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SOME NEW RELIABILITY BOUNDS FOR SUMS OF NBUE RANDOM VARIABLES

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In this article, we discuss some new upper and lower bounds for the survivor function of the sum of n independent random variables each of which has an NBUE (new better than used in expectation) distribution. In some cases, only the means of the random variables are assumed known. These bounds are compared to the sharp bounds given in Cheng and Lam [6], which requires both means and variances known. Although the new bounds are not sharp, they often produce better upper bounds for the survivor function in the extreme right tail of many NBUE lifetime distributions, an important special case in applications. Moreover, a lower bound exists in one case not handled by the lower bounds of Theorem 3 in Cheng and Lam [6]. Numerical studies are presented along with theoretical discussions.

1. INTRODUCTION

In many applications of reliability theory, we are often interested in the distribution of the sum of independent random variables. Even if the distributions of these random variables are completely known, the computation of this distribution is difficult except in a few cases. Often, these random variables are lifetimes of components and/or systems and we are interested in bounds on survival probabilities involving these components and/or systems. For many such component/system lifetimes, the assumption of membership of the lifetime distributions in certain reliability subclasses such as increasing failure rate (IFR), increasing failure rate average (IFRA), new better than used (NBU) or new better than used in expectation (NBUE) is quite reasonable, especially for components exhibiting wear with age. See Barlow and Proschan [4] for a discussion of these subclasses of lifetime distributions.

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In this article we will be especially interested in the sizes of extreme tails of the probability distributions of sums of random variables. Upper bounds for these tail sizes are very important in applications. Since large deviation theory is also concerned with extreme tails, we will briefly discuss a few results from this theory that are the most relevant to the reliability theory context of this article. For a general discussion of large deviation theory, see Varadhan [22,23], Nagaev [17], and Petrov [18]. More specific results especially pertinent to the problem at hand can be found in Bahadur and Ranga Rao [3], Sievers [21], Plachky [19], and Plachky and Steinebach [20]. Theorem 1.1 can be found in Plachky and Steinebach [20] and is a generalization of results found in Chernoff [7], Sievers [21], and Plachky [19]. It will be needed later to point out connections between the results of this article and large deviation theory.

THEOREM 1.1: (Plachky and Steinebach [20, p. 343]). *Let* $\{W_n\}_{n=1,2,\ldots}$ *be a sequence of real-valued random variables on a probability space with probability measure P, which satisfies the following assumptions*:

(*i*) $m_n(t) = \int e^{tW_n} dP < \infty$ *for all* $t \in [0, T_1)$, $T_1 > 0$.

(*ii*) *Let* $\psi_n(t) = Ln(m_n(t))$. *Then* $\psi_n(t)/n \to c_0(t)$ *as* $n \to \infty$, *where* $c_0(t)$ *is real-valued for all* $t \in (T_0, T_1)$, $0 \le T_0 < T_1$. *Then for any real sequence* $\{a_n\}_{n=1,2,\ldots}$ *with* $a_n \to a \in A = \{c_0'(h) | c_0'$ *exists and is continuous on the right and strictly monotonic for* $h \in (T_0, T_1)\}$, *it holds that* $[P(W_n > na_n)]^{1/n} \to \exp(c_0(h) - ha)$, *where the limit is equal to*

$$\inf_{t>0}\{\exp[c_0(t) - ta]\}.$$

In this article, we are especially interested in the special case where $W_n = X_1 + X_2 + \cdots + X_n$, the X_i's are all independent but not necessarily identically distributed. (Later, X_i will also be assumed to be a member of some reliability subclass of probability distributions.) In this special case, we obtain Chernoff's theorem as a special case of Theorem 1.1.

CHERNOFF'S THEOREM (Chernoff [7]): *Let* $X_1, X_2, \ldots, X_n$ *be independent and identically distributed* (i.i.d.) *random variables with* $\mu = E(X_i)$. *Let* $\bar{X} = 1/n \sum_{i=1}^{n} X_i$. *Let* $\lambda \ge 0$. *Then*

$$\frac{\mathrm{Ln}[P(\bar{X} \ge \lambda + \mu)]}{n} \to \mathrm{Ln}(p) \quad as \ \ n \to \infty,$$

where

$$p = \inf_{t>0}[e^{-(\lambda+\mu)t} M(t)] \quad \text{and} \ \ M(t) = E(e^{tX_i})$$

is the moment-generating function of X_i.

For large n, it follows that if X_i is exponential with mean μ, then $p = (1 + \lambda/\mu)e^{-\lambda/\mu}$ and $P(\bar{X} \ge \mu + \lambda) \sim (1 + \lambda/\mu)^n e^{-n\lambda/\mu}$. This will be needed later.

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The results of this article differ from large deviation theory results in that we will use upper and lower bounds on the moment-generating function (mgf) of $W_n = \sum_{i=1}^{n} X_i$ and not the actual mgf of W_n, since this is often unknown. However, we will assume each X_i is a member of various subclasses of reliability distributions with one or more moments known.

In this article we will present new upper and lower bounds on the survivor function of the sum (or sample mean, equivalently) of n independent lifetime random variables $X_1, X_2, \ldots, X_n$ in the case where each X_i has an NBUE distribution, whose definition is given below.

DEFINITION 1.2: *A lifetime random variable X has a new better than used in expectation (NBUE) distribution if*

(*a*) *X has finite mean $\mu = E(X) = \int_0^\infty \bar{F}(x)\,dx$, where $\bar{F}(x)$ is the survivor function $\bar{F}(x) = P(X > x)$, $x \geq 0$, and*

(*b*) *$\bar{F}(x) \geq 1/\mu \int_x^\infty \bar{F}(w)\,dw$ for all $x \geq 0$.*

We focus on the NBUE subclass of lifetime distributions since it is neither a too small or too large a subclass. First, we consider the case where only $\mu_i = E(X_i)$, $i = 1, 2, \ldots, n$, are known. Then we consider the case where $m_i = E(X_i^2)$, $i = 1, \ldots, n$ — the second moments—are known. In Section 2 we will obtain upper bounds on the survivor of $\bar{X} = (\sum_{i=1}^{n} X_i)/n$ which are often better (smaller) than those given Theorem 3 of Cheng and Lam [6] in the important special case of small extreme right tail areas of the distribution of $\bar{X}$. This case occurs often in statistical applications. In addition, in Section 3 we present a lower bound on the survivor in one case that is not handled by the lower bounds of Cheng and Lam [6]. Some numerical comparisons are presented in Section 4 along with relevant conclusions.

2. UPPER BOUNDS

Let $X_1, X_2, \ldots, X_n$ be n independent lifetime random variables with finite means $\mu_i = E(X_i)$, $i = 1, 2, \ldots, n$. Let $\mu = \bar{\mu} = (\sum_{i=1}^{n} \mu_i)/n$ and let $\bar{X} = (\sum_{i=1}^{n} X_i)/n$. We are interested in upper bounds for survivor of $\bar{X}$ in the form

$$\theta_\lambda = P(\bar{X} \geq \mu + \lambda)\,, \tag{2.1}$$

where $\lambda > 0$ is real. Expecially important in applications is the case where λ is "large." In this case, θ is an extreme right tail probability. In this case, exponentially fast decreasing bounds are desirable even though these bounds are not usually sharp bounds. The trade-off is poor bounds for "small" $\lambda > 0$. The bounds of Cheng and Lam [6] are exponential only in one part of their (t, p) space, and because they are sharp bounds, they sacrifice some quality in the extreme right tail.

To derive upper bounds of θ, we need the following lemma.

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LEMMA 2.1: *Let X have an NBUE lifetime distribution with $\mu_X = E(X) < \infty$. Then the moment-generating function of X is $M_X(t) = E(e^{tX}) < \infty$ for $-\infty < t < 1/\mu_X$ and*

$$M_X(t) \le \frac{1}{1 - t\mu_X}, \qquad t < \frac{1}{\mu_X}.$$

For a proof, see Theorem 2.1 of Ahmad and Mugdadi [1].

Remark: The upper bound on the mgf of Lemma 2.1 also holds for the most general harmonic new better than used (HNBUE) and $\mathcal{L}$ or Laplace order classes discussed in Alzaid [2], Klefsjo [13], Klar and Müller [12], and Li [14]. These classes all contain the NBUE class as a subclass. Thus, Theorems 2.2, 2.3, 2.4, and 2.5 also hold for these more general classes of distributions, since these theorems assume Lemma 2.1 inequality holds.

Next, we present a family of upper bounds for θ.

THEOREM 2.2: *Let $X_1, X_2, \ldots, X_n$ be independent NBUE lifetime random variables with finite means $\mu_i = E(X_i)$, $i = 1, 2, \ldots, n$. Then for all $t \ge 0$ with $t < 1/\mu_{(n)}$, we have*

$$\theta \le e^{-n(\mu+\lambda)t} \prod_{i=1}^{n} \frac{1}{1 - \mu_i t} \stackrel{\text{def}}{=} U(t), \tag{2.2}$$

where $\mu_{(n)} = \max(\mu_1, \mu_2, \ldots, \mu_n)$.

PROOF:

$$\begin{aligned}
\theta &= P(\bar{X} \ge \mu + \lambda) = P\left(e^{t\sum_{i=1}^{n} X_i} \ge e^{n(\mu+\lambda)t}\right) \\
&\le e^{-n(\mu+\lambda)t} E\left(e^{t\sum_{i=1}^{n} X_i}\right) \\
&= e^{-n(\mu+\lambda)t} \prod_{i=1}^{n} \frac{1}{1 - \mu_i t},
\end{aligned}$$

upon application of Markov's inequality and Lemma 2.1. ■

The question remains of how to select t in Theorem 2.2. A closed-form solution for such a t exists only in some cases. Let t^* denote any value of t minimizing the right-hand side of (2.2) in Theorem 2.2. For $n = 1$, it is easily shown that the choice $t^* = \lambda/\mu_1(\mu_1 + \lambda)$ is best producing the inequality $\theta \le (1 + \lambda/\mu)e^{-\lambda/\mu}$. This is worse than the upper bound $\theta \le e^{-\lambda/\mu}$ given in Klefsjo [13]. However, for $n > 1$, the upper bound of Theorem 2.2 using $t = t^*$ is often better than the Klefsjo [13] bound, as we will see in the numerical comparisons of Section 4 later. Another case in which t^* is of closed form is the i.i.d. case. Then $t^* = \lambda/(\mu(\mu + \lambda))$ is optimal producing

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$\theta \le (1+\lambda/\mu)^n e^{-n\lambda/\mu}$. This is the same bound as the Chernoff bound discussed in Section 1. In general, a numerical root finding procedure is needed to find t^*. We will prove that t^* is always the unique root of a certain equation in a predetermined bounded interval and give lower bound approximations to t^* of closed form in subsequent theorems. Let $U = U(t, \mu_1, \ldots, \mu_n, \lambda)$ denote the upper bound for θ of Theorem 2.2 considered as a function of t, μ_i and λ. Then

$$U = U(t^*, \mu_1, \ldots, \mu_n, \lambda) = \min_{0 \le t < t_u} U(t, \mu_1, \ldots, \mu_n, \lambda)\,,$$

where

$$t_u = \frac{1}{\mu_{(n)}}\,, \qquad \mu_{(n)} = \max(\mu_1, \ldots, \mu_n)\,.$$

Theorem 2.3 below states that t^* is unique.

THEOREM 2.3: *There is a unique value $t = t^*$ in $(0, 1/\mu_{(n)})$, that minimizes the upper bound $U(t, \mu_1, \ldots, \mu_n, \lambda)$, $\lambda > 0$.*

PROOF: Taking the natural logarithm of U,

$$W = \operatorname{Ln} U = -n(\mu+\lambda)t - \sum_{i=1}^{n} \operatorname{Ln}(1-\mu_i t)\,.$$

Differentiating W with respect to t gives $W'(t) = -n(\mu+\lambda) + \sum_{i=1}^{n}[\mu_i/(1-\mu_i t)]$. Now, $W'(0) = -n\lambda < 0$. Also, $W'(1/\mu_{(n)}) = \infty$. So by continuity of W, there is a t^* in $(0, 1/\mu_{(n)})$ such that $W'(t^*) = 0$. The second derivative of W is $W''(t) = \sum_{i=1}^{n}(\mu_i/(1-\mu_i t))^2$, so $W''(t^*) > 0$. So $W(t)$ has a relative and global minimum at $t = t^*$. This completes the proof. ■

Theorem 2.3 shows that $0 < U(t^*, \mu_1, \ldots, \mu_n, \lambda) < 1$, so that the upper bound is never trivial, unlike the upper bounds in Theorem 3 of Cheng and Lam [6]. Additionally, t^* satisfies the equation, $0 < t^* < \frac{1}{\mu_{(n)}}$,

$$\sum_{i=1}^{n} \frac{\mu_i}{1-\mu_i t^*} = n(\mu+\lambda)\,. \tag{2.3}$$

Theorems 2.4 and 2.5 give nontrivial (nonzero) lower bounds for t^* in Theorem 2.3. These are of closed form and often produce corresponding upper bounds for θ of closed form.

THEOREM 2.4: *A lower bound for t^* is*

$$t_L^{(1)} = \min\left(\frac{\lambda}{\mu_1(\mu_1+\lambda)}, \frac{\lambda}{\mu_2(\mu_2+\lambda)}, \ldots, \frac{\lambda}{\mu_n(\mu_n+\lambda)}\right)$$
$$= \frac{\lambda}{\mu_{(n)}(\mu_{(n)}+\lambda)}\,,$$

where $\mu_{(n)} = \max(\mu_1, \mu_2, \ldots \mu_n)$.

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PROOF: The upper bound $U = U(t, \mu_1, \ldots, \mu_n, \lambda)$ can be written as

$$U = \prod_{i=1}^{n} g_i(t) \quad \text{where} \quad g_i(t) = e^{-(\mu_i+\lambda)t} \frac{1}{1-\mu_i t}.$$

$g_i(t)$ is minimized at $t_i = \lambda/(\mu_i(\mu_i + \lambda))$ and $g_i(t)$ is decreasing for $0 \le t \le t_i$, $i = 1, 2, \ldots, n$. Since $g_i(t)$ is nonnegative, U is decreasing on $[0, t_L^{(1)}]$. By the proof of Theorem 2.3, we have $t_L^{(1)} \le t^*$. This completes the proof. ■

Remark 1: Let $\mu_{(1)} = \min(\mu_1, \ldots, \mu_n)$. An upper bound for t^* is $t_u = \lambda/\mu_{(1)}(\mu_{(1)} + \lambda)$. However, this might or might not be better than the upper bound $1/\mu_{(n)}$. In any case, $\min(1/\mu_{(n)},\ \lambda/\mu_{(1)}(\mu_{(1)} + \lambda)) \ge t^*$.

Remark 2: Also considered for a lower bound for t^*, along the lines of the proof of Theorem 2.4, was the following scheme:

$$U = \prod_{i=1}^{n} g_i(t), \quad \text{where instead} \quad g_i(t) = \frac{e^{-(a_i+\lambda)t}}{1-\mu_i t},$$

where $a_1, a_2, \ldots, a_n$ are real variables satisfying the constraints: $\sum_{i=1}^n a_i = \sum_{i=1}^n \mu_i$, $\mu_i \le a_i + \lambda, i = 1, 2, \ldots, n$. Then in the proof of Theorem 2.4 we would obtain instead

$$t_i = \frac{a_i + \lambda - \mu_i}{(a_i + \lambda)\mu_i}, \qquad i = 1, 2, \ldots, n,$$

$$t_L = \min(t_1, t_2, \ldots, t_n).$$

(In Theorem 2.4, we chose $a_i = \mu_i$, $i = 1, 2, \ldots, n$, but this is not optimal!) Thus, the best upper bound for θ corresponds to the best lower bound for t_L, considered as a function of feasible $a_1, a_2, \ldots, a_n$. Hence, we are led to the nonlinear optimization problem

$$\text{Maximize}_{a_1, a_2, \ldots, a_n} \left\{ \min_{1 \le i \le n} \frac{1}{\mu_i} \left(1 - \frac{\mu_i}{a_i + \lambda} \right) \right\} \tag{2.4}$$

subject to the constraints $a_i \ge u_i - \lambda$, $\sum_{j=1}^n a_j = \sum_{j=1}^n \mu_j$. We will solve this problem in Section 4 in one numerical case. The numerical solution of this problem is more difficult than finding the root of (2.3) to obtain the exact t^* value, so we will not pursue this nonlinear optimization problem in this article, since the emphasis is on easily computable bounds. However, problem (2.4) is interesting in its own right. In passing, it should be mentioned that for $n = 2$, a closed-form solution to (2.4) has been obtained. However, for $n = 2$, a closed-form solution to t^* is possible also (details omitted), so it is better to find t^* using (2.3) instead. It is an open problem how to efficiently solve (2.4) for larger n.

THEOREM 2.5: *Suppose n is even, $n = 2k$. Let $\mu_{(1)} \le \mu_{(2)} \le \cdots \le \mu_{(n)}$ denote the values of $\mu_1, \ldots, \mu_n$ ordered from smallest to largest. Let $w_i = \mu_{(i)} + \mu_{(i+1)} + 2\lambda$, $i =$*

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$1, 2, \ldots, n-1$. *Let* $a_i = w_i\mu_{(i)} + \mu_{(i+1)}$, $b_i = 2\mu_{(i)}\mu_{(i+1)} - w_i(\mu_{(i)} + \mu_{(i+1)})$, *and* $c_i = 2\lambda$, $i = 1, 2, \ldots, n-1$. *Let*

$$\hat{t}_i = \frac{-b_i + \sqrt{b_i^2 - 4a_ic_i}}{2a_i}, \quad i = 1, 2, \ldots, n-1\,.$$

Let

$$\hat{t}_L^{(2)} = \min(\hat{t}_1, \hat{t}_2, \ldots, \hat{t}_{n-1})\,.$$

Then $t^* \geq t_L^{(2)}$. *Additionally,* $t_L^{(2)}$ *is as good or better than* $t_L^{(1)}$ *lower bound given in Theorem* 2.4 (*i.e.*, $t_L^{(2)} \geq t_L^{(1)}$).

PROOF: Let

$$h_i(t) = e^{-(\mu_{(i)}+\mu_{(i+1)}+2\lambda)t}\frac{1}{1-\mu_{(i)}t}\frac{1}{1-\mu_{(i+1)}t}, 0 \leq t < \frac{1}{\mu_{(i+1)}}, \qquad i = 1, 3, 5, \ldots, n-1.$$

Then

$$U = U(t; \mu_1, \ldots \mu_n, \lambda) = \prod_{i\in S} h_i(t),$$

where $S = \{1, 3, 5, \ldots, 2k-1\}$. Simple differentiation of $h_i(t)$ shows that $h_i(t)$ is decreasing in t, $0 \leq t \leq \hat{t}_i$. Moreover, $h_i'(t) = 0$ if and only if

$$\frac{\mu_{(i)}}{1-\mu_{(i)}t} + \frac{\mu_{(i+1)}}{1-\mu_{(i+1)}t} = w_i, \qquad i \in S\,.$$

Algebra produces the equivalent equation

$$a_it^2 + b_it + c_i = 0, \qquad i \in S\,. \tag{2.5}$$

So $h_i'(t) = 0$ implies $a_it^2 + b_it + c_i = 0, i \in S$. Now $h_i'(0) = -2\lambda < 0, h_i'(1/\mu_{(i+1)}) = \infty$. Additionally, $a_i > 0$ and $b_i < 0$. Thus, one of the roots of the quadratic equation (2.5) must satisfy $h_i'(t) = 0$ and be positive. Thus, $\hat{t}_i = (-b_i + \sqrt{b_i^2 - 4a_ic_i})/2a_i$, since the other potential root of (2.5) will produce a negative root. So $h_i(t)$ is decreasing on $[0, \hat{t}_L^{(2)}]$, $i \in S$. Since $h_i(t) \geq 0$, $U = U(t, \mu_i, \ldots, \mu_n, \lambda)$ is nondecreasing on $[0, \hat{t}_L^{(2)}]$. Thus, $t^* \geq t_L^{(2)}$ by Theorem 2.3.

To prove $t_L^{(2)} \geq t_L^{(1)}$, let

$$R_i(t) = \frac{\mu_{(i)}}{1-\mu_{(i)}t} + \frac{\mu_{(i+1)}}{1-\mu_{(i+1)}t} - w_i, \qquad i \in S\,.$$

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Let

$$\tilde{t}_i = \frac{\lambda}{\mu_{(i)}(\mu_{(i)} + \lambda)}, \qquad \tilde{t}_{i+1} = \frac{\lambda}{\mu_{(i+1)}(\mu_{(i+1)} + \lambda)}, \qquad i \in S\,.$$

Then

$$\begin{aligned} R_i(\tilde{t}_i) &= \mu_{(i)} + \lambda + \frac{\mu_{(i+1)}}{1 - \mu_{(i+1)}\tilde{t}_i} \\ &\geq \mu_{(i)} + \lambda + \frac{\mu_{(i+1)}}{1 - \mu_{(i+1)}\tilde{t}_{i+1}} = w_i, \qquad i = 1, 2, \ldots, n-1\,. \end{aligned}$$

Similarly, $R_i(\tilde{t}_i) \leq w_i$. Since $R_i(\hat{t}_i) = w_i$ and $R_i(t)$ is increasing in t, we must have

$$\tilde{t}_{i+1} \leq \hat{t}_i \leq \tilde{t}_i, \qquad i = 1, 2, \ldots, n-1\,.$$

So $\min(\hat{t}_1, \hat{t}_2, \ldots, \hat{t}_{n-1}) \geq \min(\tilde{t}_1, \ldots, \tilde{t}_n) = \tilde{t}_n$; that is,

$$t_L^{(2)} \geq t_L^{(1)}\,.$$

This completes the proof of Theorem 2.5. ■

Remark 2.6: If n is odd, we can also produce an improved lower bound for t^*. We would use instead an additional

$$h_n(t) = e^{-(\mu_{(n)}+\lambda)t}\frac{1}{1 - \mu_{(n)}t} \text{ and}$$

$$U = \left(\prod_{i \in S} h_i(t)\right) h_n(t)$$

instead. Then $\hat{t}_L^{(2)} = \min(\hat{t}_1, \ldots, \hat{t}_{n-1}, \hat{t}_n)$, where $\hat{t}_n = \lambda/\mu_{(n)}(\mu_{(n)} + \lambda)$.

Remark 2.7: Note $U_1 = U(t_L^{(1)}, \mu_1, \ldots, \mu_n, \lambda)$ and $U^{(2)} = U(t_L^{(2)}, \mu_1, \ldots, \mu_n, \lambda)$ are both upper bounds for θ of closed form with $0 < U_2 \leq U_1 < 1$.

Next, we consider some upper bounds that also use the information about second moments of X_i or, equivalently, the variances of X_i. First, we need to define the NBRUE subclass of reliability distributions.

DEFINITION 2.8: *A lifetime random variable X with finite mean $E(X)$ is said to be new is better than renewal used in expectation (NBRUE) if for all $w > 0$,*

$$\int_w^\infty \int_u^\infty \bar{F}(y)\, dy\, du \leq E(X) \int_w^\infty \bar{F}(u)\, du\,.$$

It can be shown that an NBUE distribution is an NBRUE distribution, but the converse is false, in general. See Bhattacharje, Abouammoh, Ahmed, and Barry [5] and Ahmad and Mugdadi [1] for example.

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To derive a new upper bound for θ using both means and second moments of X_i, we need the following lemma.

LEMMA 2.9: *If X has an NBRUE distribution and if the second moment $\mu_2' = E(X^2) < \infty$, then the moment-generating function $M_X(t) = E(e^{tX})$ exists for all t with $0 \le t < 1/\mu_X$, where $\mu_X = E(X)$. Additionally,*

$$M_X(t) \le \frac{2 + \mu_2' t^2 - 2(\mu_X)^2 t^2}{2(1 - t\mu_X)}, \qquad 0 \le t < \frac{1}{\mu_X}.$$

PROOF: See Theorem 2.2 of Ahmad and Mugdadi [1]. ■

Remark 2.10: The condition $t \ge 0$ is not stated in Theorem 2.2 of Ahmad and Mugdadi [1] but is clearly necessary since the numerator of the upper bound of Lemma 2.9 can approach $-\infty$ as t approaches $-\infty$, since NBUE distributions must have $\mu_2' \le 2(\mu_X)^2$, with strict inequality possible.

THEOREM 2.11: *Let $X_1, X_2, \ldots, X_n$ be independent NBRUE lifetime random variables with finite means $\mu_i = E(X_i)$ and finite second moments $m_i = E(X_i^2)$, $i = 1, 2, \ldots, n$. Then for all t with $0 \le t < 1/\mu_{(n)}$, we have*

$$\theta \le e^{-n(\mu+\lambda)t} \prod_{i=1}^{n} \left(\frac{2 + m_i t^2 - 2\mu_i^2 t^2}{2 - 2\mu_i t} \right). \tag{2.6}$$

PROOF: Mimic the proof of Theorem 2.2 using Lemma 2.9 in a straightforward way. ■

The determination of an optimal value of t in (2.6) is not trivial. We now discuss a procedure for doing this. Let t^{**} denote such an optimal value of t minimizing the upper bound

$$\begin{aligned} B &= B(t, \mu_1, \ldots, \mu_n, \ m_1, \ldots, m_n, \lambda) \stackrel{\text{def}}{=} B(t) \\ &= e^{-n(\mu+\lambda)t} \prod_{i=1}^{n} \left(\frac{2 + m_i t^2 - 2\mu_i^2 t^2}{2 - 2\mu_i t} \right). \end{aligned} \tag{2.7}$$

THEOREM 2.12: *There is a unique value $t = t^{**}$ minimizing $B(t, \mu_1, \ldots, \mu_n, m_1, \ldots, m_n, \lambda)$. Additionally, t^{**} is the unique solution of*

$$\sum_{i=1}^{n} \frac{2m_i t - m_i \mu_i t^2 - 4\mu_i^2 t + 2\mu_i^3 t^2 + 2\mu_i}{(2 + m_i t^2 - 2\mu_i^2 t^2) \cdot (1 - \mu_i t)} = n(\mu + \lambda). \tag{2.8}$$

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PROOF: Differentiating Ln B with respect to t in (2.7) shows that t^{**} is a solution to (2.8), since $B'(0) = -n\lambda < 0$ and $B'(1/\mu_{(n)}) = \infty$. Thus, $B'(t^{**}) = 0$ and t^{**} satisfies (2.8). To prove uniqueness of a solution to (2.8), let

$$P_i(t) = \frac{2m_i t - m_i\mu_i t^2 - 4\mu_i^2 t + 2\mu_i^3 t^2 + 2\mu_i}{1 - \mu_i t},$$

$$Q_i(t) = \frac{1}{2 + m_i t^2 - 2\mu_i^2 t^2}.$$

Let $L(t) = \sum_{i=1}^{n} P_i(t)Q_i(t)$. Then (2.8) can be written as $L(t) = n(\mu + \lambda)$. Let us show that $L(t)$ is strictly increasing in t, $0 \le t < 1/\mu_{(n)}$. Clearly, $Q_i(t) \ge 0$ and $Q_i(t)$ is (strictly) increasing in t. Let us prove that $P_i(t)$ is positive and increasing in t. This will then be sufficient to prove that $L(t)$ is strictly increasing in t, since $L(0) = n\mu$, $L(1/\mu_{(n)}) = \infty$ and $n\mu < n(\mu + \lambda) < \infty$. Now, $P_i'(t) = 0$ if and only if

$$(-2m_i + 4\mu_i^2) + (2m_i\mu_i - 4\mu_i^3)t + (2\mu_i^4 - m_i\mu_i^2)t^2 = 0,$$

which can be written in the form ($\alpha_i = -2m_i + 4\mu_i^2 \ge 0$)

$$\left(\frac{1}{2}\mu_i^2\alpha_i\right)t^2 + (-\alpha_i\mu_i)t + (\alpha_i) = 0.$$

This quadratic has no real roots if $\alpha_i > 0$. If $\alpha_i = 0$, $P_i(t)Q_i(t) = (1/1 - \mu_i t)$ is clearly strictly increasing and positive, $i = 1, 2, \ldots, n$. So $P_i'(t)$ is strictly of one sign, $0 \le t < 1/\mu_i$. Since $P_i(0) = 2\mu_i > 0$ and $P_i'(0) = 2(m_i - \mu_i^2) \ge 0$, it follows that $P_i(t)$ is increasing and positive on $[0, 1/\mu_i)$. Thus, $L(t)$ is strictly increasing and positive on $[0, 1/\mu_{(n)})$. This completes the proof. ■

Next, we show that the upper bound of Theorems 2.11 and 2.12 using $t = t^{**}$ is at least as good as the upper bound in Theorem 2.2 using $t = t^*$, if X_i is NBUE.

THEOREM 2.13: *Let X_i be NBUE. Under the assumptions of Theorem 2.2, we have $B(t^{**}) \le U(t^*)$, where $U(t)$ and $B(t)$ are given by* (2.2) *and* (2.7), *respectively.*

PROOF: First, note that

$$\frac{2 + m_i t^2 - 2\mu_i^2 t^2}{2 - 2\mu_i t} \le \frac{1}{1 - \mu_i t}$$

holds, since $m_i \le 2\mu_i^2$ for any NBUE distribution, $i = 1, 2, \ldots, n$. Thus, $B(t) \le U(t)$ holds for all t with $0 \le t < 1/\mu_{(n)}$. Taking minima of both $B(t)$ and $U(t)$ with respect to t, we obtain $B(t^{**}) \le U(t^*)$. ■

THEOREM 2.14: *Suppose $X_1, X_2, \ldots, X_n$ are independent NBUE lifetime random variables with finite means $\mu_i = E(X_i) < \infty$ and finite second moments $m_i = E(X_i^2) < \infty$, $i = 1, 2, \ldots, n$. Let t^* and t^{**} be as given in Theorems 2.3 and 2.12, respectively. Then $t^{**} \ge t^*$.*

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PROOF: Let

$$S_1(t) = \sum_{i=1}^{n} \frac{\mu_i}{1-\mu_i t},$$

$$S_2(t) = \sum_{i=1}^{n} \left[\frac{2m_i t - m_i\mu_i t^2 - 4\mu_i^2 t + 2\mu_i^3 t^2 + 2\mu_i}{(2 + m_i t^2 - 2\mu_i^2 t^2)(1-\mu_i t)} \right].$$

Then $S_1(t^*) = S_2(t^{**}) = n(\mu+\lambda)$. Let $Z(t) = S_2(t) - S_1(t)$. Then $Z(0) = S_2(0) - S_1(0) = n\mu - n\mu = 0$. Additionally, straightforward but messy differentiation gives, after cancellation and simplification,

$$Z'(t) = S_2'(t) - S_1'(t) = \sum_{i=1}^{n} \frac{-2[(m_i - 2\mu_i^2)^2 t^2 + (4\mu_i^2 - 2m_i)]}{(-2 - m_i t^2 + 2\mu_i^2 t^2)^2} \le 0,$$

since each term in the sum is nonpositive and since $m_i \le 2\mu_i^2$ for the NBUE distribution of X_i, $i = 1, 2, \ldots, n$. Thus, $Z'(t) \le 0$, $0 \le t < 1/\mu_{(n)}$. Combined with $Z(0) = 0$, we see that $Z(t) \le 0$, $0 \le t < 1/\mu_{(n)}$, upon application of the mean value theorem. Since $S_1(0) = S_2(0) = n\mu$ and $Z'(t) \le 0$, we have $S_2(t) \le S_1(t)$, $0 \le t < 1/\mu_{(n)}$. So $n(\mu + \lambda) = S_2(t^{**}) \le S_1(t^{**})$. Thus, $S_1(t^{**}) \ge n(\mu+\lambda)$. However, $S_1(t^*) = n(\mu+\lambda)$. So $S_1(t^{**}) \ge S_1(t^*)$. Since $S_1(t)$ is increasing in t, we must have $t^{**} \ge t^*$. This completes the proof. ■

Remark 2.15: From Theorem 2.14 above, the lower bounds $t_L^{(1)}$ and $t_L^{(2)}$ of Theorems 2.4 and 2.5, respectively, are also lower bounds for t^{**}. Thus, the determination of t^{**} is easily accomplished computationally on a bounded interval for t (since $t^{**} < 1/\mu_{(n)}$).

Next, we present two upper bounds based upon application of the bound of Klefsjo [13].

Applying the Klefsjo [13] upper bound to $\bar{X}$, we obtain

$$P(\bar{X} \ge \mu + \lambda) \le e^{-\lambda/\mu}, \qquad \lambda > 0. \tag{2.9}$$

Note that (2.9) does not use the *individual* Klefsjo bound

$$P(X_i \ge \mu_i + \lambda) \le e^{-\lambda/\mu_i}, \qquad \lambda > 0, \quad i = 1, 2, \ldots, n. \tag{2.10}$$

Note that (2.10) states that $X_i - \mu_i \overset{\text{st}}{\le} Y_i$, $i = 1, 2, \ldots, n$, where st means "stochastically smaller than," and $Y_1, \ldots, Y_n$ are independent exponential random variables with means $E(Y_i) = \mu_i$, $i = 1, 2, \ldots, n$. Thus, $\bar{X} - \mu \overset{\text{st}}{\le} \bar{Y} = (\sum_{i=1}^{n} Y_i)/n$ and we obtain

$$P(\bar{X} \ge \mu + \lambda) \le P\left(\sum_{i=1}^{n} Y_i \ge n\lambda \right). \tag{2.11}$$

The right-hand side of (2.11) can be computed in several ways. We may apply the results and algorithms in Mathai [16] and the papers referenced therein. We will see that neither (2.9) nor (2.11) is superior to the other in all cases.

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Remark 2.16: Suppose each X_i has an exponential distribution with possibly different means in Theorems 2.2–2.5. By Theorem 1.1, the upper bound of Theorems 2.2 and 2.3 is "asymptotically" as $n \to \infty$, the "best possible" bounds, since the limiting value, in Theorem 1.1, of $[P(W_n > na_n)]^{1/n}$ equals $\exp(c_0(h) - ha)$, which is easily seen to be the $(1/n)^{\text{th}}$ power of the bound discussed in Theorems 2.2 and 2.3 with $a = \mu + \lambda$ and $h = t^*$ of Theorem 2.3, where $c_0(h)$ has been replaced by $c_0(t^*) = \prod_{i=1}^{n}((1 - \mu_i t)^{-1})$ computationally, since n must be finite in applications. Of course, if all means are equal, we obtain the bound of Chernoff's theorem given in Section 1. Additionally, the bounds of Theorems 2.2–2.5 are exponentially decaying as is the case with most large deviation bounds of extreme upper tail probabilities. Thus, the new bounds will be very good for large n and in the extreme right tail, especially for distributions that are at least somewhat exponential. These conclusions and observations will be borne out by the numerical examples of Section 4.

In the next section we will numerically compare all bounds discussed in this article. We are especially interested in upper bounds for $\theta = P(\bar{X} \geq \mu + \lambda)$ in the important special case of "large" λ and small θ, since extreme right tail probabilities are very important in practical applications.

3. LOWER BOUNDS

In this section, we will give lower bounds for $\alpha = P(\bar{X} \geq \mu - \lambda)$, where $0 \leq \lambda \leq \mu$, $\mu = (\sum_{i=1}^{n} \mu_i)/n$. First, we need the following lemma found in Ahmad and Mugdadi [1, 578].

LEMMA 3.1: *Let X have an NBUE lifetime distribution with $E(X) < \infty$. Let $M_X^*(\beta) = E(e^{-\beta X})$ be the Laplace transform. Then*

$$M_X^*(\beta) \leq \frac{1}{1 + \beta \cdot E(X)}, \qquad \beta \geq 0.$$

Theorem 3.2 presents a family of lower bounds for α.

THEOREM 3.2: *Let $X_1, X_2, \ldots, X_n$ be independent NBUE lifetime random variables with finite means $E(X_i) = \mu_i$, $i = 1, 2, \ldots, n$. Let $\mu = \bar{\mu} = (\sum_{i=1}^{n} \mu_i)/n$. Then*

$$\alpha \geq 1 - \left[(\exp(-n\lambda\beta + n\mu\beta)) \left(\prod_{i=1}^{n} \frac{1}{1 + \beta\mu_i} \right) \right]$$

$$\stackrel{\text{def}}{=} L(\beta, \mu_1, \ldots, \mu_n, \lambda) = L(\beta). \tag{3.1}$$

PROOF: By Lemma 3.1, $M_{X_i}^*(\beta) \leq 1/(1 + \mu_i\beta)$, $\beta \geq 0$, $i = 1, 2, \ldots, n$. So $1 - \alpha \leq P(\bar{X} \leq \mu - \lambda) = P\left(\sum_{i=1}^{n} X_i \leq n(\mu - \lambda)\right)$. Let χ be the characteristic (indicator)

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function on $[0, \infty)$. Then

$$\begin{aligned}
\mathrm{Ln}(1-\alpha) &= \mathrm{Ln}\left[P\left(\sum_{i=1}^{n} X_i \le n(\mu-\lambda)\right)\right] \\
&= \mathrm{Ln}\left(E\left[\chi\left(-n\lambda + n\mu - \sum_{i=1}^{n} X_i\right)\right]\right) \\
&\le \mathrm{Ln}\, E\left[\exp\left(\beta\left(-n\lambda + n\mu - \sum_{i=1}^{n} X_i\right)\right)\right] \\
&= n\beta(\mu-\lambda) + \sum_{i=1}^{n} Ln(M^*_{X_i}(\beta))
\end{aligned}$$

or

$$\mathrm{Ln}(1-\alpha) \le n\beta(\mu-\lambda) - \sum_{i=1}^{n} \mathrm{Ln}(1+\beta\mu_i).$$

Solving for α, we obtain

$$\begin{aligned}
\alpha &\ge 1 - \exp\left(-n\lambda\beta + n\mu\beta - \sum_{i=1}^{n} \mathrm{Ln}(1+\beta\mu_i)\right) \\
&= 1 - \left[(\exp(-n\lambda\beta + n\mu\beta))\left(\prod_{i=1}^{n} \frac{1}{1+\beta\mu_i}\right)\right],
\end{aligned}$$

the desired result. ■

The next question to consider is how to optimally select the $\beta \ge 0$ parameter. We will also give bounds on this optimal value of β, denoted by β^*.

THEOREM 3.3: *In Theorem 3.2, β^*, the choice of $\beta \ge 0$ maximizing the lower bound $L(\beta)$ in (3.1), is unique and satisfies*

$$\sum_{i=1}^{n} \frac{\mu_i}{1+\beta\mu_i} = n(\mu-\lambda). \tag{3.2}$$

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PROOF: From the proof of Theorem 3.2, we have

$$\mathrm{Ln}(1 - L(\beta)) = -n\lambda\beta + n\mu\beta - \sum_{i=1}^{n} \mathrm{Ln}(1 + \beta\mu_i).$$

So $L(0) = 0$ and $L(\infty) = -\infty$. Now, let $G(\beta) = Ln(1 - L(\beta))$. Then

$$G'(\beta) = -n\lambda + n\mu - \sum_{i=1}^{n} \frac{\mu_i}{1 + \beta\mu_i} = 0$$

if and only if (3.2) holds. Additionally, $G'(0) = -n\lambda < 0$ and $G'(\infty) = n\mu - n\lambda > 0$. Since $G''(\beta) > 0$, (3.2) has a unique solution. Thus, $L'(\beta) = G'(\beta)(L(\beta) - 1)$ is maximized if and only if $G'(\beta) = 0$, which occurs at the unique solution to (3.2). This completes the proof. ■

Next, we present upper and lower bounds for β^* that maximizes the lower bound $L(\beta)$ in Theorem 3.3.

THEOREM 3.4: *Under the condition of Theorem* 3.2*, we have the following:*

(a) a lower bound for β^* *is* β_L*, where*

$$\beta_L = \frac{\lambda}{\mu_{(n)}(\mu - \lambda)}, \quad \textit{where } \mu_{(n)} = \max(\mu_1, \ldots, \mu_n); \tag{3.3}$$

(b) an upper bound for β^* *is* β_u*, where*

$$\beta_u = \frac{\lambda}{\mu(\mu - \lambda)}. \tag{3.4}$$

PROOF: From (3.2) of Theorem 3.3, β^* satisfies

$$R_1 = \frac{1}{n} \sum_{i=1}^{n} \frac{\mu_i}{1 + \beta^*\mu_i} = \mu - \lambda\,. \tag{3.5}$$

Simple algebra shows that β_L satisfies

$$R_2 = \frac{1}{n} \sum_{i=1}^{n} \frac{\mu_i}{1 + \beta_L\mu_{(n)}} = \mu - \lambda\,. \tag{3.6}$$

Since $\mu_i \le \mu_{(n)}$, $R_1 = R_2$, and since $\mu_i/1 + \beta\mu_i$ is decreasing in β, we must have $\beta^* \ge \beta_L$. This proves (a).

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To prove (b), Jensen's inequality gives, since $\mu = \bar{\mu} = (\sum_{i=1}^{n} \mu_i)/n$,

$$\frac{1}{n}\sum_{i=1}^{n} \frac{\mu_i}{1+\beta^*\mu_i} \le \frac{\mu}{1+\beta^*\mu}.$$

From (3.5), we have $\mu - \lambda \le \mu/(1+\beta^*\mu)$. Solving for β^*, we obtain $\beta^* \le \beta_L = \lambda/(\mu(\mu-\lambda))$. This completes the proof of part (b). ■

Remark 3.5: In the particular case that $X_1, X_2, \ldots, X_n$ are i.i.d., or, more generally, when all means are equal ($\mu_1 = \mu_2 = \cdots = \mu_n$), it is easily shown that $\beta^* = \beta_u = \beta_L = \lambda/(\mu(\mu-\lambda))$. In many cases, unless the distributions of the X_is are quite diverse, $\beta^* \approx \beta_u$ is a good approximation, although the corresponding lower bound $L(\beta_u)$ might be zero or negative (trivial). However, it is always the case that $L(\beta_L)$ is a nontrivial bound of closed form ($L(\beta_L) > 0$). This will also be seen in Section 4.

4. NUMERICAL STUDY

In this section we present a numerical study to compare the upper bounds for $\theta = P(\bar{X} \ge \mu + \lambda)$ and to compare the lower bounds for $\alpha = P(\bar{X} \ge \mu - \lambda)$. No single upper/lower bound is superior in all cases to all the other upper/lower bounds. Some definite guidelines will be given, however, concerning in what situations each bound can be epxected to work best.

First, we compare the upper bounds on $\theta = P(\bar{X} \ge \mu + \lambda)$. We are especially interested in large values of λ (small values of θ) in practical statistical applications. The notations for the upper bounds are

$$\begin{aligned} U_1 &= \text{Klefsjo bound (2.9)} \\ U_2 &= \text{Bound (2.11)} \\ U_{\mathrm{CL}} &= \text{Upper bound of Cheng and Lam [6, p. 795] given by} \\ &\qquad M(t,p) \text{ of Theorem 3 applied to } X = \bar{X}. \end{aligned}$$

Recall that, $U(t^*)$, $U(t_L^{(1)})$, and $U(t_L^{(2)})$, where $U(t)$ is given in Theorem 2.2; t^*, $t_L^{(1)}$, and $t_L^{(2)}$ are given in (2.3), Theorem 2.4, and Theorem 2.5, respectively.

Also considered was $B(t^{**})$, where $B(t)$ is given in (2.7) and t^{**} is given in Theorem 2.12. Note that Section 2 theorems give

$$B(t^{**}) \le U(t^*) \le U(t_L^{(2)}) \le U(t_L^{(1)}).$$

Note that $B(t^{**})$ and U_{CL} requires both the means and variances of X_i, whereas the others require only the means of X_i. Additionally, the closed-form upper bounds (those that do not sometimes require numerical solution of a nonlinear equation) are $U_1, U_2, U(t_L^{(1)})$, and $U(t_L^{(2)})$.

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TABLE 1. Upper Bounds.

λ	U_1	U_2	U_{CL}	$U(t_L^{(1)})$	$U(t_L^{(2)})$	$U(t^*)$	$B(t^{**})$
$n = 4$: $(\theta_1, \theta_2) = (1,1), (1,2), (1,3), (1,4)$. $\mu = 0.9214$, $\sigma = 0.2942$							
0.20	0.8049	0.9880	0.6839	0.9226	0.9213	0.9210	0.8658
1.00	0.3378	0.3697	0.0796	0.2543	0.2490	0.2477	0.1627
2.00	0.1141	0.0267	0.0181	0.0189	0.0178	0.0176	0.0086
3.00	0.03855	0.00105	0.00548	0.00088	0.00079	0.00077	0.00031
$n = 4$: $(\theta_1, \theta_2) = (1,1), (2,1), (3,1), (4,1)$							
1.00	0.6703	0.9038	0.6516	0.8541	0.8348	0.8136	0.8136
5.00	0.1353	0.0549	0.0603	0.0962	0.0657	0.0642	0.0642
10.00	0.0183	0.0005	0.0071	0.0019	0.0012	0.0009	0.0009
$n = 20$: $(\theta_1, \theta_2) = (1,2)$ for $X_1, \ldots, X_{20}$, $\mu = 0.8862$, $\sigma = 0.2146$							
0.10	0.8933	0.9999^+	0.5176	0.8883	0.8883	0.8883	0.7538
0.50	0.5688	0.9881	0.0412	0.0967	0.0967	0.0967	0.0367
1.00	0.3236	0.2662	0.0107	0.00058	0.00058	0.00058	0.00004

Generic bounds of Chebyshev type were also considered but perform poorly in cases of small θ (large λ), so we do not present these results here.

Table 1 presents these upper bounds in the case X_i has the Weibull distribution with survivor

$$P(X_i \geq x) = \exp(-(x/\theta_1)^{\theta_2}), \qquad x > 0, \ \ \theta_1 > 0, \ \ \theta_2 > 0.$$

Many other families of survivor functions besides the Weibull were considered, but the conclusions given later for the Weibull distribution hold true in these other families as well. We also give the values of $\mu = \sum_{i=1}^n \mu_i = E(\bar{X})$ and the standard deviation of $\bar{X} = \sigma = (1/n)\sqrt{\sum_{i=1}^n \sigma_i^2}$, where $\sigma_i^2 = m_i - \mu_i^2$, $i = 1, 2, \ldots, n$. The values of t^*, t^{**}, $t_L^{(1)}$, and $t_L^{(2)}$ are available upon request.

For $\lambda = 5.00$, we also computed the upper bound corresponding to nonlinear optimization problem (2.4). This bound was 0.0678 corresponding to $a_1 = -3.78$, $a_2 = 1.98$, $a_3 = 3.06$, and $a_4 = 8.75$ with the (2.4) value of 0.177. So this bound improves on $U(t_L^{(1)}) = 0.0962$ but is not as good as $U(t_L^{(2)}) = 0.0657$, which is also of closed form. In any case, for larger λ, U_2 is the best bound in this case, probably because all pdfs are exponential distributions.

λ	U_1	U_2	U_{CL}	$U(t_L^{(1)})$	$U(t_L^{(2)})$	$U(t^*)$	$B(t^{**})$
$n = 6$: $(\theta_1, \theta_2) = (1,1), (1,2), (1,3), (4,1), (10,1), (20,1)$							
1.00	0.8495	0.9926	0.7667	0.9873	0.9816	0.9699	0.9698
10.00	0.1957	0.1380	0.1159	0.4027	0.3052	0.2214	0.2211
30.00	0.0075	0.0004	0.0036	0.0066	0.0028	0.0015	0.0015

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TABLE 2. Lower Bounds.

λ	L_{CL}	$L(\beta^*)$	$L(\beta_L)$	$L(\beta_u)$
$n = 4$: $(\theta_1, \theta_2) = (1,1), (2,1), (3,1), (2,2)$. $\mu = 1.9431$, $\sigma = 0.9637$				
0.10	0.00	0.00450	0.00450	0.00447
0.20	0.00	0.0200	0.0186	0.0197
0.75	0.3860	0.3085	0.2909	0.3071
1.50	0.7720	0.9301	0.9190	0.9299

Many other combinations of n, (θ_1, θ_2) and λ were used. The above cases are representative of the following conclusions:

1. For small λ, the bound of Cheng and Lam [6], U_{CL}, is best.
2. For large λ, the new bounds discussed in this article are the best, especially for larger values of n.
3. For cases of "diversity" of means and variances and smaller n values, U_2 is best for large λ.
4. For large n, U_2 is very poor. The new bounds are the best for this case for large λ values, an important special case in statistical applications. In this case, all of the new bounds, except $B(t^{**})$, use only the means of X_i and yet still do better than the bounds of Cheng and Lam [6].
5. For small values of n, the bound of Cheng and Lam [6] performs well, even when λ is large.

Next, we present a numerical comparison of the lower bounds discussed in this article (Table 2). Let $\alpha = P(\bar{X} \geq \mu - \lambda)$, $0 < \mu < \lambda$.

Let L_{CL} denote the lower bound $m(t,p)$ given in Theorem 3 of Cheng and Lam [6] applied to $X = \bar{X}$. Let $L(\beta)$ denote the lower bound of Theorem 3.2. We will compare L_{CL} to $L(\beta^*)$, $L(\beta_L)$ and $L(\beta_u)$, where β^*, β_L, and β_u are given in Theorems 3.2 and 3.3.

In this case, the Cheng and Lam [6] lower bound is not applicable (only the trivial lower bound of zero) for smaller values of λ. The L_{CL} bound is trivial for $0 \leq \lambda \leq 0.25$. Additionally, the L_{CL} bound is best and is nontrivial (nonzero) for $0.26 \leq \lambda \leq 0.98$. For $\lambda \geq 0.98$, the new bounds are best. Note that the $L(\beta)$ bounds $L(\beta^*)$ and $L(\beta_L)$ are always nontrivial (nonzero), unlike L_{CL}. Thus, the new lower bounds $L(\beta^*)$ and $L(\beta_L)$ are more generally applicable than L_{CL}. Moreover, they use only the means of the X_is. In addition, $L(\beta_L)$ is of closed form. This case shows that for values of λ with, $\lambda > \mu/2$ (roughly) and $\lambda < \mu$, the new lower bounds $L(\beta^*)$ and $L(\beta_L)$ are better than L_{CL}. If L_{CL} is nontrivial and $\lambda < \mu/2$, then L_{CL} is better than the new lower bounds. This has been numerically verified in tens of other examples considered but not presented here. Many other values of n, λ, and (θ_1, θ_2) were tried and, this seems to be the case. These cases seem to lend support to the claim of Cheng and Lam [6] that their lower bound $m(t,p)$ is more generally applicable than they were able to demonstrate.

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An attempt was made to get a new lower bound that also uses m_i values (or equivalently, the variances of X_i). However, we could not obtain an upper bound on $E(e^{-\beta X_i})$ for $\beta > 0$ in this case.

5. COMMENTS

The connections between large deviation theory and the results of this article have previously been discussed. However, there are also connections of the results in this article to the area of saddlepoint approximations. For pioneering work on saddlepoint approximations, see Daniels [8,9]. For a good work on saddlepoint approximations to distributions of sums of i.i.d. random variables, see Goutis and Case11a [10] or Lugannani and Rice [15]. A good review of saddlepoint approximations is Jensen [11].

The method of saddlepoint approximation to the density of $\bar{X}$ is

$$f_{\bar{X}}(\bar{X}) \approx \left[\frac{1}{2\pi K''(\hat{t}(\bar{X}))}\right]^{1/2} \exp[K(\hat{t}(\bar{X})) - \hat{t}(\bar{X}) \cdot \bar{X}], \tag{5.1}$$

where $K(t)$ is the cumulant generating function (cgf) of $\bar{X}$; that is,

$$K(t) = \mathrm{Ln}(M_{\bar{X}}(t)) = \mathrm{Ln}\left(\prod_{i=1}^{n} M_{X_i}(t)\right),$$

where $M_{X_i}(t)$ is the mgf of X_i and $\hat{t}(\bar{X})$ is the value of t satisfying $K'(t) = \bar{X}$. See, for example, Goutis and Casella [10, Sect. 3]. Approximation (5.1) does not produce an easily computable upper or lower bound to the cumulative distribution function (cdf) of $\bar{X}$; however, upon integration, since as accurate as saddlepoint approximations may be, it is often very difficult to obtain bounds on the approximation errors of saddlepoint approximations.

In this article, the cgf of X_i is unknown, as we only have upper bounds on the mgfs. In the case of only the means of X_i being known, we could substitute the exponential cgf into (5.1), but integration of the resulting density (after normalizing it) will not necessarily produce bounds on the cdf of $\bar{X}$, since (5.1) is only an approximation and error bounds are not easily found. It is an open problem how to use saddlepoint approximations to obtain upper or lower bounds on the cdf of $\bar{X}$ for the reliability classes of distributions considered in this article.

There is an interesting connection between the computation of the upper bound in Theorem 2.3 and the computation of the saddlepoint approximation (5.1). If the X_i random variables are exponential with means μ_i, then the equation $K'(t) = \bar{X}$ has the same solution as the equation

$$\sum_{i=1}^{n} \frac{\mu_i}{1 - \mu_i t} = n(\mu + \lambda) \quad \text{of (2.3)}$$

in Theorem 2.3, if $\bar{X} = \mu + \lambda$; that is, $\hat{t}(\mu + \lambda) = t^*$. This is the only value of $\bar{X}$ that needs to be considered when solving $K'(t) = \bar{X}$ if the cdf of $\bar{X}$ at $\bar{X} = \mu + \lambda$ is desired.

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So the *same* value of t^* is involved in *both* the saddlepoint approximation method and the method of this article in Theorem 2.3. Of course, the saddlepoint method is not needed in this case. In nonexponential NBUE cases, it is easly seen from Theorem 1.1 that the value of $t = t(a)$ minimizing $\inf_{t>0}\{\exp[c_0(t) - ta]\}$ is the same as the value of $t = t^*$ minimizing the nonexponential NBUE version of the upper bound of Theorem 2.3 if $c_0(t) = \psi_n(t)/n$ exactly. Otherwise, these two values of t agree only "asymptotically" as $n \to \infty$. The latter two statements, of course, assume a known mgf (and hence cgf) of X_i. In applications, this is often not the case in a reliability setting in which only membership in a reliability subclass of distributions is known, not the actual distributions of the X_i.

Other connections between large deviation theory and saddlepoint approximations can be found in Lugannani and Rice [15] and Petrov [18, Ch. 8].

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