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A Simple Method to Solve the Instantaneous Kinematics of the 5-RPUR Parallel Manipulator

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SUMMARY
In this work a simple method to solve the kinematics of the 5-RPUR parallel manipulator is introduced. Dealing with the displacement analysis, the kinematic constraint equations required to address the forward–inverse displacement analysis are established according to linear combinations of two vectors attached to the moving platform. Then, besides the solution of the inverse displacement analysis two strategies are proposed in order to solve the forward position analysis. Finally, the input–output equations of velocity and acceleration are systematically obtained by resorting to reciprocal-screw theory. Numerical examples are provided with the purpose to illustrate the proposed method. Furthermore, the numerical results obtained by means of screw theory are confirmed with the aid of commercially available software.

KEYWORDS: Parallel manipulators; Limited mobility; Klein form; Screw theory; Kinematics.

1. Introduction

Parallel manipulators with limited mobility have been extensively investigated due to their positive impact in both academia and industry. In that trend, translational and spherical parallel manipulators have attracted more attention compared with five-degrees-of-freedom parallel manipulators (5-dof PMs). The reason for this choice is easy to explain. While in the industry context applications of translational and spherical robot manipulators are clear and easy to implement, applications of 5-dof PMs require, among other issues, a full comprehension of their mobility owing the existence of mixed coupled motions. Consider for instance that almost two decades ago it had been believed that 5-dof PMs with identical limb structures cannot be constructed,[1,2] an exaggerated presumption. However, few years later such assumption was abolished with the introduction of symmetrical 5-dof PMs. Since then the publication of theoretical papers dealing with the study and applications of 5-dof PMS has increased considerably. Consider for instance that Fang and Tsai[3] reported the synthesis of symmetrical 5-dof PMs based on the reciprocity of the involved twist and wrench systems. Lu and Hu[4] studied the stiffness analysis of 5-dof PMs composed of SPR-type and RPS-type limbs by means of the principle of virtual work. Unless otherwise stated in the rest of the contribution R, P, S, and U stand for Revolute, Prismatic, Spherical, and Universal joints, respectively. Gallardo-Alvarado et al.[5]

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approached the acceleration analysis of a decoupled 5-dof PM by means of the theory of screws. Masouleh et al.[6] investigated the forward kinematic problem of the 5-RPUR parallel manipulator based on algebraic geometry[7] where study parameters are employed in order to elucidate the issue in a seven-dimensional kinematic space, an in-deep contribution. Sangveraphunsiri and Chooprasird[8] addressed the dynamics and control of a H-4-type parallel manipulator equipped with a rotary table, a viable option to simplify the kinematics of this class of parallel manipulators. The singularity analysis of parallel manipulator generators of the 3T2R motion was investigated by Amine et al.[9] by combining Grassmann–Cayley algebra with Grassmann geometry. Pisla et al.[10] introduced a hybrid parallel manipulator for laparoscopic surgery, an option for substituting the classical serial robots recursively employed in surgery tasks. Song et al.[11] introduced a 5-dof PM where the Klein form associated to the wrench and twist screws combined with a multi-objective dimensional synthesis is employed with the purpose to optimize the design. Ding et al.[12] developed a software for the semi-automated structure synthesis of 5-dof symmetrical and asymmetrical parallel manipulators. Loizaga et al.[13] shown that the synthesis of 5-dof partially decoupled manipulators can be achieved combining the idea of multi-level manipulators[14,15] and the concept of plain leg surfaces. Wang et al.[16] introduced a 5-dof PM able to perform the 3T2R motion showing a high-rotational capability, an attribute preferred in this kind of robots.[17] Cao et al.[18] introduced an approach devoted to investigate the compliance modeling overconstrained parallel manipulators performing the 3T2R motion. Kang et al.[19] applied redundancy in statically balanced 5-dof PMs composed of pantographs whose results provided interesting abilities to the robot like increased load-carrying capacity, capability to avoid singularities, and the possibility to perform in sequence auxiliary grasping tasks. Overall, the advantages of using the 5-dof PM as one structure were pointed out by Shayya et al.[20]

This work is devoted to investigate the kinematics of the 5-RPUR parallel manipulator. In that concern it is opportune to mention that while considerable effort has been devoted to solve the kinematic analysis of the Gough–Stewart platform,[21] see for instance refs.[22–26] on the whole it must be recognized that there are fewer contributions conducted on the kinematic analysis of the 5-RPUR parallel manipulator which are credited to a reduced group of researchers.[6,9,27] Furthermore, the acceleration analysis of the 5-RPUR parallel manipulator has not been reported in previous works, and then it occurs that those contributions are devoted mainly to solve the forward kinematic problem and the singularity analysis of the robot. The rest of the contribution is organized as follows. In Section 2 the architecture of the parallel manipulator is explained, remarking the conditions of mechanical assembly of the joints in each leg in order to satisfactorily achieve the 3T2R motion. In Section 3 the finite kinematics of the robot is approached establishing simple closure equations obtained upon linear combinations of two unit vectors attached to the moving platform. Then, two strategies are proposed to solve the forward displacement analysis. In Section 4 the input–output equations of velocity and acceleration of the parallel manipulator are systematically obtained by resorting to reciprocal-screw theory. With the purpose to illustrate the method, in Section 5 numerical examples covering most of the topics treated in the contribution are provided. Finally, some conclusions are given at the end of the paper.

2. Topology of the Robot Manipulator

In this section the architecture of the parallel manipulator at hand is outlined. With reference to Fig. 1, the parallel manipulator under study comprises a moving platform (m) connected to a fixed platform (0) by means of five identical RPUR-type limbs where the prismatic joints P are actuated. Meanwhile, revolute and universal joints, notated as R and U respectively, assume the role of passive kinematic pairs.

In order to explain the geometry of the robot, let O_XYZ be a reference frame attached to the fixed platform with the Z-axis normal to the plane of the fixed platform. The lower revolute joints connecting the limbs to the fixed platform are characterized by points A_i, located by vectors $\boldsymbol{a}_i$, and a unit vector $\hat{\boldsymbol{u}}$ along the X-axis denoting the orientation of the lower revolute joints. Said otherwise, the lower revolute joints have parallel axes. Points A_i define the shape's vertices of a pentagon of sides a, b, and c, where point A_1 is chosen as the origin O of the fixed reference frame. Following the lower ith revolute joint, we have a prismatic joint which is perpendicular to the vector $\hat{\boldsymbol{u}}$ and it is related with the generalized coordinate q_i. The next kinematic pair of the ith limb is a universal joint simulated with two concurrent revolute joints whose axes, defined by $\hat{\boldsymbol{u}}$ and $\hat{\boldsymbol{v}}$, intersect at point B_i which is located by vector $\boldsymbol{b}_i$. On the other hand, the legs of the robot are connected to the

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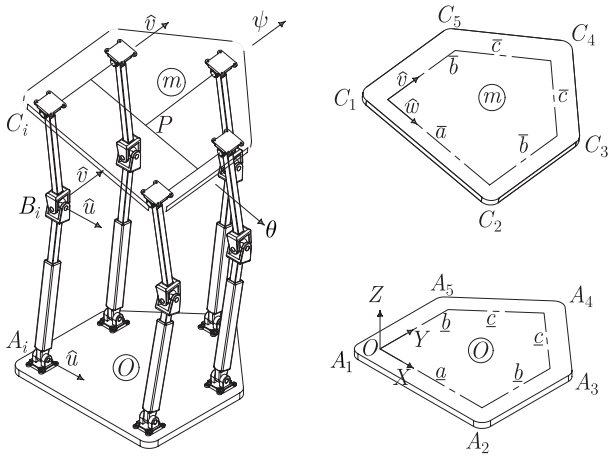


Fig. 1. Topology and geometry of the 5-RPUR parallel manipulator.

moving platform through revolute joints whose nominal positions are notated by points C_i, located by vectors $\boldsymbol{c}_i$. The upper revolute joints have parallel axes defined by the unit vector $\hat{\boldsymbol{v}}$. With this arrangement of the universal and revolute joints, the parallel manipulator loses one freedom, that is, the moving platform cannot rotate about any axis parallel to $\hat{\boldsymbol{u}} \times \hat{\boldsymbol{v}}$.[27] Similar to the fixed platform, points C_i form a pentagon of sides $\bar{a}$, $\bar{b}$, and $\bar{c}$. Furthermore, d denotes the signed distance between points B_i and C_i. With this architecture, the orientation of the moving platform may be described according to the angles θ and ψ. In that concern, it is interesting to note that the orientation of the moving platform can be also specified according to the unit vectors $\hat{\boldsymbol{v}}$ and $\hat{\boldsymbol{w}}$, where $\hat{\boldsymbol{w}}$ is directed from C_1 to C_2. Hence, conveniently the pose of the moving platform may be expressed as observed from the fixed platform according to the sets $\{\boldsymbol{c}_1, \theta, \psi\}$ or $\{\boldsymbol{c}_1, \hat{\boldsymbol{v}}, \hat{\boldsymbol{w}}\}$.

The topology of the parallel manipulator under study is easy to explain. However, the solution of its kinematics is a complex task due to the necessity to compute intermediate variables.

3. Displacement Analysis

In this section the displacement analysis of the parallel manipulator is presented. In that concern it is opportune to emphasize that the displacement analysis, in particular the forward kinematic problem, is a formidable task that has been carefully solved by Masouleh et al.[6] by resorting to study parameters, an elegant approach. This section is an attempt to simplify the formulation of the closure equations associated to the displacement analysis of the 5-RPUR parallel manipulator.

3.1. Kinematic constraint equations

With reference to Fig. 1 let us consider that the position of the moving platform m, as observed from the fixed platform 0, is determined by the vector $\boldsymbol{c}_1$ while its orientation is defined by unit vectors $\hat{\boldsymbol{w}}$ and $\hat{\boldsymbol{v}}$ located on the plane of the moving platform. Then, the vectors $\boldsymbol{c}_i$ may be expressed as follows:

$$\boldsymbol{c}_i = \boldsymbol{c}_1 + \delta_i \hat{\boldsymbol{v}} + \lambda_i \hat{\boldsymbol{w}} \quad \lambda_i, \delta_i \in \Re \quad i = 1, 2, \ldots, 5 \tag{1}$$

Therein $\hat{\boldsymbol{w}}$ is a unit vector pointed from C_1 to C_2 whereas $\hat{\boldsymbol{v}}$ is a unit vector pointed from C_1 to C_5. Meanwhile, it is evident that

$$\hat{\boldsymbol{v}} \cdot \hat{\boldsymbol{v}} = 1, \quad \hat{\boldsymbol{w}} \cdot \hat{\boldsymbol{w}} = 1 \tag{2}$$

Furthermore, taking into account that $\hat{\boldsymbol{w}}$ and $\hat{\boldsymbol{v}}$ are orthonormal vectors, it follows that

$$\hat{\boldsymbol{w}} \cdot \hat{\boldsymbol{v}} = 0 \tag{3}$$

On the other hand, the revolute joints constraint the mobility of the limbs of the robot in such a way that owing the peculiar assembly of universal and revolute joints we have

$$(\boldsymbol{b}_i - \boldsymbol{a}_i) \cdot \hat{\boldsymbol{u}} = 0 \tag{4}$$

and

$$(\boldsymbol{c}_i - \boldsymbol{b}_i) \cdot \hat{\boldsymbol{v}} = 0 \tag{5}$$

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Meanwhile based on the generalized coordinates q_i it is evident that

$$(\boldsymbol{b}_i - \boldsymbol{a}_i) \cdot (\boldsymbol{b}_i - \boldsymbol{a}_i) = q_i^2 \tag{6}$$

Similarly, one can consider that

$$(\boldsymbol{c}_i - \boldsymbol{b}_i) \cdot (\boldsymbol{c}_i - \boldsymbol{b}_i) = d^2 \tag{7}$$

The closure equations are completed taking into account the limited mobility of the parallel manipulator. In fact, it is straightforward to demonstrate that the component of $\hat{\boldsymbol{v}}$ along the $X-$axis vanishes.

After, the orientation of the moving platform may be described building its rotation matrix R based on the angles θ and ψ, see Fig. 1, as follows:

$$\mathrm{R} = \begin{bmatrix} \cos(\psi) & 0 & \sin(\psi) \\ \sin(\theta)\sin(\psi) & \cos(\theta) & -\sin(\theta)\cos(\psi) \\ -\cos(\theta)\sin(\psi) & \sin(\theta) & \cos(\theta)\cos(\psi) \end{bmatrix} \tag{8}$$

Furthermore, the rotation matrix can be also obtained as follows:

$$\mathrm{R} = \left[\hat{\boldsymbol{w}}\ \hat{\boldsymbol{v}}\ \hat{\boldsymbol{w}} \times \hat{\boldsymbol{v}}\right] \tag{9}$$

Hence equating (8) and (9) it follows that

$$\hat{\boldsymbol{w}} = \cos(\psi)\hat{\boldsymbol{i}} + \sin(\theta)\sin(\psi)\hat{\boldsymbol{j}} - \cos(\theta)\sin(\psi)\hat{\boldsymbol{k}} \tag{10}$$

and

$$\hat{\boldsymbol{v}} = \cos(\theta)\hat{\boldsymbol{j}} + \sin(\theta)\hat{\boldsymbol{k}} \tag{11}$$

Expressions (1)–(11) are sufficient to solve the inverse–forward displacement analysis of the parallel manipulator.

3.2. Forward displacement analysis

The forward displacement analysis consists of determining the pose, position, and orientation of the moving platform as observed from the fixed platform for a given set of generalized coordinates $q_i (i = 1, 2, \ldots, 5)$, that is, it is required to compute the vector $\boldsymbol{c}_1$ and the orientating angles θ and ψ. This analysis is a rather complicated task due to the necessity to calculate intermediate variables, for example, it is necessary to compute the coordinates of points B_i. Eqs. (1)–(11) bring us more than one combination to solve the forward displacement analysis. In what follows two strategies are briefly explained.

3.2.1. Forward displacement analysis (Method 1). The first method is composed of the next steps:

1. with the purpose to express the vectors $\boldsymbol{c}_i$ in terms of $\boldsymbol{c}_1$, θ, and ψ, substitute Eqs. (10) and (11) into Eq. (1);
2. resort to Eqs. (4)–(7) to formulate a system of 20 equations in 20 unknowns which correspond to the vectors $\boldsymbol{b}_i$, $\boldsymbol{c}_1$, θ, and ψ;
3. consider $\sin(\theta)$, $\cos(\theta)$, $\sin(\psi)$, and $\cos(\psi)$ as new variables;
4. the last step implies that we have 20 equations in 22 unknowns. Complete the 22 equations taking into account that $\sin^2(\theta) + \cos^2(\theta) = 1$ and $\sin^2(\psi) + \cos^2(\psi) = 1$;
5. solve the system of 22 equations by applying homotopy continuation method[28–30] or the Newton–Raphson technique if only one solution is needed.

3.2.2. Forward displacement analysis (Method 2). The second method comprises the following steps:

1. express the vectors $\boldsymbol{c}_i$ in terms of $\boldsymbol{c}_1$, $\hat{\boldsymbol{v}}$, and $\hat{\boldsymbol{w}}$, see Eq. (1);
2. based on Eqs. (2)–(7) generate a system of 23 equations in 23 unknowns which are precisely the unknown components of $\boldsymbol{c}_1$, $\hat{\boldsymbol{v}}$, $\hat{\boldsymbol{w}}$, and $\boldsymbol{b}_i$. In that regard consider that the component of $\hat{\boldsymbol{v}}$ along the X-axis vanishes;

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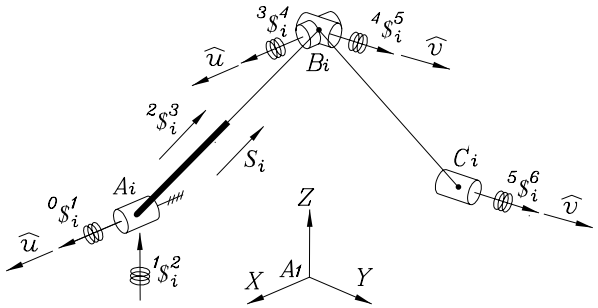


Fig. 2. Screws of one limb of the parallel manipulator.

3. solve the system of 23 equations by applying homotopy continuation method[28–30] or the Newton–Raphson technique if only one solution is required;
4. once the unit vectors $\hat{\boldsymbol{v}}$ and $\hat{\boldsymbol{w}}$ are computed, the angles θ and ψ may be obtained upon Eqs. (10) and (11), for example, $\cos(\psi) = \hat{\boldsymbol{w}} \cdot \hat{\boldsymbol{i}}$ and $\cos(\theta) = \hat{\boldsymbol{v}} \cdot \hat{\boldsymbol{j}}$.

The two methods here, provided for solving the forward displacement analysis of the 5-RPUR parallel manipulator, are easy to follow. However, the complete solution of this kind of equations is in reality a formidable task, for example, Masouleh et al.[6] concluded that the forward displacement analysis of the robot at hand admits 1680 finite solutions, real and complex, when the free available software Bertini[30] is employed. Finally, it is opportune to mention that in order to simplify the forward displacement analysis we can introduce coalescence conditions of mechanical assembly,[27] an available option also for six-degrees-of-freedom parallel manipulators.[31] In that regard, to successfully achieve a physical prototype it is necessary to consider additional conditions of manufacture in order to avoid uncertainties in the kinematic model yielding parasitic perturbation forces that cannot be compensated by the controls.[32]

3.3. Inverse displacement analysis

The inverse displacement analysis consists of finding the generalized coordinates q_i of the robot given the pose of the moving platform. In other words, given the position vector $\boldsymbol{c}_1$ and the angles θ and ψ it is required to compute the generalized coordinates q_i. The analysis is straightforward and it is included here only for the sake of completeness. In fact, the unit vectors $\hat{\boldsymbol{w}}$ and $\hat{\boldsymbol{v}}$ may be computed upon Eqs. (10) and (11), respectively. After, the vectors $\boldsymbol{c}_i$ are computed using Eq. (1). Gathering these results, the vectors $\boldsymbol{b}_i$ are determined solving simultaneously Eqs. (4), (5), and (7), whereas the generalized coordinates q_i are computed by means of Eq. (6), naturally according to the computed vectors $\boldsymbol{b}_i$. This procedure implies that we can find two feasible locations of B_i for each leg of the parallel manipulator.

4. Instantaneous Kinematics

In this section the instantaneous kinematics of the parallel manipulator is approached by means of the theory of screws. The screws representing the kinematic pairs of one limb of the robot are sketched in Fig. 2. For details of this section the reader is referred to Gallardo-Alvarado.[33]

In what follows a brief explanation of the screws is given. The screw $^0\$_i^1$ is associated with the passive revolute joint connecting the limb to the fixed platform while $^1\$_i^2$ denotes a pseudo revolute joint introduced with the purpose to avoid the handling of deficient rank Jacobian matrices which emerge from the velocity and acceleration analyses if this consideration is ignored. Furthermore, $^2\$_i^3$ is related with the actuable prismatic joint. On the other hand, the third and fourth revolute joints simulate the universal joint and are represented by the screws $^3\$_i^4$ and $^4\$_i^5$. Finally the screw $^5\$_i^6$ is associated with the passive revolute joint connecting the limb to the moving platform.

4.1. Input–output equation of velocity

The velocity analysis of the parallel manipulator at hand was approached by means of screw theory by Gallardo-Alvarado et al.[34] An improved strategy based also in the theory of screws is included here for the sake of completeness.

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Let ${}^0\boldsymbol{\omega}^m$ be the angular velocity vector of the moving platform m as observed from the fixed platform 0. Furthermore, let $\boldsymbol{v}_P$ be the velocity vector of a point P of the moving platform. Then, the velocity state of the moving platform as measured from the fixed platform, the vector $\boldsymbol{V}_P = ({}^0\boldsymbol{\omega}^m, \boldsymbol{v}_P)$, may be expressed in screw form through any of the five limbs of the manipulator as follows:

$$\boldsymbol{V}_P = \sum_{j=0}^{5} {}_j\omega_{j+1}^i \, {}^j\$_i^{j+1} \quad i = 1, 2, \ldots, 5 \tag{12}$$

Naturally the infinitesimal screws are computed considering point P as the reference pole. In what follows the input–output equation of velocity of the parallel manipulator is obtained by resorting to reciprocal-screw theory. In that regard the Klein form of the Lie algebra $se(3)$ of the Euclidean group $SE(3)$, notated as $\{*; *\}$, plays a central role.

Note that the screw ${}^3\$_i^4$ is reciprocal to all the screws in the same limb excepting the screws ${}^1\$_i^2$, ${}^2\$_i^3$, and ${}^5\$_i^6$. Hence taking into account that ${}_1\omega_2{}^i = 0$ due to the fact that it is associated to a pseudo revolute joint, then the application of the Klein form of the screw ${}^3\$_i^4$ to both sides of Eq. (12) with the reduction of terms we have

$$\dot{q}_i\{{}^3\$_i^4; {}^2\$_i^3\} + {}_5\omega_6^i\{{}^3\$_i^4; {}^5\$_i^6\} = \{{}^3\$_i^4; \boldsymbol{V}_P\} \tag{13}$$

On the other hand, let us consider that $\boldsymbol{S}_i$ is a line in Plücker coordinates directed from A_i to B_i. Then the application of the Klein form of $\boldsymbol{S}_i$ to both sides of Eq. (12) with the cancellation of terms we have

$$\dot{q}_i + {}_5\omega_6^i\{\boldsymbol{S}_i; {}^5\$_i^6\} = \{\boldsymbol{S}_i; \boldsymbol{V}_P\} \tag{14}$$

Eliminating ${}_5\omega_6^i$ from Eqs. (13) and (14) it follows that

$$\{\boldsymbol{G}_i; \boldsymbol{V}_P\} = g_i\dot{q}_i \tag{15}$$

where $\boldsymbol{G}_i = \{\boldsymbol{S}_i; {}^5\$_i^6\}{}^3\$_i^4 - \{{}^3\$_i^4; {}^5\$_i^6\}\boldsymbol{S}_i$ and $g_i = \{{}^3\$_i^4; {}^2\$_i^3\}\{\boldsymbol{S}_i; {}^5\$_i^6\} - \{{}^3\$_i^4; {}^5\$_i^6\}$.

On the other hand, let us consider that $\boldsymbol{G}_6 = (\boldsymbol{0}, \hat{\boldsymbol{u}} \times \hat{\boldsymbol{v}})$. Clearly $\boldsymbol{G}_6$ is a constraint wrench system reciprocal to all the screws of the parallel manipulator. Hence in order to avoid the handling of deficient rank Jacobian matrices one can consider that

$$\{\boldsymbol{G}_6; \boldsymbol{V}_P\} = 0 \tag{16}$$

Expression (16) is a confirmation that the moving platform cannot rotate any axis parallel to $\hat{\boldsymbol{u}} \times \hat{\boldsymbol{v}}$. Indeed, $\{\boldsymbol{G}_6; \boldsymbol{V}_P\} = {}^0\boldsymbol{\omega}^m \cdot (\hat{\boldsymbol{u}} \times \hat{\boldsymbol{v}}) = 0$.

Casting into a matrix–vector form Eqs. (15) and (16), the input–output equation of velocity of the robot manipulator results in

$$\mathbf{A}\boldsymbol{V}_O = \mathbf{Q}_v \tag{17}$$

where $\mathbf{A} = \mathbf{J}^T\Delta$ is named the first-order coefficients matrix of the manipulator in which $\mathbf{J} = \begin{bmatrix}\boldsymbol{G}_1 & \boldsymbol{G}_2 & \boldsymbol{G}_3 & \boldsymbol{G}_4 & \boldsymbol{G}_5 & \boldsymbol{G}_6\end{bmatrix}$ is the Jacobian matrix of the robot manipulator while $\Delta = \begin{bmatrix}\mathbf{0} & \mathbf{I} \\ \mathbf{I} & \mathbf{0}\end{bmatrix}$ is an operator of polarity. Furthermore, $\mathbf{Q}_v = \begin{bmatrix}g_1\dot{q}_1 & g_2\dot{q}_2 & g_3\dot{q}_3 & g_4\dot{q}_4 & g_5\dot{q}_5 & 0\end{bmatrix}^T$ is named the first-order driver matrix of the parallel manipulator.

Finally, the input–output equation of velocity may be used to explore the singular configurations of the parallel manipulator, that is, the linear dependency between the rows of the Jacobian matrix gives us $\det(\mathbf{J}) = 0$ and the moving platform cannot perform the velocity state declared by the generalized coordinates, for example, if the limbs of the robot are coplanar then the dual parts of $\boldsymbol{G}_1, \ldots, \boldsymbol{G}_5$ are collinear causing the singularity of $\mathbf{J}$. For an in-deep singularity analysis of the robot, the reader is referred to Amine *et al.*[9]

4.2. Input–output equation of acceleration

In what follows the input–output equation of acceleration of the robot manipulator is obtained based on the strategy outlined for the velocity analysis.

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Let ${}^0\boldsymbol{\alpha}^m$ be the angular acceleration vector of body m as observed from body 0. Furthermore, let $\boldsymbol{a}_P$ be the acceleration vector of a point P of the moving platform. Then, the reduced acceleration state of the moving platform as measured from the fixed platform, the vector $\boldsymbol{A}_P = ({}^0\boldsymbol{\alpha}^m, \boldsymbol{a}_P - {}^0\boldsymbol{\omega}^m \times \boldsymbol{v}_P)$, may be expressed in screw form through any of the five limbs of the parallel manipulator as follows:

$$\boldsymbol{A}_P = \sum_{j=0}^{5} {}_j\alpha_{j+1}^i \; {}^j\$_i^{j+1} + \boldsymbol{L}_i \quad i = 1, 2, \ldots, 5 \tag{18}$$

Therein ${}_*\alpha_*^i = \frac{d}{dt}({}_*\omega_*^i)$ is the joint acceleration rate between two adjacent links of the ith limb. Furthermore, $\boldsymbol{L}_i$ is the ith Lie screw of acceleration which is computed as

$$\boldsymbol{L}_i = \sum_{j=0}^{4} \left[{}_j\omega_{j+1}^i \; {}^j\$_i^{j+1} \qquad \sum_{k=j+1}^{5} {}_k\omega_{k+1}^i \; {}^k\$_i^{k+1} \right] \tag{19}$$

where the brackets $\left[* \quad *\right]$ denote the Lie product of the Lie algebra $se(e)$ of the Euclidean group $SE(3)$. Furthermore, the joint velocity rates of each limb are determined upon Eq. (12) as follows:

$$\Omega_i = \mathbf{J}_i^{-1} \, \boldsymbol{V}_O \tag{20}$$

where $\Omega_i = \left[{}_0\omega_1^i \; {}_1\omega_2^i \; {}_2\omega_3^i \; {}_3\omega_4^i \; {}_4\omega_5^i \; {}_5\omega_6^i \right]^T$, while $\mathbf{J}_i = \left[{}^0\$_i^1 \; {}^1\$_i^2 \; {}^2\$_i^3 \; {}^3\$_i^4 \; {}^4\$_i^5 \; {}^5\$_i^6 \right]$ is the local screw-coordinate Jacobian matrix of the ith limb.

Following the trend of the velocity analysis, the input–output equation of acceleration of the parallel manipulator results in

$$\mathbf{A}\boldsymbol{a}_O = \mathbf{Q}_a + \mathbf{H} \tag{21}$$

where $\mathbf{Q}_a = \left[g_1\ddot{q}_1 \; g_2\ddot{q}_2 \; g_3\ddot{q}_3 \; g_4\ddot{q}_4 \; g_5\ddot{q}_5 \; 0 \right]^T$ is named the second-order driver matrix of the parallel manipulator. Meanwhile

$$\mathbf{H} = \begin{bmatrix} \left\{ {}^3\$_1^4; \boldsymbol{L}_1 \right\} \left\{ \boldsymbol{S}_1; {}^5\$_1^6 \right\} - \left\{ {}^3\$_1^4; {}^5\$_1^6 \right\} \left\{ \boldsymbol{S}_1; \boldsymbol{L}_1 \right\} \\ \left\{ {}^3\$_2^4; \boldsymbol{L}_2 \right\} \left\{ \boldsymbol{S}_2; {}^5\$_2^6 \right\} - \left\{ {}^3\$_2^4; {}^5\$_2^6 \right\} \left\{ \boldsymbol{S}_2; \boldsymbol{L}_2 \right\} \\ \left\{ {}^3\$_3^4; \boldsymbol{L}_3 \right\} \left\{ \boldsymbol{S}_3; {}^5\$_3^6 \right\} - \left\{ {}^3\$_3^4; {}^5\$_3^6 \right\} \left\{ \boldsymbol{S}_3; \boldsymbol{L}_3 \right\} \\ \left\{ {}^3\$_4^4; \boldsymbol{L}_4 \right\} \left\{ \boldsymbol{S}_4; {}^5\$_4^6 \right\} - \left\{ {}^3\$_4^4; {}^5\$_4^6 \right\} \left\{ \boldsymbol{S}_4; \boldsymbol{L}_4 \right\} \\ \left\{ {}^3\$_5^4; \boldsymbol{L}_5 \right\} \left\{ \boldsymbol{S}_5; {}^5\$_5^6 \right\} - \left\{ {}^3\$_5^4; {}^5\$_5^6 \right\} \left\{ \boldsymbol{S}_5; \boldsymbol{L}_5 \right\} \\ 0 \end{bmatrix} \tag{22}$$

is the complementary matrix of acceleration which contains the Coriolis and centripetal terms of the acceleration analysis.

5. Numerical Applications

In order to show the application of the method, in this section numerical examples comprising most of the topics treated in the contribution are provided. To this aim consider that the dimensions of the fixed and moving platforms are given by $\underline{a} = \bar{a} = 0.6\,\text{m}$, $\underline{b} = \bar{b} = 0.4\,\text{m}$, $\underline{c} = \bar{c} = 0.4\,\text{m}$, while $d = 0.5\,\text{m}$. Hence the coordinates of the corners of the fixed platform results in $A_1 = (0, 0, 0)\text{m}$, $A_2 = (0.6, 0, 0)\text{m}$, $A_3 = (0.6, 0.4, 0)\text{m}$, $A_4 = (0.3, 0.664, 0.0)\text{m}$, and $A_5 = (0.0, 0.4, 0.0)\text{m}$.

The first part of the case study is devoted to apply the method of the inverse displacement analysis mainly with the purpose to establish the reference configuration of the parallel manipulator. To this end consider that the orientation of the moving platform is defined according to the angles $\theta = 15^\circ$ and $\psi = 10^\circ$ whereas its position is defined by $C_1 = (0.0, 0.0, 1.0)$. With these data the resulting reachable configurations of the parallel manipulator are reported in Table I.

The next part of the exercise consists of solving the instantaneous kinematics of the parallel manipulator. To this end consider that the reference configuration of the robot corresponds to one solution of the inverse displacement analysis which is given in Table II.

In what follows the method outlined to solve the instantaneous kinematics of the parallel manipulator is exemplified by means of two numerical applications. To achieve this task, method 2 is applied

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Table I. Complete solutions of the inverse displacement analysis.

i	$\{B_i, q_i\}$	C_i
1	{(0.0,0.1294,0.5170),0.5329}{(0.0,−0.1294,1.4829),1.4885}	(0.0,0.0,1.0)
2	{(0.6,0.1563,0.4164),0.4448}{(0.6,−0.1024,1.3822),1.3860}	(0.5908,0.0269,0.8993)
3	{(0.6,0.2839,1.4857),1.4902}{(0.6,0.5427,0.5200),0.5392}	(0.5908,0.4133,1.0028)
4	{(0.3,0.5260,1.6046),1.6105}{(0.3,0.7848,0.6387),0.6499}	(0.2954,0.6554,1.1216)
5	{(0.0,0.2569,1.5864),1.5929}{(0.0,0.5157,0.6205),0.6312}	(0.0,0.3863,1.1035)

Table II. Reference configuration of the parallel manipulator.

i	A_i	B_i	C_i	q_i
1	(0.0,0.0,0.0)	(0.0,−0.1294,1.4829)	(0.0,0.0,1.0)	1.4885
2	(0.6,0.0,0.0)	(0.6,−0.1024,1.3822)	(0.5908,0.0269,0.8993)	1.3860
3	(0.6,0.4,0.0)	(0.6,0.5427,0.5200)	(0.5908,0.4133,1.0028)	0.5392
4	(0.3,0.6645,0.0)	(0.3,0.7848,0.6387)	(0.2954,0.6554,1.1216)	0.6499
5	(0.0,0.4,0.0)	(0.0,0.5157,0.6205)	(0.0,0.3863,1.1035)	0.6312

to solve the forward kinematic problem. Firstly, provided that

$$\left\{\begin{array}{l} \boldsymbol{c}_1 = w_1\hat{\boldsymbol{i}} + w_2\hat{\boldsymbol{j}} + w_3\hat{\boldsymbol{k}} \\ \hat{\boldsymbol{v}} = w_4\hat{\boldsymbol{j}} + w_5\hat{\boldsymbol{k}} \\ \hat{\boldsymbol{w}} = w_6\hat{\boldsymbol{i}} + w_7\hat{\boldsymbol{j}} + w_8\hat{\boldsymbol{k}} \end{array}\right\}$$

while $\hat{\boldsymbol{b}}_1 = w_9\hat{\boldsymbol{i}} + w_{10}\hat{\boldsymbol{j}} + w_{11}\hat{\boldsymbol{k}}, \ldots, \hat{\boldsymbol{b}}_5 = w_{21}\hat{\boldsymbol{i}} + w_{22}\hat{\boldsymbol{j}} + w_{23}\hat{\boldsymbol{k}}$. Apply Eq. (1) to obtain

$$\left\{\begin{array}{l} \hat{\boldsymbol{c}}_2 = (w_1 + 0.6w_6)\hat{\boldsymbol{i}} + (w_2 + 0.6w_7)\hat{\boldsymbol{j}} + (w_3 + 0.6w_8)\hat{\boldsymbol{k}} \\ \hat{\boldsymbol{c}}_3 = (w_1 + 0.6w_6)\hat{\boldsymbol{i}} + (w_2 + 0.6w_7 + 0.4w_4)\hat{\boldsymbol{j}} + (w_3 + 0.6w_8 + 0.4w_5)\hat{\boldsymbol{k}} \\ \hat{\boldsymbol{c}}_4 = (w_1 + 0.3w_6)\hat{\boldsymbol{i}} + (w_2 + 0.3w_7 + 0.664w_4)\hat{\boldsymbol{j}} + (w_3 + 0.3w_8 + 0.664w_5)\hat{\boldsymbol{k}} \\ \hat{\boldsymbol{c}}_5 = w_1\hat{\boldsymbol{i}} + (w_2 + 0.4w_4)\hat{\boldsymbol{j}} + (w_3 + 0.4w_5)\hat{\boldsymbol{k}} \end{array}\right\}$$

After, the 23 equations required to solve the forward kinematic problem are obtained as follows.

1. Closure equations credited to expressions (2) and (3):

$$\begin{aligned} f_1 &= w_4^2 + w_5^2 - 1.0 \\ f_2 &= w_6^2 + w_7^2 + w_8^2 - 1.0 \\ f_3 &= w_7w_4 + w_8w_5 \end{aligned}$$

2. Closure equations credited to expressions (4):

$$\begin{aligned} f_4 &= 1.0w_9 \\ f_5 &= 1.0w_{12} - 0.6 \\ f_6 &= 1.0w_{15} - 0.6 \\ f_7 &= 1.0w_{18} - 0.3 \\ f_8 &= 1.0w_{21} \end{aligned}$$

3. Closure equations credited to expressions (5):

$$\begin{aligned} f_9 &= w_4w_2 - w_4w_{10} + w_5w_3 - w_5w_{11} \\ f_{10} &= w_4w_2 + 0.6w_7w_4 - w_4w_{13} + w_5w_3 + 0.6w_8w_5 - w_5w_{14} \\ f_{11} &= w_4w_2 + 0.6w_7w_4 + 0.4w_4^2 - w_4w_{16} + w_5w_3 + 0.6w_8w_5 + 0.4w_5^2 - w_5w_{17} \\ f_{12} &= w_4w_2 + 0.30w_7w_4 + 0.664w_4^2 - w_4w_{19} + w_5w_3 + 0.30w_8w_5 + 0.664w_5^2 - w_5w_{20} \\ f_{13} &= w_4w_2 + 0.4w_4^2 - w_4w_{22} + w_5w_3 + 0.4w_5^2 - w_5w_{23} \end{aligned}$$

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4. Closure equations credited to expressions (6):

$$
\begin{aligned}
f_{14} &= w_9^2 + w_{10}^2 + w_{11}^2 - q_1^2 \\
f_{15} &= 0.36 - 1.2w_{12} + w_{12}^2 + w_{13}^2 + w_{14}^2 - q_2^2 \\
f_{16} &= 0.52 - 1.2w_{15} + w_{15}^2 - 0.8w_{16} + w_{16}^2 + w_{17}^2 - q_3^2 \\
f_{17} &= 0.531 - 0.6w_{18} + w_{18}^2 - 1.329w_{19} + w_{19}^2 + w_{20}^2 - q_4^2 \\
f_{18} &= w_{21}^2 + 0.16 - 0.8w_{22} + w_{22}^2 + w_{23}^2 - q_5^2
\end{aligned}
$$

5. Closure equations credited to expressions (7):

$$
\begin{aligned}
f_{19} &= w_9^2 - 2w_9w_1 + w_1^2 + w_{10}^2 - 2w_{10}w_2 + w_2^2 + w_{11}^2 - 2w_{11}w_3 + w_3^2 - 0.25 \\
f_{20} &= w_{12}^2 - 2w_{12}w_1 - 1.2w_{12}w_6 + w_1^2 + 1.2w_1w_6 + 0.36w_6^2 + w_{13}^2 - 2w_{13}w_2 - \\
&\quad 1.2w_{13}w_7 + w_2^2 + 1.2w_2w_7 + 0.36w_7^2 + w_{14}^2 - 2w_{14}w_3 - 1.2w_{14}w_8 + w_3^2 + \\
&\quad 1.2w_3w_8 + 0.36w_8^2 - 0.25 \\
f_{21} &= -0.8w_4w_{16} - 0.8w_5w_{17} + 0.48w_7w_4 + 0.48w_8w_5 + 0.8w_4w_2 + 0.8w_5w_3 + \\
&\quad 0.36w_6^2 + 0.36w_7^2 + 0.36w_8^2 + 0.16w_4^2 + 0.16w_5^2 + 1.2w_1w_6 + 1.2w_2w_7 + 1.2w_3w_8 - \\
&\quad 2w_{15}w_1 - 1.2w_{15}w_6 - 2w_{16}w_2 - 1.2w_{16}w_7 - 2w_{17}w_3 - 1.2w_{17}w_8 + w_{17}^2 + \\
&\quad w_{15}^2 + w_{16}^2 + w_1^2 + w_2^2 + w_3^2 - 0.25 \\
f_{22} &= 0.398w_7w_4 + 0.398w_8w_5 + 1.329w_4w_2 + 1.329w_5w_3 + 0.09w_6^2 + \\
&\quad 0.09w_7^2 + 0.09w_8^2 + 0.441w_4^2 + 0.441w_5^2 + 0.6w_1w_6 + 0.6w_2w_7 + \\
&\quad 0.6w_3w_8 - 2w_{18}w_1 - 0.6w_{18}w_6 - 2w_{19}w_2 - 0.6w_{19}w_7 - 2w_{20}w_3 - \\
&\quad 0.6w_{20}w_8 + w_{20}^2 + w_{18}^2 + w_{19}^2 + w_1^2 + w_2^2 + w_3^2 - 1.329w_4w_{19} - \\
&\quad 1.329w_5w_{20} - 0.25 \\
f_{23} &= w_{21}^2 - 2w_{21}w_1 + w_1^2 + w_{22}^2 - 2w_{22}w_2 - 0.8w_4w_{22} + w_2^2 + 0.8w_4w_2 + \\
&\quad 0.16w_4^2 + w_{23}^2 - 2w_{23}w_3 - 0.8w_5w_{23} + w_3^2 + 0.8w_5w_3 + 0.16w_5^2 - 0.25
\end{aligned}
$$

In the aforementioned equations the numerical values were truncated for brevity to three decimals but the computations were achieved taking into account 10 decimals.

5.1. Instantaneous kinematics of the moving platform (Example 1)

Let us consider that the generalized coordinates are conducted to follow periodical functions given by

$$
\begin{aligned}
q_1 &= 1.4885 - 0.1[\cos^2(2.0t - 0.5\pi) + \sin(0.5t)] \\
q_2 &= 1.3860 - 0.1[\cos^2(2.0t - 0.5\pi) + 2\sin(0.5t)] \\
q_3 &= 0.5392 + 0.1[\cos^2(2.0t - 0.5\pi) + 2\sin(0.5t)] \\
q_4 &= 0.6499 + 0.1[\cos^2(2.0t - 0.5\pi) + 2\sin(0.5t)] - 0.05\sin(0.5t)\cos(t + 0.5\pi) \\
q_5 &= 0.6312 + 0.1[\cos^2(2.0t - 0.5\pi) + 2\sin(0.5t)] - 0.05\sin(t)\cos(t + 0.5\pi)
\end{aligned}
$$

where the time t is given in the interval $0 \leq t \leq 2\pi$(s). For clarity, plots of the generalized coordinates are shown in Fig. 3.

The resulting temporal behavior of the pose of the moving platform assuming that point P is the center of the rectangle $C_1C_2C_3C_5$, for example, P is the middle point between points C_1 and C_3, is shown in Fig. 4.

On the other hand, the resulting time history of the components of the angular velocity and acceleration of the moving platform as measured from the fixed platform is provided in Fig. 5. Meanwhile, the resulting time history of the components of the velocity and acceleration of point P is depicted in Fig. 6. Furthermore, the numerical results obtained by means of the theory of screws are verified with the aid of special software like ADAMS.™The corresponding plots are provided in the same figures.

Finally, the restricted mobility of the moving platform may be evidenced by considering the expression $error = \boldsymbol{\omega} \cdot (\boldsymbol{u} \times \boldsymbol{v})$. After, the time history of *error* is provided in Fig. 7.

According to Fig. 7 in fact the moving platform cannot rotate any axis parallel to $\boldsymbol{u} \times \boldsymbol{v}$.

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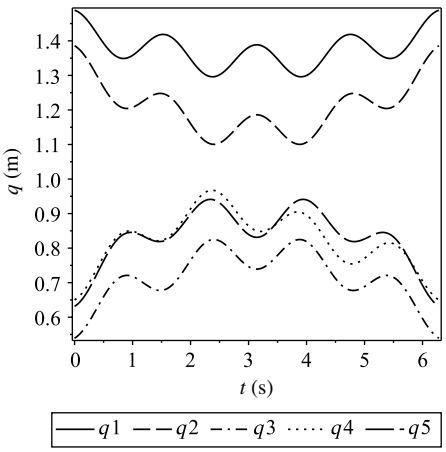


Fig. 3. Generalized coordinates (Example 1).

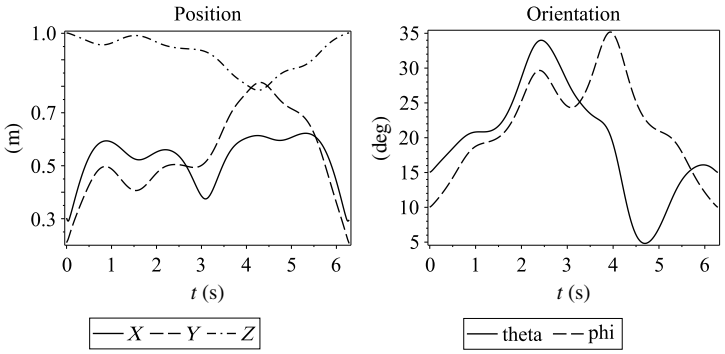


Fig. 4. Temporal behavior of the pose of the moving platform (Example 1).

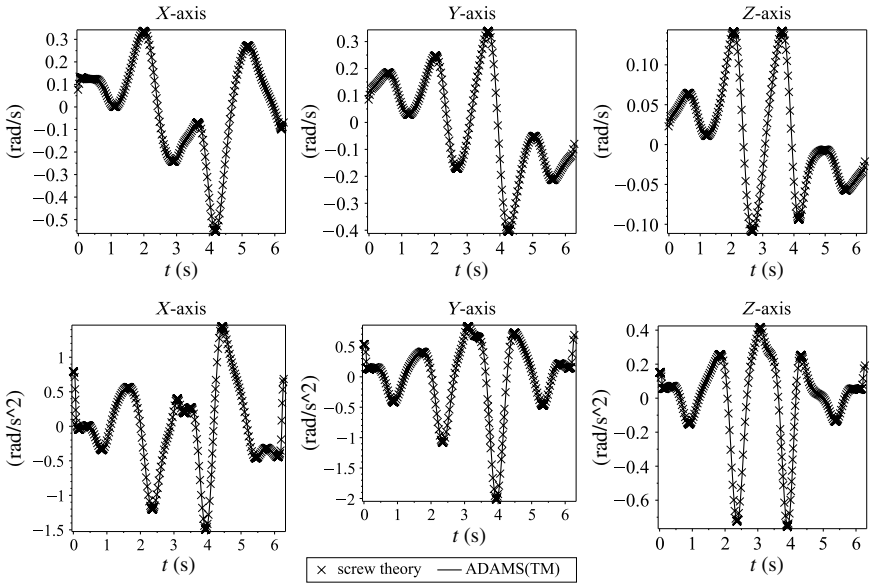


Fig. 5. Temporal behavior of the angular velocity and acceleration of the moving platform (Example 1).

5.2. Instantaneous kinematics of the moving platform (Example 2)

Let us consider that the parallel manipulator is at its reference configuration, the second application consists of solving the kinematics of the parallel manipulator assuming that at the beginning of the analysis the parallel manipulator is motionless and is required to move it in a smooth

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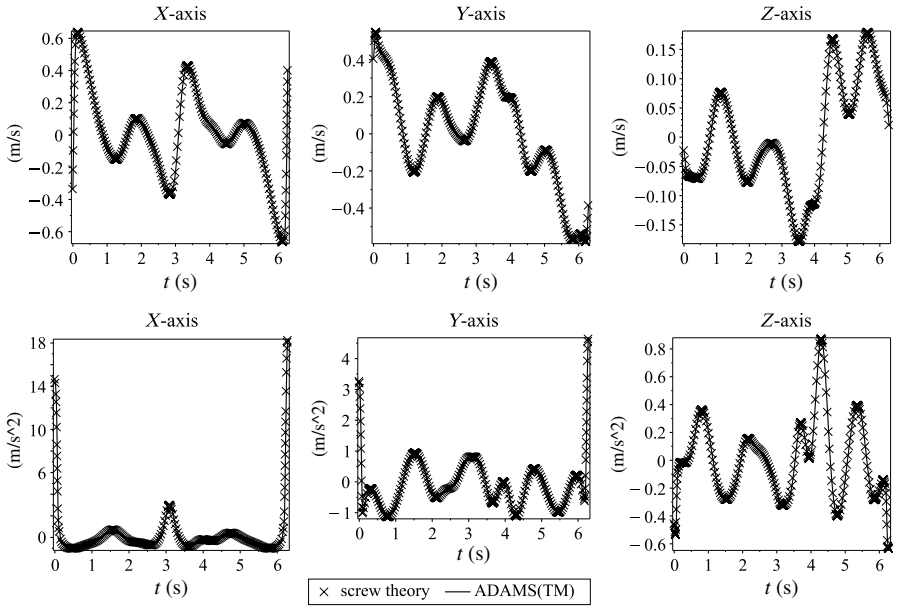


Fig. 6. Temporal behavior of the velocity and acceleration of point P (Example 1).

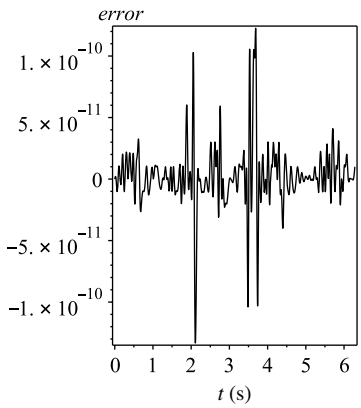


Fig. 7. Numerical validation of the loss mobility of the parallel manipulator (Example 1).

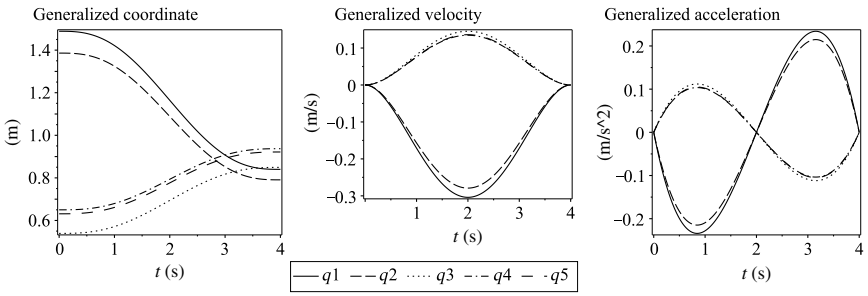


Fig. 8. Generalized coordinates and their time derivatives (Example 2).

manner in such a way that after 4 s the robot returns to rest and the pose of the moving platform is defined according to the orientation angles $\theta = 38.5°$ and $\psi = 32°$, and the coordinates $C_1 = (0.485, 0.433, 0.665)$m. Quintic polynomial expressions of the form $q(t) = c_0 + c_1 t + c_2 t^2 + c_3 t^3 + c_4 t^4 + c_5 t^5$ are appropriated to achieve this task. Plots of the generalized coordinates are provided in Fig. 8.

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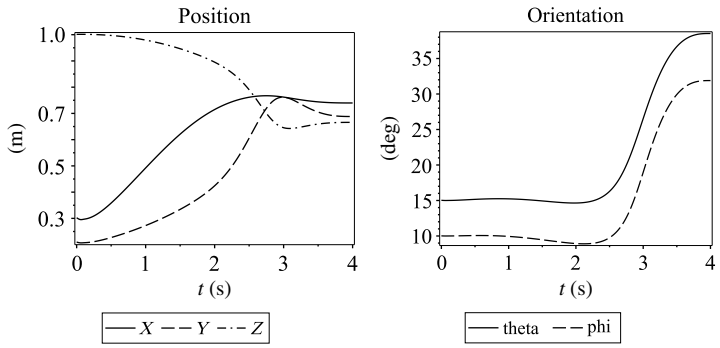


Fig. 9. Temporal behavior of the pose of the moving platform (Example 2).

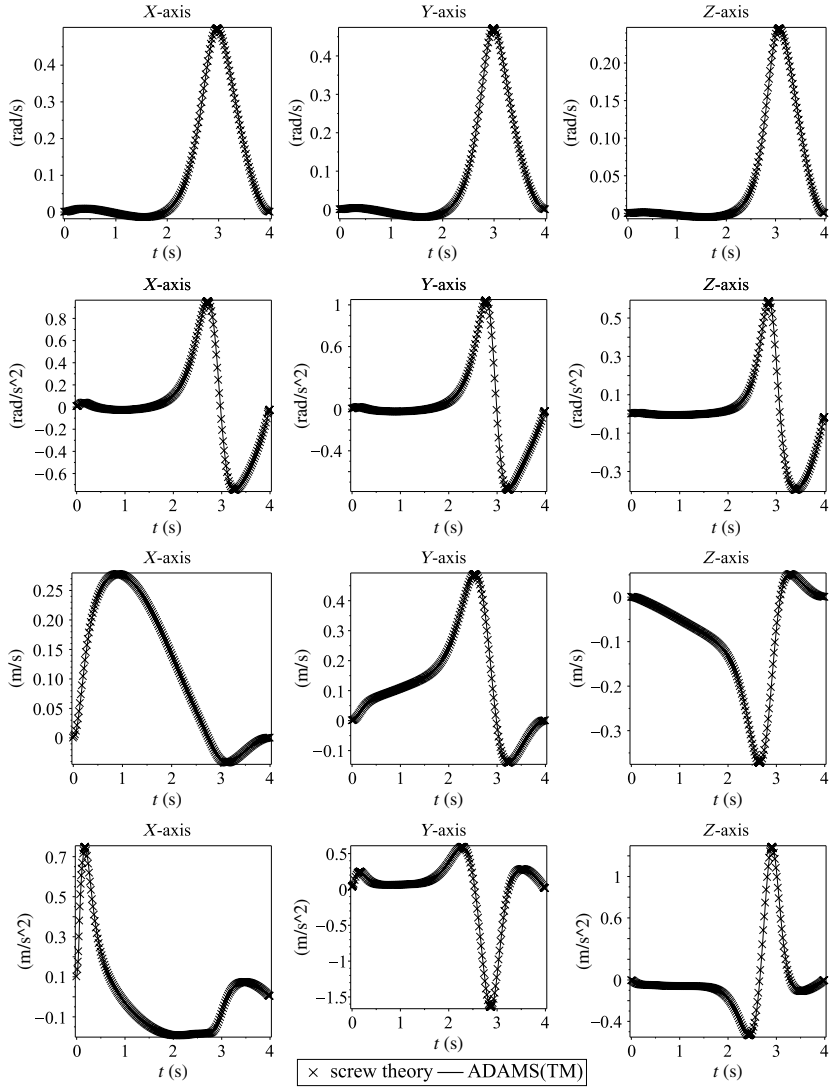


Fig. 10. Temporal behavior of the instantaneous kinematics of point P of the moving platform (Example 2).

Then, the resulting temporal behavior of the pose of the moving platform as measured from the fixed platform concerned with point P is shown in Fig. 9.

Finally, the resulting time history of the instantaneous kinematics of the moving platform dealing with its point P is given in Fig. 10. Furthermore, for the sake of completeness the numerical results of Example 2 were also confirmed with simulations obtained with ADAMS.TM

Finally, the restricted rotation of the moving platform of Example 2 is evidenced in Fig. 11.

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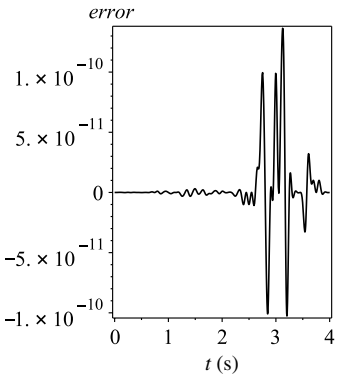


Fig. 11. Numerical validation of the loss mobility of the parallel manipulator (Example 2).

6. Conclusions

The motivation of this work is the proposal of a simple method to approach the forward kinematic problem as well as the solution of the velocity and acceleration analyses of the 5-RPUR parallel manipulator. The latter is the main contribution of the paper.

Dealing with the forward kinematic problem, the closure equations associated to the displacement analysis of the parallel manipulator are formulated using the three-dimensional Euclidean space. Simple kinematic constraints are derived taking into account two elementary features: (i) the particular kinematic assembly of the joints in each leg, and (ii) the Euclidean distance between selected points of the robot. Following this strategy 23 closure equations are quickly obtained. After, two methods for solving the forward kinematic problem can be implemented. The difference between both methods is how the coordinates C_i denoting the nominal positions of the revolute joints mounted on the moving platform are expressed. In the first method points C_i are expressed in terms of a position vector and two angles denoting the orientation of the moving platform, while in the second method points C_i are expressed according to a position vector and two unit vectors located on the plane of the moving platform. Then, the handling of the closure equations yields a system of equations in 22 or 23 variables, depending on the chosen method, which may be solved by means of homotopy continuation method or the Newton–Raphson method, if only one solution is required. Compared with existing methods based on study parameters, the proposed method in the contribution is easy to follow. In fact, the application of Study's kinematic mapping for solving the forward kinematic problem of the robot at hand demands, among other things, the handling of D-H parameters, Gröbner basis and a linear implicitization algorithm in order to generate homogeneous quadruple and study quadric expressions. Meanwhile, the method exposed in the contribution resorts to the three-dimensional Euclidean space and therefore only simple operations of the usual three-dimensional vectorial algebra are required.

Dealing with the instantaneous kinematics of the 5-RPUR parallel manipulator, the input–output equation of velocity of the robot is obtained by resorting to the theory of screws. To this end, the velocity state of the moving platform is expressed in screw form through the five limbs of the robot considering a pseudo revolute joint for each leg with the purpose to satisfy an algebraic requirement. On the other hand, the loss rotation of the moving platform gives us a sixth expression. Afterwards, the input–output equation of velocity of the robot is obtained by a systematic application of the Klein form. Similar to the displacement analysis, the velocity analysis of the parallel manipulator demands the computation of intermediate variables, for example, it is necessary to compute one passive velocity joint for each limb. Following this trend, the input–output equation of acceleration of the parallel manipulator is obtained. With the purpose to illustrate the benefits of the method, numerical examples comprising most of the topics treated in the contribution are provided. In that concern the loss rotation of the parallel manipulator is numerically evidenced through the computation of the inner product between the angular velocity vector of the moving platform and the common vector normal to the axes of the universal joints. Furthermore, the numerical results obtained using the theory of screws are confirmed by means of special software like ADAMS.™

Finally, as it is pointed out by Gosselin et al.[35] the full potential of parallel mechanisms has not yet been exploited. We are sure that a key strategy to overcome such assumption is precisely the improvement of existing methods of kinematic analysis or the introduction of new methods.

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Conflicts of Interest
The authors declare that there are no conflicts of interest regarding the publication of this paper.

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