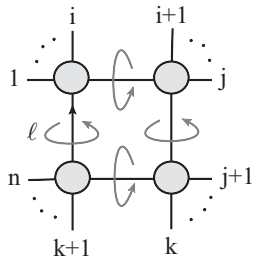
Leading Singularities and on-shell diagrams 8

Chapter 6 introduced unitarity cuts. In D-dimensions, we can at most cut D propagators per loop since each loop-momentum only has D components. When the maximum number of propagators, $D \times L$, is cut, the unitarity cut is called a ***maximal cut*** [117, 118]. The maximal cuts are very useful for determining the integrand, in particular in 4d planar $\mathcal{N} = 4$ SYM.

▷ *Example.* As an example of a maximal cut, consider the quadruple cut of the 1-loop n-point amplitude in 4d:

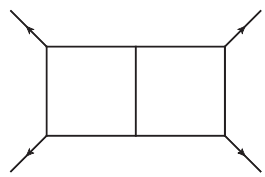


Four propagators are put on-shell:

$$\ell^2 = (\ell - K_1)^2 = (\ell - K_2)^2 = (\ell - K_3)^2 = 0\,. \tag{8.1}$$

Here $K_1 = p_1 + \cdots + p_i$, $K_2 = p_{i+1} + \cdots + p_j$, $K_3 = p_{j+1} + \cdots + p_k$, and $K_4 = p_{k+1} + \cdots + p_n$. No further propagators can be put on-shell in 4d since the loop-momentum only has four components. ◁

The result of a maximal cut is a product of on-shell tree amplitudes, $\mathcal{A}^{\text{tree}}_{n_1} \cdots \mathcal{A}^{\text{tree}}_{n_j}$, appropriately summed over all possible intermediate states, with the loop-momenta evaluated on the solutions to the cut constraints. For example, for the 1-loop box in 4d $\mathcal{N} = 4$ SYM, the value of the maximal cut (8.1) is $\int d^4\eta_I\, \mathcal{A}^{\text{tree}}_{n_1} \mathcal{A}^{\text{tree}}_{n_2} \mathcal{A}^{\text{tree}}_{n_3} \mathcal{A}^{\text{tree}}_{n_4}$ evaluated on the two solutions to the quadratic loop-momentum constraints. At $L > 1$, the cut constraints generically have 2^L distinct solutions, however, there are situations where there are not enough propagators to cut; a simple example in 4d is the following 2-loop double-box integral



. (8.2)

It has only 7 propagators, but we need to take $2 \times 4 = 8$ propagators on-shell for a maximal cut. For such cases, the solution space for the loop-momenta is not a set of isolated points but rather a continuous multi-dimensional manifold. However, if we impose the cut constraints

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on one loop-momentum at a time, new poles appear and they can be used to fix the remaining degrees of freedom, again leaving us with a set of isolated solutions for the loop-momenta. We demonstrate this explicitly for the double-box (8.2) in Section 8.2.

The method of generalized unitarity is to find an integrand that reproduces all the unitarity cuts, including of course all the maximal cuts. But how exactly do we treat the distinct solutions to the maximal cut constraints? There are two ways to proceed:[1]

1. *Appropriate sampling over all solutions*. In one approach, we require that the correct integrand matches the maximal cut evaluated on a sampling of all 2^L solutions, with each solution given an appropriate weight. At 1-loop there are just two solutions and the proper weight is $1/2$ for both, thus in effect averaging over the two solutions, as in (6.31) and (6.32). For higher-loops, one starts with a set of integrals that integrate to zero. The appropriate weight for each solution is determined by the requirement that their contributions to the vanishing integrals need to sum to zero. Explicit examples and further discussions at 2-loop order can be found in [96].
2. *Match each solution.* In the other approach, we require the integrand to reproduce *each* of the cut solutions individually. The individual cut solutions are treated as independent entities and the resulting value for the maximal cut evaluated on each solution is called a ***Leading Singularity (LS)***. The name reflects that these objects are the residues of the most singular configuration of the loop-integrand (for generic external kinematics). Note that, despite the name, these contributions are finite.

We focus here on the second approach. A major motivation is that all planar loop-integrands of $\mathcal{N} = 4$ SYM can be written as a linear combination of dual conformal invariant "unit Leading Singularity integrands" (to be introduced below) [119]. The characterization of Leading Singularities turns out to be a quite interesting mathematical problem [120]. The Leading Singularities offer insight into the structure of planar $\mathcal{N} = 4$ SYM loop-integrands at all loop orders, but note that to obtain the actual amplitudes, one still needs to perform the challenging loop-integrations. While the notion of dual conformal invariance is only applicable in the planar limit, it is a well-defined (and open) question whether the full non-planar loop-integrands of $\mathcal{N} = 4$ SYM can also be determined by the Leading Singularities.

We begin our study of the Leading Singularities at 1-loop order. All amplitudes in this chapter are in 4d planar $\mathcal{N} = 4$ SYM.

8.1 1-loop Leading Singularities

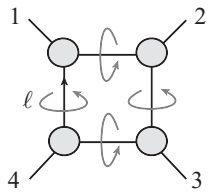
To build intuition for the Leading Singularities at 1-loop order, we start with the simplest case, namely the 4-point integrand, then consider the new features at 5-point, and finally generalize to n-point.

[1] The maximal cut was formulated in [118] for 1-loop amplitudes of $\mathcal{N} = 4$ SYM and generalized to multi-loop amplitudes in [117]. A more detailed review is offered in [68].

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4-point

For $n = 4$, the maximal cut conditions (8.1) are simply



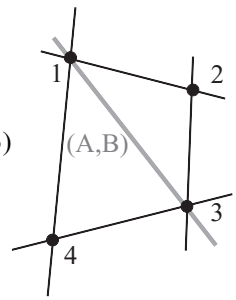
$$\ell^2 = (\ell - p_1)^2 = (\ell - p_1 - p_2)^2 = (\ell - p_1 - p_2 - p_3)^2 = 0\,. \quad (8.3)$$

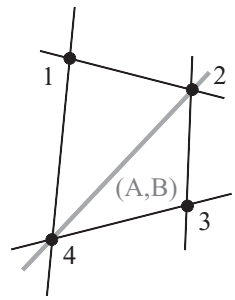
In dual variables, this is simply $y_{01}^2 = y_{02}^2 = y_{03}^2 = y_{04}^2 = 0$ (see Section 7.1). And translating that to momentum twistor space (as in the early part of Section 7.4), we have

$$\langle AB12\rangle = \langle AB23\rangle = \langle AB34\rangle = \langle AB41\rangle = 0\,. \quad (8.4)$$

The geometric interpretation of the cut constraints (8.4) is that (A, B) is a line in $\mathbb{CP}^3$ that intersects each of the four lines $(1, 2)$, $(2, 3)$, $(3, 4)$, and $(4, 1)$. As anticipated from the quadratic nature of the constraint (8.3), there are two independent solutions. This is rather obvious geometrically:

(a) $(A, B) = (1, 3)$

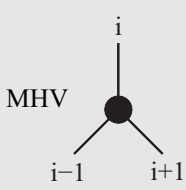




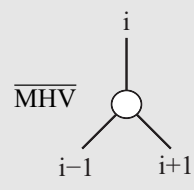
(b) $(A, B) = (2, 4)$.

(8.5)

Now, on the quadruple cut, the 1-loop integrand factorizes into a product of four tree-level 3-point amplitudes, which can be either MHV or anti-MHV. Recall that special kinematics, (2.58) and (2.59), apply to the 3-point amplitudes – we summarize it here:



$$|i-1] \propto |i] \propto |i+1]\,,$$



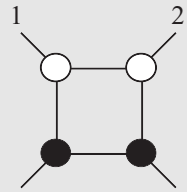
$$|i-1\rangle \propto |i\rangle \propto |i+1\rangle\,.$$

(8.6)

We use a black blob to indicate an MHV 3-point subamplitude (or vertex), and a white blob for 3-point $\overline{\text{MHV}}$ = anti-MHV.

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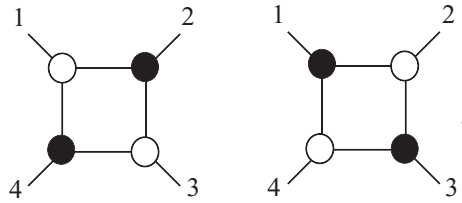
A configuration where two MHV subamplitudes with external legs are adjacent vanishes:



$= 0\,.$ (8.7)

This is because special kinematics (8.6) implies $|1\rangle \propto |2\rangle$, so that $s_{12} = -(p_1 + p_2)^2 = \langle 12\rangle[12] = 0$. Of course this does not hold true for generic momenta p_1 and p_2. Hence, for generic external momenta, we are not allowed to have helicity configurations, such as (8.7), where two MHV or two anti-MHV subamplitudes with external legs are adjacent.

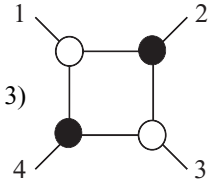
The only helicity options for the 4-point quadruple cut are therefore

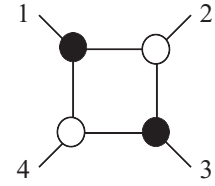


. (8.8)

Now these two diagrams have to be evaluated on the kinematic solutions (8.5). For the solution (a) where the line (A, B) is $(1, 3)$, we can simply pick the momentum twistor of the loop line ℓ to be $Z_1 = (\,|1\rangle, [\mu_1|\,)$. This means that $|\ell\rangle \propto |1\rangle$ and that selects the kinematics where the vertex that line 1 attaches to is anti-MHV, i.e. this picks the first helicity configuration in (8.8). Likewise, solution (b) with $(A, B) = (2, 4)$ selects the opposite helicity configuration. So we conclude that the maximal cuts have the two solutions:

(a) $(A, B) = (1, 3)$





(b) $(A, B) = (2, 4)\,.$ (8.9)

These two diagrams encode the ***Leading Singularities*** for the 4-point 1-loop amplitude. The Leading Singularities, $\mathrm{LS}_{(a)}$ and $\mathrm{LS}_{(b)}$, are calculated as the product of the four subamplitudes summed over all intermediate states and evaluated on the respective solutions (a) and (b), *times a Jacobian factor*. The Jacobian factor $1/J$ comes from a change of variables that converts the associated loop-integral over $\mathbb{R}^4$ to a contour integration with four contours encircling each of the four propagator poles in the quadruple cut.[2] The conversion of the

[2] Since the solutions to the cut conditions may be complex-valued, we should really consider the loop-integral as an integral over the real section in $\mathbb{C}^4$; that makes it more natural to convert to a contour integral encircling the poles corresponding to the on-shell propagators.

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integral can be done via a change of variables $u_i = y_{0i}^2$, for $i = 1, 2, 3, 4$, giving

$$\int \frac{d^4 y_0}{y_{01}^2 y_{02}^2 y_{03}^2 y_{04}^2} = \int \frac{du_1}{u_1}\frac{du_2}{u_2}\frac{du_3}{u_3}\frac{du_4}{u_4} J\,, \tag{8.10}$$

where $J = \det(\partial y_0^\mu / \partial u_i)$ is the Jacobian. As we show explicitly in the example below, the Jacobian is

$$J = \frac{1}{y_{13}^2 y_{24}^2} = -\frac{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}{\langle 1234\rangle^2}\,. \tag{8.11}$$

The Leading Singularity for the 4-point 1-loop amplitude is then

$$\mathrm{LS}_{(\mathrm{a})} = J \int \left[\prod_{i=1}^{4} d^4\eta_{\ell_i}\right] \Big(\mathcal{A}_3^{\overline{\mathrm{MHV}}}(-\ell_4, p_1, \ell_1)\, \mathcal{A}_3^{\mathrm{MHV}}(-\ell_1, p_2, \ell_2) \times \mathcal{A}_3^{\overline{\mathrm{MHV}}}(-\ell_2, p_3, \ell_3)\, \mathcal{A}_3^{\mathrm{MHV}}(-\ell_3, p_4, \ell_4)\Big)\Big|_{\ell=\ell^{(a)}}\,. \tag{8.12}$$

A similar expression is found for $\mathrm{LS}_{(\mathrm{b})}$. Evaluating the $\mathrm{LS}_{(\mathrm{a})}$ and $\mathrm{LS}_{(\mathrm{b})}$, one finds

$$\mathrm{LS}_{(\mathrm{a})} = \mathrm{LS}_{(\mathrm{b})} = \mathcal{A}_4^{\mathrm{tree}}\,. \tag{8.13}$$

▶ **Exercise 8.1**

Evaluate the RHS of (8.12) to show that $\mathrm{LS}_{(\mathrm{a})} = \mathcal{A}_4^{\mathrm{tree}}$.

Now before exploring the Leading Singularities further, let us illustrate how the Jacobian is obtained. It can of course be calculated brute-force, but in the example below we carry out the calculation via a tour to momentum twistor:

▷ *Example.* We calculate J in (8.10) via the momentum twistor formulation. From (7.40)–(7.43), we read off

$$\int \frac{d^4 y_0}{y_{01}^2 y_{02}^2 y_{03}^2 y_{04}^2} = \int \frac{d^4 Z_A d^4 Z_B}{\mathrm{vol}(GL(2))} \frac{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}{\langle AB12\rangle\langle AB23\rangle\langle AB34\rangle\langle AB41\rangle}\,. \tag{8.14}$$

The loop momentum twistors Z_A and Z_B can be expanded on the basis of four external line momentum twistors as

$$Z_A = a_1 Z_1 + a_2 Z_2 + a_3 Z_3 + a_4 Z_4\,, \qquad Z_B = b_1 Z_1 + b_2 Z_2 + b_3 Z_3 + b_4 Z_4\,. \tag{8.15}$$

This linear transformation gives $d^4 Z_A\, d^4 Z_B = \langle 1234\rangle^2\, d^4 a_i d^4 b_i$. The 4-brackets evaluate to $\langle A, B, i-1, i\rangle = \langle 1234\rangle M_{i+1}$, where M_j is the jth minor of the 2×4 matrix

$$\begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \end{pmatrix}\,. \tag{8.16}$$

For example, $\langle AB34\rangle = \langle 1234\rangle M_1 = \langle 1234\rangle(a_1 b_2 - a_2 b_1)$. Now, consider a $GL(2)$-transformation (7.55) of Z_A, Z_B. We can use it to set $a_4 = b_2 = 0$ and $a_2 = b_4 = 1$.

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► **Exercise 8.2**

Convince yourself that a $GL(2)$ rotation of Z_A, Z_B allows you to make the above choice of parameters, but that setting $a_4 = b_4 = 0$ would be illegal.

In this gauge, we have $\langle AB12\rangle = \langle 1234\rangle\, a_3$, etc., and the integrand then has no dependence (obviously) on a_4, b_2, a_2, b_4. This means that the $GL(2)$-volume factor cancels and we are then left with

$$\int \frac{d^4y_0}{y_{01}^2 y_{02}^2 y_{03}^2 y_{04}^2} = \int \frac{da_1}{a_1}\frac{db_1}{b_1}\frac{da_3}{a_3}\frac{db_3}{b_3}\frac{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}{-\langle 1234\rangle^2}\,. \tag{8.17}$$

This way we have brought the loop-integral to the form on the LHS of (8.10) and we see that the Jacobian is indeed (8.11).

Now our integration variables a_i and b_i in (8.17) are not exactly the $u_i = y_{0i}^2$ that we introduced above (8.10): for example $u_1 = y_{01}^2 = \frac{\langle 41AB\rangle}{\langle 41\rangle\langle AB\rangle} = -\frac{\langle 1234\rangle}{\langle 41\rangle\langle AB\rangle}\, b_3$. So the u_is are proportional to the $a_{1,3}$ and $b_{1,3}$, but the factors of proportionality drop out of the du_i/u_i measure. ◁

Recall that we are studying the Leading Singularities in order to find an integrand that faithfully reproduces both $\mathrm{LS}_{(a)}$ and $\mathrm{LS}_{(b)}$. The integrand that we already know for the 4-point 1-loop amplitude does the job – let us see how. We have previously found (see (6.39)) that

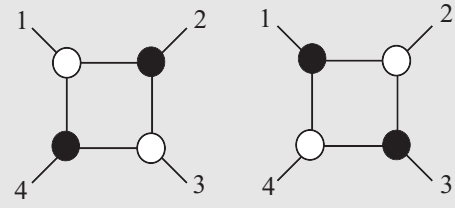
$$\mathcal{A}_4^{\text{1-loop}} = \mathcal{A}_4^{\text{tree}}\, y_{13}^2 y_{24}^2 \int \frac{d^4y_0}{y_{01}^2 y_{02}^2 y_{03}^2 y_{04}^2}\,. \tag{8.18}$$

When we convert this to the contour integral, the prefactor $y_{13}^2 y_{24}^2$ exactly cancels the Jacobian (8.11), so we are left with

$$\mathcal{A}_4^{\text{1-loop}} = \mathcal{A}_4^{\text{tree}} \int \frac{da_1}{a_1}\frac{db_1}{b_1}\frac{da_3}{a_3}\frac{db_3}{b_3}\,. \tag{8.19}$$

From this, we can directly read off the residue at the propagator poles $a_1 = b_1 = a_3 = b_3 = 0$. The result is independent of which of the two solutions (a) or (b) we use to localize the loop-integral, so the quadruple cuts of the integral match exactly with the Leading Singularities, $\mathrm{LS}_{(a)} = \mathrm{LS}_{(b)} = \mathcal{A}_4^{\text{tree}}$. This may not shock you, but once we venture beyond 4-point amplitudes, things are not so simple.

The result $\mathrm{LS}_{(a)} = \mathrm{LS}_{(b)}$ is special for the 4-point case. It can be represented diagrammatically as



(8.20)

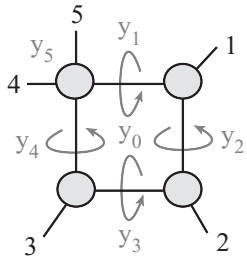
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This identity is called the ***square move*** and it will show up later in our discussions of higher-loop Leading Singularities and on-shell diagrams.

5-point

At this stage, we have constructed the 4-point 1-loop amplitude of $\mathcal{N}=4$ SYM in three different ways: generalized unitarity, loop-level BCFW, and Leading Singularities. It is time to move ahead.

We consider a specific maximal cut of the 5-point 1-loop amplitude:

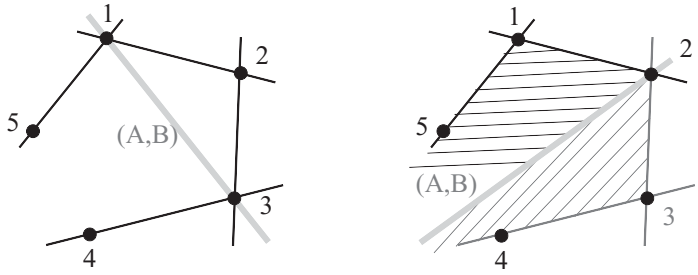


$$y_{01}^2 = y_{02}^2 = y_{03}^2 = y_{04}^2 = 0\,. \tag{8.21}$$

The cut constraints for this maximal cut can be written in momentum twistor space as

$$\langle AB12\rangle = \langle AB23\rangle = \langle AB34\rangle = \langle AB51\rangle = 0\,. \tag{8.22}$$

This says that $(A,\,B)$ is a line that intersects the four lines $(1,\,2)$, $(2,\,3)$, $(3,\,4)$, and $(5,\,1)$. There are two solutions:



(a) $(A,\,B\,) = (1,\,3)$ (b) $(A,\,B\,) = (5,\,1,\,2) \bigcap (2,\,3,\,4)\,.$ (8.23)

It is straightforward to see that (a) is a solution to (8.22). As for (b), note that generically two planes in $\mathbb{CP}^3$ intersect in a line. Any points A and B on the intersection of the two planes in (b) will be linearly dependent with any two points in either plane. This establishes that (b) is a solution to (8.22).

Since intersections of planes in $\mathbb{CP}^3$ may not feel as natural to you as brushing your teeth (hopefully), let us make the solutions (8.23) explicit in momentum space. With $\ell = y_{10}$, the constraints are $\ell^2 = (\ell - p_1)^2 = (\ell - p_1 - p_2)^2 = (\ell - p_1 - p_2 - p_3)^2 = 0$ and it is not hard to verify that the two solutions for the loop-momentum ℓ can be written

$$\ell^{(1)} = -|1\rangle\left([1| + \frac{\langle 23\rangle}{\langle 13\rangle}[2|\right)\,, \qquad \ell^{(2)} = -\left(|1\rangle + \frac{[23]}{[13]}|2\rangle\right)[1|\,. \tag{8.24}$$

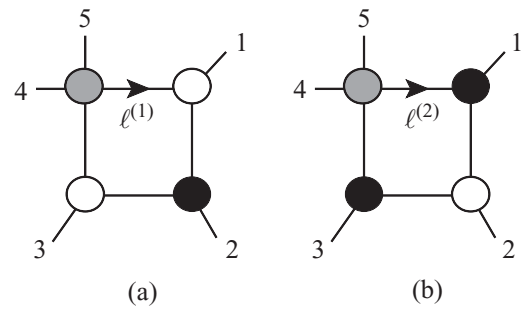
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Note that even though $\ell^{(1)}$ is formally the complex conjugate of $\ell^{(2)}$, their geometric interpretations in momentum twistor space are quite different. This is because momentum twistors are chiral objects (only $|i\rangle$ appears, not $|i]$).

▶ **Exercise 8.3**

Show that $\ell^{(1)}$ and $\ell^{(2)}$ in (8.24) solve the cut constraints. Check little group scaling. Then show that two solutions, $\ell^{(1)}$ and $\ell^{(2)}$, correspond to the two geometric solutions (a) and (b) of (8.23), respectively, in momentum twistor space.

The solution $\ell^{(1)}$ has $|\ell\rangle \propto |1\rangle$ and by momentum conservation these are also proportional to the angle spinor of $(\ell - p_1)$. This means that the special 3-point kinematics forces the vertex where line 1 attaches to be anti-MHV. Likewise, the solution $\ell^{(2)}$ forces the same vertex to be anti-MHV. By (8.7), the rest of the helicity structure is then fixed, and we see that the two solutions (a) and (b) correspond to the two options



. (8.25)

The gray blob for the 4-point vertex does not have specific helicity designation because 4-point on-shell amplitude is simultaneously MHV and anti-MHV.

Now, let us count the number of Grassmann ηs of these blob-diagrams. MHV has 8 ηs and 3-point anti-MHV has 4 ηs. For each of the four internal lines we have to do a $d^4\eta$-integral. For diagram (a), this then gives $8 + 4 + 4 + 8 - 4 \times 4 = 8$ corresponding to the 5-point MHV sector. For diagram (b): $8 + 8 + 8 + 4 - 4 \times 4 = 12$ which identifies it as belonging to the NMHV sector. Including the appropriate Jacobians, the results for the two diagrams (8.25) are the respective MHV and NMHV 5-point tree-level amplitudes, respectively.

The above discussion tells us that for a given MHV or NMHV 1-loop 5-point amplitude, only one of these solutions to the maximal cut conditions (8.21) is relevant. If we focus on the MHV sector, only diagram (a) in (8.25) matters, and it equals $\mathcal{A}_5^{\text{tree}}$ for solution $\ell^{(1)}$ and is zero when evaluated on solution $\ell^{(2)}$. This leads us to the crux of the problem we mentioned in the beginning of this chapter: whether the integral basis Ansatz we write for the integrand faithfully reproduces all Leading Singularities. Let us illustrate this explicitly. Consider the scalar box-integral whose propagators are those considered in the maximal cut (8.21):

$$I_{5,\text{box}}(1,2,3,4) = \frac{\langle 5123\rangle\langle 1234\rangle}{\langle AB51\rangle\langle AB12\rangle\langle AB23\rangle\langle AB34\rangle}\,. \tag{8.26}$$

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We use the labels on the n-point box-integral $I_{n,\text{box}}(i,\,j,\,k,\,l)$ to specify the first external leg on each of the vertices. For example, for the arrangement in (8.1), the corresponding scalar box-integral would be labeled $I_n(1, i+1, j+1, k+1)$.

When evaluating the quadruple cut for the integral $I_{5,\text{box}}(1,2,3,4)$, the Jacobian cancels the numerator factor $\langle 5123\rangle\langle 1234\rangle$, and since there is no other dependence on the loop-momenta than the four propagators we are cutting, this integral produces the same answer, namely 1, no matter if we evaluate it on solution $\ell^{(1)}$ or $\ell^{(2)}$: i.e.

$$I_{5,\text{box}}\big|_{(1)} = I_{5,\text{box}}\big|_{(2)} = 1\,. \tag{8.27}$$

On the other hand, we now know that for the MHV sector the corresponding Leading Singularities of diagram (a) in (8.25) are

$$\text{LS}_{(1)} = \mathcal{A}^{\text{tree}}_{5,\text{MHV}}\,, \qquad \text{LS}_{(2)} = 0\,. \tag{8.28}$$

This means that the Ansatz

$$\mathcal{A}^{\text{1-loop}}_{5,\text{MHV}} = \mathcal{A}^{\text{tree}}_{5,\text{MHV}} \times \Big(I_{5,\text{box}}(1,2,3,4) + \text{other box-integrals}\Big) \tag{8.29}$$

does *not* produce the Leading Singularities faithfully. However, it does produce the average of the two maximal cuts correctly because, when multiplied by a factor of $\mathcal{A}^{\text{tree}}_{5,\text{MHV}}$,

$$\frac{1}{2}\Big(I_{5,\text{box}}\big|_{(1)} + I_{5,\text{box}}\big|_{(2)}\Big) = \frac{1}{2}(1+1) = 1 \tag{8.30}$$

equals the sum of the Leading Singularities $\text{LS}_{(1)} + \text{LS}_{(2)}$. The message is that the integrand Ansatz (8.29) can produce the correct maximal cut when one averages over the two constraints (as is usually done in applications of the generalized unitarity method), but it does not produce each Leading Singularity honestly. If you just want an answer for the amplitude, you don't have to care. But let us try to be caring people and see where it takes us.

We have learned now that we need something else in the Ansatz (8.29) in order to match the Leading Singularities. That something else turns out to be the ***pentagon integral***

$$I_{5,\text{pentagon}} = \frac{\langle A,\,B|(1,2,3)\bigcap(3,4,5)\rangle\langle 2451\rangle}{\langle AB12\rangle\langle AB23\rangle\langle AB34\rangle\langle AB45\rangle\langle AB51\rangle}\,. \tag{8.31}$$

The numerator includes the bi-twistor $(1,2,3)\bigcap(3,4,5)$ that characterizes the line of intersection between the planes $(1,2,3)$ and $(3,4,5)$; the intersection formula was given in (5.67). When we evaluate the maximal cut (8.21) of the pentagon, the residue depends on the loop-momentum and hence on which solution (8.24) we choose to evaluate it. Including the Jacobian $J = \langle 5123\rangle\langle 1234\rangle$, one finds

$$I_{5,\text{pentagon}}\big|_{(a)} = 0, \qquad I_{5,\text{pentagon}}\big|_{(b)} = -1\,. \tag{8.32}$$

This is good news, because now the improved Ansatz

$$\mathcal{A}^{\text{1-loop}}_{5,\text{MHV}} = \mathcal{A}^{\text{tree}}_{5,\text{MHV}} \times \Big(I_{5,\text{box}}(1,2,3,4) + I_{5,\text{pentagon}} + \text{other}\Big) \tag{8.33}$$

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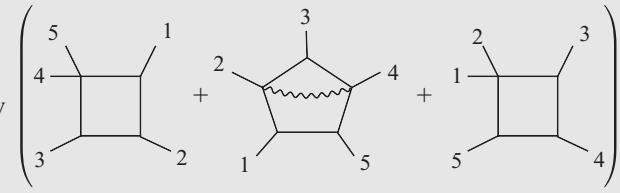
(where "other" is assumed not to contribute to our quadruple cut) has the following maximal cut (8.21): on two solutions, $\ell^{(1)}$ and $\ell^{(2)}$, it gives

$$\begin{aligned}\mathcal{A}^{\text{tree}}_{5,\text{MHV}}\big(I_{5,\text{box}}\big|_{(1)} + I_{5,\text{pentagon}}\big|_{(1)}\big) &= \mathcal{A}^{\text{tree}}_{5,\text{MHV}} \times (1+0) = \mathcal{A}^{\text{tree}}_{5,\text{MHV}} = \text{LS}_{(1)}\,,\\ \mathcal{A}^{\text{tree}}_{5,\text{MHV}}\big(I_{5,\text{box}}\big|_{(2)} + I_{5,\text{pentagon}}\big|_{(2)}\big) &= \mathcal{A}^{\text{tree}}_{5,\text{MHV}} \times (1-1) = 0 = \text{LS}_{(2)}\,.\end{aligned} \tag{8.34}$$

So it produces the correct Leading Singularities for the cut (8.21)!

Now, unfortunately we are not done yet, because we have to worry about all the other cuts: there are a total of $2 \times 5 = 10$ Leading Singularities for the 5-point 1-loop amplitude. With just the box diagram and the pentagon diagram in (8.33), there is no chance that this can be the full answer: the reason is simply that the sum of those two integrals is not cyclically invariant. It takes just one more integral to achieve cyclic invariance, namely the box integral $I_{5,\text{box}}(3,4,5,1)$. Diagrammatically, we can express the final answer as

$$\begin{aligned}\mathcal{A}^{\text{1-loop}}_{5,\text{MHV}} &= \mathcal{A}^{\text{tree}}_{5,\text{MHV}} \times \Big(I_{5,\text{box}}(1,2,3,4) + I_{5,\text{pentagon}} + I_{5,\text{box}}(3,4,5,1)\Big)\\ &= \mathcal{A}^{\text{tree}}_{5,\text{MHV}} \Bigg(\quad + \quad + \quad\Bigg).\end{aligned} \tag{8.35}$$



The diagrammatic notation for the pentagon integral has a wavy line indicating that the bi-twistor $(1,2,3)\bigcap(3,4,5)$ goes in the numerator in (8.31). Our previous results plus cyclic invariance then guarantee that (8.35) produces all ten Leading Singularities correctly.

▶ **Exercise 8.4**

Show that (8.32) is true.

▶ **Exercise 8.5**

Show that (8.35) is invariant under cyclic permutations of the external lines.

We have introduced here box and pentagon 1-loop integrands, (8.26) and (8.31), whose quadruple cuts evaluate to either $+1$, -1 or 0. Such integrands are called ***unit Leading Singularity integrands***.

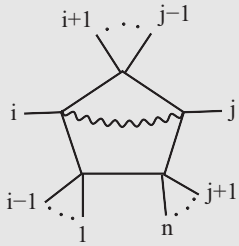
It has been shown [119] that all planar loop amplitudes of $\mathcal{N}=4$ SYM can be obtained as a linear combination of unit Leading Singularity integrands (times a tree amplitude). The coefficients in front of the unit Leading Singularity integrands are determined by the Leading Singularity, thus knowing them is sufficient to determine the entire amplitude. For planar amplitudes, we need unit Leading Singularity integrands that are also dual conformal invariant and local, and this is a rather restrictive class of integrands. We have seen the Leading Singularity method at work for 4- and 5-point 1-loop amplitudes. The structure generalizes to higher points, as we now outline.

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6-point and beyond

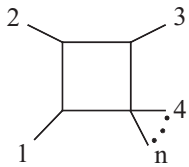
The 1-loop n-point MHV amplitude is given by a simple generalization of the 5-point result:

$$\mathcal{A}^{\text{1-loop}}_{n,\text{MHV}} = \mathcal{A}^{\text{tree}}_{n,\text{MHV}} \left(\sum_{1<i<j<n} \right.$$



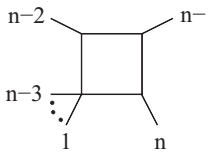
$$\left. \right) = \int_{A,B} \sum_{1<i<j<n} \frac{\langle A,B|(i-1,i,i+1)\bigcap(j-1,j,j+1)\rangle\langle i,j,n,1\rangle}{\langle A,Bi,i-1\rangle\langle AB,i,i+1\rangle\langle A,B,j-1,j\rangle\langle A,Bj,j+1\rangle\langle A,B,n,1\rangle}\,. \tag{8.36}$$

In the sum, there are two boundary cases: $i = 2$, $j = 3$ and $i = n-2$, $j = n-1$. These correspond to box integrals whose numerators are simply the Jacobian coming from cutting all four propagators. More precisely, we have



$$= \int_{A,B} \frac{\langle n123\rangle\langle 1234\rangle}{\langle AB12\rangle\langle AB23\rangle\langle AB34\rangle\langle ABn1\rangle} \tag{8.37}$$

and



$$= \int_{A,B} \frac{\langle n-3,n-2,n-1,n\rangle\langle n-2,n-1,n,1\rangle}{\langle A,B,n-3,n-2\rangle\langle A,B,n-2,n-1\rangle\langle A,B,n-1,n\rangle\langle A,B,n,1\rangle}\,. \tag{8.38}$$

In conclusion, the two Leading Singularities of arbitrary 1-loop MHV amplitudes can be reproduced by including the simple combination of tensorial (due to the loop-momentum dependence in the numerator) pentagon integrals. These are all local unit Leading Singularity integrands. Note that these integrands can be used as part of the basis for 1-loop amplitudes in any massless quantum field theory. The special situation for $\mathcal{N} = 4$ SYM is that these integrals provide the entire answer, whereas for a generic QFT, one needs in addition the various lower-gon integrals that are not captured by the maximal cuts.

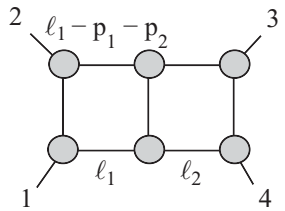
You may (and should) be puzzled that in the beginning of Section 6.2, we stated that the 1-loop amplitudes in a unitary 4d quantum field theory can be expanded on a basis of scalar box-, triangle-, and bubble-integrals with possible additional input from rational terms. This was summarized in equation (6.26), and we noted that in $\mathcal{N} = 4$ SYM, the

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only non-vanishing contributions were the box-integrals. There were no pentagons in that story! So what is the deal? The point of the pentagon integrals in the present section is that they allow us to write the $\mathcal{N}=4$ SYM 1-loop integrand in a form that reproduces each Leading Singularity faithfully. On the other hand, (6.26) determines the 1-loop $\mathcal{N}=4$ SYM amplitudes as a sum of box-integrals whose coefficients are evaluated by quadruple cuts, evaluated as the *average* of the two loop-constraint solutions. More box-diagrams contribute in (6.26) than in (8.36). So what *is* the deal? Well, the two representations of the integrand must yield the same answer for the amplitude. The integrals have to be regulated, and if one uses dimensional regularization $4-2\epsilon$, the difference between the two representations is only in the $O(\epsilon)$-terms. Specifically, the pentagon integrals contain the information about the "missing" boxes plus $O(\epsilon)$ [121]. Thus the two procedures yield the same integrated answer.

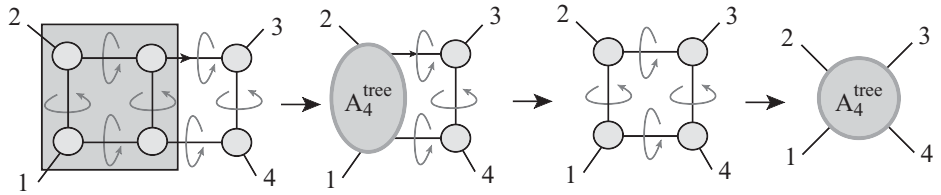
8.2 2-loop Leading Singularities

At the start of this chapter we noted that not all loop-diagrams appear to have enough propagators available for a maximal cut of $4L$-lines. A representative example is the double-box diagram of the 2-loop 4-point amplitude:



(8.39)

With seven propagators, we can only localize seven of the eight components of the two loop-momenta, leaving behind a 1-dimensional loop-integral. However, when the seven propagators are on-shell, the 4-point 1-loop analysis tells us that the lefthand box in (8.39) is a Leading Singularity that equals the 4-point tree amplitude, $A_4^{\text{tree}}[\ell_2, p_1, p_2, \ell_2 - p_1 - p_2]$, where ℓ_2 parameterizes the loop-momentum in the righthand box. But this tree amplitude has a propagator $1/(\ell_2 - p_1)^2$ that can now be used to localize the final component of the loop-momenta, thus providing a maximal cut. Moreover, on this pole, the 4-point tree amplitude factorizes into two 3-point amplitudes, and therefore the result is simply an on-shell 4-point 1-loop box diagram. The procedure is illustrated here:



(8.40)

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The last step uses the fact that the 1-loop 4-point Leading Singularity is equal to the tree amplitude $A_4^{\text{tree}}[1234]$. In conclusion:

(i) the Leading Singularities are well-defined for the 2-loop 4-point amplitude, even though the double-box only has seven loop propagators, and
(ii) the 4-point "double-box Leading Singularity" equals the 4-point tree amplitude.

A Leading Singularity that involves a "pole-under-a-pole" is called a ***composite Leading Singularity***. A prototype of such a composite object is the 3-variable contour integral

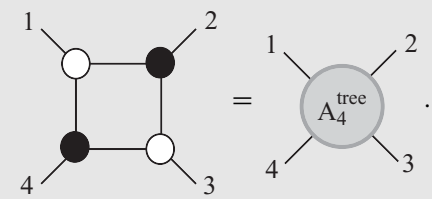
$$\oint dx\, dy\, dz\, \frac{1}{x(x+yz)}\,. \tag{8.41}$$

This integrand appears to have only two poles, insufficient to localize the 3d integral. However, if the x-contour encircles the pole at $x = 0$, then an additional pole emerges in the form $1/(yz)$ and this can then localize the remaining two integrals, giving the residue 1 (ignoring $2\pi i$s).

The idea of composite Leading Singularities resolves the subtlety about defining maximal cuts and Leading Singularities for higher-loop amplitudes. Henceforth we work with the understanding that the Leading Singularities of multi-loop amplitudes are always well-defined.

8.3 On-shell diagrams

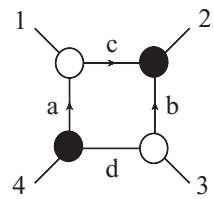
We have found in Section 8.1 that the 4-point 1-loop Leading Singularity is equal to the 4-point tree amplitude:



(8.42)

This looks rather peculiar since the LHS is a 1-loop diagram while the RHS is a tree amplitude. It actually turns out that the LHS can be interpreted as a super-BCFW diagram! We now show how.

Consider the top two vertices in the Leading Singularity diagram



$$|a\rangle \propto |c\rangle \propto |1\rangle\,,$$
$$|b] \propto |c] \propto |2]\,. \tag{8.43}$$

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The MHV and anti-MHV designations imply the indicated special 3-particle kinematics. Up to an overall factor z, this determines p_c to be $p_c = -z|1\rangle[2|$. Momentum conservation then fixes p_a and p_b to be

$$p_a = -|1\rangle\big([1| + z[2|\big) \quad \text{and} \quad p_b = -\big(|2\rangle - z|1\rangle\big)[2|\,. \tag{8.44}$$

We recognize p_a and p_b as BCFW $[1,2\rangle$-shifted momenta $\hat{p}_1$ and $\hat{p}_2$!

What about the Grassmann variables? Let us carry out the η_c-integral in the product of the Grassmann delta functions of the first two vertices,

$$\begin{aligned}&\int d^4\eta_a\, d^4\eta_b\, d^4\eta_c\, \delta^{(4)}\big([1c]\eta_a + [ca]\eta_1 + [a1]\eta_c\big)\, \delta^{(8)}\big(|2\rangle\eta_2 - |b\rangle\eta_b - |c\rangle\eta_c\big)\\ &\propto \int d^4\eta_a\, d^4\eta_b\, \delta^{(8)}\Big(|1\rangle\big(\eta_a - (\eta_1 - z\eta_b)\big) - |2\rangle\big(\eta_b - \eta_2\big)\Big)\,.\end{aligned} \tag{8.45}$$

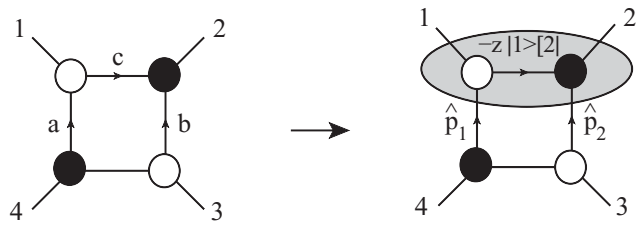
The last integral localizes η_a and η_b to be

$$\eta_a = \eta_1 - z\eta_2 \quad \text{and} \quad \eta_b = \eta_2\,. \tag{8.46}$$

This is exactly the shift of the Grassmann variables associated with the supersymmetrization of the BCFW shift (8.44).

Finally, let us see how the internal line d in (8.43) fixes z. The on-shell condition is $0 = p_d^2 = (p_3 + p_b)^2 = (\langle 23\rangle - z\langle 13\rangle)[23]$, i.e. $z = \langle 23\rangle/\langle 13\rangle$. This corresponds exactly to the pole where the propagator $1/\hat{p}_{23}$ in the $[1,2\rangle$-shifted 4-point tree amplitude goes on-shell. As we know from the super-BCFW calculation (4.56), this is exactly the factorization pole that allows us to determine the full 4-point tree amplitude in $\mathcal{N} = 4$ SYM from the $\text{MHV}_3 \times$ anti-MHV_3 super-BCFW diagram.

We have established the connection between the Leading Singularity diagram on the LHS of (8.42) and the 4-point super-BCFW diagram (4.56), and this allows us to understand why the 4-point 1-loop Leading Singularity is just the 4-point tree superamplitude. The connection is summarized diagrammatically as the ***BCFW bridge***

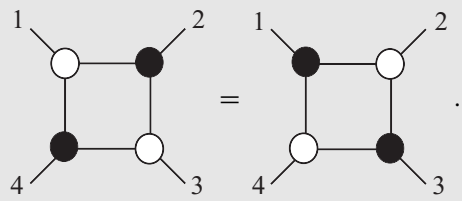


(8.47)

where the upper two vertices, surrounded by the shaded region, are the "bridge." The bridge provides the BCFW super-shift.

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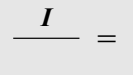
Exchanging black and white dots in the BCFW bridge simply corresponds to the conjugate BCFW shift. This also gives another meaning to the ***square move*** (8.20):



(8.48)

It says that the two BCFW super-shifts $[1, 2\rangle$ and $[2, 1\rangle$ give the same 4-point amplitude.

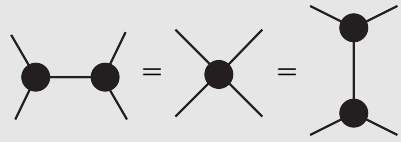
With the square move and the BCFW shift, it becomes fun to calculate Leading Singularities. Starting from the fundamental 3-point vertices, we can build ***on-shell diagrams*** that contain information about the higher-loop amplitudes. Each 3-point vertex represents the MHV or anti-MHV 3-point amplitude, along with the implication that the square or angle spinors of its legs are proportional. The vertices are glued together by "on-shell propagators" whose rules can be written



$$= \int \frac{d^2|I\rangle d^2|I] d^4\eta_I}{U(1)} . \tag{8.49}$$

The η_I integral is the usual state sum. The integration over the momentum variables will be localized by the momentum conservation delta function on both sides of the propagator.

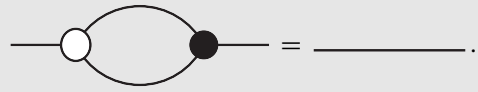
In addition to the square move (8.48), there are two rules that help us simplify complicated on-shell diagrams. The first rule follows from the observation that each MHV 3-vertex imposes that the square spinors of the associated lines are proportional, so two consecutive MHV vertices imply that all four square spinors are proportional. This gives the rule



(8.50)

There is of course an equivalent rule for anti-MHV. The black 4-vertex blob imposes, per definition, that the four lines have proportional square spinors. This blob does not represent a 4-point MHV tree amplitude; it is just a short-hand notation for the double-blob diagrams.

The second rule is

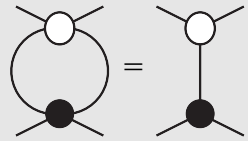


(8.51)

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The 3-particle kinematics forces the internal lines in the bubble to be collinear, and this collapses the bubble. This formally eliminates a loop-integral.

Combining the two rules (8.50) and (8.51) gives

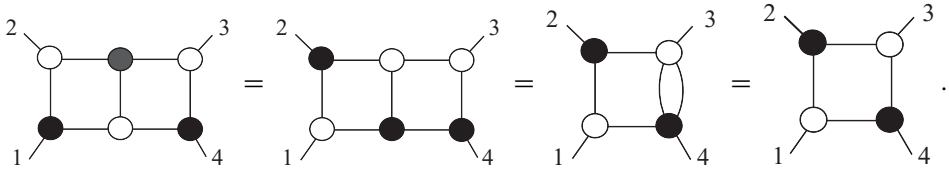


(8.52)

► **Exercise 8.6**

Show that the internal lines in the bubble (8.51) are collinear and that (8.52) follows from (8.50) and (8.51).

The point of these rules is to simplify the evaluation of on-shell diagrams. For the on-shell diagram of the 2-loop 4-point Leading Singularity, we first apply the square move (8.48) and then the collapse-moves (8.50) and (8.52) to get



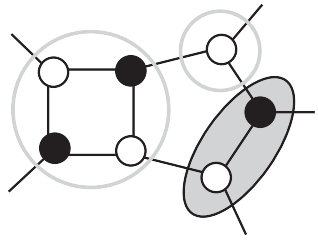
(8.53)

This shows that the 4-point 2-loop Leading Singularity equals the 4-point 1-loop Leading Singularity, which in turn is just the 4-point tree amplitude. We had already found this result in Section 8.2 by evaluating the composite Leading Singularity. The rules for the on-shell diagrams offer a simpler diagrammatical derivation.

► **Exercise 8.7**

Write down 3-loop on-shell diagrams for the 4-point MHV amplitude and show that they reduce to the 1-loop result.

Using the BCFW bridge, we can begin to build up more complicated on-shell diagrams. For example, we can use the BCFW bridge to interpret the on-shell diagram:



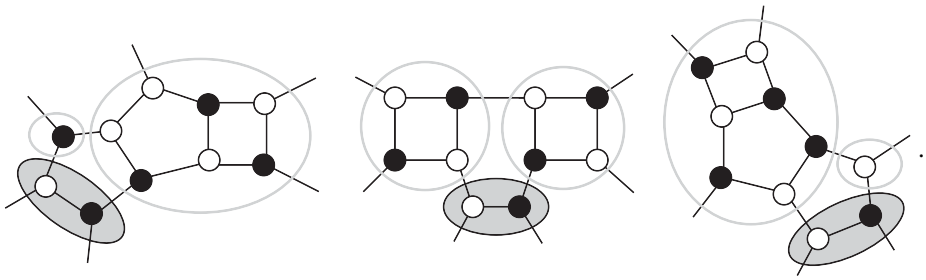
(8.54)

The circles highlight the 4-point MHV tree amplitude and 3-point anti-MHV tree subamplitudes. The BCFW bridge, indicated with the shaded area, induces the BCFW shift on the

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two affected lines, so this on-shell diagram represents the BCFW diagram for the 5-point MHV tree-level superamplitude.

The super-BCFW recursion relations for the 6-point NMHV tree amplitude can be represented with on-shell diagrams as



(8.55)

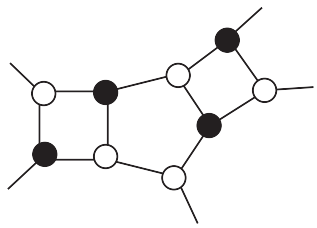
The left-hand diagram is the $\text{MHV}_3 \times$ anti-MHV_5 BCFW diagram, the middle diagram is the BCFW diagram with two 4-point MHV tree subamplitudes and the right-hand diagram is the $\text{MHV}_5 \times$ anti-MHV_3 BCFW diagram.

▶ Exercise 8.8

Do the η-counting to show that (8.54) represents an on-shell diagram for an MHV amplitude and (8.55) an NMHV amplitude.

▶ Exercise 8.9

Interpret the effect of the rule (8.50) on the on-shell diagram (8.54). Show that the 3-loop diagram



(8.56)

is equivalent to (8.54). What does that tell you about Leading Singularities?

The concept of the Leading Singularity is well-defined for lower-$\mathcal{N}$ SYM and for non-supersymmetric theories, and so are the on-shell diagrams. The only distinction is that the edges need arrows because the on-shell states split into two CPT conjugate multiplets, one with the positive helicity gluon and the other with the negative helicity gluon. ($\mathcal{N} = 4$ SYM is special in that its supermultiplet is CPT self-conjugate.) Opposite helicity multiplets must sit at different ends of the propagators, so they are dressed with arrows to indicate the assignment.

In the next section, we discuss a formula that reproduces all Leading Singularities in planar $\mathcal{N} = 4$ SYM. This formula will tell us that the number of distinct Leading Singularities for given n and given N^KMHV-level is fixed. For example, for $K = 0$ (MHV) there

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is only one Leading Singularity: that is why both the 1- and 2-loop Leading Singularities for the 4-point amplitude evaluate to the same value, namely the 4-point tree amplitude. That this pattern continues is rather remarkable, since it says that on-shell diagrams with four external lines when evaluated all give the 4-point tree amplitude, no matter how many hundreds of loops we add in.

The problem of determining and classifying all distinct on-shell diagrams, under the equivalence-moves, turns out to be an interesting mathematical problem that has become an exciting research topic [120].

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