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Electron dynamics inside a vacuum tube diode through linear differential equations

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In this paper we analyze the motion of charged particles in a vacuum tube diode by solving linear differential equations. Our analysis is based on expressing the volume charge density as a function of the current density and coordinates only, i.e. $\rho = \rho(J, z)$, while in the usual scheme the volume charge density is expressed as a function of the current density and electrostatic potential, i.e. $\rho = \rho(J, V)$. We show that, in the case of slow varying charge density, the space-charge-limited current is reduced up to 50%. Our approach gives the well-known behavior of the classical current density proportional to the three-halves power of the bias potential and inversely proportional to the square of the gap distance between electrodes, and does not require the solution of the nonlinear differential equation normally associated with the Child–Langmuir formulation.

1. Introduction

The motion of charged particles accelerated across a gap is of wide interest in the fields such as high power diodes and vacuum microelectronics. Child and Langmuir (Child 1911; Langmuir 1913) first studied the space-charge-limited emission for two infinite parallel plane electrodes at a fixed voltage V_0 in vacuum separated by a distance D. The charges produced at the cathode are made to increase (e.g. by increasing the temperature in a thermionic diode or by increasing the power of a laser in case of a photocathode). Interestingly, a saturation is observed and the current is said to be space-charge-limited. A useful approximation for the amount of current flowing in such cases is the Child–Langmuir expression for transmitted current. For electrodes having a potential difference V and separated by a distance D, the Child–Langmuir law is obtained by solving Poisson's equation,

$$\frac{d^2V}{dz^2} = -\frac{\rho}{\epsilon_0}, \tag{1.1}$$

where V is the electrostatic potential, ρ is the volume charge density, and ϵ_0 is the permittivity of free space (Pollak and Stump 2002). Since this diode is assumed to have infinite extension in x–y directions, we can define current density by

$$J(z) = \rho(z)v(z) = -J_{CL}, \tag{1.2}$$

where v is the velocity of the electrons. By charge conservation the current density cannot vary with z, hence the current density is constant. Now we can find the velocity

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of electrons by the conservation of energy,

$$\frac{mv^2}{2} - eV = 0, \tag{1.3}$$

where m and e are the electron's mass and charge, respectively. In (1.3) we have assumed that the electron is initially at rest in the grounded cathode. Solving (1.3) for the velocity and substituting in (1.2), we obtain the volume charge density as a function of current density and electrostatic potential,

$$\rho(z) = -\frac{J_{CL}}{\sqrt{2eV/m}}. \tag{1.4}$$

Substituting (1.4) into (1.1), we have a second-order nonlinear differential equation for the electrostatic potential

$$\frac{d^2V}{dz^2} = \frac{J_{CL}}{\epsilon_0\sqrt{2eV/m}} \tag{1.5}$$

with the following boundary conditions,

$$\left.\frac{dV}{dz}\right|_{z=0} = 0 \quad \text{and} \quad \left.V(z)\right|_{z=0} = 0. \tag{1.6}$$

Using the following ansatz for (1.5),

$$V(z) = V_0\left(\frac{z}{D}\right)^p \tag{1.7}$$

and substituting (1.7) into (1.5), we find that the non-linear differential equation is satisfied for

$$p = \frac{4}{3} \quad \text{and} \quad J_{CL} = \frac{4\epsilon_0}{9D^2}\sqrt{\frac{2e}{m}}V_0^{3/2}. \tag{1.8}$$

The volume charge density in the gap is determined by (1.4),

$$\rho(z) = -\frac{4\epsilon_0 V_0}{9D^2}\left(\frac{D}{z}\right)^{2/3}. \tag{1.9}$$

The main result is the Child–Langmuir law, which states that the behavior of the current density is proportional to the three-halves power of the bias potential and inversely proportional to the square of the gap distance between electrodes.

Since the derivation of this fundamental law, many important and useful variations on the classical Child–Langmuir law have been investigated to account for special geometries (Langmuir and Blodgett 1923, 1924; Page and Adams 1945), relativistic electron energies (Jory and Trivelpiece 1969), non-zero initial electron velocities (Langmuir 1923; Jaffé 1944), quantum mechanical effects (Lau et al. 1991; Ang et al. 2003; Jensen et al. 2012), non-zero electric field on the cathode surface (Barbour et al. 1953), and using vacuum capacitance (Umstattd et al. 2005). In technology, space-charge effects can help to stabilize emission current and to inhibit sharp focusing of particle beams (Forbes 2008; Rokhlenko et al. 2010).

Given the wide interest and applicability in this topic, we present the analysis of the motion of charged particles in a vacuum tube diode by solving linear differential equations. This paper aims to present an alternative mathematical treatment of the basic one-dimensional Child–Langmuir model. Our derivation is based on the linearization of Poisson's equation and involves the introduction of a new concept

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that we call *adiabatic* charge density. We show that, in the case of slow varying charge density, the space-charge-limited current is reduced up to 50%.

2. Alternative approach

In the Child–Langmuir derivation we have expressed the volume charge density as a function of current density and electrostatic potential; let us now try to express the volume charge density as a function of current density and spatial coordinate. The velocity of a charged particle subjected to an electric field is governed by Newton's second law,

$$m\frac{dv}{dt} = -eE(z), \tag{2.1}$$

where $e = 1.6 \times 10^{-19}$ C and $E(z)$ is the electric field. We can obtain the electric field inside the gap by integrating Gauss's law from $z = 0$ to an arbitrary position in the gap,

$$E(z) = \frac{1}{\epsilon_0}\int_0^z \rho(\zeta)d\zeta, \tag{2.2}$$

where we have used the fact that $E(z = 0) = 0$. Integrating by parts (2.2), we have

$$E(z) = \frac{\rho}{\epsilon_0}z - \frac{1}{\epsilon_0}\int_0^z \zeta\frac{d\rho}{d\zeta}d\zeta. \tag{2.3}$$

Let us now work in the adiabatic regime and neglect the contribution of $\rho'(\zeta)$ in (2.3), i.e. the volume charge density is slowly varying in space. The validity of this approximation is given in Appendix A. Substituting (2.3) into (2.1) we have

$$m\frac{dv}{dt} = \frac{d}{dz}\left(\frac{mv^2}{2}\right) = -\frac{e\rho}{\epsilon_0}z. \tag{2.4}$$

Integrating (2.4) from $z = 0$ to an arbitrary position, we have

$$\frac{mv^2}{2} = -\frac{e\rho}{2\epsilon_0}z^2 + \frac{e}{2\epsilon_0}\int_0^z \zeta^2\frac{d\rho}{d\zeta}d\zeta, \tag{2.5}$$

where we have integrated by parts the right-hand term. Applying the adiabatic condition to (2.5), we obtain

$$\frac{mv^2}{2} = -\frac{e\rho}{2\epsilon_0}z^2. \tag{2.6}$$

Solving for the velocity in (1.2) and substituting into (2.6), we have

$$\frac{mJ^2}{2\rho^2} = -\frac{e\rho}{2\epsilon_0}z^2. \tag{2.7}$$

Using (2.7) we can express the volume charge density as a function of current density and coordinates as

$$\rho(z) = -\left(\frac{mJ^2\epsilon_0}{ez^2}\right)^{1/3}. \tag{2.8}$$

Substituting (2.8) into (1.1), we have

$$\frac{d^2V}{dz^2} = \frac{1}{\epsilon_0}\left(\frac{mJ^2\epsilon_0}{ez^2}\right)^{1/3}. \tag{2.9}$$

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Equation (2.9) is a linear second-order differential equation which can be solved by quadratures. Integrating (2.9) twice and using the initial conditions given in (1.6), we obtain solution for electrostatic potential as

$$V(z) = \frac{9}{4\epsilon_0}\left(\frac{mJ^2\epsilon_0 z^4}{e}\right)^{1/3}. \tag{2.10}$$

If we apply the boundary condition of the fixed potential anode $V(z = D) = V_0$ in (2.10), we can find the current density in the adiabatic regime, which is given by

$$|J| = \frac{8\epsilon_0}{27D^2}\sqrt{\frac{e}{m}}V_0^{3/2}. \tag{2.11}$$

Comparing (2.11) with (1.8), we have $|J| \approx 0.47\ J_{CL}$, this result shows that the space-charge-limited current is reduced up to 50%. Substituting (2.11) into (2.8) and (2.10), we find

$$\rho(z) = -\frac{4\epsilon_0 V_0}{9D^2}\left(\frac{D}{z}\right)^{2/3} \quad \text{and} \quad V(z) = V_0\left(\frac{z}{D}\right)^{4/3}. \tag{2.12}$$

Equations (2.12) are the same exact results derived with the Child–Langmuir formulation.

If we want to go beyond the adiabatic approximation, we integrate again by parts (2.3) to obtain

$$E(z) = \frac{\rho}{\epsilon_0}z - \frac{\rho'}{2\epsilon_0}z^2 + \frac{1}{2\epsilon_0}\int_0^z \zeta^2\frac{d^2\rho}{d\zeta^2}d\zeta. \tag{2.13}$$

Assuming now that we can neglect the last term in (2.13) (the validity of this approximation is given in Appendix B), we have by Newton's second law,

$$m\frac{dv}{dt} = \frac{d}{dz}\left(\frac{mv^2}{2}\right) = -\frac{e\rho}{\epsilon_0}z + \frac{e\rho'}{2\epsilon_0}z^2. \tag{2.14}$$

Integrating by parts (2.14), we have

$$\frac{mv^2}{2} = -\frac{e\rho}{2\epsilon_0}z^2 + \frac{e}{3\epsilon_0}z^3\frac{d\rho}{dz} - \frac{e}{3\epsilon_0}\int_0^z \zeta^3\frac{d^2\rho}{d\zeta^2}d\zeta. \tag{2.15}$$

Neglecting the last term in (2.15), we have

$$\frac{mv^2}{2} = -\frac{e\rho}{2\epsilon_0}z^2 + \frac{e}{3\epsilon_0}z^3\frac{d\rho}{dz}. \tag{2.16}$$

Solving for the velocity in (1.2) and substituting into (2.16) we end up with

$$\frac{mJ^2}{2\rho^2} = -\frac{e\rho}{2\epsilon_0}z^2 + \frac{e}{3\epsilon_0}z^3\frac{d\rho}{dz}. \tag{2.17}$$

Equation (2.17) is a linear differential equation over ρ^3. Solving (2.17) for the charge density, we find

$$\rho^3(z) = -\frac{9m\epsilon_0 J^2}{13ez^2} + Cz^{9/2}, \tag{2.18}$$

where C is a constant of integration. If we apply the following condition $\rho'(z = D) = 0$ in (2.18), i.e. the charge density is minimum at the anode, we have the following charge

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distribution in the vacuum tube diode,

$$\rho(z) = -\left(\frac{m\epsilon_0 J^2}{13ez^2}\right)^{1/3}\left[9+4\left(\frac{z}{D}\right)^{13/2}\right]^{1/3}. \tag{2.19}$$

Substituting (2.19) into (1.1), we have

$$\frac{d^2V}{dz^2} = \frac{1}{\epsilon_0}\left(\frac{m\epsilon_0 J^2}{13ez^2}\right)^{1/3}\left[9+4\left(\frac{z}{D}\right)^{13/2}\right]^{1/3}. \tag{2.20}$$

Equation (2.20) is a linear differential equation which can be solved by quadratures. Integrating (2.20) with Mathematica we find

$$V(z) = \frac{1}{\epsilon_0}\left(\frac{243m\epsilon_0 J^2}{13e}\right)^{1/3}\left\{z^{4/3}\left[{}_2F_1\left(-\frac{1}{3},\frac{2}{39};\frac{41}{39};-\frac{4}{9}\left(\frac{z}{D}\right)^{13/2}\right)\right.\right.$$
$$\left.\left.-\frac{1}{4}{}_2F_1\left(-\frac{1}{3},\frac{8}{39};\frac{47}{39};-\frac{4}{9}\left(\frac{z}{D}\right)^{13/2}\right)\right]\right\}, \tag{2.21}$$

where ${}_2F_1(a,b;c;z)$ is a hypergeometric function (Lebedev 1972). If we apply the boundary condition of the fixed potential anode $V(z=D)=V_0$ in (2.21), we can find the current density, which is given by

$$|J| = \frac{\epsilon_0}{D^2}\sqrt{\frac{13e}{243m\left[{}_2F_1\left(-\frac{1}{3},\frac{2}{39};\frac{41}{39};-\frac{4}{9}\right)-\frac{1}{4}{}_2F_1\left(-\frac{1}{3},\frac{8}{39};\frac{47}{39};-\frac{4}{9}\right)\right]^3}}V_0^{3/2}. \tag{2.22}$$

Evaluating (2.22) numerically we find that $|J| \approx 0.57\ J_{CL}$. Our approach can be easily extended to include the next order derivatives in the volume charge density to improve the accuracy of the current charge density; unfortunately this will lead us to solve non-linear differential equations to express the volume charge density as a function of current density and spatial coordinate, and the expressions for electrostatic potential and charge distribution become far more complex.

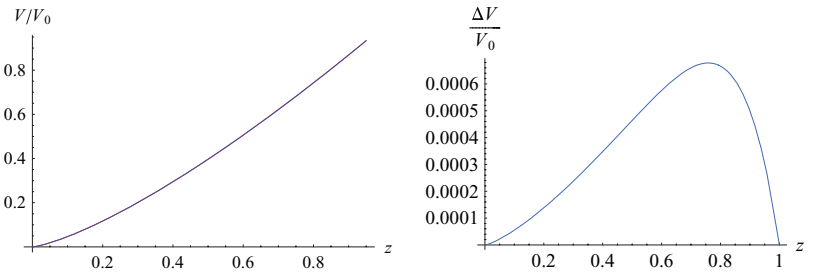
In Fig. 1 we have plotted the electrostatic potential in the adiabatic regime, and also the difference between electrostatic potential in adiabatic and non-adiabatic regimes, respectively. Note how there is no difference at $z=0$ and $z=D$ for the electrostatic potential function in order for the applied bias to be V_0 in the vacuum tube diode. One can see from Fig. 1(b) that the electrostatic potential is always greater for the non-adiabatic regime for $0<z<D$, which means that the electric field in this regime increases, therefore the difference between the electrostatic potential in adiabatic and non-adiabatic regimes has one maxima.

In Fig. 2 we have plotted the volume charge density for adiabatic and non-adiabatic regimes, where we can see that the functions are practically identical except at $z=D$, where the volume charge density for the non-adiabatic regime has a horizontal slope.

3. Conclusions

In this paper we develop an alternative method of deriving the $V^{3/2}/D^2$ scaling of the Child–Langmuir law which avoids the need of solving a nonlinear differential equation and the use of conservation of energy. The key features of this paper have been the linearization of Poisson's equation and the introduction of the concept of adiabatic volume charge distribution. We have obtained solutions for electrostatic potential and volume charge density in adiabatic and non-adiabatic approximations by using a linear differential form of Poisson's equation. Within our formulation, we

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(a) (b)

FIGURE 1. (Colour online) Plots showing (a) the electrostatic potential in the adiabatic approximation with $D = 1$, (b) difference between electrostatic potential in adiabatic and non-adiabatic approximations. Note how the functions for electrostatic potential have the same value at $z = 0$ and $z = D$ such that the applied bias in the vacuum tube diode is V_0, and are really close to each other for $0 < z < D$.

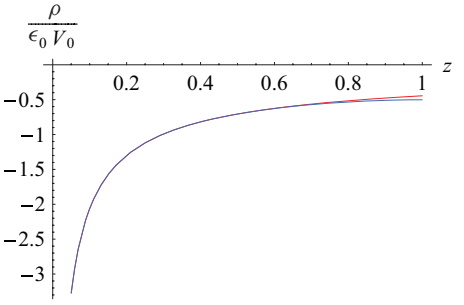


FIGURE 2. (Colour online) We have plotted the graphs of the volume charge density for adiabatic and non-adiabatic regimes with $D = 1$. Note how the functions superimposed except at $z = D$, where the volume charge density for the non-adiabatic regime has a horizontal slope.

have shown that for the adiabatic regime the space charge current density is reduced to at most 50%. The theory applies to any form of field-induced charged-particle emission, provided that all particles are the same and have a zero emission velocity. These results should contribute primarily to a better basic understanding of the space-charge-limited current in a vacuum tube diode with planar geometry.

Appendix A. Validity of the adiabatic approximation

We know that the current density in a planar diode can be written in terms of the particles velocity by

$$J = \rho v. \tag{A 1}$$

Integrating (A 1) with respect to time, we have

$$\int_0^t J dt = \int_0^z \rho(\zeta) d\zeta = \epsilon_0 E, \tag{A 2}$$

where we have used Gauss's law in the last step and assumed that the electric field is zero at $z = 0$. We conclude from (A 2) that $E = Jt/\epsilon_0$, therefore from Newton's law

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we get

$$m\frac{d^2z}{dt^2} = -eE = -\frac{eJ}{\epsilon_0}t. \tag{A 3}$$

Differentiating both sides of (A 3) with respect to time gives the Benham–Müller–Llewellyn equation (Plonsey and Collin 1961),

$$\frac{d^3z}{dt^3} = -\frac{eJ}{m\epsilon_0}. \tag{A 4}$$

Equation (A 4) is a dynamic equation of motion for the electron in the planar vacuum tube diode. If we approximate the electric field by $E = \rho z/\epsilon_0$ and insert it in Newton's law, we have

$$m\frac{d^2z}{dt^2} = -eE = -\frac{e\rho}{\epsilon_0}z. \tag{A 5}$$

If we differentiate both sides of (A 5) with respect to time, we get

$$\frac{d^3z}{dt^3} = -\frac{e}{m\epsilon_0}\left(\rho\frac{dz}{dt} + z\frac{d\rho}{dt}\right) = -\frac{e}{m\epsilon_0}\left(J + z\frac{d\rho}{dt}\right). \tag{A 6}$$

In order to recover the Benham–Müller–Llewellyn equation, the following condition must hold

$$|J| \gg \left|z\frac{d\rho}{dt}\right|. \tag{A 7}$$

Using the maximum value for $z = D$ in (A 7), we find the validity of the adiabatic approximation

$$\left|\frac{d\rho}{dt}\right| \ll \left|\frac{J}{D}\right|. \tag{A 8}$$

Equation (A 8) means that the time rate of change of charge density must be small compared with the corresponding current density divided by separation between the plates in the vacuum tube diode.

Appendix B. Validity of non-adiabatic approximation

To obtain the validity of non-adiabatic approximation we insert the electric field $E = \rho z/\epsilon_0 - \rho' z^2/2\epsilon_0$ into Newton's second law, i.e.

$$m\frac{d^2z}{dt^2} = -eE = -\frac{e\rho}{\epsilon_0}z + \frac{e\rho'}{2\epsilon_0}z^2. \tag{B 1}$$

If we differentiate both sides of (B 1) with respect to time, we get

$$\frac{d^3z}{dt^3} = -\frac{e}{m\epsilon_0}\left(\rho\frac{dz}{dt} + z\frac{d\rho}{dt} - \rho' z\frac{dz}{dt} - \frac{z^2\rho''}{2}\frac{dz}{dt}\right) = -\frac{e}{m\epsilon_0}\left(J - \frac{z^2}{2}\frac{d\rho'}{dt}\right). \tag{B 2}$$

In order to recover the Benham–Müller–Llewellyn equation, the following condition must hold,

$$|J| \gg \left|\frac{z^2}{2}\frac{d\rho'}{dt}\right|. \tag{B 3}$$

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Using the maximum value for $z = D$ in (B 3), we find the validity of the non-adiabatic approximation,

$$\left|\frac{d\rho'}{dt}\right| \ll \left|\frac{2J}{D^2}\right|. \tag{B 4}$$

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