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On nonlinear perturbations of linear second order elliptic boundary value problems

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Abstract. Let $\Omega \subseteq \mathbb{R}^n$ be open and bounded, with $\partial\Omega$ smooth. Sufficient conditions on f are given in order that for $p > n$, the equation

$$L(u)(x) + f(x, u(x), \nabla u(x), D^2u(x)) = h(x) \quad \text{in} \quad \Omega,$$
$$B(u)(x) = 0 \quad \text{on} \quad \partial\Omega$$

has a solution for every $h \in L^p$; L is a second order symmetric elliptic operator and B represents either the first, second or third boundary value problems. These conditions are in terms of the asymptotic behaviour of f, in its second variable, in relation to those λ for which there is a nontrivial solution of

$$L(u)(x) + \lambda u(x) = 0 \quad \text{in} \quad \Omega,$$
$$B(u)(x) = 0 \quad \text{on} \quad \partial\Omega.$$

Introduction. Let $\Omega \subseteq \mathbb{R}^n$ be open and bounded with $\partial\Omega$ smooth and let

$$f: \bar{\Omega} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n_2} \to \mathbb{R}.$$

We are interested in the question of the existence of solutions $u: \bar{\Omega} \to \mathbb{R}$ of

$$L(u)(x) + f(x, u(x), \nabla u(x), D^2u(x)) = h(x) \quad \text{in} \quad \Omega \tag{0·1}$$
$$B(u)(x) = 0 \quad \text{on} \quad \partial\Omega,$$

where

$$L(u)(x) \equiv -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{i,j}(x)\frac{\partial u}{\partial x_j}\right)$$

is symmetric and uniformly elliptic on $\bar{\Omega}$;

$$B(u)(x) = \alpha\left(\sum_{i,j=1}^{n} a_{i,j}(x)\frac{\partial u}{\partial x_i}(x)\,\eta_j(x)\right) + \sigma(x)\,u(x),$$

with either $\alpha = 0$ and $\sigma \equiv 1$ or $\alpha = 1$ and $\sigma \geqslant 0$ and $\eta(x) = (\eta_1(x), \ldots, \eta_n(x))$ is the outward unit normal to $\partial\Omega$ at x;

$$n_2 = \#\{\alpha = (\alpha_1, \ldots, \alpha_n) \in (\mathbb{N} \cup \{0\})^n \mid |\alpha| \equiv \alpha_1 + \alpha_2 + \ldots + \alpha_n = 2\},$$

and when $u: \bar{\Omega} \to \mathbb{R}$ has second-order derivatives (in some sense), $D^2u: \bar{\Omega} \to \mathbb{R}^{n_2}$ is defined by

$$(D^2u(x))_\alpha = \frac{\partial^2}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}} u(x), \quad \text{for} \quad |\alpha| = 2 \quad \text{and} \quad x \in \bar{\Omega}.$$

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In the case when $f(x, u(x), \nabla u(x), D^2u(x)) = f(x, u(x))$ a number of results have been obtained concerning sufficient conditions on f in order to guarantee existence of solutions to (0·1) for particular h and for certain choices of B (see (3, 6, 7, 8, 9, 10, 11, 12, 15, 16, 17)) and the references contained therein). In particular when f generates an operator in a suitable function space which has sublinear growth a number of general existence results are known. The significance of the restriction that $f = f(x, u(x))$ is that in this situation one can reformulate equation (0·1) as a fixed point equation for a compact operator. No matter what nonlinear dependence on higher-order derivatives is allowed, however, it is clear that the existence of solutions of (0·1) for a reasonably large class of h will have to involve some compatibility assumption between f and those λ such that there is a non-trivial solution of

$$\begin{cases} L(u)(x) + \lambda u(x) = 0 & \text{in} \quad \Omega \\ B(u)(x) = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{0·2}$$

which we refer to as eigenvalues of (0·2).

We seek solutions of equation (0·1) in $W^{2,p}(\Omega)$, where $p > n$. Under the assumption that f is continuous, and uniformly Lipschitz with respect to these variables which compose with second partial derivatives, we prove that equation (0·1) may be reformulated as a fixed point equation for a set-condensing mapping which acts in $L^p(\Omega)$. The L^2-existence and regularity results together with the L^p-estimates of Agmon, Douglis and Nirenberg (2) and F. E. Browder (4) allow us to handle the linear part of equation (0·1) in $W^{2,p}$, while the embedding of $W^{1,p}(\Omega)$ in $C(\bar{\Omega})$ when $p > n$ allows us to handle the nonlinearity. We then prove that this condensing mapping satisfies a Leray–Schauder type boundary condition on sufficiently large spheres of $L^p(\Omega)$ provided that f satisfies either of two asymptotic conditions, the first of which is that

$$\lim_{s \to \pm\infty} f(x, s, r, t) - \lambda_1 s = \pm\infty$$

uniformly with respect to $(x, r, t) \in \Omega \times \mathbb{R}^n \times \mathbb{R}^{n_2}$,

where λ_1 is the largest eigenvalue of (0·2), the second one being that

$$\lim_{s \to \pm\infty} \frac{f(x, s, r, t)}{s} = f_{\pm}(x),$$

uniformly with respect to $(r, t) \in \mathbb{R}^n \times \mathbb{R}^{n_2}$,

where $f_-(x)$, $f_+(x) \in (\lambda_{j+1}, \lambda_j)$ and λ_j and λ_{j+1} are consecutive eigenvalues of equation (0·2).

Under the first asymptotic condition we use the simplicity and the nodal structure of the first eigenvalue to verify the Leray–Schauder boundary condition, while for the second asymptotic condition the variational characterization of (0·2) together with the Courant min–max principle allows one to verify the Leray–Schauder condition on sufficiently large balls.

In section 1 we recapitulate those facts concerning the spaces $W^{m,p}(\Omega)$ and set-condensing mappings which we will need later. Section 2 is devoted to obtaining some

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a priori bounds for solutions of certain perturbations of equation (0·1). In the final section 3 we first of all show how equation (0·1) may be reformulated as a fixed point equation for a condensing mapping in $L^p(\Omega)$, and then we show how the *a priori* bounds of section 2 can be used to verify the Leray–Schauder condition for this mapping. Thus we obtain the existence of a solution of (0·1) in $W^{2,p}(\Omega)$ for any $h \in L^p(\Omega)$.

Section 1. Let Ω be an open bounded subset of $\mathbb{R}^n$, $p \geqslant 1$, and $m \in \mathbb{N}$. We denote by $W^{m,p} \equiv W^{m,p}(\Omega)$ the space of functions on Ω whose generalized derivatives up to order m lie in $L^p \equiv L^p(\Omega)$. If $\alpha = (\alpha_1, \ldots, \alpha_n) \in (\mathbb{N} \cup \{0\})^n$, we let $|\alpha|$ denote

$$\alpha_1 + \alpha_2 + \ldots + \alpha_n \quad \text{and let} \quad D^\alpha(u) = \partial^{|\alpha|}/(\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n})\, u,$$

in the generalized sense. $W^{m,p}$ is a Banach space under the norm $\| \ \|_{m,p}$ defined by

$$\|u\|_{m,p} = \sum_{|\alpha| \leqslant m} \|D^\alpha(u)\|_{L^p}.$$

Denote by $C(\bar{\Omega})$ the Banach space of continuous real valued functions on $\bar{\Omega}$, with the norm being the maximum norm. Throughout this paper we will always assume that $\partial\Omega$ is C^2 and, this being so, let us record for later reference the following useful facts concerning the spaces $W^{m,p}$.

PROPOSITION 1·1. *Letting 'I' denote the appropriate inclusion mapping we have:*

(1·1) $I\colon W^{1,p} \to L^p$ *is compact,*

(1·2) *if* $p > n$, *then* $W^{1,p} \subseteq C(\bar{\Omega})$, *and* $I\colon W^{1,p} \to C(\bar{\Omega})$ *is continuous,*

(1·3) *if* $1 \leqslant s < n$ *and* $q(s) = ns/(n-s)$, *then* $W^{2,s} \subseteq W^{1,q(s)}$ *and* $I\colon W^{2,s} \to W^{1,q(s)}$ *is continuous.*

Let $a_{i,j} \in C^2(\bar{\Omega})$, *for* $i, j = 1, \ldots, n$, *and such that* $a_{i,j} = a_{j,i}$. *Suppose there is a* $\tau > 0$ *such that*

$$\sum_{i,j=1}^{n} a_{i,j}(x)\, \zeta_i \zeta_j \geqslant \tau \sum_{i=1}^{n} |\zeta_i|^2$$

for all $x \in \bar{\Omega}$ *and* $(\zeta_1, \ldots, \zeta_n) \in \mathbb{R}^n$. *Then the operator*

$$L(u)(x) = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{i,j}(x) \frac{\partial u}{\partial x_j}(x)\right)$$

is called uniformly elliptic. Let $\sigma \in C^1(\partial\Omega)$ *and let*

$$B(u)(x) = \alpha\left(\sum_{i,j=1}^{n} a_{i,j}(x) \frac{\partial u}{\partial x_i}(x)\, \eta_j(x)\right) + \sigma(x)\, u(x),$$

for $x \in \partial\Omega$, *where either* $\alpha = 0$ *and* $\sigma \equiv 1$ *or* $\alpha = 1$ *and* $\sigma \geqslant 0$.

(1·4) *Let* $r > 1$ *and suppose* $l \geqslant 2$. *Then there exists a constant* $c = c(r, l)$ *such that if* $u \in W^{2,r}$, $B(u) = 0$, *and* $L(u) \in W^{l-2,r}$, *then* $u \in W^{l,r}$ *and*

$$\|u\|_{l,r} \leqslant c(\|L(u)\|_{l-2,r} + \|u\|_{0,r}).$$

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(1·1), (1·2) and (1·3) are embedding results, which may be found in Adams(1), while (1·4) constitutes part of Theorem 15·2 of Agmon, Douglis and Nirenberg(2) and is also proved in Browder(4).

The topological method which we shall invoke to solve our equations is the fixed-point theory for set-condensing mappings. The basic references for this theory are Nussbaum(18) and Sadovskiĭ(21). Let us recall the basic facts from these papers which we will need in the sequel.

Suppose X is a Banach space, and $K \subseteq A$ is bounded. One defines the set-measure of noncompactness of K, which is denoted by $\gamma(K)$, to be $\inf\{r > 0 |$ there exists a finite number of sets, each of diameter less than r, whose union covers $K\}$. A continuous mapping $T: X \to X$ is called set-condensing provided that whenever $K \subseteq X$ is bounded and $\bar{K}$ is not compact, $\gamma(T(K)) < \gamma(K)$.

PROPOSITION 1·2. *Let X be a Banach space and let $T: \bar{B}(0,r) \subseteq X \to X$ be set-condensing and such that*

$$x - tT(x) \neq 0 \quad \text{when} \quad t \in [0,1] \quad \text{and} \quad \|x\| = r. \tag{1·5}$$

Then T has a fixed point in $\bar{B}(0,r)$.

The above proposition is a consequence of the degree theory for condensing mappings, developed in (18) and (21), or it may also be proved directly from the fixed-point theorem for such mappings (20) once one shows that $\rho \circ T$ is set-condensing where ρ is the radial retraction of X onto $\bar{B}(0,r)$. The fixed-point theorem for condensing mappings may be deduced from the Schauder fixed-point theorem.

The above proposition is a generalization of the Leray–Schauder fixed-point theorem for compact T. But set-condensing mappings are considerably more general than compact-mappings. In section 3 we shall reformulate equation (0·1) as a fixed-point equation for a condensing mapping in L^p. In order to verify the Leray–Schauder boundary condition (1·5) for this condensing mapping we shall first of all, in the next section, establish some *a priori* bounds for solutions of certain perturbations of equation (0·1).

Section 2. Throughout this section $\Omega \subseteq \mathbb{R}^n$ will be open and bounded, $p > n$, $\partial\Omega$ is smooth (say C^3) and L is the symmetric elliptic operator defined preceding the statement of property (1·4). For notational and conceptual simplicity we will first prove our boundedness results with a non-linearity which is simpler than that of equation (0·1). Then we will discuss the modifications necessary for the general case.

A few remarks concerning problem (0·1) are in order. The operator L, together with the boundary condition $B(u) = 0$, may be formulated variationally as an operator from $W^{1,2}$ into $W^{1,2}$, and for $\beta > 0$ this variational operator perturbed by βI is bijective, with the inverse image of L^2 lying in $W^{2,2}$. This is developed in, say, Ladyzhenskaia and Uraltseva(14). The compactness of the inverse of the above operators from L^2 to L^2 yields the completeness in L^2 of the eigenfunctions of (0·2), the eigenvalues being a discrete subset of $(-\infty, 0]$. Moreover, the largest eigenvalue λ_1 is simple and has as

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eigenfunctions functions without interior zeros. Let g_1 be the eigenfunction, with $g_1(x) > 0$ on Ω and $\|g_1\|_{L^2} = 1$. Let $Q: L^2 \to L^2$ be defined by

$$Q(u)(x) = \left(\int_\Omega g_1(y)\, u(y)\, dy\right) g_1(x) \quad \text{for} \quad u \in L^2, x \in \Omega. \tag{2·1}$$

The completeness of the eigenfunctions in L^2 implies that

$$\int_\Omega (L(u)(x) + \lambda_1 u(x))\, u(x)\, dx \geqslant -\lambda_2 \|(I-Q)(u)\|_{L^2}^2, \tag{2·2}$$

for $u \in W^{2,2}$, where λ_2 is the second largest eigenfunction of (0·2).

We recall that a mapping $f: \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is said to satisfy the Carathéodory conditions provided that for each $s \in \mathbb{R}$, $f(\cdot, s)$ is measurable on Ω, while for almost all $x \in \Omega$, $f(x, \cdot)$ is continuous on $\mathbb{R}$. If there exists an $r \geqslant 1$, $v \in L^r$ and $b > 0$ such that

$$|f(x,s)| \leqslant v(x) + b|s| \quad \text{for all} \quad (x,s) \in \overline{\Omega} \times \mathbb{R}, \tag{2·3}$$

then f generates by composition a bounded continuous operator from L^r to L^r (see Krasnoselskiĭ (13)).

PROPOSITION 2·1. *Let $f: \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ satisfy the Carathéodory conditions, let $p > n$, and assume there exist $v \in L^p$ and b such that condition* (2·3) *holds. Moreover, assume*

$$\lim_{s \to \pm\infty} (f(x,s) - \lambda_1 s) = \pm\infty, \text{ uniformly with respect to } x \in \overline{\Omega}. \tag{A}$$

Then whenever $\{h_k\} \subseteq L^p$ is bounded, $\{t_k\} \subseteq [0,1]$, and $\{u_k\} \subseteq W^{2,p}$ is such that for each $k \in N$

$$\begin{cases} L(u_k)(x) + \lambda_1 u_k(x) + t_k(f(x, u_k(x)) - \lambda_1 u_k(x)) + (1-t_k) Q(u_k)(x) = h_k(x) & \text{in} \quad \Omega, \\ B(u_k)(x) = 0 \quad \text{on} \quad \partial\Omega \end{cases} \tag{2·4}$$

it follows that $\{u_k\}$ is bounded in $W^{2,p}$.

Proof. It is sufficient to prove that $\{u_k\}$ is bounded in L^p. Indeed, if we can establish this then we will also have $\{f(x, u_k(x))\}$ and $\{Q(u_k)\}$ bounded in L^p, thus $\{L(u_k)\}$ will be bounded in L^p, so that we may invoke property (1·4). To prove the boundedness of $\{u_k\}$ in L^p we first prove boundedness in L^2.

Let $k \in \mathbb{N}$. Multiplying equation (2·4) by $u_k(x)$, for each $x \in \Omega$, integrating and using the fact that Q is an orthogonal projection on L^2 we get

$$\begin{aligned} &\int_\Omega (L(u_k)(x) + \lambda_1 u_k(x))\, u_k(x)\, dx \\ &\quad + t_k \int_\Omega (f(x, u_k(x)) - \lambda_1 u_k(x) - Q(h_k)(x))\, u_k(x)\, dx + (1-t_k) \int_\Omega Q(u_k)(x)\, u_k(x)\, dx \\ &\qquad = \int_\Omega h_k(x)\, (I-Q)(u_k)(x)\, dx + (1-t_k) \int_\Omega Q(h_k)(x)\, u_k(x)\, dx. \end{aligned}$$

Now $\{Q(h_k)\}$ are uniformly bounded continuous functions, and so, letting

$$\alpha = \max\{g_1(x) | x \in \overline{\Omega}\},$$

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we may use the asymptotic assumption of f to choose $r > 0$ such that

$$f(x,s) - \lambda_1 s - Q(h_k)(x) \geqslant \alpha + 1 \quad \text{if} \quad s > r, \quad x \in \overline{\Omega},$$

while $$f(x,s) - \lambda_1 s - Q(h_k)(x) \leqslant -\alpha - 1 \quad \text{if} \quad s \leqslant -r, \quad x \in \overline{\Omega}.$$

Then, if we let $$\Omega_k^+ = \{x \in \Omega | u_k(x) \geqslant r\},$$

$$\Omega_k^- = \{x \in \Omega | u_k(x) \leqslant -r\} \quad \text{and} \quad \Omega_k^0 = \Omega \backslash (\Omega_k^+ \cup \Omega_k^-),$$

we have $$\int_{\Omega_k^+} (f(x, u_k(x)) - \lambda_1 u_k(x) - Q(h_k)(x)) u_k(x)\, dx \geqslant \int_{\Omega_k^+} g_1(x) u_k(x)\, dx,$$

and $$\int_{\Omega_k^-} (f(x, u_k(x)) - \lambda_1 u_k(x) - Q(h_k)(x)) u_k(x)\, dx \geqslant -\int_{\Omega_k^-} g_1(x) u_k(x)\, dx.$$

Moreover, we also have both

$$\left\{\int_{\Omega_k^0} (f(x, u_k(x)) - \lambda_1 u_k(x) - Q(h_k)(x)) u_k(x)\, dx\right\}$$

and

$$\left\{\int_{\Omega_k^0} |u_k(x)|\, dx\right\}$$

bounded. Consequently, since $g_1(x) > 0$ on Ω, so that

$$\int_{\Omega_k^+} g_1(x) u_k(x)\, dx \quad \text{and} \quad -\int_{\Omega_k^-} g_1(x) u_k(x)\, dx$$

are positive, it follows that we may choose a constant a, independent of k, such that

$$\int_\Omega (f(x, u_k(x)) - \lambda_1 u_k(x) - Q(h_k)(x)) u_k(x)\, dx \geqslant \|Q(u_k)\|_{L^2} + a,$$

for each $k \in \mathbb{N}$. We can now use the above inequality, together with inequality (2·2) and the Cauchy–Schwarz inequality to conclude that for each $k \in \mathbb{N}$,

$$-\lambda_2 \|(I-Q)(u_k)\|_{L^2}^2 + t_k(a + \|Q(u_k)\|_{L^2}) + (1-t_k)\|Q(u_k)\|_{L^2}^2 \leqslant \|h_k\|_{L^2} \cdot \|(I-Q)(u_k)\|_{L^2} + (1-t_k)\|Q(h_k)\|_{L^2}\|Q(u_k)\|_{L^2}.$$

But recall that $\{h_k\}$ is bounded in L^2, and from this we claim that the above inequality implies that both $\{(I-Q)(u_k)\}$ and $\{Q(u_k)\}$ are bounded in L^2. Indeed, if $\{Q(u_k)\}$ is not bounded in L^2, then for k sufficiently large one will have

$$-\lambda_2 \|(I-Q)(u_k)\|_{L^2}^2 + t_k(a + \|Q(u_k)\|_{L^2}) + \frac{(1-t_k)}{2}\|Q(u_k)\|_{L^2}^2 \leqslant \|h_k\|_{L^2}\|(I-Q)(u_k)\|_{L^2},$$

and this implies, first that $\{(I-Q)(u_k)\}$ is bounded in L^2, and then that

$$\left\{t_k(a + \|Q(u_k)\|_{L^2}) + \frac{(1-t_k)}{2}\|Q(u_k)\|_{L^2}^2\right\}$$

is bounded, so that $\{\|Q(u_k)\|_{L^2}\}$ is bounded. Thus $\{Q(u_k)\}$ is bounded in L^2, and then by repeating the above argument one also gets the L^2 boundedness of $\{(I-Q)(u_k)\}$.

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We now show that since $\{u_k\}$ is bounded in L^2, it must also be bounded in L^p.

Now equation (2·4), together with property (1·4), implies that if $s \in (1, p]$ and $\{u_k\}$ is bounded in L^s then $\{u_k\}$ is bounded in $W^{2,s}$. An elementary calculation shows that one can find $j \in \mathbb{N}$ and $s_0 \in [1, 2]$ such that $q^j(s_0) < n$ and $q^{j+1}(s_0) = p$ where $q(s) = ns/(n-s)$, for $s \in [1, n]$. So we may invoke property (1·3), together with a bootstrap argument, to obtain the boundedness of $\{u_k\}$ in L^p. ∎

Remark 2·1. The above result can be strengthened somewhat. Consider the case when $B(u) = \partial u/\partial \eta$. If one assumes $\underline{\lim} f(x, s) \geqslant \alpha$, uniformly with respect to $x \in \Omega$ and $\overline{\lim} f(x, s) \leqslant \beta$, uniformly with respect to $x \in \Omega$, where $\beta < 0 < \alpha$, then the above proof needs only minor modification to yield the boundedness in $W^{2,p}$ of $\{u_k\}$, where each u_k satisfies equations (2·2) and (2·3) and in addition to the boundedness of $\{h_k\}$ in L^p we also assume $\{Q(h_k)\} \subseteq (\beta + \epsilon, \alpha - \epsilon)$, for some $\epsilon > 0$.

Our next boundedness result will be concerned with a nonlinearity $f(x, u(x))$ where $f(x, s)$ approaches $-\infty$ as $s \to +\infty$ while $f(x, s)$ approaches $+\infty$ as $s \to -\infty$. Of course, the Fredholm Alternative shows one does not have solvability of equation (0·1) for all right-hand sides when the above $f(x, s)$ equals βs, where β is an eigenvalue of (0·2). Bearing this in mind we will assume that f satisfies

(*B*) $\lim_{s \to \pm\infty} f(x, s)/s = f_\pm(x)$, for each $x \in \bar{\Omega}$, where there exist consecutive eigenvalues $\lambda_{j+1} < \lambda_j \leqslant 0$ of problem (0·2) and an $\epsilon > 0$ such that $f_+(x), f_-(x) \in [\lambda_{j+1} + \epsilon, \lambda_j - \epsilon]$, for each $x \in \bar{\Omega}$, λ_j denoting the jth negative eigenvalue of problem (0·2).

PROPOSITION 2·2. *Let $f: \bar{\Omega} \times \mathbb{R} \to \mathbb{R}$ satisfy the Carathéodory conditions, let $p > n$, and suppose f satisfies the condition (2·3) with $r = p$, also satisfies condition (B). Then whenever $\{u_k\} \subseteq W^{2,p}$, $\{t_k\} \subseteq [0, 1]$, and $\{h_k\} \subseteq L^p$ are such that for each $k \in \mathbb{N}$*

$$\begin{cases} L(u_k)(x) + t_k f(x, u_k(x)) + (1 - t_k)\dfrac{(\lambda_j + \lambda_{j+1})}{2} u_k(x) = h_k(x) & \text{in } \Omega \\ B(u_k)(x) = 0 & \text{on } \partial\Omega, \end{cases} \tag{2·5}$$

it follows that $\{u_k\}$ is bounded in $W^{2,p}$, if $\{h_k\}$ is bounded in L^p.

Proof. Following the arguments used in the proof of the previous proposition, it will suffice to prove the boundedness of $\{u_k\}$ in L^2. So suppose this is not the case. Then, by choosing a subsequence, if necessary, we may assume $\{\|u_k\|_{L^2}\} \to +\infty$ and $\{t_k\} \to t_0$. Letting $v_k = u_k/\|u_k\|_{L^2}$ and $w_k = h_k/\|u_k\|_{L^2}$, we have, for each $k \in \mathbb{N}$,

$$\begin{cases} L(v_k)(x) + t_k \dfrac{f(x, u_k(x))}{\|u_k\|_{L^2}} + (1 - t_k)\dfrac{(\lambda_j + \lambda_{j+1})}{2} v_k(x) = w_k(x) & \text{in } \Omega, \\ B(v_k)(x) = 0 & \text{on } \partial\Omega. \end{cases} \tag{2·6}$$

Now

$$\left\{ t_k \frac{f(x, u_k)}{\|u_k\|_{L^2}} + (1 - t_k)\frac{(\lambda_j + \lambda_{j+1})}{2} v_k - w_k \right\}$$

is bounded in L^2, as is $\{v_k\}$, of course, so property (1·4) yields the boundedness of $\{v_k\}$ in $W^{2,2}$. Consequently, property (1·1) implies that by again taking a subsequence, if necessary, we may assume $\{v_k\} \to v_0$ in $W^{1,2}$ and $\{v_k\} \to v_0$ pointwise, almost everywhere.

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Let $\Omega_+ = \{x \in \Omega | v_0(x) > 0\}$, $\Omega_- = \{x \in \Omega | v_0(x) < 0\}$, and $\Omega_0 = \Omega \backslash \{\Omega_+ \cup \Omega_-\}$, and let $\psi_k(x) = f(x, u_k(x))/\|u_k\|_{L^2}$, for each $x \in \Omega$, $k \in \mathbb{N}$. Assumption (B) implies that on Ω_+, $\{\psi_k\}$ converges to $f_+ v_0$, almost everywhere, while on Ω_-, $\{\psi_k\}$ converges to $f_- v_0$, almost everywhere. Finally, since

$$|\psi_k(x)| \leqslant \frac{|v(x)|}{\|u_k\|_{L^2}} + M \frac{|u_k(x)|}{\|u_k\|_{L^2}},$$

$\{\psi_k\}$ converges to 0, almost everywhere on Ω_0. It follows that if we define $h: \bar{\Omega} \to \mathbb{R}$ by

$$h(x) = \begin{cases} f_+(x) & \text{when} \quad x \in \Omega_+, \\ f_-(x) & \text{when} \quad x \in \Omega_-, \\ \frac{1}{2}(\lambda_j + \lambda_{j+1}) & \text{when} \quad x \in \Omega_0, \end{cases}$$

then $\{\psi_k\}$ converges pointwise to hv_0 almost everywhere on Ω, and since $\{\psi_k\}$ is bounded in L^2, $\{\psi_k\}$ also converges weakly to hv_0 (weakly in L^2).

To obtain a contradiction from the above situation we will first consider the case when B does not correspond to the Dirichlet boundary conditions.

Let $w \in W^{1,2}$. Then multiplying equation (2·6) by $w(x)$, for each $x \in \Omega$, integrating by parts and taking the limit as $k \to \infty$ we obtain

$$(2\cdot7) \quad \sum_{i,j=1}^{n} \int_\Omega a_{i,j}(x) \frac{\partial v_0}{\partial x_i}(x) \frac{\partial w}{\partial x_j}(x)\, dx + \int_{\partial\Omega} \sigma(x)\, v_0(x)\, w(x)\, ds + \int_\Omega a(x)\, v_0(x)\, w(x)\, dx = 0,$$

where σ is the coefficient function occurring in the boundary condition and

$$a(x) = t_0 h(x) + (1 - t_0)\tfrac{1}{2}(\lambda_j + \lambda_{j+1}) \quad \text{for each} \quad x \in \Omega.$$

We observe that $a \in L^\infty$ and that we may select $\eta > 0$ such that

$$a(x) \in (\lambda_{j+1} + \eta, \lambda_j - \eta) \quad \text{for} \quad x \in \Omega.$$

Now (2·7) is the variational formulation in $W^{1,2}$, when $\lambda = 1$, of

$$(2\cdot8) \qquad \begin{cases} L(u)(x) + \lambda a(x)\, u(x) = 0 & \text{in} \quad \Omega \\ B(u)(x) = 0, & \text{on} \quad \partial\Omega, \end{cases}$$

and since $v_0 \neq 0$, this means 1 is an eigenvalue of (2·8), which occurs as, say, the mth eigenvalue. But the Courant min–max principle implies that the mth eigenvalue of

$$\begin{aligned} L(u)(x) + \lambda\lambda_j u(x) = 0 \quad & \text{in} \quad \Omega \\ B(u)(x) = 0 \quad & \text{on} \quad \partial\Omega, \end{aligned}$$

namely λ_m/λ_j, is greater than 1, while the mth eigenvalue of

$$\begin{aligned} L(u)(x) + \lambda\lambda_{j+1} u(x) = 0 \quad & \text{in} \quad \Omega \\ B(u)(x) = 0, \quad & \text{on} \quad \partial\Omega, \end{aligned}$$

namely λ_m/λ_{j+1}, is less than 1. Thus, $\lambda_{j+1} < \lambda_m < \lambda_j$, which contradicts the fact that λ_j and λ_{j+1} have been chosen to be consecutive eigenvalues of equation (0·2). This contradiction proves our result when B does not correspond to the Dirichlet conditions.

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Now consider the case when $B(u) = u$. In this case by the same argument as was used above one gets that (2·7) holds for each $w \in C_c^\infty(\Omega)$. This is precisely the variational characterization of

$$\begin{aligned} L(u)(x) + a(x)u(x) &= 0 \quad \text{in} \quad \Omega \\ u(x) &= 0, \qquad\quad \text{on} \quad \partial\Omega, \end{aligned}$$

and so can follow the same argument as was used above to obtain a contradiction. ∎

As we mentioned at the beginning of this section, we shall require the boundedness results expressed in Propositions 2·1 and 2·2 for non-linearities more general than those prescribed in the statements of these results. Let us describe the more general situation.

Define n_2 to be $\#\{\alpha | \alpha \in (N \cup \{0\})^n,\ |\alpha| = 2\}$ and for $u: \bar{\Omega} \to \mathbb{R}$ let $D^2(u): \bar{\Omega} \to \mathbb{R}^{n_2}$ be defined by $(D^2(u))_\alpha = D^\alpha u(x)$, for $\alpha \in (\mathbb{N} \cup \{0\})^n$ with $|\alpha| = 2$, and $x \in \bar{\Omega}$.

(H 1) Let $f: \bar{\Omega} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n_2} \to \mathbb{R}$ satisfy the generalized Carathéodory conditions (i.e. if $(s, r, t) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n_2}$ then $f(\cdot, s, r, t)$ is measurable, and for almost all $x \in \Omega$ $f(x, \cdot, \cdot, \cdot): \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n_2} \to \mathbb{R}$ is continuous), and be such that there exists $v \in L^p$ and $M > 0$ such that

$$|f(x, s, r, 0)| \leqslant v(x) + M|s|,$$

for $(x, s, r, t) \in \Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n_2}$.* Moreover assume that there is a constant α such that

$$|f(x, s, r, t_1) - f(x, s, r, t_2)| \leqslant \alpha |t_1 - t_2|_{l^1},$$

for $(x, s, r) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$, $t_1, t_2 \in \mathbb{R}^{n_2}$, where α satisfies $\alpha\beta < 1$, β being such that

$$\sum_{|\alpha|=2} \|D^\alpha u\|_{L^s} \leqslant \beta \|L(u)\|_{L^s},$$†

for all $s \in [2, p]$ and $u \in W^{2,p}$ with $B(u) = 0$ on $\partial\Omega$.

In addition to the above assumption let us identify two separate types of asymptotic conditions.

(H 2) $\lim_{s \to \pm\infty} f(x, s, r, t) = \pm\infty$, uniformly with respect to $(x, r, t) \in \Omega \times \mathbb{R}^n \times \mathbb{R}^{n_2}$.

(H 3) $\lim_{s \to \pm\infty} f(x, s, r, t)/s = f_\pm(x)$, uniformly with respect to $(r, t) \in \mathbb{R}^n \times \mathbb{R}^{n_2}$, where there exist consecutive eigenvalues $\lambda_{j+1} < \lambda_j \leqslant 0$ of problem (0·2) and $\epsilon > 0$ such that

$$f_\pm(x) \in [\lambda_{j+1} + \epsilon, \lambda_j - \epsilon] \quad \text{for} \quad x \in \Omega.$$

THEOREM 2·1. *Let $f: \Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n_2} \to \mathbb{R}$ satisfy* (H 1) *and* (H 2). *Then whenever $\{h_k\} \subseteq L^p$ is bounded, $\{t_k\} \subseteq [0, 1]$, and $\{u_k\} \subseteq W^{2,p}$ is such that for each $k \in N$*

$$(2\cdot10) \quad \begin{cases} L(u_k)(x) + \lambda_1 u_k(x) + t_k(f(x, u_k(x), \nabla u_k(x), D^2 u_k(x)) - \lambda_1 u_k(x)) \\ \qquad\qquad + (1 - t_k) Q(u_k(x)) = h_k(x) \quad \text{in} \quad \Omega, \\ B(u_k)(x) = 0, \quad \text{on} \quad \partial\Omega, \end{cases}$$

it follows that $\{u_k\}$ is bounded in $W^{2,p}$.

Proof. Precisely the same argument as was used in proving Proposition 2·1 yields the boundedness of $\{u_k\}$ in $W^{2,2}$, so that to prove our result it suffices to show that the concluding bootstrap argument of that proposition is also valid in the present situation.

* If $z = (z_1, \ldots, z_j) \in \mathbb{R}^j$, we denote $\Sigma_{k=1}^j |z_k|$ by $|z|_{l^1}$.

† The existence of such a $\beta > 0$ follows from Proposition 3·3 and the subsequent remarks.

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But now this argument depends on knowing that if $s \in (1, p]$ and $\{u_k\}$ is bounded in $W^{1,s}$ then $\{u_k\}$ is bounded in $W^{2,s}$. But if $\{u_k\}$ is bounded in $W^{1,s}$, then from equation (2·10) together with hypotheses (H 1) one obtains that for each $k \in \mathbb{N}$

$$\begin{aligned}|L(u_k)(x)| \leqslant\ & t_k |f(x, u_k(x), \nabla u_k(x), D^2 u_k(x)) - f(x, u_k(x), \nabla u_k(x), 0)| \\ & + t_k |f(x, u_k(x), \nabla u_k(x), 0)| \\ & + |1 - t_k|\, |Q(u_k)(x)| + |h_k(x)| - \lambda_1 (1 - t_k) |u_k(x)|\end{aligned}$$

which implies, using Hölder's inequality, that

$$\begin{aligned}\|Lu_k\|_{L^s} \leqslant\ & t_k \alpha\beta \|L(u_k)\|_{L^s} + t_k \|v\|_{L^s} - \lambda_1 (1 - t_k) \|u_k\|_{L^s} \\ & + t_k M\Big(\sum_{|\alpha| \leqslant 1} \|D^\alpha u_k\|_{L^s}\Big) + (1 - t_k) \|Q(u_k)\| + \|h_k\|_{L^s}.\end{aligned}$$

But $0 \leqslant t_k \alpha\beta < \alpha\beta < 1$, for each k, and thus $\{Lu_k\}$ is bounded in $W^{2,s}$. ∎

As the proof of Proposition 2·1 was adapted to yield the previous theorem, so that of Proposition 2·2 yields

THEOREM 2·2. *Let $f: \Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n^2} \to \mathbb{R}$ satisfy* (H 1) *and* (H 3). *Then whenever $\{h_k\} \subseteq L^p$ is bounded, $\{t_k\} \subseteq [0, 1]$, and $\{u_k\} \subseteq W^{2,p}$ is such that for each $k \in \mathbb{N}$*

$$L(u_k)(x) + t_k f(x, u_k(x), \nabla u_k(x), D^2 u_k(x)) + (1 - t_k) \tfrac{1}{2}(\lambda_j + \lambda_{j+1}) u_k(x) = h_k(x) \quad \text{in} \quad \Omega$$

$$B(u_k)(x) = 0, \quad \text{on} \quad \partial\Omega,$$

it follows that $\{u_k\}$ is bounded in $W^{2,p}$.

Section 3. In this section we wish, under suitable assumptions on f, to solve equation (0·1) for each $h \in L^p$. We will do so by combining the results of section 2 with the following two abstract results.

We recall that a mapping $S \in L(X, Y)$ is called Fredholm of index 0 provided that its null-space is finite dimensional and its range is closed and has finite codimension equal to the dimension of its null-space. A mapping $S \in L(X, Y)$ is Fredholm of index 0 if and only if there is a compact mapping $H \in L(X, Y)$ such that $S + H$ is bijective (see Cotlar and Cignoli(5)). In what follows we will use the above terminology but the basic result from which our conclusions are drawn is the following: Let $S \in L(X, Y)$ and suppose there exists a compact mapping C_1 such that $S + C_1$ is a bijection. Then whenever $C_2 \in L(X, Y)$ is compact and $S + C_2$ is one-to-one, $S + C_2$ is a bijection. It may be of interest to observe that instead of employing the usual linear methods to prove this, one can give a very simple proof based upon the Schauder fixed-point theorem.

The following proposition was proved by the author in (8) by means of a degree theory argument. For completeness, we include a proof of it here, based directly upon Proposition 1·2.

PROPOSITION 3·1. *Let X and Y be Banach spaces, and let $T \in L(X, Y)$ be Fredholm of index 0. Suppose $C \in L(X, Y)$ is compact and $T + C$ is bijective. Assume $F: X \to Y$ is such that $F \circ (T + C)^{-1}: Y \to Y$ is set-condensing and the following condition holds:*

(3·1) *whenever $\{x_k\} \subseteq X$ and $\{t_k\} \subseteq [0, 1]$ are such that $\{T(x_k) + (1 - t_k) F(x_k) + t_k C(x_k)\}$ is bounded, then $\{x_k\}$ is also bounded.*

Then $(T + F)(X) = Y$.

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Proof. Let $h \in Y$. To show $h \in (T+F)(X)$ is equivalent to solving the equation

$$v = C \circ (T+C)^{-1}(v) - F \circ (T+C)^{-1}(v) + (T+C)^{-1}(h), \quad v \in Y. \tag{3·2}$$

Since the mapping $v \mapsto C \circ (T+C)^{-1}(v) + (T+C)^{-1}(h)$ is compact and $F \circ (T+C)^{-1}$ is, by assumption, set-condensing, the mapping

$$v \mapsto C \circ (T+C)^{-1}(v) - F \circ (T+C)^{-1}(v) + (T+C)^{-1}(h)$$

is also set-condensing. Hence, by Proposition 1·2 in order to solve equation (3·2) it will suffice to show that there exists an $r > 0$ such that

$$\begin{cases} v - t[C \circ (T+C)^{-1}(v) - F \circ (T+C)^{-1}(v) + (T+C)^{-1}(h)] \neq 0, \\ \qquad \text{when} \quad \|x\| = r \quad \text{and} \quad 0 \leqslant t \leqslant 1. \end{cases} \tag{3·3}$$

Indeed, if such an r does not exist then we may select $\{v_k\} \subseteq Y$ and $\{t_k\} \subseteq [0, 1]$ such that $\{v_k\}$ is unbounded and for each $k \in \mathbb{N}$,

$$v_k = t_k[C \circ (T+C)^{-1}(v_k) - F \circ (T+C)^{-1}(v_k) + (T+C)^{-1}(h)].$$

For each $k \in \mathbb{N}$ let $x_k = (T+C)^{-1}(v_k)$. Then $\{x_k\}$ is unbounded and moreover, for each k, x_k and t_k satisfy

$$T(x_k) + t_k F(x_k) + (1-t_k) C(x_k) = t_k (T+C)^{-1}(h).$$

But $\{t_k(T+C)^{-1}(h)\}$ is bounded and $\{x_k\}$ is unbounded, which is a contradiction of assumption (3·1). ∎

Proposition 3·2. *Let X and Y be Banach spaces, $Y \in L(X, Y)$ be Fredholm of index 0, and $F: X \to Y$ be continuous. Suppose there exists a compact $C_1 \in L(X, Y)$ such that $T + C_1$ is bijective and $F \circ (T+C_1)^{-1}$ is set-condensing. Then for any $C \in L(X, Y)$ which is compact and such that $T + C$ is bijective, it follows that $F \circ (T+C)^{-1}$ is set-condensing.*

Proof. For any bounded subset A of Y, with $\bar{A}$ non-compact,

$$\gamma(F \circ (T+C_1)^{-1}(A)) < \gamma(A),$$

which means that for any bounded, non-precompact subset D of X,

$$\gamma(F(D)) < \gamma((T+C_1)(D)).$$

But $\gamma((T+C_1)(D)) = \gamma(T(D)) = \gamma((T+C)(D))$. Thus $\gamma(F \circ (T+C)^{-1}(A)) < \gamma(A)$ when $A \subseteq Y$ and $\bar{A}$ is not compact. ∎

Letting $p > n$, we define X to be the closed subspace of $W^{2,p}$ consisting of those u with $B(u) = 0$ on $\partial\Omega$, and we let $Y = L^p$. Define $K: X \to Y$ by

$$K(u) = L(u), \quad u \in X.$$

Proposition 3·3. *$K \in L(X, Y)$ is Fredholm of index zero.*

Proof. The L^2-theory (see (14)) implies that if $\beta > 0$ then

$$K + \beta I: \{u \in W^{2,2} \mid B(u) = 0 \text{ on } \partial\Omega\} \to L^2$$

is a bijection and, moreover, if $u \in W^{2,2}$, $B(u) = 0$ on $\partial\Omega$, and

$$K(u) + \beta u = w, \quad \text{where} \quad w \in C^1(\bar{\Omega}), \quad \text{then} \quad u \in C^2(\bar{\Omega}) \subseteq W^{2,p}.$$

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Now $K+\beta I: X \to Y$ is one-to-one, and (1·4) implies that we may choose $c' > 0$ such that

$$\|u\|_X \leqslant c'(\|K(u)+\beta(u)\|_Y + \|u\|_Y) \quad \text{for} \quad u \in X,$$

from which it follows that $(K+\beta I)(X)$ is closed. The above L^2-theory implies that $(K+\beta I)(X)$ is dense in Y. Thus $K+\beta I$ is a bijection. Since $\beta I: X \to Y$ is compact, K is Fredholm of index zero. ∎

We note that the strong maximum principle (theorem 7, (19)) means that $K \in L(X, Y)$ is one-to-one, and hence a bijection, except where $B(u) = \partial u/\partial \eta$, in which case K has a nullspace consisting of the constant functions.

Suppose $f: \Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n_2} \to \mathbb{R}$ satisfies assumption (H 1) of the previous section, and define $N: X \to Y$ by

$$N(u)(x) = f(x, u(x), \nabla u(x), D^2 u(x)) \quad \text{for} \quad u \in X \quad \text{and} \quad x \in \Omega.$$

Then N is well-defined, bounded, and continuous from X to Y. If $f: \bar{\Omega} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n_2}$ satisfies (H 1) and, in addition, f is continuous in (x, s, r, t), uniformly with respect to $t \in \mathbb{R}^{n_2}$, then we shall say that f satisfies assumption (H 1′).

PROPOSITION 3·4. *Let $p > n$ and let K, N, X and Y be as defined above. Define $Q: X \to Y$ by equation (2·1) where $g_1 \equiv 1$. In the case when $B(u) = \partial u/\partial \eta$, $K+Q$ is bijective and $N \circ (K+Q)^{-1}$ is set-condensing. In the case of boundary conditions other than the Neumann condition, K is bijective and $N \circ K^{-1}$ is set-condensing.*

Proof. We first consider the case when $B(u) = \partial u/\partial \eta$. It is clear that $K+Q$ is one-to-one and hence, by Proposition 3·3, it must be bijective.

Let $A \subseteq Y = L^p$ be bounded and such that $\bar{A}$ is not compact. Since $p > n$, and $\partial\Omega$ is smooth, $(K+Q)^{-1}(A)$ and $\nabla((K+Q)^{-1}(A))$ are precompact in $C(\bar{\Omega}, \mathbb{R})$ and $C(\bar{\Omega}, \mathbb{R}^n)$, respectively. Consequently, from the uniform continuity hypothesis in (H 1′) to prove our result it suffices to show that if $w: \bar{\Omega} \to \mathbb{R} \times \mathbb{R}^n$ is continuous and we define $S: X \to Y$ by

$$S(v)(x) = f(x, w_0(x), \ldots, w_n(x), D^2(K+Q)^{-1}(v)(x)) \quad v \in Y, \quad x \in \Omega,$$

is set-condensing. But assumption (H 1) implies S: Range $(K) \to Y$ is Lipschitz with a Lipschitz constant $\alpha\beta < 1$. Moreover, one also has $I-Q: Y \to Y$ is a projection onto Range (K) with

$$(S \circ (I-Q))(A) = S(A) \quad \text{for} \quad A \subseteq Y.$$

Consequently, if $A \subseteq Y$ is bounded and such that $\bar{A}$ is not compact,

$$\begin{aligned} \gamma(S(A)) &= \gamma((S \circ (I-Q))(A)) \\ &< \gamma((I-Q)(A)) \\ &= \gamma(A), \end{aligned}$$

so that S is set-condensing.

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The above argument also suffices for boundary conditions other than the Neumann condition. ∎

Remark 3·1. In (8) the author obtained a result similar to the above when

$$B(u) = \partial u/\partial \eta \quad \text{and} \quad L(u) = -\Delta(u),$$

but where the second-order nonlinear dependence in f was of the form

$$f(x, u(x), \nabla u(x), \Delta u(x)).$$

We may now combine the previous propositions of this section with Theorems 2·1 and 2·2 to prove our existence result.

THEOREM 3·1. *Let $p > n$ and let $f: \Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n^2} \to \mathbb{R}$ satisfy hypothesis* (H 1′). *Moreover, suppose f satisfies either hypothesis* (H 2) *or* (H 3). *Then given any $h \in L^p$ there exists $u \in W^{2,p}$ such that*

$$\begin{cases} L(u)(x) + f(x, u(x), \nabla u(x), D^2(u)(x)) = h(x) & \text{in} \quad \Omega, \\ B(u)(x) = 0 \quad \text{on} \quad \partial\Omega. \end{cases} \tag{3·3}$$

Proof. First consider the case where f satisfies (H 1′) and (H 2).

Let λ_1 be the largest eigenvalue of (0·2), and letting g_1 be the positive eigenfunction corresponding to λ_1 with $\|g_1\|_{L^2} = 1$ define Q by equation (2·1). Then it is easy to see that $K + \lambda_1 I + Q$ is one-to-one so, since K is Fredholm of index 0 and $\lambda_1 I + Q$ is compact, $K + \lambda_1 I + Q$ is a bijection. We now wish to appeal to Proposition 3·1 where $T = K + \lambda_1 I$, $F = N - \lambda_1 I$ and $C = Q$. To show that $F \circ (T + C)^{-1}$ is set-condensing we just combine Proposition 3·2 with Proposition 3·4, letting $Q - \lambda_1 I$ play the role of C_1 when one has Neumann boundary conditions, and letting $C_1 = -\lambda_1 I$ otherwise. We now have to verify condition (3·1). But with the above choice of T, F, and C Theorem 2·1 implies condition (3·1). Thus $(K + N)(X) = Y$ when (H 1′) and (H 2) are satisfied.

Now consider the case when (H 1′) and (H 3) are satisfied. In this case we again appeal to Proposition 3·1, but this time with $T = K$, $F = N$ and

$$C = \tfrac{1}{2}(\lambda_j + \lambda_{j+1}) I : X \to Y.$$

By the definition of λ_j and λ_{j+1}, $T + C$ is one-to-one, and thus it is a bijection. Arguing as above $F \circ (T + C)^{-1}$ is set-condensing, and, this time, Theorem 2·2 guarantees that condition (3·1) is satisfied. ∎

Remark 3·1. If one is interested in solving equation (3·2) for a particular $h_0 \in Y$ it is clear from the proof of Proposition 3·1 that one will obtain solvability for this h_0 if one replaces the hypothesis (3·1) with the weaker assumption that

(3·1′) whenever $\{x_k\} \subseteq X$, and $\{t_k\} \subseteq [0, 1]$ are such that

$$K(x_k) + t_k N(x_k) + (1 - t_k) C(x_k) = t_k (K + C)^{-1}(h_0),$$

it follows that $\{x_k\}$ is bounded.

Now Remark 2·1 implies that if, say with $B(u) = \partial u/\partial \eta$, one relaxes the asymptotic condition to the condition that $\varliminf_{s \to \infty} f(x, s, r, t) \geqslant \beta$, uniformly in $(x, r, t) \in \Omega \times \mathbb{R}^n \times \mathbb{R}^{n^2}$,

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while $\overline{\lim_{s\to-\infty}} f(x,s,r,t) \leqslant \alpha$, uniformly in $(x,r,t)\in\Omega\times\mathbb{R}^n\times\mathbb{R}^{n_2}$, then if $\alpha < 0 < \beta$, equation (3·3) will have a solution for all $h\in L^p$ such that

$$\frac{1}{\text{meas}\,(r)}\int_\Omega h(y)\,dy \in (\alpha,\beta).$$

When $f = f(x,u(x))$ and $L = -\Delta$ such results have been obtained, based on recession function arguments, in (3).

Remark 3·2. Our choice of $p > n$ was made in order to verify that under (H 1′) one could reformulate equation (0·1) as a fixed-point result for a condensing mapping in L^p. If f does not involve dependence on second order partial derivatives, one can reformulate the equation in L^2.

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Added in proof. The *a priori* bounds are valid under weaker growth conditions. For instance, one may weaken (2·3) by assuming that there is a σ less than $1+(4/n)$ such that

$$|f(x,s)| \leqslant v(x)+b|s|^{\sigma} \quad \text{for} \quad (x,s) \in \overline{\Omega} \times \mathbb{R}.$$

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