PSEUDO-ABELIAN VARIETIES

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It is a familiar fact that the Picard surface (or hyperelliptic surface of rank 1) admits a completely transitive permutable continuous group of ∞^2 automorphisms. There are, however, other non-scrollar surfaces which possess continuous groups of automorphisms, namely, the elliptic surfaces. Every elliptic surface V_2 contains a pencil of birationally equivalent elliptic curves, which are the trajectories of the group in question; it also contains a second, elliptic, pencil of birationally equivalent curves; the intersection number of the two pencils is an important character, known as the determinant of V_2. Just as any Picard surface can be mapped on a multiple Picard surface of divisor unity, so V_2 can be mapped on a multiple elliptic surface of determinant unity, the branch curve (if any) corresponding to a certain number of trajectories.

It is also a familiar fact that the hyperelliptic surfaces of rank greater than unity are all mapped by involutions on a Picard surface; among them we have the improperly hyperelliptic surfaces, which admit a parametric representation by means of Abelian functions of lower genus than two. These map involutions without coincidences on Picard surfaces of special type; and those which are not (elliptic) scrollar are elliptic surfaces with a virtual canonical curve of order zero.

Both the Picard and the elliptic surfaces have absolute linear genus $p^{(1)} = 1$, arithmetic genus $p_a = -1$, and Zeuthen-Segre invariant $I = -4$; i.e. their canonical series of equivalence have order zero.

The above results have already been extended to threefolds in previous work; thus, corresponding to the elliptic surfaces, we have the elliptic and hyperelliptic threefolds (8, 9), which possess continuous groups of automorphisms of dimensions 1 and 2 respectively. Again, the principal types of improperly Abelian threefolds, which map superficially irregular involutions, other than those of irregularity 3, on a Picard threefold, have been classified in (10); these are all elliptic or hyperelliptic. Finally, it has been shown that the elliptic and hyperelliptic threefolds, like the Picard threefold itself, have canonical series of order zero.

The pseudo-Abelian varieties W_p of genus p, which are studied in the present work, are a natural generalization of the elliptic and hyperelliptic threefolds. A pseudo-Abelian variety W_p of *type* q contains a congruence $\{V_q\}$ (∞^{p-q} system of index 1) of birationally equivalent Picard varieties V_q which are the trajectories of the group of ∞^q automorphisms of W_p; in the present paper it is assumed, for the sake of simplicity, that V_q has general moduli. To begin with, we show that W_p contains a second congruence $\{V_{p-q}\}$, birationally equivalent to a Picard variety, of varieties V_{p-q} which are themselves birationally equivalent and which cut the generic V_q in sets of an involution i_d ($d \geqslant 1$) without coincidences: the number d is called the determinant of W_p. We may then map W_p on a d-ple variety W_p^* of determinant unity, the coincidence locus (if any)

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of the representation consisting of a number of varieties, possibly of various dimensions, generated by trajectories V_q. From this mapping we deduce an equivalence for the canonical system $|X_{p-1}|$ of W_p, and a lower limit to the superficial irregularity; we also obtain an analytical representation for W_p in the case where the involution i_d is cyclic.

We next consider the canonical systems $\{X_k\}$ of dimensions $k\,(k = 0, 1, \ldots, p-2)$, showing that, like $|X_{p-1}|$, these are either generated by trajectories or else have order zero. One consequence of this fact is that all the canonical invariants of W_p are zero.

It has been shown by Lefschetz that, for a Picard variety V_p, the arithmetic genus (on the postulational definition) is equal to $(-1)^{p-1}$; from this we deduce that, when $d = 1$, the same result holds for W_p; on the hypothesis that the arithmetic genus is an enumerative character, we show that it holds also for $d > 1$.

In a subsequent note† we propose to consider the *improperly Abelian* varieties W_p of type q, i.e. the varieties which map irreducible involutions of superficial irregularity $q < p$ on V_p. Using the results of §2, one can show that all such varieties are pseudo-Abelian of type q. We shall also examine the *para-Abelian* varieties of genus p, which bear the same relation to the pseudo-Abelian varieties as the paraelliptic surfaces (5, 10) do to the elliptic surfaces.

1. *Picard varieties.* We begin by recalling the main properties of Abelian and, more particularly, Picard varieties, which will be required in this work, referring to Conforto (3), Lefschetz (7) and Scorza (11) for further details.

An Abelian variety W_p is an irreducible variety the coordinates of whose generic point are expressible as Abelian functions, of genus p, of p independent variables $u_1, u_2, \ldots, u_p$. The *rank* r of W_p is defined as the number of points in the primitive period parallelepiped U_p which correspond to the generic point of W_p. In the case $r = 1$, W_p is called a Picard variety.

Assuming that the Picard variety can be cleared of singularities, we can show that it is possible to remove also any exceptional subvarieties, so that the correspondence between the points of the variety and those of U_p is (1, 1) and unexceptional. We thus obtain a model which we denote by V_p and which will be used in all that follows;‡ and the symbol V_q will be used with a similar meaning.

Every Picard variety possesses two types of automorphism, the so-called ordinary transformations; and if it is *general* (i.e. has general moduli) it admits only these. The transformations of the first kind are represented by the equations

$$u_i' = u_i + a_i \quad (i = 1, 2, \ldots, p), \tag{1}$$

where the a_i are arbitrary constants. It is clear that these transformations form a completely transitive permutable continuous ∞^p group. On the other hand, the transformations of the second kind, expressed by the equations

$$u_i' = -u_i + a_i \quad (i = 1, 2, \ldots, p) \tag{2}$$

do not form a group; for any given set of constants a_i they define an involution I_2 on V_p which possesses 2^{2p} coincidences.

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‡ All varieties considered in the sequel will be assumed to be non-singular, unless the contrary is stated.

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More generally, we shall be led to consider series of sets of m points on V_p, given by equations of the form

$$u_i^{(1)} + u_i^{(2)} + \ldots + u_i^{(m)} = c_i \quad (i = 1, 2, \ldots, p). \tag{3}$$

Such series were noticed by Castelnuovo(2); it follows from Severi's work†(15) on series of equivalence that any series of linear circulation zero and hence, in particular, any series of intersection, is representable by equations of type (3).

It is known that a general V_p contains no Picard varieties V_q $(q < p)$. It can be shown that, if V_p contains one such variety V_q, it must contain a *congruence* $\{V_q\}$ (system of freedom‡ $p-q \geqslant 1$ and index 1) of varieties V_q, the congruence itself being of Picard type; and, further, that V_p must contain a second Picard congruence $\{V_{p-q}\}$ of Picard varieties V_{p-q}. We shall call such a variety V_p *special of type* q (or $p-q$); in the most general case, both V_q and V_{p-q} are general.

It follows readily from the definition that every Abelian variety W_p of rank $r > 1$ can be mapped by an involution of order r on a Picard variety V_p; the coincidences of such an involution may be ∞^q $(0 \leqslant q \leqslant p-1)$. Particular interest attaches to those cases in which there are no coincidences at all; it may then be shown that, if V_p is general, W_p must be a Picard variety, and that the involution is generated by transformations of the first kind; when, however, V_p is special, other types can arise (see §8).

2. *Characteristic properties of Picard varieties.* We have remarked that V_p possesses a transitive permutable continuous group of ∞^p automorphisms; conversely, it may be proved that any variety W_p which admits such a group must be a Picard variety of genus p.

From this result Enriques(4) has deduced that *any variety which can be mapped on a multiple* V_p *without branch points is itself a Picard variety.*

We now show that

Any variety W_p *of superficial irregularity* p, *with a pure canonical hypersurface*§ *of order zero, and not containing a congruence of irregularity*‖ p, *is a Picard variety.*

The method of proof is on the lines of Severi's generalization(16) of Castelnuovo's mapping(2) of linear systems of a surface by points of a Picard variety. By hypothesis, the variety W_p in question contains infinitely many continuous systems $\{A\}$ of hypersurfaces, each consisting of ∞^p disequivalent linear systems $|A|$, which may be mapped by the points of a Picard variety V_p. We thereby set up a correspondence between W_p and V_p in which a point of W_p, or rather the ∞^{p-1} system of hypersurfaces A passing through it, is mapped by a hypersurface B on V_p. The aggregate of hypersurfaces B form an ∞^p continuous system $\{B\}$; these cannot all be equivalent, for then the hypersurfaces A passing through a given point of W_p would cut equivalent sub-

† Severi deals only with simple integrals of the first kind on a surface; but his result extends at once to varieties of any dimension.

‡ Thus we use the term *congruence* to include *pencil*; on the other hand, *pencil* will retain its usual meaning.

§ Variety of dimension $p-1$ on W_p.

‖ I.e. a congruence of superficial irregularity p, or a pencil of genus p, according as the freedom of the system is greater than or equal to unity. For the superficial irregularity of a variety, see Severi(17).

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varieties on any fixed member A_0 of the system, and hence would themselves be equivalent.

We have now two possibilities to consider:

(i) The hypersurfaces of $\{B\}$ fall into ∞^i systems $(1 \leqslant i \leqslant p-1)$, each containing ∞^{p-i} equivalent members.

(ii) There is only a finite number $n\,(\geqslant 1)$ of hypersurfaces equivalent to a given hypersurface B.

In case (i) we can show, exactly as in Severi's work, that W_p must contain a congruence of irregularity p, which is contrary to our hypothesis. In case (ii) W_p is mapped on the n-ple variety V_p; and since, by hypothesis, W_p has a canonical hypersurface of order zero, the representation must obviously be free from branch points. Hence, by the above result of Enriques, W_p must be a Picard variety.

We turn now to the canonical systems (§ 6) of V_p. It has been remarked by Severi (18) that each of the systems $X_k(V_p)\,(k = 0, 1, \ldots, p-1)$ is of order zero—as is easily seen in the case where V_p is special of type 1, containing a pencil of varieties V_{p-1} which are also special of type 1, and so on. (Severi has conjectured that such a property characterizes the Picard varieties.) It follows then that all the canonical invariants of V_p are zero; in particular, the Zeuthen-Segre invariant I has the value $(-1)^{p-1}\,2p$.

Finally, we note that Lefschetz (7), using the postulational definition, has shown that the arithmetic genus of V_p is equal to $(-1)^{p-1}$.

3. *Pseudo-Abelian varieties.* Consider now a non-singular variety W_p which admits a continuous permutable group $\mathcal{G}$ of ∞^q automorphisms $(1 \leqslant q \leqslant p-1)$; we call W_p *a pseudo-Abelian variety of type* q. The trajectories of $\mathcal{G}$ constitute a congruence $\{V_q\}$ of varieties V_q, the generic member of which is irreducible; and all irreducible members are birationally equivalent to $\mathcal{G}$ and hence to each other. Each such variety V_q is invariant under $\mathcal{G}$ and no two trajectories intersect. If we assume that $\mathcal{G}$ acts transitively, without exceptions, on the generic V_q, it follows from § 2 that V_q must be a Picard variety; if, further, we assume that V_q represents biunivocally and unexceptionally the transformations of $\mathcal{G}$, then V_q is non-singular and free from exceptional varieties. We shall suppose throughout that V_q is general, so that, by § 1, the operations of $\mathcal{G}$ are transformations of the first kind on V_q itself.

It will appear from § 4 that, as a hypothesis of generality, we may assume that every irreducible trajectory is non-singular; but it will also appear that we may expect $\{V_q\}$ to contain a number, finite or infinite, of reducible members, each consisting of an irreducible non-singular Picard variety counted with a certain multiplicity.

We first prove that W_p *contains a Picard congruence* $\{V_{p-q}\}$ *of* ∞^q *birationally equivalent varieties* V_{p-q}. For, consider the series of intersection cut on V_q by the primals of any given order in the ambient space of W_p; to any set of this series, which is represented by equations of the form (3), there corresponds a unique set of the analogous series cut on any other irreducible trajectory V'_q, and vice versa. Now it follows from (3) that there is a finite number† n, say, of transformations of the first kind which leaves each series invariant and which transforms V_q birationally into V'_q; to each point P of V_q we

† The transformations $u'_i = u_i \pm a_i$ leave the given series invariant if, and only if, $ma_i \equiv 0$, where m is the order of the series; thus $n = 2m^{2q}$.

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may thus make correspond a unique set of n points of V'_q, while to each point P' of V'_q there corresponds a unique set of n points of V, such that P (or P') belongs to a unique set of such points. As V'_q varies in $\{V_q\}$, the set describes a variety of dimension $p-q$, possibly reducible; if it is irreducible, denote it by V_{p-q}, while if it is reducible, let V_{p-q} be any one of its irreducible components. Then the transforms of V_{p-q} under $\mathcal{G}$ form an irreducible system of index 1, which is the congruence $\{V_{p-q}\}$ in question; the members of this congruence cut V_q in sets of an involution which is without coincidences and which, since V_q is general, is therefore of Picard type (§1). It follows that $\{V_{p-q}\}$ is itself a Picard congruence. Evidently the congruence is free from base points and also from singular points, through which pass an infinity of members. We shall assume that the generic V_{p-q} is non-singular, from which, by virtue of §4, we are led to suppose that every V_{p-q} is non-singular.

It is not, of course, asserted that $\{V_{p-q}\}$ is the sole Picard congruence lying on W_p; in fact, one can easily construct examples of varieties which contain two or more such congruences. However, it can be proved that *any variety W_p which contains two congruences $\{V_q\}$ and $\{V_{p-q}\}$ defined as above is necessarily pseudo-Abelian of type q*; the proof of this result, which is similar to that of the analogous theorem in the theory of the elliptic surfaces (4), depends on the fact that transformations of the first kind are transferred from one trajectory to another, and rationally determinable, by means of the involutions which $\{V_{p-q}\}$ cuts on the trajectories.

The intersection number $[V_q V_{p-q}] = d$, which we call the *determinant* of W_p, is an important character of the variety, somewhat analogous to the divisors of V_p. We remark that *each variety V_{p-q} is invariant under a permutable group $\mathcal{G}_d$ of automorphisms*, namely, the transformations of $\mathcal{G}$ which leave each V_{p-q} invariant: in other words, the involution cut by $\{V_{p-q}\}$ on V_q can be generated by a group of d automorphisms. Both these properties are of fundamental importance in the classification of W_p.

4. *Representation on a variety of determinant unity.* In the case where $d = 1$, the congruences $\{V_q\}$, $\{V_{p-q}\}$ are birationally equivalent to V_{p-q} and V_q respectively; we may therefore map W_p on the product $V_q \times V_{p-q}$ of V_q and V_{p-q} and, by the assumptions made in §3, the representation is free from exceptional elements.

To obtain a representation in the case $d > 1$, we first construct the variety $W_p^* = V_q^* \times V_{p-q}^*$, where V_q^* and V_{p-q}^* are birationally equivalent, without exceptions, to $\{V_{p-q}\}$ and $\{V_q\}$ respectively; such a variety contains two congruences which, without risk of confusion, we may denote by $\{V_q^*\}$ and $\{V_{p-q}^*\}$, the members of $\{V_q^*\}$ being the trajectories of W_p^*. We now make correspond to the generic point of W_p^* the set of d points common to the generic V_q and V_{p-q}, and thus obtain a mapping of W_p on the d-ple variety W_p^*.

In this mapping each trajectory V_q corresponds to a d-ple Picard variety V_q^*, the representation being without branch points, since the involution i_d cut by $\{V_{p-q}\}$ on the generic V_q is free from coincidences. Hence the branch locus is either lacking altogether or else consists of a number of irreducible varieties belonging to the congruence $\{V_q^*\}$, i.e. generated by V_q^*'s; such varieties, which we shall suppose to be non-singular, may have any dimension varying from q to $p-1$, those of dimension q consisting of isolated V_q^*'s. To each generator $V_{q,s}^*$, say, of an $(s-1)$-fold component of the branch locus,

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there corresponds a variety $V_{q,s}$ which is an $(s-1)$-fold element of the coincidence locus of the involution I_d defined by the intersections of $\{V_q\}$ and $\{V_{p-q}\}$, and which is such that $sV_{q,s} \equiv V_q$; the numbers $s\,(2 \leqslant s \leqslant d)$ may *a priori* be any divisors of d.

Since $V_{q,s}$ is mapped on a d/s-ple Picard variety $V^*_{q,s}$ without the intervention of branch points, it follows that $V_{q,s}$ must itself be a Picard variety (§2); and since, by hypothesis, V_q has general moduli, so also has $V_{q,s}$. As a hypothesis of generality we may assume that each variety $V_{q,s}$ is non-singular; in that case the representation of W_p on W^*_p has no other exceptional features than those already described; in particular, no trajectories other than the varieties $sV_{q,s}$ are reducible.

In the case where the correspondence between W_p and W^*_p is biregular, the coincidence locus of I_d is pure, consisting of one or more irreducible hypersurfaces; we may then take W^*_p to be a non-singular variety and suppose that the correspondence is free from fundamental elements. When, however, the coincidence locus contains components of dimension less than $p-1$, it follows from a result of Severi(19) that, if we require W^*_p to be non-singular, each such component must be mapped by a hypersurface on W^*_p, in which case the correspondence possesses fundamental elements.†

We now show that *the superficial irregularity q_2 of W_p satisfies the inequality $q_2 \geqslant q^*_2 + q$*, where q^*_2 denotes the irregularity of the congruence $\{V_q\}$. For the superficial irregularity of W^*_p, which is equal to the number of linearly independent Picard integrals of the first kind, is evidently the sum of the superficial irregularities of V^*_q and V^*_{p-q}. Again, a generic surface (in the sense of Castelnuovo and Enriques) on W_p is mapped on a multiple generic surface of W^*_p; the required result then follows from the remark, due to Bagnera and De Franchis(1), that the irregularity of the former surface is at least equal to that of the latter.

We next obtain an equivalence for the virtual canonical system $|X_{p-1}|$ of W_p. In the first place, Todd(21) has shown that the analogous system $|X^*_{p-1}|$ of the product W^*_p is given by

$$X^*_{p-1} \equiv X_{p-q-1}(V^*_{p-q}) \times V^*_q + V^*_{p-q} \times X_{q-1}(V^*_q). \tag{4}$$

Since, by §2, $X_{q-1}(V^*_q)$ is of order zero, we have

$$X^*_{p-1} \equiv X_{p-q-1}(V^*_{p-q}) \times V^*_q. \tag{5}$$

Then, if the composite branch hypersurface on W^*_p is denoted by $\Sigma(s-1)\,W^*_{p-1}$, it follows from the transformation law for canonical systems that

$$X_{p-1} \equiv \bar{X}^*_{p-1} + \Sigma(s-1)\,\bar{W}^*_{p-1}, \tag{6}$$

where the bar over a symbol denotes the transform of the variety in question. Thus $|X_{p-1}|$ *contains as $(s-1)$-ple component every hypersurface W_{p-1} of the coincidence locus of I_d which is generated by $V_{q,s}$'s, and passes $(s-1)$-ply through each variety of lower dimension which is generated by $V_{q,s}$'s.* Hence, in all cases, $|X_{p-1}|$ belongs to the congruence $\{V_q\}$.

† Consider, for example, the elliptic threefold W_3, of determinant 2, containing a Kummer congruence of elliptic curves and an elliptic pencil of Jacobi surfaces; in this case there are 16 isolated curves $V_{1,2}$, each corresponding to an elliptic scrollar component of the branch surface on W^*. For systems of dimension exceeding 3, it is not known at present whether such a mapping can always be obtained.

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5. *Analytical representation of W_p.* In order to construct a projective model of W_p, it is first necessary to obtain a mapping of V_q on the d-ple variety V_q^*. The general problem involved here has not so far been studied for $q > 1$; if, however, we restrict ourselves to the case where the representation on V_q^* is cyclic, the analytical representation of W_p is easily obtained, and gives some idea of the possible forms for V_{p-q}.

In the first place, it is clear that there exist cyclic involutions $i_d\,(d \geqslant 2)$ without coincidences on V_q; to obtain such an involution we have merely to choose the constants a_i in (1) so that $a_i d \equiv 0$, modulo the periods. Now suppose that V_q^* is situated in S_{q+1} and endowed with ordinary singularities, and that the equation of V_q^*, in non-homogeneous coordinates, is $F(x_1, x_2, \ldots, x_{q+1}) = 0$; then V_q is represented by the equations

$$v^d = G(x_1, x_2, \ldots, x_{q+1}), \quad F(x_1, x_2, \ldots, x_{q+1}) = 0,$$

where $G(x_1, x_2, \ldots, x_{q+1})$ is a suitably chosen polynomial; it must satisfy the two requirements that the representation of V_q on V_q^* should be without branch points, and that V_q should be irreducible. To satisfy the first condition, we suppose that the equation $G = 0$ represents a d-adjoint primal of V_q^* whose order is a multiple of d—in which case there will be no branch points at infinity—and which has d-point intersection with V_q^* at all common points (if any) outside the double hypersurface. To satisfy the second, we require that the locus of such common points should not be the hypersurface of d'-point intersection with V_q^* for some d'-adjoint primal, where d' is any divisor of d. The existence of the cyclic involutions i_d implies that of the d-adjoint primals in question.

Now let V_{p-q}^* be situated in S_{p-q+1}, and represented by the equation

$$f(y_1, y_2, \ldots, y_{p-q+1}) = 0;$$

and suppose that V_{p-q}, which must be mapped on the cyclic d-ple variety V_{p-q}^*, is represented by the equations

$$v^d = Ag(y_1, y_2, \ldots, y_{p-q+1}), \quad f(y_1, y_2, \ldots, y_{p-q+1}) = 0,$$

where A is an arbitrary constant, and where the polynomial g is chosen so that the variety V_{p-q} thus represented is irreducible. Then W_p is represented by the equations

$$v^d = G(x_1, x_2, \ldots, x_{q+1})\, g(y_1, y_2, \ldots, y_{p-q+1}), \quad F = 0, \quad f = 0.$$

6. *The canonical systems.* We begin by recalling the definition of the canonical† varieties $X_k(W_p)\,(k = 0, 1, \ldots, p-1)$ of a given variety W_p. First, let $k = 0$; and let S be a hypersurface of a rational pencil $|S|$, of general character, on W_p, which possesses δ nodal members; then X_0 is defined inductively, for $p = 2, 3, \ldots$, by the equivalence

$$X_0(W_p) \equiv \delta - 2X_0(S) - X_0(S^2). \tag{7}$$

Precisely as in the theory of surfaces, we may now show that, if instead S belongs to an irrational pencil $\{S\}$, of genus $\rho > 0$, and of general character, then

$$X_0(W_p) \equiv \delta + 2(\rho - 1)\, X_0(S). \tag{8}$$

The order of the (Severi) series of equivalence $\{X_0(W_p)\}$ is equal to $I + (-1)^p\, 2p$, where I is the Zeuthen-Segre invariant of W_p.

† These are the classical canonical varieties, introduced by B. Segre (12) in the case $p = 3$, and subsequently generalized, for $p > 3$, by Eger and Todd (20, 22). For a more recent concept of canonical systems, see B. Segre (14). A complete bibliography will be found in the latter work.

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In the case where $k > 0$, the varieties $X_k(W_p)$ are defined, by induction on k and p, by means of the equivalence

$$X_k(W_p) \equiv A_k(S) - X_k(S) \quad (k = 1, 2, \ldots, p-1), \tag{9}$$

where S is *any* hypersurface of W_p, and $A_k(S)$ is an *adjoint* variety to S, cutting S in a $X_{k-1}(S)$; here $X_{p-1}(S) = S$.

We now prove three preliminary results which will be required in the sequel.

LEMMA 1. *If two varieties W_p, W_p^* are in a $(d, 1)$ correspondence which is free from branch elements and fundamental elements, then* $[X_0(W_p)] = d[X_0(W_p^*)]$.

This result has been established in the cases $p = 1, 2$ (cf. B. Segre(13)); we prove it by induction on p. Consider a general pencil $|S^*|$ on W_p^*, having δ^* nodal members; to it there corresponds on W_p a general pencil $|S|$ with a Jacobian set of order $d\delta^*$. By the inductive hypothesis, the numbers $[X_0(S)]$, $[X_0(S^2)]$ are respectively equal to d times the numbers $[X_0(S^*)]$, $[X_0(S^{*2})]$; whence, by (7), the required result.

LEMMA 2. *Suppose that the correspondence between W_p and W_p^* is free from fundamental elements, but that there is a branch locus consisting of a number of irreducible non-singular and non-intersecting hypersurfaces $B_{(s)}^*$ such that to every point of $B_{(s)}^*$ there corresponds a set of d/s distinct points whose locus is an irreducible non-singular hypersurface $B_{(s)}$: then*

$$[X_0(W_p)] = d[X_0(W_p^*)] + \Sigma(s-1)\,[X_0(B_{(s)})]. \tag{10}$$

This result is known to hold when $p = 1$, and is easily proved in the case $p = 2$. Assuming the truth of (10) for varieties of dimension less than p, we observe that the Jacobian set of the pencil $|S|$ is the transform of the Jacobian set of $|S^*|$, together with the transforms of the Jacobian sets of the pencils cut by $|S^*|$ on the hypersurfaces $B_{(s)}^*$, counted $(s-1)$ times. Now the number of points in a typical set of the latter kind is $[X_0(B_{(s)}^*)] + 2[X_0(S^*B_{(s)}^*)] + [X_0\{(S^*B_{(s)}^*)^2\}]$, and each of these points gives rise to $d(s-1)/s$ nodes on members of $|S|$. Also, by Lemma 1, we have

$$[X_0(SB_{(s)})] = \frac{d}{s}[X_0(S^*B_{(s)}^*)], \quad [X_0(SB_{(s)})^2] = \frac{d}{s}[X_0(S^*B_{(s)}^*)^2], \quad [X_0(B_{(s)})] = \frac{d}{s}[X_0(B_{(s)}^*)].$$

Substituting these numbers in (7), we obtain (10).

We have assumed that no two of the hypersurfaces $B_{(s)}$ intersect; however, in the cases with which we shall be concerned, $B_{(s)}$ belongs to the congruence $\{V_q\}$ of trajectories of a pseudo-Abelian variety W_p, and then two such hypersurfaces, corresponding to the same value of s, may possibly have a common part. But this also belongs to $\{V_q\}$ and is thus a pseudo-Abelian or Picard variety (possibly reducible); and for the hypersurfaces $B_{(s)}$ in question it is easily shown that the character $X_0(B_{(s)})$ is additive, so that (10) can still be applied.

We next have

LEMMA 3. *Let $|S|$ be an irreducible pencil of W_p, with irreducible simple base variety S^2, and let S_0 be a member of $|S|$ whose only singularity is a double irreducible non-singular variety W_r $(1 \leqslant r \leqslant p-2)$ which does not meet S^2; then the contribution made by S_0 to the number $[X_0(W_p)]$, calculated from $|S|$, is $[X_0(W_r)]$.*

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This lemma, which generalizes a well-known result in the theory of surfaces(4), is proved in an analogous manner; we introduce an auxiliary pencil $|T|$ of general character, and invoke the inductive hypothesis. For the details we may refer to Todd(23), where a similar result is established.

The method of proof shows that, if S_0 contains a finite number of such varieties W_r, no two of which intersect, the total contribution made to I is $\Sigma[X_0(W_r)]$. Further, the result can be extended to either or both of the cases where (*a*) S_0 contains W_r to a higher multiplicity, (*b*) $r = p-1$, and the variety W_r, counted with a certain multiplicity, belongs to a pencil, rational or irrational; in this last case we use (8). These two extensions would cover all the requirements of §7; however, it has seemed worth while to give Lemma 2 for its own sake.

7. *Applications to pseudo-Abelian varieties.* We now show that

The canonical systems $\{X_k(W_p)\}$ $(k = 0, 1, \ldots, p-1)$ *of any pseudo-Abelian variety* W_p *either belong to the congruence* $\{V_q\}$ *of trajectories or else have order zero.*

This property is familiar in the case $p = 2$, and it is known to hold when $p = 3$(8, 9); we make the inductive hypothesis that it is true for every pseudo-Abelian variety of dimension less than p.

Consider first the case $q = p-1$; here we have a pencil $\{V_{p-1}\}$, without base points, of Picard varieties on W_p, containing a certain number of multiple members each of which consists of a Picard variety $V_{p-1\,s}$ counted s times; and in the mapping of W_p on W_p^*, each variety $V_{p-1,s}$ corresponds to an $(s-1)$-ple branch hypersurface $V^*_{p-1,s}$ of the pencil $\{V^*_{p-1}\}$. Now Todd(21) has shown that, for any product $W_p^* = V_q^* \times V^*_{p-q}$, we have $[X_0(W_p^*)] = [X_0(V_q^*)] \times [X_0(V^*_{p-q})]$; and since, in the present case, $[X_0(V_q^*)] = 0$ (§2), it follows that $[X_0(W_p^*)] = 0$. Hence, by Lemma 2, $[X_0(W_p)] = 0$. Again, by §4, we know that $|X_{p-1}|$ is compounded of the pencil $\{V_{p-1}\}$; and in the present case there is nothing more to prove.

Next, suppose that $1 \leqslant q \leqslant p-2$; here, as in the previous case, we have $[X_0(W_p^*)] = 0$, so that, if the correspondence between W_p and W_p^* is biregular, we deduce from Lemma 2 that $[X_0(W_p)] = 0$. But, in any case, we can use Lemma 3; thus, corresponding to a general pencil of hypersurfaces on W_p^* which belong to the congruence $\{V_q^*\}$, we have, on W_p, a general pencil $|S|$ of hypersurfaces belonging to the congruence $\{V_q\}$; such a pencil has an irreducible non-singular base variety S^2, but, instead of possessing members with isolated nodes, it includes a certain number which are each endowed with d ordinary double varieties V_q, trajectories of W_p. Now, by Lemma 3, such double varieties make no contribution to the value of I, calculated from $|S|$; and since both S and S^2 are pseudo-Abelian (of type q) it follows from the inductive hypothesis and equation (7) that $[X_0(W_p)] = 0$.

The result for $k > 0$ now follows from the inductive hypothesis and equation (9), where it is again assumed that S is any hypersurface belonging to $\{V_q\}$; it is clear that, for $k \geqslant q$, the system $\{X_k(W_p)\}$ is either generated by trajectories or else has order zero, and that, for $k < q$, it has order zero.

We now define the *canonical invariants* of W_p as the intersection numbers $[X_{i_1} X_{i_2} \ldots X_{i_r}]$ of the various canonical systems; here the suffixes $i_1, i_2, \ldots$ may take any values such that $i_1 + i_2 + \ldots + i_r = (r-1)p$, repetitions of any symbol being allowed. Since the

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varieties $X_{i_1}, X_{i_2}, \ldots$ either belong to the congruence $\{V_q\}$ or else have order zero, we see that *the canonical invariants of any pseudo-Abelian variety W_p are all zero*. Thus, in particular, the invariant I is equal to $(-1)^{p-1}\,2p$.

We consider next the arithmetic genus P_a of W_p, defined by means of the postulation formula. First we remark that, in the case of a Picard variety V_p, Lefschetz(7) has shown that P_a is equal to $(-1)^{p-1}$; using this result, we may show that, *when $d = 1$, the arithmetic genus of W_p is* $(-1)^{p-1}$. For Gaeta(6) has proved that the arithmetic genus P_a of any variety $W_p = V_q \times V_{p-q}$ is given by the formula

$$P_a + (-1)^p = \{P'_a + (-1)^q\}\{P''_a + (-1)^{p-q}\},$$

where P'_a and P''_a are the arithmetic genera of V_q and V_{p-q} respectively.

In the case where $d > 1$, we may obtain a similar result by invoking an additional hypothesis:

With the assumption that the arithmetic genus of W_p is an enumerative character of the variety, its value is $(-1)^{p-1}$.

For, with this assumption, Todd(21) has proved that $P_a + (-1)^p$ is expressible as a homogeneous linear function of the canonical invariants.

8. *Notes and examples.* In this section we briefly illustrate and comment upon the preceding results.

(i) It has been assumed throughout that the Picard variety V_q has general moduli; if this is not the case, V_q may possess other groups of automorphisms than the group of transformations of the first kind. When $q = 1$, V_q can only be either general or harmonic or equianharmonic, and no great difficulty is then encountered; but the examination of the transformations in the cases $q \geqslant 2$ is very complicated. The problem of determining all possible automorphisms of the Picard surfaces has been solved by Scorza(11), but, for $q > 2$, no general result is yet known.

(ii) It is natural to inquire on what hypotheses the variety W_p, as specified in § 4, will be free from exceptional varieties. Our assumptions that the congruence $\{V_q\}$ contains no nodal members, and that the only reducible members are the varieties $V_{q,s}$, counted s times, secure that the invariant $[X_0(W_p)]$, calculated from a pencil $|S|$ belonging to $\{V_q\}$, should be zero. In the case $p = 2$, these assumptions certainly lead to the conclusion that the surface in question is free from exceptional curves; if, however, $p > 2$, the analogous result does not necessarily hold. Thus, supposing that $d = 1$, and that V_{p-q} contains exceptional subvarieties, then so also will $W_p = V_q \times V_{p-q}$, and any pseudo-Abelian variety mapped on W_p in the usual way. In this connexion we may note Gaeta's conjecture(6) that any product variety will be free from exceptional subvarieties provided that each factor of the product is similarly free.

(iii) The problem of classifying the pseudo-Abelian varieties W_p of type q may be described in general terms as follows: given the invariant characters of the congruence $\{V_q\}$, to determine the possible forms of variety V_{p-q} and the corresponding values of d. (On the assumption that V_q has general moduli, the involution cut by $\{V_{p-q}\}$ on V_q presents no remarkable features.) The process of classification is based on the result (§ 3) that each V_{p-q} is invariant under a permutable group $\mathscr{G}_d$ of automorphisms; it is this group that generates the involution cut on V_{p-q} by $\{V_q\}$. We have

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thus to obtain (a) the varieties V_{p-q} which admit such groups of automorphisms, and (b) the subclass of these varieties which can carry involutions birationally equivalent to $\{V_q\}$. The only complete solutions of this problem which have so far been found are for the case $p-q=1$, where the curves V_1 have genus 1, 2 or 3.

(iv) We have seen in §4 that the canonical system $|X_{p-1}|$, when effective, in general possesses fixed multiple components and multiple base varieties, all belonging to the congruence $\{V_q\}$; evidently the dimension P_g-1 of $|X_{p-1}|$ depends not only on these base varieties but also on the nature of $\{V_q\}$. The simplest case is that for which $q=p-1$; W_p then possesses a pencil $\{V_{p-1}\}$ of trajectories, and the only base points of $|X_{p-1}|$ are the $(s-1)$-ple varieties $V_{p-1,s}$. In this case the calculation of the geometric genus P_g and the plurigenera $P_i\,(i=2,3,\ldots)$ is the same as for the elliptic surfaces; the details, for threefolds, will be found in (9). Another result of general character which can be established in the case $q=p-1\,(p\geqslant 3)$, concerns the congruence $\{V_1\}$; we may show that *a necessary and sufficient condition that W_p should contain a congruence of rational curves is* $P_{12}=0$. (For the proof cf. (9).)

In this case and, more generally, that in which V_{p-q} is a variety of geometric genus and plurigenera zero, we must have $P_g=P_i=0$. For, to begin with, the locus of ∞^1 varieties V_{p-q} must obviously have geometric genus and plurigenera zero; and, proceeding step by step, we may show that an analogous result holds for any hypersurface belonging to $\{V_{p-q}\}$; hence, in conclusion, it holds for W_p also.

More interesting is the case where V_{p-q} is a Picard variety, which we suppose to be free from exceptional subvarieties. Reasoning as before, we may show that either $P_g=P_i=0$, or else W_p possesses a pure canonical hypersurface (effective or virtual) of order zero. If, in particular, $\{V_q\}$ is a Picard congruence, then W_p is itself a Picard variety (special of type q). Other important varieties of this class map certain species of involutions on special Picard varieties.

(v) The analytical representation given in §5 applies only to the cyclic pseudo-Abelian varieties; in order to discuss the non-cyclic types which, as shown in (9), certainly exist, it would first be necessary to obtain projective models of the various kinds of involution cut by $\{V_{p-q}\}$ on the generic V_q. At present general results are known only for elliptic curves ($q=1$); in addition to the cyclic form which is of course deducible from §5, there is a type corresponding to an Abelian group $\mathscr{G}_d$ of base 2; this can arise only when d is composite, say $d=mn$. Thus, supposing that $F(x_1,x_2)=0$ is the equation of a non-singular plane cubic, we may show that the representation in question is of the form

$$v=\{G(x_1,x_2)\}^{1/m}+\{H(x_1,x_2)\}^{1/n},$$

where $G=0$, $H=0$ represent curves of orders m, n respectively, having respectively m-point and n-point intersection with $F=0$ at all common points; in order that the elliptic curve so represented should be irreducible, these intersections must satisfy certain conditions, for which reference may be made to (4). It then follows that W_p is representable by the equations

$$v=\{G(x_1,x_2)\,g(y_1,y_2,\ldots,y_p)\}^{1/m}+\{H(x_1,x_2)\,h(y_1,y_2,\ldots,y_p)\}^{1/n},\quad F=0,\quad f=0,$$

where, as in §5, the equation $f=0$ represents the variety V^*_{p-1}.

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(vi) The invariant characters of W_p include the determinant d and the various numbers s, and the continuously varying parameters known as moduli. Evidently the number of moduli is equal to the sum of the $q(q+1)/2$ moduli of V_q and the moduli of the congruence $\{V_q\}$; hence, in the particular case where $\{V_q\}$ is birational, the number is simply $q(q+1)/2$.

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