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Global cross sections for Anosov flows

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Abstract. We provide a new criterion for the existence of a global cross section to a volume-preserving Anosov flow. The criterion is expressed in terms of expansion and contraction rates of the flow and generalizes known results of this type.

1. *Introduction*

Henri Poincaré introduced the idea of a cross section to a flow to study the 3-body problem. A global cross section to a flow Φ on a manifold M is a codimension-one submanifold Σ of M such that Σ intersects every orbit of Φ transversely. It is natural to ask whether any given non-singular flow admits one.

If Σ is a global cross section for Φ, it is not hard to check that every orbit which starts on Σ returns to Σ after some positive time, defining the Poincaré first-return map $g : \Sigma \to \Sigma$. The analysis of Φ can then be reduced to the study of the map g, which in principle can be an easier task. The flow can be reconstructed from the Poincaré map by suspending it (cf. [**KH95**]).

The object of this paper is to investigate the existence of global cross sections to volume-preserving Anosov flows.

Recall that a non-singular flow $\Phi = \{f_t\}$ on a closed (compact and without boundary) Riemannian manifold M is called *Anosov* if there exists an invariant splitting $TM = E^{ss} \oplus E^c \oplus E^{uu}$ of the tangent bundle of M and uniform constants $c > 0$, $0 < \mu_- \leq \mu_+ < 1$ and $\lambda_+ \geq \lambda_- > 1$ such that the *center bundle* E^c is spanned by the infinitesimal generator X of the flow and for all $v \in E^{ss}$, $w \in E^{uu}$, and $t \geq 0$, we have

$$\frac{1}{c}\mu_-^t \|v\| \leq \|Tf_t(v)\| \leq c\mu_+^t \|v\| \tag{1.1}$$

and

$$\frac{1}{c}\lambda_-^t \|w\| \leq \|Tf_t(v)\| \leq c\lambda_+^t \|w\|, \tag{1.2}$$

where Tf_t denotes the derivative (or tangent map) of f_t. We call E^{ss} and E^{uu} the *strong stable* and *strong unstable bundles*; $E^{cs} = E^c \oplus E^{ss}$ and $E^{cu} = E^c \oplus E^{uu}$ are called

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the *center stable* and *center unstable bundles*. It is well known [**HPS77**, **Has94**] that all of them are Hölder continuous and uniquely integrable [**Ano67**]. The corresponding foliations will be denoted by W^{ss}, W^{uu}, W^{cs}, and W^{cu}. They are also Hölder continuous in the sense that each one admits Hölder foliation charts. This means that if W^{σ} ($\sigma \in \{ss, uu, cs, cu\}$) is C^{θ}, then every point in M lies in a C^{θ} chart (U, φ) such that in U the local W^{σ}-leaves are given by $\varphi_{k+1} = \text{constant}, \ldots, \varphi_n = \text{constant}$, where $\varphi = (\varphi_1, \ldots, \varphi_n)$ is a C^{θ} homeomorphism and k is the dimension of W^{σ}. The leaves of all invariant foliations are as smooth as the flow. See also [**PSW97**] for a discussion of regularity of Hölder foliations.

1.1. *Related work.* For general flows on closed manifolds Schwartzman [**Sch57**] showed that there exists a global cross section if and only if there is *no* fully supported flow invariant ergodic probability measure whose asymptotic cycle is trivial. Fried [**Fri82**] reformulated the work of Schwartzman in the language of so-called homology directions and essentially showed that a general C^1 flow on a closed manifold admits a global cross section if and only if all the homology classes (modulo torsion) are represented by closed orbits lying in an open half space in $H_1(M, \mathbb{R})$. Such results specializing to Anosov flows were first proved by Plante in [**Pla72**]. He showed that if $E^{ss} \oplus E^{uu}$ is a uniquely integrable distribution or, equivalently, if the foliations W^{ss} and W^{uu} are jointly integrable†, then the Anosov flow admits a global cross section. Sharp [**Sha93**] showed that a transitive Anosov flow admits a global cross section if and only if the functional $\beta : H^1_{\text{de Rham}}(M, \mathbb{R}) \to \mathbb{R}$ defined by

$$\beta(\omega) = \sup\left\{ h(\mu) + \int_M \omega(X)\, d\mu : \mu \text{ is an invariant probability measure} \right\}$$

is unbounded from below. (Here $h(\mu)$ denotes the entropy of μ and X is the infinitesimal generator of the flow.) Along different lines Bonatti and Guelman [**BG09**] showed that if the time-one map of an Anosov flow can be C^1 approximated by Axiom A diffeomorphisms, then the flow is topologically equivalent to the suspension of an Anosov diffeomorphism.

Let $n_s = \dim E^{ss}$ and $n_u = \dim E^{uu}$. If $n_u = 1$ or $n_s = 1$, the Anosov flow is said to be of codimension one. In this paragraph we always assume that $n_u = 1$. In [**Ghy89**] Ghys proved the existence of global cross sections for codimension-one Anosov flows in the following cases: (1) if $E^{su} = E^{ss} \oplus E^{uu}$ is C^1 and $n \geq 4$ (in this case the global cross section has constant return time); (2) if the flow is volume-preserving, $n \geq 4$ and W^{cs} is of class C^2. This was generalized by the author in [**Sim96**, **Sim97**] where we showed that a codimension-one Anosov flow admits a global cross section if any of the following assumptions is satisfied: (1) E^{su} is Lipschitz (in the sense that it is locally spanned by Lipschitz vector fields) and $n \geq 4$; (2) the flow is volume-preserving, $n \geq 4$, and E^{su} is C^{θ}-Hölder for *all* $\theta < 1$ (3) the flow is volume-preserving, $n \geq 4$, and E^{cs} is of class $C^{1+\theta}$ for *all* $\theta < 1$. Note that all the regularity assumptions above require that the invariant bundles be smoother than they usually are: E^{su} is generically only Hölder continuous and,

† This means that locally speaking the W^{uu}-holonomy between local W^{cs}-leaves takes local W^{ss}-leaves to W^{ss}-leaves.

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in the codimension-one case, E^{cs} is generically only $C^{1+\theta}$ for some small $0 < \theta < 1$. See [**Has94**, **Has97**, **HW99**].

The goal of this paper is to establish the following result.

THEOREM. *Let $\Phi = \{f_t\}$ be a volume-preserving Anosov flow on a closed Riemannian manifold M and let $0 < \theta \leq 1$ be the smaller of the Hölder exponents of W^{ss} and W^{cs}. If*

$$\mu_+^{(n_s-1)\theta}\lambda_+^{(n_u-1)\theta} < \mu_-^{2(1-\theta)}, \tag{1.3}$$

then Φ admits a global cross section.

Remarks.

(a) Condition (1.3) has a chance of being satisfied only if n_u is much smaller than n_s. If $n_u > n_s$, then by reversing time it is easy to show that

$$\lambda_+^{2(1-\alpha)} < \lambda_-^{(n_u-1)\alpha}\mu_-^{(n_s-1)\alpha}$$

also implies the existence of a global cross section, where α is the minimum of the Hölder exponents of W^{uu} and W^{cs}.

(b) If the flow is of codimension one with $n_u = 1$, then (1.3) reduces to

$$\mu_+^{(n-3)\theta} < \mu_-^{2(1-\theta)}. \tag{1.4}$$

It is well known (cf. [**Has94**, **HPS77**]) that the center stable bundle E^{cs} and strong unstable bundle E^{uu} of a volume-preserving Anosov flow in dimensions $n \geq 4$ are both $C^{1+\text{Hölder}}$. Thus if E^{su} is Lipschitz as in [**Sim96**] or C^θ, for all $\theta < 1$, as in [**Sim97**], then (1.4) is clearly satisfied. If E^{cs} is $C^{1+\theta}$ for all $\theta < 1$ as in [**Sim97**], then it is not hard to show that E^{ss} is necessarily of class C^θ for all $\theta < 1$, which again implies (1.4). Therefore, in the case of volume-preserving codimension-one Anosov flows, our result implies all the previously known criteria for the existence of global cross sections.

(c) In the early 1970s, prompted by a dearth of examples, Verjovsky conjectured [**Ver74**] that every codimension-one Anosov flow in dimensions $n \geq 4$ admits a global cross section. The importance of Verjovsky's conjecture stems from the fact that codimension-one Anosov diffeomorphisms were classified by Franks [**Fra70**] and Newhouse [**New70**] who showed that every such diffeomorphism is topologically conjugate to a linear hyperbolic automorphism of a torus. Therefore, the affirmation of Verjovsky's conjecture would yield a complete classification of codimension-one Anosov flows in dimensions $n \geq 4$.

Progress towards Verjovsky's conjecture was made in the early 1980s by Plante [**Pla81**, **Pla83**] and Armendariz [**Arm82**] who showed that the conjecture holds if the fundamental group of the manifold is solvable. By the work of Asaoka [**Asa08**] it follows that it suffices to prove the conjecture for volume-preserving flows, since any topologically transitive codimension-one Anosov flow is topologically equivalent to a volume-preserving one. As of this writing, the conjecture remains open.

Throughout this paper smooth will mean of class C^∞.

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1.2. *Outline of the proof.* The main idea of the proof of the theorem is to find a smooth closed 1-form η such that $\eta(X) > 0$, where X is the infinitesimal generator of the Anosov flow. It is not hard to see that this immediately implies the existence of a global cross section (cf. §3). To construct η, we use the fact that for any k-form ξ, the C^0-distance from ξ to the space of closed k-forms is bounded above by the C^0 norm $\|d\xi\|$; see Proposition 2.4. It therefore suffices to construct a smooth 1-form ξ such that $|\xi(X_p)| > \|d\xi\|$, for all $p \in M$, since Proposition 2.4 then yields a smooth closed 1-form η such that $\eta(X) > 0$. We will actually construct a smooth 1-form ξ such that $\xi(X) = 1$ and $\|d\xi\| < 1$.

The construction of ξ is divided into two steps. In the first step, we find an initial candidate for ξ such that the norm of its exterior derivative restricted to E^{cs} is small, while its total norm blows up in a way controlled by the Hölder exponent θ. More precisely, for each $\varepsilon > 0$ we construct a smooth 1-form ξ_0^ε on M such that $\xi_0^\varepsilon(X) = 1$, $\|d\xi_0^\varepsilon \restriction_{E^{cs}}\| \le D\varepsilon^\theta$, and $\|d\xi_0^\varepsilon\| \le K\varepsilon^{\theta-1}$, where D and K are positive constants independent of ε. This is achieved by carefully building smooth local cross sections and the corresponding flow boxes; cf. §2.2.

In the second step, we pull back ξ_0^ε by f_{-t} for suitable $t > 0$ to make the norm of its exterior derivative small in the remaining directions. For this, we use an estimate (see Lemma 2.5) on the growth of $\|Tf_{-t}(v \wedge w)\|$, for $t > 0$, $v \in E^{ss}$ and $w \in E^{uu}$. Assuming (1.3), we then show that there exist $\varepsilon > 0$ and $t > 0$ such that $\xi = f_{-t}^* \xi_0^\varepsilon$ has the desired properties.

2. *Preliminaries*

2.1. *Regularization.* Here we recall a standard technique for approximating locally integrable functions by smooth ones called *regularization* or *mollification* (see [**Eva98**, **Ste70**]). Define the standard mollifier $\eta : \mathbb{R}^n \to \mathbb{R}$ by

$$\eta(x) = \begin{cases} A_0 \exp\left(\dfrac{1}{|x|^2 - 1}\right) & \text{if } |x| < 1, \\ 0 & \text{if } |x| \ge 1, \end{cases}$$

where A_0 is chosen so that $\int \eta \, dx = 1$. For every $\varepsilon > 0$, set $\eta_\varepsilon(x) = \varepsilon^{-n}\eta(x/\varepsilon)$. Note that the support of η_ε is contained in the ball $B(0, \varepsilon)$ of radius ε centered at the origin and that $\int \eta_\varepsilon \, dx = 1$. For a locally integrable function $u : \mathbb{R}^n \to \mathbb{R}$ define

$$u^\varepsilon(x) = (u * \eta_\varepsilon)(x) = \int_{\mathbb{R}^n} u(y)\eta_\varepsilon(x - y)\, dy = \int_{\mathbb{R}^n} u(x - y)\eta_\varepsilon(y)\, dy.$$

PROPOSITION 2.1. *Assume that $u : \mathbb{R}^n \to \mathbb{R}$ is locally integrable. Then:*

(a) $u^\varepsilon \in C^\infty(\mathbb{R}^n)$.

(b) *If $u \in L^\infty$, then $\|u^\varepsilon\|_{L^\infty} \le \|u\|_{L^\infty}$.*

(c) *If u is continuous, then $u^\varepsilon \to u$, uniformly as $\varepsilon \to 0$. If $u \in C^\theta$ $(0 < \theta \le 1)$, then $\|u^\varepsilon - u\|_{C^0} \le \|u\|_{C^\theta}\varepsilon^\theta$, where the C^θ-norm is the sum of its sup-norm and its best Hölder constant.*

(d) *If $u \in C^\theta$, then*

$$\|du^\varepsilon\| \le \|d\eta\|_{L^1}\|u\|_{C^\theta}\varepsilon^{\theta-1}$$

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where $\|d\eta\|_{L^1} = \max_i \int_{\mathbb{R}^n} |\partial\eta/\partial x_i|\, dx$ *and* $\|du^\varepsilon\|$ *denotes the maximum of the sup-norms of the partial derivatives of* u^ε.

Proof. Proofs of (a), (b) and the first part of (c) can be found in [**Eva98**]. For the second part of (c), we have

$$\begin{aligned} |u^\varepsilon(x) - u(x)| &= \left| \int_{B(0,\varepsilon)} \eta_\varepsilon(y)[u(x-y) - u(x)]\, dy \right| \\ &\le \|u\| \varepsilon^\theta \int_{B(0,\varepsilon)} \eta_\varepsilon(y)\, dy \\ &= \|u\|_{C^\theta} \varepsilon^\theta. \end{aligned}$$

If $u \in C^1$, then the same estimates hold with θ replaced by 1.

Observe that since η_ε has compact support,

$$\int_{\mathbb{R}^n} \frac{\partial \eta_\varepsilon}{\partial x_i}(y)\, dy = 0, \tag{2.1}$$

for $1 \le i \le n$. Note also that

$$\frac{\partial \eta_\varepsilon}{\partial x_i}(x) = \frac{1}{\varepsilon^{n+1}} \frac{\partial \eta}{\partial x_i}\left(\frac{x}{\varepsilon}\right).$$

Assuming $u \in C^\theta$, we obtain (d):

$$\begin{aligned} \left| \frac{\partial u^\varepsilon}{\partial x_i}(x) \right| &= \left| \int_{\mathbb{R}^n} u(x-y) \frac{\partial \eta_\varepsilon}{\partial x_i}(y)\, dy \right| \\ &\overset{\text{by (2.1)}}{=} \left| \int_{B(0,\varepsilon)} [u(x-y) - u(x)] \frac{\partial \eta_\varepsilon}{\partial x_i}(y)\, dy \right|, \\ &\le \|u\|_{C^\theta}\, \varepsilon^\theta \int_{B(0,\varepsilon)} \left| \frac{\partial \eta_\varepsilon}{\partial x_i}(y) \right| dy \\ &= \|u\|_{C^\theta}\, \varepsilon^\theta \int_{B(0,\varepsilon)} \frac{1}{\varepsilon^{n+1}} \left| \frac{\partial \eta}{\partial x_i}\left(\frac{y}{\varepsilon}\right) \right| dy \\ &\overset{z=\frac{y}{\varepsilon}}{=} \|u\|_{C^\theta}\, \varepsilon^\theta \cdot \frac{1}{\varepsilon} \int_{B(0,1)} \left| \frac{\partial \eta}{\partial x_i}(z) \right| dz, \\ &\le \|d\eta\|_{L^1} \|u\|_{C^\theta}\, \varepsilon^{\theta-1}. \end{aligned}$$

□

COROLLARY 2.2. *Let* M *be a compact manifold without boundary. Fix a finite atlas* $\mathscr{A} = \{(U_i, \varphi_i) : i \in I\}$ *of* M. *If* $u : U \to \mathbb{R}$ *is* C^θ *and* $U \subset U_j$ *for some* $j \in I$, *then there exist* $\varepsilon_* > 0$ *and a family of smooth approximations* u^ε $(0 < \varepsilon < \varepsilon_*)$ *of* u *such that:*

(a) u^ε *is defined on* $U^\varepsilon \subset U$ *where* $U^\varepsilon = \varphi_j^{-1}\{x \in \varphi_j(U) : d(x, \partial\varphi_j(U)) > \varepsilon\}$;

(b) $\|u^\varepsilon - u\|_{C^0} \le \kappa\varepsilon^\theta$;

(c) $\|du^\varepsilon\|_{C^0} \le \kappa\varepsilon^{\theta-1}$.

Here $\kappa > 0$ *depends only on* $\mathscr{A}$ *and* $\|u\|_{C^\theta}$.

Proof. The family $u^\varepsilon = [(u \circ \varphi_j^{-1}) * \eta_\varepsilon] \circ \varphi_j$ has the desired properties. We can take ε_* to be any positive number such that for $\varepsilon < \varepsilon_*$, the sets U^ε defined above are non-empty. □

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2.2. *Construction of local cross sections.* In this section we construct a finite covering of the manifold by smooth flow boxes defined by carefully chosen smooth local cross sections.

Let $p \in M$ be arbitrary and choose a W^{ss}-foliation chart (U, φ) containing p. This means that $\varphi : U \to \mathbb{R}^n$ is a C^θ-homeomorphism such that the local W^{ss}-leaves in U are given by

$$\varphi_{n_s+1} = \text{constant}, \ldots, \varphi_n = \text{constant},$$

where $\varphi = (\varphi_1, \ldots, \varphi_n)$. We can also arrange that the local W^{cs}-leaves in U are defined by

$$\varphi_{n_s+1} = \text{constant}, \ldots, \varphi_{n-1} = \text{constant},$$

so that, in each local W^{cs}-leaf in U, W^{ss} is given by $\varphi_n = \text{constant}$. Furthermore, we can take U so that its closure is contained in a *smooth* chart for M. This condition will allow us to modify continuous functions defined on U without having to shrink the domain.

Even though φ is only Hölder, the flow invariance of W^{ss} implies that each φ_i is differentiable with respect to X. Since X is tangent to the W^{cs} leaves, it follows that $X\varphi_i = 0$, for $n_s + 1 \le i \le n - 1$. Furthermore, the restriction of W^{ss} to W^{cs}-leaves is as smooth as the flow, so φ_n is smooth on the local W^{cs}-leaves in U. Since X is uniformly transverse to W^{ss} it is clear that $X\varphi_n \neq 0$ and by continuity there exists $\delta > 0$ such that $|X\varphi_n| \ge \delta$ on U.

Define

$$\Sigma_0 = \bigcup_{x \in W^{uu}_{\text{loc}}(p)} W^{ss}_{\text{loc}}(x),$$

where $W^\sigma_{\text{loc}}(x)$ denotes the local W^σ-leaf in U (for $\sigma \in \{uu, ss\}$). Then Σ_0 is a Hölder continuous local cross section for the flow. (This makes sense, since the intersection of Σ_0 with each local W^{cs}-leaf is a local W^{ss}-leaf, hence smooth and transverse to the flow.) Let Σ be a slightly smaller compact subset of Σ_0 containing p.

We claim that there exists $T > 0$, depending on Σ and U, such that

$$V \stackrel{\text{def}}{=} \bigcup_{|t|<T} f_t(\Sigma)$$

is a Hölder continuous flow box contained in U. Assume the contrary; it follows that the map $(t, q) \mapsto f_t(q)$ fails to be one-to-one on $(-T, T) \times \Sigma$, for any $T > 0$. Thus there exist sequences (q_k) and (T_k) such that $q_k \in \Sigma$, $T_k > 0$, $T_k \to 0$ and $f_{T_k}(q_k) \in \Sigma$, for all k. Since $f_{T_k}(q_k)$ and q_k lie in the same local W^{cs}-leaf, it follows that $\varphi_n(f_{T_k}(q_k)) = \varphi_n(q_k)$. On the other hand, by compactness of Σ, (q_k) has a subsequence (q_{k_j}) which converges to some $q \in \Sigma_0$. This implies $X\varphi_n(q) = 0$, which contradicts $|X\varphi_n| \ge \delta$. Therefore, T exists; we will call it the *length* of the continuous flow box V (although a more appropriate name would be half-length).

Define $\tau : V \to \mathbb{R}$ by

$$\tau(f_t q) = t,$$

for $q \in \Sigma$ and $|t| < T$. It is clear that τ is C^θ, $X\tau = 1$ and τ is constant on the local W^{ss}-leaves in V.

Next we approximate Σ, τ and V by smooth objects with similar properties.

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LEMMA 2.3. *There exists $\varepsilon_* > 0$ such that for every $0 < \varepsilon < \varepsilon_*$ there exist an open set $V^\varepsilon \subset V$ and a smooth function $\tau^\varepsilon : V^\varepsilon \to \mathbb{R}$ with the following properties:*

(a) $X\tau^\varepsilon = 1$.

(b) $|d_q\tau^\varepsilon(v)| \le A\varepsilon^\theta\|v\|$, *for all $q \in V^\varepsilon$ and $v \in E^{ss}$, where A is a constant independent of ε, q, and v.*

(c) $\|d\tau^\varepsilon\| \le B\varepsilon^{\theta-1}$, *where B is a constant independent of ε.*

(d) *The Hausdorff distance between V and V^ε tends to zero, as $\varepsilon \to 0$.*

Proof. For the sake of notational simplicity, we will write $f(x) \lesssim g(x)$ $(x \in S)$ to mean that there exists a constant C independent of $x \in S$ such that $f(x) \le Cg(x)$, for all $x \in S$.

Since U is contained in a smooth chart for M, by Corollary 2.2 there exists ε_* such that for each $i > n_s$ there is a family φ_i^ε $(0 < \varepsilon < \varepsilon_*)$ of smooth approximations of φ_i satisfying

$$\|\varphi_i^\varepsilon - \varphi_i\|_{C^0} \lesssim \varepsilon^\theta \quad \text{and} \quad \|d\varphi_i^\varepsilon\|_{C^0} \lesssim \varepsilon^{\theta-1}. \tag{2.2}$$

Note that u^ε is defined on all of U. Denote by $\mathscr{F}_\varepsilon$ the foliation of U^ε defined by

$$\varphi_{n_s+1}^\varepsilon = \text{constant}, \ldots, \varphi_n^\varepsilon = \text{constant}.$$

It is easy to see that $\mathscr{F}_\varepsilon$ is smooth and the (largest principal) angle between W^{ss}_{loc} and $\mathscr{F}_\varepsilon$ is $\lesssim \varepsilon^\theta$. Let

$$\Sigma^\varepsilon = \bigcup_{x \in W^{uu}_{\text{loc}}(p)} \mathscr{F}_\varepsilon(x).$$

Then Σ^ε is a smooth local cross section and there exists $T_\varepsilon > 0$ such that the set

$$V^\varepsilon = \bigcup_{|t|<T_\varepsilon} f_t(\Sigma^\varepsilon)$$

is a smooth flow box for X contained in U^ε. It is clear that the length T_ε of V^ε is close to the length T of V; we can also take $T_\varepsilon \le T$.

Define $\tau^\varepsilon : V^\varepsilon \to \mathbb{R}$ by

$$\tau^\varepsilon(f_t q) = t,$$

for all $q \in \Sigma^\varepsilon$ and $-T_\varepsilon < t < T_\varepsilon$. Clearly, τ^ε is smooth and $X\tau^\varepsilon = 1$, proving (a).

We will first show that (b) holds along Σ^ε. Let $q \in \Sigma^\varepsilon$ be arbitrary. Since the angle between $\mathscr{F}_\varepsilon(q)$ and $W^{ss}_{\text{loc}}(q)$ is $\lesssim \varepsilon^\theta$, it follows that the angle between $\Sigma^\varepsilon \cap W^{cs}_{\text{loc}}(q)$ and $W^{ss}_{\text{loc}}(q)$ is also $\lesssim \varepsilon^\theta$. See Figure 1. Then the fact that $d\tau^\varepsilon = 0$ on $\Sigma^\varepsilon \cap W^{cs}_{\text{loc}}(q)$ implies $\|d_q\tau^\varepsilon \restriction_{E^{ss}}\| \lesssim \varepsilon^\theta$, as desired. To extend this to all points in V^ε we will use the invariance of $d\tau^\varepsilon$ with respect to the flow: $f_t^*(d\tau^\varepsilon) = d\tau^\varepsilon$, whenever both sides are defined. For $q \in \Sigma^\varepsilon$, $|t| < T_\varepsilon$ and $v \in E^{ss}_{f_t q}$, we have

$$\begin{aligned}
|d_{f_t q}\tau^\varepsilon(v)| &= |d_q\tau^\varepsilon(Tf_{-t}(v))| \\
&\le \|d_q\tau^\varepsilon \restriction_{E^{ss}}\|\|Tf_{-t}(v)\| \\
&\lesssim \varepsilon^\theta \mu_-^{-t}\|v\| \\
&\le \varepsilon^\theta \mu_-^{-T_\varepsilon}\|v\| \\
&\le \varepsilon^\theta \mu_-^{-T}\|v\|,
\end{aligned}$$

where we have used $T_\varepsilon \le T$. This proves (b) for all $q \in V^\varepsilon$.

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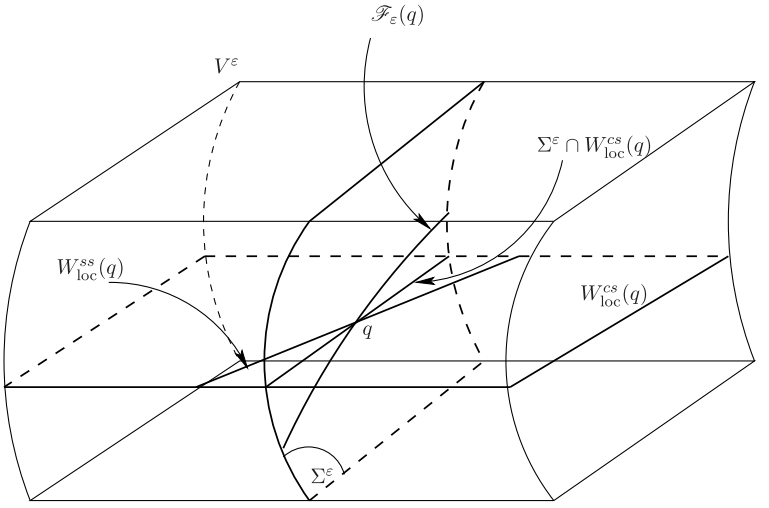


FIGURE 1. The flow box V^ε with local cross section Σ^ε.

To prove (c), we work in a smooth coordinate system in which $X = \partial/\partial x_1$. Since Σ is a C^θ hypersurface, it is locally the graph of a C^θ function g such that $\tau(g(z), z) = 0$, for all z in some open set in $\mathbb{R}^{n-1}$. Similarly, Σ^ε is locally the graph of a smooth function g^ε such that $\tau^\varepsilon(g^\varepsilon(z), z) = 0$, for all z in some open set in $\mathbb{R}^{n-1}$. Since $g^\varepsilon \to g$, as $\varepsilon \to 0$, (2.2) forces $\|dg^\varepsilon\| \lesssim \varepsilon^{\theta-1}$. Differentiating $\tau^\varepsilon(g^\varepsilon(z), z) = 0$ with respect to z_i for $i > 1$ and using $\partial\tau^\varepsilon/\partial z_1 = X\tau^\varepsilon = 1$, we obtain

$$\left|\frac{\partial \tau^\varepsilon}{\partial z_i}\right| = \left|\frac{\partial g^\varepsilon}{\partial z_i}\right| \lesssim \varepsilon^{\theta-1}.$$

Thus (c) holds on Σ^ε; we can extend it to V^ε using the flow invariance of $d\tau^\varepsilon$ as in the proof of (b).

Part (d) holds by construction. □

By the above analysis and compactness, we can cover M by finitely many Hölder flow boxes $V_1, \ldots, V_\ell$, each of which is equipped with a local Hölder cross section Σ_i. We can approximate each Σ_i by a smooth cross section Σ_i^ε as above and obtain smooth flow boxes $V_i^\varepsilon \subset V_i$ and smooth functions $\tau_i^\varepsilon : V_i^\varepsilon \to \mathbb{R}$ satisfying the properties from Lemma 2.3; namely,

$$X\tau_i^\varepsilon = 1, \quad \|d\tau_i^\varepsilon \restriction_{E^{ss}}\| \le A_i\varepsilon^\theta, \quad \|d\tau_i^\varepsilon\| \le B_i\varepsilon^{\theta-1}, \tag{2.3}$$

where the constants A_i, B_i are independent of ε. Let $A = \max A_i$, $B = \max B_i$ (not to be confused with the constants A, B in Lemma 2.3). By Lemma 2.3(d), there exists $\varepsilon_* > 0$ such that for all $0 < \varepsilon < \varepsilon_*$ the sets $V_1^\varepsilon, \ldots, V_\ell^\varepsilon$ cover M.

2.3. *Distance to the space of closed forms.* We will consider C^r differential k-forms ξ, with $r \ge 1$. We denote the C^0 norm of ξ on M by $\|\xi\|$:

$$\|\xi\| = \sup_{p\in M} |\xi_p|, \tag{2.4}$$

where $|\xi_p|$ is the operator norm of ξ_p as a k-linear map $T_pM \times \cdots \times T_pM \to \mathbb{R}$.

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Consider first a differential form ω on $M \times [0, 1]$, where t is the coordinate in $[0, 1]$. Denote by $\pi_M : M \times [0, 1] \to M$ and $\pi_I : M \times [0, 1] \to [0, 1]$ the obvious projections. Since

$$T_{(p,t)}(M \times [0, 1]) = T_p M \oplus T_t[0, 1],$$

any differential k-form on $M \times [0, 1]$ can be uniquely written as

$$\omega = \omega_0 + dt \wedge \eta,$$

where $\omega_0(v_1, \ldots, v_k) = 0$ if some v_i is in the kernel of $(\pi_M)_*$ and η is a $(k-1)$-form with the analogous property (i.e. $i_v\omega_0 = i_v\eta = 0$, for every 'vertical' vector $v \in T(M \times [0, 1])$, where i_v denotes contraction by v).

Define a $(k-1)$-form $\mathscr{H}(\xi)$ on M by

$$\mathscr{H}(\omega)_p = \int_0^1 j_t^* \eta_{(p,t)}\, dt,$$

where $j_t : M \to M \times [0, 1]$ is defined by $j_t(p) = (p, t)$. It is well known (cf. [**Spi05**]) that

$$j_1^*\omega - j_0^*\omega = d(\mathscr{H}\omega) + \mathscr{H}(d\omega). \tag{2.5}$$

Considering $\mathscr{H}$ as a linear operator from $\Omega^k(M \times [0, 1])$ to $\Omega^{k-1}(M)$, both equipped with the C^0 norm, it is not hard to see that

$$\|\mathscr{H}\| = 1. \tag{2.6}$$

We make the following claim.

PROPOSITION 2.4. *Let ξ be a C^r differential k-form ($r \geq 1$) on a closed manifold M. Then*

$$\inf\{\|\xi - \eta\| : \eta \in C^r,\ d\eta = 0\} \leq \|d\xi\|.$$

In other words, $\|d\xi\|$ is an upper bound on the distance from ξ to the space of closed forms. The inequality also holds for continuous forms which admit a continuous exterior derivative.

Proof. First, let us show that the result holds on any manifold M which is smoothly contractible to a point p_0 via $H : M \times [0, 1] \to M$, where $H(p, 0) = p_0$ and $H(p, 1) = p$, for all $p \in M$. Since $H \circ j_1$ is the identity map of M and $H \circ j_0$ is the constant map p_0, it follows that

$$\xi = (H \circ j_1)^*\xi = j_1^*(H^*\xi) \quad \text{and} \quad 0 = (H \circ j_0)^*\xi = j_0^*(H^*\xi).$$

Applying (2.5) to $H^*\xi$, we obtain

$$\begin{aligned} \xi &= \xi - 0 \\ &= j_1^*(H^*\xi) - j_0^*(H^*\xi) \\ &= d\mathscr{H}(\xi) + \mathscr{H}(d\xi). \end{aligned}$$

Using (2.6), we obtain

$$\|\xi - d\mathscr{H}(\xi)\| = \|\mathscr{H}(d\xi)\| \leq \|d\xi\|.$$

Therefore, the statement of the proposition holds for contractible M.

Let M now be any closed manifold and ξ a C^r k-form on M, $r \geq 1$. Cover M by contractible open sets $U_1, \ldots, U_m$. Denote the operator $\mathscr{H}$ restricted to forms on U_i by

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$\mathscr{H}_i$ and let ξ_i be the restriction of ξ to U_i. Define a k-form η on M by requiring that the restriction of η to U_i be equal to $d\mathscr{H}_i(\xi_i)$. We claim that η is well defined and closed.

Indeed, $\xi_i = \xi_j$ on $U_i \cap U_j$, and $\mathscr{H}_i(\omega) = \mathscr{H}_j(\omega)$ for every k-form ω defined on $(U_i \cap U_j) \times [0, 1]$. Thus on $U_i \cap U_j$, we have $d\mathscr{H}_i(\xi_i) = d\mathscr{H}_j(\xi_j)$, so η is well defined. Since η is locally exact, it follows that it is closed. By (2.4), we obtain

$$\|\xi - \eta\| \le \max_{1\le i\le m} \|\xi_i - d\mathscr{H}_i(\xi_i)\| \le \max_{1\le i\le m} \|d\xi_i\| \le \|d\xi\|.$$

This completes the proof of the proposition. □

2.4. *Change of Riemannian metric.* Denote by Ω the smooth volume form preserved by the flow and let $\mathscr{R}$ be the Riemannian metric which induces Ω.

Our goal is to show that, relative to some Riemannian metric, the area of the parallelogram $Tf_{-t}(v \wedge w)$ grows as $(\mu_+^{n_s-1}\lambda_+^{n_u-1})^t$, where $v \in E^{ss}$, $w \in E^{uu}$, and $t \ge 0$. To do this, it will be convenient to switch from the original Riemannian metric $\mathscr{R}$ to a new metric $\mathscr{R}'$ with respect to which X is a unit vector and $E^c \oplus E^{ss} \oplus E^{uu}$ is an orthogonal splitting. This metric can in general be only continuous and the corresponding volume form Ω' may not be invariant with respect to the flow. We will show that this does not present a problem.

Let $\mathscr{R}'$ be as above and let Ω' be the Riemannian volume form induced by $\mathscr{R}'$. Since Ω and Ω' are both volume forms, there exists a positive continuous function ϕ such that $\Omega' = \phi\Omega$. Let

$$L = \frac{\max_M \phi}{\min_M \phi}.$$

Denote the norms of tangent vectors (and their wedge products) with respect to $\mathscr{R}$ and $\mathscr{R}'$ by $\|\cdot\|$ and $\|\cdot\|'$, respectively. By compactness of M there exist $b_-, b_+ > 0$ such that

$$b_-\|v\| \le \|v\|' \le b_+\|v\|,$$

for all $v \in TM$. Observe that for $v \in E^{ss}$, $w \in E^{uu}$, and $t \ge 0$, we have

$$\|Tf_t(v)\|' \le bc\mu_+^t\|v\|' \quad \text{and} \quad \|Tf_t(w)\|' \le bc\lambda_+^t\|w\|',$$

where $b = b_+/b_-$. It is easy to check that

$$f_t^*\Omega' = \frac{\phi \circ f_t}{\phi}\Omega',$$

for all $t \in \mathbb{R}$. Thus for any n-dimensional parallelepiped Π in a tangent space to M and $t \in \mathbb{R}$, we have

$$\|\Pi\|' \le L\|Tf_t(\Pi)\|'$$

LEMMA 2.5. *If $\Phi = \{f_t\}$ is a volume-preserving Anosov flow with constants defined in (1.1) and (1.2), then*

$$\|Tf_{-t}(v \wedge w)\|' \le L(bc)^{n-3}(\mu_+^{n_s-1}\lambda_+^{n_u-1})^t\|v \wedge w\|',$$

for all $v \in E^{ss}$, $w \in E^{uu}$ and $t \ge 0$.

Proof. Let $v \in E^{ss}$, $w \in E^{uu}$ and $t \ge 0$ be arbitrary. Set $v_1 = Tf_{-t}(v)$ and $w_1 = Tf_{-t}(w)$. Choose vectors $v_2, \ldots, v_{n_s} \in E^{ss}$ and $w_2, \ldots, w_{n_u} \in E^{uu}$ such that, relative to $\mathscr{R}'$,

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$(X, v_1, \dots, v_{n_s}, w_1, \dots, w_{n_u})$ is an orthogonal basis of the corresponding tangent space and v_i, w_i are all of unit length, for $i \geq 2$. Then

$$\begin{aligned}
\|Tf_{-t}(v \wedge w)\|' &= \|v_1 \wedge w_1\|' \\
&= \|X \wedge v_1 \wedge \cdots \wedge v_{n_s} \wedge w_1 \wedge \cdots \wedge w_{n_u}\|' \\
&= \|\Pi\|' \\
&\leq L\|Tf_t(\Pi)\|' \\
&= L\|Tf_t(X \wedge v_1 \wedge \cdots \wedge v_{n_s} \wedge w_1 \wedge \cdots \wedge w_{n_u})\|' \\
&\leq L\|Tf_t(v_1 \wedge w_1)\|'\|Tf_t(X \wedge v_2 \wedge \cdots \wedge v_{n_s} \wedge w_2 \wedge \cdots \wedge w_{n_u})\|' \\
&\leq L\|v \wedge w\|'(bc)^{n_s-1}\mu_+^{(n_s-1)t} \cdot (bc)^{n_u-1}\lambda_+^{(n_u-1)t} \\
&= L(bc)^{n-3}\mu_+^{(n_s-1)t}\lambda_+^{(n_u-1)t}\|v \wedge w\|'. \qquad \square
\end{aligned}$$

In the remainder of the paper we will always be working with $\mathscr{R}'$ as the underlying Riemannian metric on M; the norms of tangent vectors and differential forms are taken relative to $\mathscr{R}'$ and will be denoted by the symbol $\|\cdot\|$ (thus slightly abusing notation for the sake of keeping it less cumbersome).

3. *Proof of the main theorem*

We will construct a smooth closed 1-form η such that $u = \eta(X) > 0$. Assuming for a moment that such a form has been found, the proof can be completed as follows. Define

$$\tilde{X} = \frac{1}{u}X.$$

Then $\tilde{X}$ is an Anosov vector field [**AS67**] and $\eta(\tilde{X}) = 1$. Thus the Lie derivative of η with respect to $\tilde{X}$ satisfies

$$L_{\tilde{X}}\eta = (di_{\tilde{X}} + i_{\tilde{X}}d)\eta = 0,$$

which implies that η is invariant with respect to the flow. It follows that its kernel $\mathrm{Ker}(\eta)$ is an invariant codimension-one distribution transverse to the flow, so $\mathrm{Ker}(\eta)$ is forced to be the sum $\tilde{E}^{su} = \tilde{E}^{ss} \oplus \tilde{E}^{uu}$ of the strong stable and strong unstable bundles of $\tilde{X}$. Since η is closed, $\tilde{E}^{su}$ is uniquely integrable, so by Plante [**Pla72**], $\tilde{X}$ admits a global cross section Σ, which is also a global cross section for X.

So it remains to construct a smooth 1-form η with $\eta(X) > 0$, which will be done in three steps. In the first two steps we construct a smooth 1-form ξ such that

$$\xi(X) = 1 \quad \text{and} \quad \|d\xi\| < 1.$$

The third step consists of approximating ξ by a smooth closed 1-form using Proposition 2.4.

Step 1. Let $\varepsilon > 0$ be arbitrary. As in §2.2, for $\varepsilon < \varepsilon_*$, we can cover M by smooth flow boxes $V_1^\varepsilon, \dots, V_\ell^\varepsilon$, with ℓ independent of ε, such that with respect to the Hausdorff distance each V_i^ε is close to a fixed open set V_i. In addition, we have smooth functions $\tau_i^\varepsilon : V_i^\varepsilon \to \mathbb{R}$ such that

$$X\tau_i^\varepsilon = 1, \quad \|d\tau_i^\varepsilon \restriction_{E^{ss}}\| \leq A\varepsilon^\theta \quad \text{and} \quad \|d\tau_i^\varepsilon\| \leq B\varepsilon^{\theta-1}, \tag{3.1}$$

where A, B are constants independent of ε.

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Let $\{\psi_i^\varepsilon\}$ be a smooth partition of unity subordinate to the cover $\{V_i^\varepsilon\}$. Since ℓ and the sizes of the sets V_i^ε are independent of ε, there is a constant $C > 0$ also independent of ε such that

$$\sum_{i=1}^{\ell} \|d\psi_i^\varepsilon\| \le C, \tag{3.2}$$

for all $0 < \varepsilon < \varepsilon_*$. Define

$$\xi_0^\varepsilon = \sum_{i=1}^{\ell} \psi_i^\varepsilon \, d\tau_i^\varepsilon.$$

LEMMA 3.1. *ξ_0^ε is a smooth 1-form on M with the following properties:*

(a) $\xi_0^\varepsilon(X) = 1$;

(b) $\|\xi_0^\varepsilon \restriction_{E^{ss}}\| \le A\varepsilon^\theta$, *for all* $0 < \varepsilon < \varepsilon_*$;

(c) $|d\xi_0^\varepsilon(v, w)| \le D\varepsilon^\theta$, *for all unit vectors* $v, w \in E^{cs}$ *and* $0 < \varepsilon < \varepsilon_*$, *where D is a constant independent of ε;*

(d) $\|d\xi_0^\varepsilon\| \le K\varepsilon^{\theta-1}$, *for every* $0 < \varepsilon < \varepsilon_*$, *where K is a constant independent of ε.*

Proof. Part (a) is clear. Part (b) follows easily from the second inequality in (3.1). To prove (c), first note that if β is a bilinear form on an inner product space E which splits into two orthogonal subspaces E_1 and E_2 and $\beta_{ij} = \beta \restriction_{E_i \times E_j}$, then

$$\|\beta\| \le \|\beta_{11}\| + 2\|\beta_{12}\| + \|\beta_{22}\|. \tag{3.3}$$

We fix $p \in M$ and take $\beta = d_p\xi_0^\varepsilon$, $E_1 = E_p^c$, and $E_2 = E_p^{ss}$. Since

$$d\xi_0^\varepsilon = \sum_{i=1}^{\ell} d\psi_i^\varepsilon \wedge d\tau_i^\varepsilon,$$

if $v, w \in E^{ss}$ are unit vectors, then

$$|d\xi_0^\varepsilon(v, w)| \le 2C\varepsilon^\theta$$

and

$$|d\xi_0^\varepsilon(X, v)| = \left|\sum_i d\psi_i^\varepsilon(X)\, d\tau_i^\varepsilon(v) - d\psi_i^\varepsilon(v)\, d\tau_i^\varepsilon(X)\right| = \left|\sum_i d\psi_i^\varepsilon(X)\, d\tau_i^\varepsilon(v)\right| \le AC\varepsilon^\theta,$$

where we used $d\tau_i^\varepsilon(X) = 1$ and $\sum_i d\psi_i^\varepsilon(v) = d(\sum_i \psi_i^\varepsilon)(v) = 0$. Thus by (3.3) we can take $D = (A + 2)C\varepsilon$ in (c). Finally, (3.2) and $\|d\tau_i^\varepsilon\| \le B\varepsilon^{\theta-1}$ imply

$$\|d\xi_0^\varepsilon\| = \left\|\sum_{i=1}^{\ell} d\psi_i^\varepsilon \wedge d\tau_i^\varepsilon\right\| \le BC\varepsilon^{\theta-1},$$

so (d) holds with $K = BC$. □

In summary, for every $0 < \varepsilon < \varepsilon_*$ we have a 1-form which is small on E^{ss}, whose exterior derivative is small when restricted to E^{cs}, but whose overall norm grows as $\varepsilon^{\theta-1}$, as $\varepsilon \to 0$.

Step 2. To remedy the problem represented by part (d) of the previous lemma, we flow backwards and use Lemma 2.5. Let $t > 0$ be large (how large will be specified shortly) and set

$$\xi_t^\varepsilon = f_{-t}^* \xi_0^\varepsilon.$$

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LEMMA 3.2. *There exists a constant $H > 0$ independent of ε and t such that*

$$\|d\xi_t^\varepsilon\| \le \max\{H\varepsilon^{\theta-1}(\mu_+^{n_s-1}\lambda_+^{n_u-1})^t,\ H\varepsilon^\theta\mu_-^{-2t}\}.$$

Proof. Observe that if β is a bilinear form as in the proof of the previous lemma and $\|\beta_{22}\| \le \|\beta_{ij}\|$, for all $i,\ j = 1, 2$, then

$$\|\beta\| \le 4\max(\|\beta_{11}\|, \|\beta_{12}\|).$$

Fix $p \in M$ and take $\beta = d_p\xi_t^\varepsilon$, $E = T_pM$, $E_1 = E_p^{cs}$, and $E_2 = E_p^{uu}$. Since $d\xi_t^\varepsilon = f_{-t}^*(d\xi_0^\varepsilon)$ and $\|Tf_{-t} \restriction_{E^{cs}}\| \le c\mu_-^{-t}$, for $t > 0$, Lemmas 3.1 and 2.5 imply

$$\|\beta_{11}\| \le c^2 D\varepsilon^\theta\mu_-^{-2t} \quad \text{and} \quad \|\beta_{12}\| \le KL\varepsilon^{\theta-1}(bc)^{n-3}(\mu_+^{n_s-1}\lambda_+^{n_u-1})^t.$$

Note that the second inequality holds because if $v \in E^{cs}$ and $w \in E^{uu}$, then $|d\xi_t^\varepsilon(v, w)| = |d\xi_0(T_{-t}(v \wedge w))|$ is largest if $v \in E^{ss}$. Observe also that $\|\beta_{22}\|$ is smaller than $\|\beta_{ij}\|$ $(i \ne 2,\ j \ne 2)$ since the flow contracts strong unstable vectors in negative time.

The proof of the lemma is now complete with $H = 4\max(c^2D,\ KL(bc)^{n-3})$. □

Since $\xi_t^\varepsilon(X) = 1$, for all $0 < \varepsilon < \varepsilon_*$ and $t > 0$, the following lemma will complete the construction of the desired form ξ.

LEMMA 3.3. *There exist $0 < \varepsilon < \varepsilon_*$ and $t > 0$ such that $\|d\xi_t^\varepsilon\| < 1$.*

Proof. By the previous lemma, it suffices to show that the system of inequalities

$$H\varepsilon^{\theta-1}(\mu_+^{n_s-1}\lambda_+^{n_u-1})^t < 1,$$
$$H\varepsilon^\theta\mu_-^{-2t} < 1$$

admits a solution $\varepsilon,\ t > 0$. Solving the first inequality for t, we obtain

$$t > \frac{(1-\theta)\log\varepsilon - \log H}{(n_s-1)\log\mu_+ + (n_u-1)\log\lambda_+}. \tag{3.4}$$

The second inequality is equivalent to

$$t < \frac{\log H + \theta\log\varepsilon}{2\log\mu_-}. \tag{3.5}$$

There exists t with these properties if and only if we can find $0 < \varepsilon < \varepsilon_*$ such that

$$\frac{(1-\theta)\log\varepsilon - \log H}{(n_s-1)\log\mu_+ + (n_u-1)\log\lambda_+} < \frac{\log H + \theta\log\varepsilon}{2\log\mu_-}.$$

Since H is fixed and ε is small, this is possible if

$$\frac{(1-\theta)\log\varepsilon}{(n_s-1)\log\mu_+ + (n_u-1)\log\lambda_+} < \frac{\theta\log\varepsilon}{2\log\mu_-},$$

which is equivalent to (1.3). □

Step 3. Choose $0 < \varepsilon < \varepsilon_*$ and $t > 0$ such that $\|d\xi_t^\varepsilon\| < 1$ and set

$$\xi = \xi_t^\varepsilon.$$

By Proposition 2.4 there exists a smooth closed 1-form η such that

$$\|\xi - \eta\| \le \|d\xi\| < 1.$$

Since $\xi(X) = 1$, it follows that $\eta(X) > 0$. This completes the construction of η and concludes the proof of the theorem. □

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