbf{n}}| = \lambda_i(\mathbf{n}) = \sum_{i\in i_1,\ldots,i_{r_S},i_0} c_{ij}\lambda_i(\mathbf{n})$ is within distance $(\max_{j\in\langle S\rangle\cup I_{[\lambda]}}\sum_{i\in\{i_1,\ldots,i_{r_S},i_0\}}|c_{ij}|)\cdot\theta$ from $\rho\dot{\lambda}_j$. Thus as θ can be arbitrarily small, we can choose $\mathbf{n}$ to make $\zeta_i^{\mathbf{n}}$ arbitrarily close to $e^{\rho(\dot{\lambda}_j+\mathrm{i}\cdot\dot{\beta}_j)}$ simultaneously for all $j \in \langle S\rangle \cup I_{[\lambda]}$.

In particular, observe that $\mathbf{n} \in (H')_{(\epsilon-\epsilon_0)/2,S} \subset H_{(\epsilon-\epsilon_0)/2,S}$. Actually, for all $i \in S$ and $j \in I$, because of the way we chose $\mathbf{n} \in H'$, the Lyapunov exponent $\lambda_i(\mathbf{n})$ and the complex argument $\beta_j(\mathbf{n})$ are respectively arbitrarily close to $\rho\dot{\lambda}_i$ and $\rho\dot{\beta}_i$. As both $|\rho\dot{\lambda}_i|$ and $|\rho\dot{\beta}_j|$ are bounded by

$$\frac{\delta}{6dC_1C_2^2R}\cdot|\dot{\lambda}| \leq \frac{\delta}{6dC_1C_2^2R}\cdot C_2 \leq \frac{\delta}{6dC_1C_2R} \leq \frac{\epsilon-\epsilon_0}{6}$$

by (3.33),

$$|\lambda_i(\mathbf{n})| < \frac{\epsilon-\epsilon_0}{2} \text{ for all } i \in S, \quad |\beta_j(\mathbf{n})| < \frac{\epsilon-\epsilon_0}{2} \text{ for all } j \in I, \tag{3.49}$$

that is, $\mathbf{n} \in (H')_{(\epsilon-\epsilon_0)/2,S}$.

Remark that

$$\begin{aligned}\zeta^{\mathbf{n}}\cdot y &= \zeta^{\mathbf{n}}\cdot y_* + \zeta^{\mathbf{n}}\cdot v_{\langle S\rangle} + \zeta^{\mathbf{n}}\cdot v\\ &= y_* + \sum_{j\in\langle S\rangle}\zeta_j^{\mathbf{n}}v_j + \sum_{j\in I_{[\lambda]}}\zeta_j^{\mathbf{n}}v_j,\end{aligned} \tag{3.50}$$

by the decomposition of $v_{\langle S\rangle}$ and v and the fact that $\mathbf{n}$ is in the stabilizer H' of y_*.

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In consequence, since $\zeta_j^{\mathbf{n}}$ is arbitrarily close to $e^{\rho(\dot\lambda_j+\mathrm{i}\cdot\dot\beta_j)}$, in order to prove (ii) it suffices to show that both

$$\left|\sum_{j\in\langle S\rangle} e^{\rho(\dot\lambda_j+\mathrm{i}\cdot\dot\beta_j)}v_j - v_{\langle S\rangle}\right| \tag{3.51}$$

and

$$\left|\sum_{j\in I_{[\lambda]}} e^{\rho(\dot\lambda_j+\mathrm{i}\cdot\dot\beta_j)}v_j - v - \rho w\right| \tag{3.52}$$

are bounded by $\delta/3$.

Remark that while $|\alpha|\le 1$, $|e^\alpha-1|<2|\alpha|$ and $|e^\alpha-1-\alpha|<|\alpha|^2$. By (3.33),

$$|\rho(\dot\lambda_j+\mathrm{i}\cdot\dot\beta_j)|\le\frac{\delta}{6dC_1C_2^2R}\cdot C_3\le\frac{\delta}{6dC_1C_2^2R}\cdot C_2=\frac{\delta}{6dC_1C_2R}\le 1.$$

So we have

$$|e^{\rho(\dot\lambda_j+\mathrm{i}\cdot\dot\beta_j)}-1|\le 2|\rho(\dot\lambda_j+\mathrm{i}\cdot\dot\beta_j)|\le\frac{\delta}{3dC_1C_2R} \tag{3.53}$$

and

$$|e^{\rho(\dot\lambda_j+\mathrm{i}\cdot\dot\beta_j)}-1-\rho(\dot\lambda_j+\mathrm{i}\cdot\dot\beta_j)|\le|\rho(\dot\lambda_j+\mathrm{i}\cdot\dot\beta_j)|^2\le\left(\frac{\delta}{6dC_1C_2R}\right)^2. \tag{3.54}$$

Thus by (3.34), (3.35) and (3.46),

$$\begin{aligned}(3.51)&=\left|\sum_{j\in\langle S\rangle}(e^{\rho(\dot\lambda_j+\mathrm{i}\cdot\dot\beta_j)}-1)v_j\right|\le\frac{\delta}{3dC_1C_2R}|v_{\langle S\rangle}|\\&\le\frac{\delta}{3C_1C_2R}\cdot C_1\le\frac{\delta}{3}\end{aligned} \tag{3.55}$$

and

$$\begin{aligned}(3.52)&=\left|\sum_{j\in I_{[\lambda]}}(e^{\rho(\dot\lambda_j+\mathrm{i}\cdot\dot\beta_j)}-1-\rho(\dot\lambda_j+\mathrm{i}\cdot\dot\beta_j))v_j\right|\\&\le\left(\frac{\delta}{6dC_1C_2R}\right)^2\cdot|v|\le\left(\frac{\delta}{6dC_1C_2R}\right)^2\cdot\frac{3d^2C_1^2C_2^2R^2}{\delta}\\&=\frac{\delta}{12}.\end{aligned} \tag{3.56}$$

This completes the proof of (ii) and thus of the lemma. □

Proposition 3.10 is now established by the argument following Lemma 3.12.

3.4. *Density of lines in the torus.* Proposition 3.10 reduces Proposition 3.2 to the following claim.

PROPOSITION 3.14. *Let L be a line through the origin inside $\mathbb{R}^{r_1}\oplus\mathbb{C}^{r_2}$. Then for all $\epsilon,\delta>0$, there exists $\mathbf{n}\in H_{\epsilon,S}$ such that $\pi(\zeta^{\mathbf{n}}\cdot L)$ is δ-dense in X, where $\pi:\mathbb{R}^{r_1}\oplus\mathbb{C}^{r_2}\mapsto X$ is the natural projection.*

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By 'δ-dense' we mean that $\pi(\zeta^{\mathbf{n}} \cdot L)$ is a δ-net of the ambient space X.

In order to prove Proposition 3.14, we will make use of the total irreducibility assumption from Theorem 1.8.

Recall that $d = r_1 + 2r_2$ and $\sigma_1 \ldots, \sigma_{r_1}$ are real embeddings of K while $\sigma_{r_1+1}, \ldots, \sigma_d$ are complex embeddings where $\sigma_{r_1+r_2+j} = \overline{\sigma_{r_1+j}}$.

LEMMA 3.15. *Suppose that* $\{\alpha^{\mathbf{n}} : \mathbf{n} \in G_{\epsilon,S}\}$ *contains a totally irreducible toral automorphism for all* $\epsilon > 0$*, as assumed in Theorem 1.8. Let* $i_0, i_1, \ldots, i_m$ *be distinct elements from* $\{1, \ldots, d\}$*. Suppose that* $m \geq 1$ *and let* $\Psi : \mathbb{Z}^r \mapsto (\mathbb{C}^\times)^m$ *be the group morphism*

$$\Psi(\mathbf{n}) = \left(\frac{\zeta_{i_1}^{\mathbf{n}}}{\zeta_{i_0}^{\mathbf{n}}}, \ldots, \frac{\zeta_{i_m}^{\mathbf{n}}}{\zeta_{i_0}^{\mathbf{n}}} \right). \tag{3.57}$$

Then the image $\Psi(H_{\epsilon,S})$ *has infinite size for all* $\epsilon > 0$*.*

Proof. Suppose, for the sake of obtaining a contradiction, that $\Psi(H_{\epsilon,S})$ has finite size N for some ϵ. Let q be the index of H in $G = \mathbb{Z}^r$. By assumption, there exists $\mathbf{n} \in G_{\epsilon/Nq,S}$ such that $\alpha^{k\mathbf{n}}$ is irreducible for all $k \neq 0$.

By the pigeonhole principle, there are two distinct $k, k' \in \{0, 1, \ldots, q\}$ such that $k\mathbf{n}$, $k'\mathbf{n}$ belong to the same coset of H. Then $(k - k')\mathbf{n}$ is in both H and $G_{\epsilon/N,S}$, hence in $H_{\epsilon/N,S}$.

Then $h(k - k')\mathbf{n} \in H_{\epsilon,S}$, for all $h \in \{0, 1, \ldots, N\}$. As $|\Psi(H_{\epsilon,S})| = N$, there are $h \neq h'$ in $\{0, 1, \ldots, N\}$ such that $\Psi(h(k - k')\mathbf{n}) = \Psi(h'(k - k')\mathbf{n})$. So $(h - h')(k - k')\mathbf{n} \in H$ and $\Psi((h - h')(k - k')\mathbf{n}) = (1, 1, \ldots, 1)$, which, in particular, implies that

$$\zeta_{i_1}^{(h-h')(k-k')\mathbf{n}} = \zeta_{i_0}^{(h-h')(k-k')\mathbf{n}}. \tag{3.58}$$

Recall that $\zeta_{i_1}^{(h-h')(k-k')\mathbf{n}}$ and $\zeta_{i_0}^{(h-h')(k-k')\mathbf{n}}$ are respectively algebraic conjugates of $\zeta^{(h-h')(k-k')\mathbf{n}}$ by σ_{i_1} and σ_{i_0}, two different embeddings of the number field K. Equation (3.58) actually shows that $\zeta^{(h-h')(k-k')\mathbf{n}}$ belongs to some proper subfield of K. Thus by Remark 1.3, $\alpha^{\mathbf{n}}$ is not a totally irreducible toral automorphism on $\mathbb{T}^d$ because $(h - h')(k - k') \neq 0$, which contradicts our assumption. The proof is complete. □

The dual group $\hat{X}$ of $X = (\mathbb{R}^{r_1} \oplus \mathbb{C}^{r_2})/\Gamma$ consists of all real linear functionals $\xi : \mathbb{R}^{r_1} \oplus \mathbb{C}^{r_2} \mapsto \mathbb{R}$ such that $\xi(\Gamma) \subset \mathbb{Z}$ and the character $\xi : X \mapsto \mathbb{R}/\mathbb{Z}$ is defined by $\xi(\pi(v)) = (\xi(v) \mod \mathbb{Z})$, for all $v \in \mathbb{R}^{r_1} \oplus \mathbb{C}^{r_2}$.

As a linear functional, ξ may be expressed as

$$\xi(x) = \sum_{i=1}^{r_1} \xi_i x_i + \sum_{i=r_1+1}^{r_1+r_2} (\xi_i x_i + \xi_{i+r_2} \overline{x_i}), \tag{3.59}$$

where $x = (x_1, \ldots, x_{r_1+r_2})$ with $x_1, \ldots, x_{r_1} \in \mathbb{R}$, $x_{r_1+1}, \ldots, x_{r_1+r_2} \in \mathbb{C}$. In order that ξ takes real values, $\xi_1, \ldots, \xi_{r_1} \in \mathbb{R}$ and $\xi_{i+r_2} = \overline{\xi_i}$ for $i = r_1 + 1, \ldots, r_1 + r_2$.

LEMMA 3.16. *If* $\xi \neq 0$ *then* $\xi_i \neq 0$*, for all* $i = 1, \ldots, d$*.*

Proof. Suppose that $\xi_i = 0$ for some i. If $i \leq r_1$ then $V_i \cong \mathbb{R}$ and, by (3.59), $\xi|_{V_i} = 0$. Suppose $i > r_1$, then $\overline{\xi_i} = 0$ as well, and therefore, since $\xi_{i+r_2} = \bar{\xi}_i$, we may assume

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that $r_1 + 1 \le i \le r_1 + r_2$. So in this case once again we have $\xi|_{V_i} = 0$ by (3.59). Because ξ is non-trivial, $\xi^\perp = \{x \in X : \xi(x) = 0\}$ is a non-trivial proper closed subgroup of X. Since $\xi|_{V_i} = 0$, $\xi^\perp$ contains $\pi(V_i)$; so $\overline{\pi(V_i)}$ is a proper closed subgroup in X as well. However, for $\mathbf{n} \in \mathbb{Z}^r$, the multiplicative action by $\zeta^{\mathbf{n}}$ preserves $V_i \subset \mathbb{R}^{r_1} \oplus \mathbb{C}^{r_2}$, hence also preserves $\pi(V_i)$ and $\overline{\pi(V_i)}$. Therefore $\overline{\pi(V_i)} \subset X$ is a non-trivial proper connected closed subgroup invariant under the $\mathbb{Z}^r$-action $\zeta^{\mathbf{n}} \curvearrowright X$, which is conjugate to the $\mathbb{Z}^r$-action α on $\mathbb{T}^d$. Thus α admits a non-trivial proper connected invariant closed subgroup in $\mathbb{T}^d$, which is necessarily a subtorus. This violates the assumption that α contains irreducible toral automorphisms. Hence $\xi_i \neq 0$, for all $i = 1, \ldots, d$. □

We borrow our next lemma from Berend's original proof.

LEMMA 3.17. [**Ber83**, Lemma 4.7] *Let X be a compact abelian metric group and $\hat{X}$ its Pontryagin dual. Suppose that $\{X_k\}_{k=1}^\infty$ is a sequence of proper closed subgroups in X such that, for all $\xi \in \hat{X}\backslash\{0\}$, $X_k \notin \xi^\perp$ for sufficiently large k. Then for any $\delta > 0$, X_k is δ-dense for sufficiently large k.*

We are now ready to show Proposition 3.14.

Proof of Proposition 3.14. Denote the line L by $\mathbb{R}v$ for some vector $v = (x_i)_{i\in I}$ where $x_i \in \mathbb{R}$ or $\mathbb{C}$ according to whether $i \le r_1$ or not. Denote $x_{r_1+r_2+j} = \overline{x_{r_1+j}}$ for $1 \le j \le r_2$ and

$$I_L = \{i : 1 \le i \le d, x_i \neq 0\}; \tag{3.60}$$

then I_L is not empty. Write $I_L = \{i_0, \ldots, i_m\}$. Then, for all $\xi \in \hat{X}\backslash\{0\}$, $\xi(v) = \sum_{h=0}^m \xi_{i_h} x_{i_h}$ by (3.59).

Case 1. If $m = 0$ then take $\mathbf{n} = 0 \in H_{\epsilon,S}$. Note that $\overline{\pi(L)} \subset X$ is a closed subgroup of X. For all $\xi \in \hat{X}\backslash\{0\}$, we have $\xi(v) = \xi_{i_0} x_{i_0} \neq 0$ thanks to Lemma 3.16, so $\xi(\pi(av)) = (a\xi(v) \mod \mathbb{Z}) \neq 0$ for almost every $a \in \mathbb{R}$. Hence as $\pi(av) \in \overline{\pi(L)}$, $\overline{\pi(L)} \notin \xi^\perp$ for all non-trivial ξ. Therefore $\overline{\pi(L)} = X$.

Case 2. If $m \ge 1$ then by applying Lemma 3.15 we obtain a sequence $\{\mathbf{n}_k\}_{k=1}^\infty \subset H_{\epsilon,S}$ such that the $\Psi(\mathbf{n}_k)$s are all distinct where Ψ is the group morphism in (3.57).

Now fix $\xi \in \hat{X}\backslash\{0\}$ and consider $\xi(\zeta^{\mathbf{n}_k} \cdot v) = \sum_{h=0}^m \xi_{i_h} \zeta_{i_h}^{\mathbf{n}_k} x_{i_h}$. Here $\xi_{i_h} x_{i_h} \neq 0$, for all h, by Lemma 3.16 and the choices of $i_0, \ldots, i_m$. If $\xi(\zeta^{\mathbf{n}_k} \cdot v) = 0$ then

$$\xi_{i_0} x_{i_0} + \sum_{h=1}^m \xi_{i_h} x_{i_h} \frac{\zeta_{i_h}^{\mathbf{n}_k}}{\zeta_{i_0}^{\mathbf{n}_k}} = 0. \tag{3.61}$$

Equation (3.61) is a linear equation in the variable

$$\Psi(\mathbf{n}_k) = \left(\frac{\zeta_{i_h}^{\mathbf{n}_k}}{\zeta_{i_0}^{\mathbf{n}_k}}\right)_{h=1}^m \in (\mathbb{C}^\times)^m.$$

Since Ψ is a group morphism, $\Psi(\mathbb{Z}^r)$ is a subgroup of finite rank in $(\mathbb{C}^\times)^m$. It follows from the estimate [**ESS02**, Theorem 1.1] by Evertse *et al* of number of solutions to unit equations in a multiplicative group, that there are only finitely many solutions to (3.61) in $\Psi(\mathbb{Z}^r)$.

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Because all the $\Psi(\mathbf{n}_k)$s are different, when k is sufficiently large (3.61) fails, i.e. $\xi(\zeta^{\mathbf{n}_k} \cdot v) \neq 0$.

Therefore when k is large enough, for almost all $a \in \mathbb{R}$ the expression $\xi(\pi(a\zeta^{\mathbf{n}_k} \cdot v)) = (a\xi(\zeta^{\mathbf{n}_k} \cdot v) \bmod \mathbb{Z})$ does not vanish. As $a\pi(\zeta^{\mathbf{n}_k} \cdot v) \in \pi(\zeta^{\mathbf{n}_k} \cdot L)$, the closed subgroup $\overline{\pi(\zeta^{\mathbf{n}_k} \cdot L)} \subset X$ is not contained in $\xi^{\perp}$. Since ξ is an arbitrary non-trivial character, Lemma 3.17 claims, for all $\delta > 0$, that $\overline{\pi(\zeta^{\mathbf{n}_k} \cdot L)}$ is δ-dense in X when k is large enough, and thus so is $\pi(\zeta^{\mathbf{n}_k} \cdot L)$ itself. Proposition 3.14 is proved. □

Finally, we are in a position to give the proof of Proposition 3.2.

Proof of Proposition 3.2. If A_{ϵ_0} contains a $V_{\langle S\rangle}$-pattern for some $\epsilon_0 < \epsilon$, then by Proposition 3.10, $A_{(\epsilon_0+\epsilon)/2}$ contains $y + L$ where $y \in X$, and L is a line in $\mathbb{R}^{r_1} \oplus \mathbb{C}^{r_2}$. Take $\mathbf{n} \in H_{(\epsilon-\epsilon_0)/2,S}$ such that $\pi(\zeta^{\mathbf{n}} \cdot L)$ is δ-dense in X. Then $\zeta^{\mathbf{n}} \cdot (y + L)$, which is the translate of $\pi(\zeta^{\mathbf{n}} \cdot L)$ by $\zeta^{\mathbf{n}} \cdot y \in X$, is δ-dense as well; and therefore so is A_ϵ because $A_\epsilon \supset H_{(\epsilon-\epsilon_0)/2,S}.A_{(\epsilon_0+\epsilon)/2} \supset \zeta^{\mathbf{n}} \cdot (y + L)$. A_ϵ is actually dense in X because δ can be arbitrarily small. So $A_\epsilon = X$ by closedness. □

4. *Proof of the main results*

4.1. *Characterization of minimal (G, S)-invariant sets.*

In order to be able to use what was shown in the previous section, we hope to show that every (G, S)-invariant set contains a $V_{\langle S\rangle}$-translated torsion point.

We need two lemmas to establish this fact. The first is quite simple and the second relies on Proposition 2.11.

LEMMA 4.1. *Suppose that $\mathbf{m}, \mathbf{n} \in \mathbb{Z}^r$ are not equal.*

(i) *If $x \in X$ satisfies $\zeta^{\mathbf{m}} \cdot x = \zeta^{\mathbf{n}} \cdot x$, then x is a torsion point.*

(ii) *If $\zeta^{\mathbf{m}} \cdot x = \zeta^{\mathbf{n}} \cdot x + v$ where $v \in V_{\langle S\rangle}$, then x is a $V_{\langle S\rangle}$-translated torsion point.*

Proof. (i) Write $x = \pi(\tilde{x})$ where $\tilde{x} \in \mathbb{R}^{r_1} \oplus \mathbb{C}^{r_2}$. Then $\zeta^{\mathbf{m}} \cdot \tilde{x} - \zeta^{\mathbf{n}} \cdot \tilde{x} \in \Gamma$. Recall that $\mathbf{m} \mapsto \zeta^{\mathbf{m}}$ is a group embedding of $\mathbb{Z}^r$ into the group of units U_K. So if $\mathbf{m} \neq \mathbf{n}$ then $\zeta^{\mathbf{m}} \neq \zeta^{\mathbf{n}}$ and we have $(\zeta^{\mathbf{m}} - \zeta^{\mathbf{n}})^{-1} \in K$. So $\tilde{x} \in (\zeta^{\mathbf{m}} - \zeta^{\mathbf{n}})^{-1} \cdot \Gamma$ where the multiplication is given by (1.2). Recall that $\mathbb{R}^{r_1} \oplus \mathbb{C}^{r_2} \cong K \otimes_{\mathbb{Q}} \mathbb{R}$ via the map σ given in (1.1), and $\Gamma \subset \sigma(K)$ is a full-rank sublattice of $\mathbb{R}^{r_1} \oplus \mathbb{C}^{r_2}$. Thus $\tilde{x} \in (\zeta^{\mathbf{m}} - \zeta^{\mathbf{n}})^{-1} \cdot \sigma(K) = \sigma((\zeta^{\mathbf{m}} - \zeta^{\mathbf{n}})^{-1} \cdot K) = \sigma(K)$ where we have used (1.3). Because Γ has full rank, its $\mathbb{Q}$-span is $\sigma(K)$ and contains $\tilde{x}$. Therefore there is an integer q such that $q\tilde{x} \in \Gamma$. So $qx = 0$ in $X = (\mathbb{R}^{r_1} \oplus \mathbb{C}^{r_2})/\Gamma$.

(ii) Let $x' = x + (\zeta^{\mathbf{m}} - \zeta^{\mathbf{n}})^{-1} \cdot v \in X$; then $\zeta^{\mathbf{m}} \cdot x' = \zeta^{\mathbf{n}} \cdot x'$. By (i), x' is of torsion. Moreover, $(\zeta^{\mathbf{m}} - \zeta^{\mathbf{n}})^{-1} \cdot v \in V_{\langle S\rangle}$. Thus x is a $V_{\langle S\rangle}$-translated torsion point. □

LEMMA 4.2. *There does not exist a pair of minimal (G, S)-invariant closed sets M, M' such that $M + M' = X$.*

Proof. Assume there is such a pair of sets. By Lemma 2.10 we may write $M = \bigcap_{\epsilon>0} \overline{G_{\epsilon,S} \cdot x} \subset X$ and $M' = \bigcap_{\epsilon>0} \overline{G_{\epsilon,S} \cdot x'} \subset X$.

Fix a prime number p. Define a sequence of congruence subgroups

$$G^{(f)} = \{\mathbf{n} \in \mathbb{Z}^r : \zeta^{\mathbf{n}}\gamma \equiv \gamma \,(\mathrm{mod}\ p^f\Gamma),\ \forall \gamma \in \Gamma\} \tag{4.1}$$

for all $f \geq 0$. Then $G = G^{(0)} > G^{(1)} > G^{(2)} > \cdots$.

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Furthermore, $G^{(f)}$ is of finite index in G for all $f \geq 0$. To see this, observe that multiplication by any $\zeta^{\mathbf{n}}$ preserves $p^f\Gamma$ since it preserves Γ. Thus the multiplicative action ζ induces a $\mathbb{Z}^r$-action on the finite abelian group $\Gamma/p^f\Gamma$ (in other words, a group morphism from $G = \mathbb{Z}^r$ to $\mathrm{Aut}(\Gamma/p^f\Gamma)$). $G^{(f)}$ is just the kernel of this morphism, which has finite index as $|\mathrm{Aut}(\Gamma/p^f\Gamma)| < \infty$.

CLAIM. *There exists a decreasing sequence of non-empty closed sets $M = M^{(0)} \supset M^{(1)} \supset M^{(2)} \supset \cdots$ such that, for all $f \geq 0$, $M^{(f)}$ is a minimal $(G^{(f)}, S)$-invariant closed set and $M^{(f)} + M' = X$.*

We prove the claim by induction. When $f = 0$ the claim is part of the condition in the lemma. Suppose that there is already a finite sequence $M = M^{(0)} \supset M^{(1)} \supset \cdots \supset M^{(f)}$ satisfying the claim. We wish to construct $M^{(f+1)} \subset M^{(f)}$.

As $G^{(f+1)}$ is of finite index in $G^{(f)}$, Proposition 2.11 applies and produces a finite collection of minimal $(G^{(f+1)}, S)$-invariant closed sets $M_1^{(f+1)}, \ldots, M_N^{(f+1)} \subset M^{(f)}$, whose union is $M^{(f)}$. Applying Proposition 2.11 to G and $G^{(f+1)}$, we can also find another finite collection of minimal $(G^{(f+1)}, S)$-invariant closed sets $(M')_1^{(f+1)}, \ldots, (M')_{N'}^{(f+1)} \subset M'$ whose union is M'. It is already known that $M^{(f)} + M' = X$. Hence

$$\bigcup_{\substack{n=1,\ldots,N \\ n'=1,\ldots,N'}} (M_n^{(f+1)} + (M')_{n'}^{(f+1)}) = X.$$

In consequence, because all the $(M_n^{(f+1)} + (M')_{n'}^{(f+1)})$s are closed there exists a pair (n, n') such that $M_n^{(f+1)} + (M')_{n'}^{(f+1)}$ has non-empty interior. In particular, $M_n^{(f+1)} + (M')_{n'}^{(f+1)}$ contains a torsion point and an open neighborhood of it.

Therefore $M_n^{(f+1)} + (M')_{n'}^{(f+1)}$ contains a $V_{\langle S\rangle}$-pattern. By Lemma 2.10,

$$M_n^{(f+1)} = \bigcap_{\epsilon>0} \overline{(G^{(f+1)})_{\epsilon,S} \cdot z} \quad \text{and} \quad (M')_{n'}^{(f+1)} = \bigcap_{\epsilon>0} \overline{(G^{(f+1)})_{\epsilon,S} \cdot z'}$$

for some $z, z' \in X$. Therefore $A = M_n^{(f+1)} + (M')_{n'}^{(f+1)}$ satisfies condition (3.1) for $H = G^{(f+1)}$. By Proposition 3.2, $M_n^{(f+1)} + (M')_{n'}^{(f+1)} = X$. In particular, $M_n^{(f+1)} + M' = X$ as $(M')_{n'}^{(f+1)} \subset M'$. The inductive step is proved by setting $M^{(f+1)} = M_n^{(f+1)}$; this proves the claim.

We fix a point x_∞ in the limit set $\bigcap_{f=0}^{\infty} M^{(f)}$, which is non-empty by compactness of X.

For any given $f \in \mathbb{N}$, take a point $y \in X$ of the form

$$y = \pi(\tilde{y}) \quad \text{where } \tilde{y} \in p^{-f}\Gamma \subset \mathbb{R}^{r_1} \oplus \mathbb{C}^{r_2}. \tag{4.2}$$

Then $p^f y = 0$ in X and y is a torsion point. By the claim, $y = u + u'$ where $u \in M^{(f)}$, $u' \in M'$. Since $M^{(f)}$ is minimal and contains x_∞, $x_\infty \in \bigcap_{\epsilon>0} \overline{(G^{(f)})_{\epsilon,S} \cdot u}$ by Lemma 2.10. According to Remark 2.6, there is a sequence $\{\mathbf{n}_k\}_{k=1}^{\infty}$ such that $\mathbf{n}_k \in G^{(f)}_{\epsilon_k,S}$, where $\lim_{k\to\infty} \epsilon_k = 0$, and

$$\lim_{k\to\infty} \zeta^{\mathbf{n}_k} \cdot u = x_\infty. \tag{4.3}$$

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Suppose that $\tilde{y} = p^{-f}\gamma$ with $\gamma \in \Gamma$, then $\zeta^{\mathbf{n}_k} \cdot \tilde{y} = p^{-f}\zeta^{\mathbf{n}_k}\gamma$. Since $\mathbf{n}_k \in G^{(f)}$, $\zeta^{\mathbf{n}_k} \cdot \gamma \equiv \gamma (\mathrm{mod}\ p^f\Gamma)$, therefore $\zeta^{\mathbf{n}_k} \cdot \tilde{y} \equiv p^{-f}\gamma = \tilde{y} (\mathrm{mod}\ \Gamma)$ and the projection $y = \pi(\tilde{y})$ satisfies

$$\zeta^{\mathbf{n}_k} \cdot y = y. \tag{4.4}$$

Taking the difference between (4.3) and (4.4), we obtain

$$\lim_{k\to\infty} \zeta^{\mathbf{n}_k} \cdot u' = y - x_\infty. \tag{4.5}$$

As $\mathbf{n}_k \in G^{(f)}_{\epsilon_k,S}$ with $\epsilon_k \to 0$, it follows from Remark 2.6 that $y - x_\infty$ belongs to $\bigcap_{\epsilon>0} \overline{(G^{(f)})_{\epsilon,S} \cdot u'}$. As M' is (G, S)-invariant, it is also $(G^{(f)}, S)$-invariant; so $\bigcap_{\epsilon>0} \overline{(G^{(f)})_{\epsilon,S} \cdot u'} \subset M'$ by definition. Thus $y - x_\infty \in M'$ for all points y of the form (4.2). However, when f tends to ∞, because $p^{-f}\Gamma$ becomes dense in $\mathbb{R}^{r_1} \oplus \mathbb{C}^{r_2}$, points y of the form (4.2) become dense in X, and so do the $(y - x_\infty)$s since x_∞ is fixed. Hence M' is dense and, as a closed set, it must be X. This contradicts the minimality of M' as a (G, S)-invariant closed set because X contains proper (G, S)-invariant closed subsets. The proof is complete. □

PROPOSITION 4.3. *Suppose that $M \subset X$ is a minimal (G, S)-invariant closed set. Then M contains a $V_{\langle S\rangle}$-translated torsion point.*

Remark that, together with Lemma 2.10, the proposition implies that $M = \bigcap_{\epsilon>0} \overline{G_{\epsilon,S} \cdot x}$, where x is a $V_{\langle S\rangle}$-translated torsion point. As a result, M is finite by Lemma 3.3.

Proof of Proposition 4.3. By Lemma 2.10, $M = \bigcap_{\epsilon>0} \overline{G_{\epsilon,S} \cdot x}$ for some $x \in X$. We distinguish between three cases.

Case 1. If there exists $\epsilon > 0$ such that $G_{\epsilon,S} \cdot x$ is finite then x is a torsion point. Actually, because it follows from Lemma 3.6 that $G_{\epsilon,S}$ is infinite, $\zeta^{\mathbf{m}} \cdot x = \zeta^{\mathbf{n}} \cdot x$ for some pair $\mathbf{m} \neq \mathbf{n}$ from $G_{\epsilon,S}$. By Lemma 4.1, x is of torsion so we are done.

Case 2. Suppose that there exists $\epsilon > 0$ such that $G_{\epsilon,S} \cdot x$ is infinite but for any converging sequence $\{y_k\}_{k=1}^\infty \subset \overline{G_{\epsilon,S} \cdot x}$, whose limit we denote by y, $y_k - y \in V_{\langle S\rangle}$ for sufficiently large k. (Here as in Definition 3.1, $y_k - y$ is regarded as a very short vector in $\mathbb{R}^{r_1} \oplus \mathbb{C}^{r_2}$ when y_k is sufficiently close to y and $y_k - y \to 0$ as $k \to \infty$.) In this case we claim that x is a $V_{\langle S\rangle}$-translated torsion point.

In fact since $G_{\epsilon,S} \cdot x$ is infinite and X is compact, there is always such a converging sequence $\{y_k\}_{k=1}^\infty$ inside $G_{\epsilon,S} \cdot x$ where the y_ks are all distinct. By assumption for very large k, $y_k - y \in V_{\langle S\rangle}$ where $y = \lim_{k\to\infty} y_k \in \overline{G_{\epsilon,S} \cdot x}$. Thus for k, k' both sufficiently large, $y_k = y_{k'} + v$ for some $v \in V_{\langle S\rangle}$. Since $y_k = \zeta^{\mathbf{n}_k} \cdot x$ and $y_{k'} = \zeta^{\mathbf{n}_{k'}} \cdot x$ for some $\mathbf{n}_k, \mathbf{n}_{k'} \in G_{\epsilon,S}$, x is a $V_{\langle S\rangle}$-translated torsion point by Lemma 4.1.

Case 3. If neither Case 1 nor Case 2 holds, then for all given $\epsilon > 0$, $G_{\epsilon,S} \cdot x$ is infinite and, moreover, there is a sequence $\{y_k\}_{k=1}^\infty \subset \overline{G_{\epsilon,S} \cdot x}$ converging to some $y \in \overline{G_{\epsilon,S} \cdot x}$ such that $y_k - y \notin V_{\langle S\rangle}$, for all k.

The opposite operator $x \mapsto -x$ on X commutes with the $\mathbb{Z}^r$-action. Therefore as M is a minimal (G, S)-invariant closed set, so is $-M = \{-y : y \in M\} = \bigcap_{\epsilon>0} \overline{G_{\epsilon,S} \cdot (-x)}$.

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By Lemmas 2.7 and 2.8, $A = M - M$ is (G, S)-invariant and is equal to $\bigcap_{\epsilon>0}(\overline{G_{\epsilon,S} \cdot x} - \overline{G_{\epsilon,S} \cdot x})$.

Let $A_\epsilon = \overline{G_{\epsilon,S} \cdot x} - \overline{G_{\epsilon,S} \cdot x}$. Remark that, for $\{y_k\}_{k=1}^\infty$ and y above, $\pi(y_k - y) \in A_\epsilon$ for all k. The earlier characterization of $\{y_k\}_{k=1}^\infty$ and y actually says that A_ϵ contains a $V_{\langle S\rangle}$-pattern at 0. It follows from Proposition 3.2 that $A_\epsilon = X$ for all $\epsilon > 0$ and therefore $M - M = \bigcap_{\epsilon>0} A_\epsilon = X$, which contradicts Lemma 4.2. So Case 3 cannot occur. This concludes the proof of Proposition 4.3. □

4.2. *Proof of the main theorems.*

Proof of Theorem 1.10. Theorem 1.10(i) is already covered by Lemma 3.3, so we only need to prove part (ii)—that, for any $x \in X$ which is not a $V_{\langle S\rangle}$-translated torsion point, the set $A_\epsilon = \overline{G_{\epsilon,S} \cdot x}$ is equal to X for all ϵ.

Let $A = \bigcap_{\epsilon>0} \overline{G_{\epsilon,S} \cdot x}$, which is (G, S)-invariant by Corollary 2.5. It contains a minimal (G, S)-invariant set by Lemma 2.9, thus contains a $V_{\langle S\rangle}$-translated torsion point y by Proposition 4.3.

Fix an arbitrary $\epsilon > 0$. We claim that $A_{\epsilon/2}$ contains a $V_{\langle S\rangle}$-pattern near the given point y; i.e. a sequence $\{y_k\}_{k=1}^\infty$ converging to y, such that $y_k - y \notin V_{\langle S\rangle}$. Suppose otherwise; then there are only two possibilities.

Case 1. $y \in G_{\epsilon/2,S} \cdot x$. Hence $x \in G_{\epsilon/2,S} \cdot y$ because it is clear that $\mathbf{n} \in G_{\epsilon/2,S}$ if and only if $-\mathbf{n} \in G_{\epsilon/2,S}$. Therefore by Lemma 3.3, x is a $V_{\langle S\rangle}$-translated torsion point itself, giving a contradiction.

Case 2. $y \notin G_{\epsilon/2,S} \cdot x$ (in other words, y is an accumulation point of $G_{\epsilon/2,S} \cdot x$). Since $V_{\langle S\rangle}$-pattern is not allowed by assumption, there must be a convergent sequence $\{y_k\}_{k=1}^\infty \subset G_{\epsilon/2,S} \cdot x$ whose limit is y, such that y_k is in the $V_{\langle S\rangle}$-foliation through y for sufficiently large k (or equivalently, for all k by choosing a subsequence). Hence the point y_k itself is a $V_{\langle S\rangle}$-translated torsion point because y is. But $y_k \in G_{\epsilon/2,S} \cdot x$. Again by Lemma 3.3, $x \in G_{\epsilon/2,S} \cdot y_k$ is a $V_{\langle S\rangle}$-translated torsion point and this contradicts the assumption on x.

Therefore the claim that $A_{\epsilon/2}$ contains a $V_{\langle S\rangle}$-pattern is proved. But then it follows from Proposition 3.2 that $A_\epsilon = X$. This establishes Theorem 1.10. □

Proof of Theorem 1.8. As via the transformation ψ, the actions $\alpha : \mathbb{Z}^r \curvearrowright \mathbb{T}^d$ and $\zeta : \mathbb{Z}^r \curvearrowright X$ are equivalent, it suffices to prove the part concerning X in Theorem 1.8 and the analogous claim about $\mathbb{T}^d$ follows.

If $x \in X$ is a $V_{\langle S\rangle}$-translated torsion point, then by Corollary 3.5, the set (1.11) is not dense.

On the other hand, if x is not a $V_{\langle S\rangle}$-translated torsion point, then by Theorem 1.10, $\overline{G_{\epsilon,S} \cdot x} = X$. Because the set (1.11) contains $G_{\epsilon,S} \cdot x$ it is also dense in X, which completes the proof. □

Acknowledgements. I wish to thank my thesis advisor, Professor Elon Lindenstrauss, for initiating the larger project of which this paper is a part, and for his constant support. Thanks also go to the referee for helpful comments. I am grateful to the Hebrew University of Jerusalem for its hospitality.

https://doi.org/10.1017/S0143385711000484 Published online by Cambridge University Press

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